$$\vec{l} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix},$$

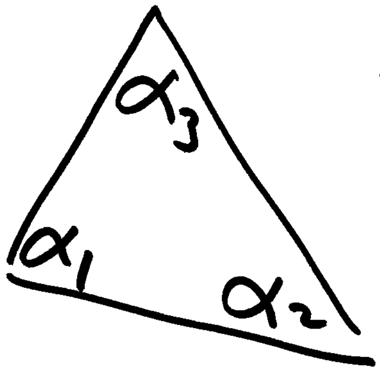
need correction to each observation

$v$ : residual

$$l_i + v_i = \hat{l}_i \quad ✓$$

our sign convention

$$l_i - \cancel{v_i} = \hat{l}_i$$




relate observations, additional unknowns + math model by

**condition equations**

**observations only**

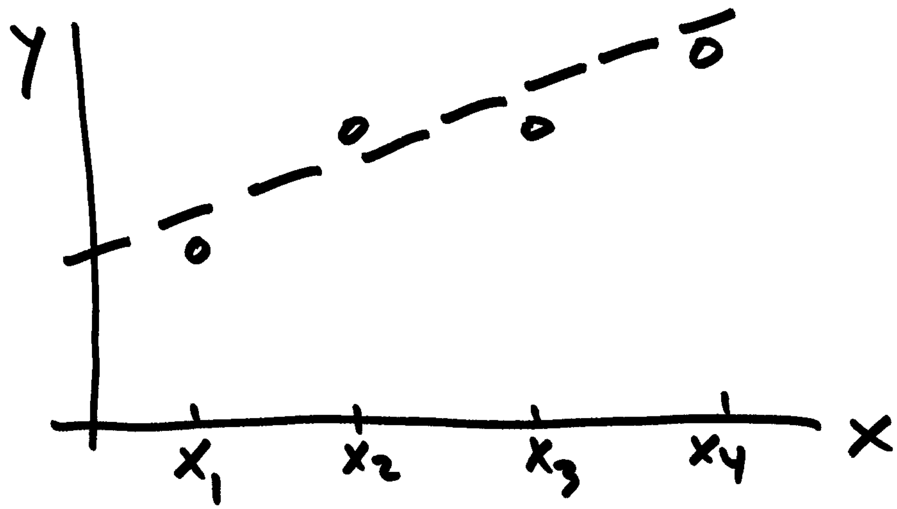
$$c = r = 1$$

$$n = 3$$
$$n_0 = 2$$
$$r = 1$$

$$\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 = 180^\circ$$

if we want new unknowns (parameters)

then **indirect observations**




$n = 4$

$n_0 = 2$

$r = 2$

indirect observations

new unknowns $m, b$

slope = $m$

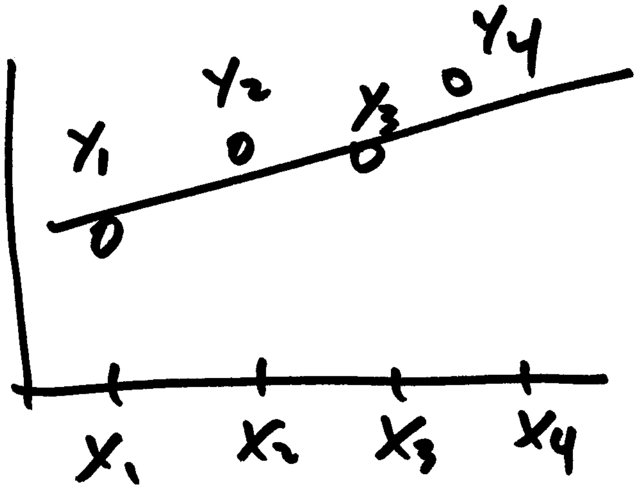
y-intercept = $b$

choose $n_0 = 2$ new parameters

write $n$ condition equations

$l_i + v_i = f(x)$

$$y_1 + v_1 = m \cdot x_1 + b$$
$$y_2 + v_2 = m \cdot x_2 + b$$
$$y_3 + v_3 = m x_3 + b$$
$$y_4 + v_4 = m x_4 + b$$



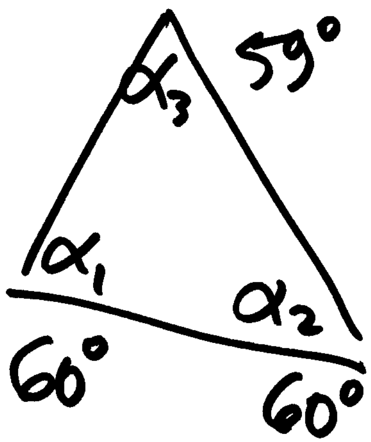

$n = 4$
$n_0 = 2$

$r = 2$

observations only $\quad C = r = 2$

$$\frac{Y_3 - Y_1}{X_3 - X_1} = \frac{Y_2 - Y_1}{X_2 - X_1}$$

$$\frac{Y_4 - Y_1}{X_4 - X_1} = \frac{Y_2 - Y_1}{X_2 - X_1}$$



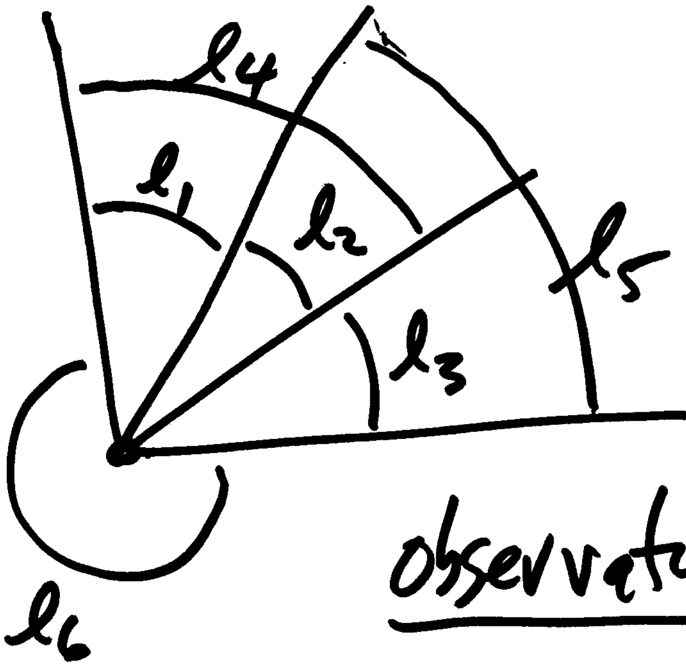

| | | |
|---|---|---|
| $V_1 = 0$ | $V_1 = 0$ | $V_1 = 1/3$ |
| $V_2 = 0$ ✓ | $V_2 = 1$ ✓ | $V_2 = 1/3$ ✓ |
| $V_3 = 1$ | $V_3 = 0$ | $V_3 = 1/3$ |

? which one to choose ?

$$\Phi = V_1^2 + V_2^2 + V_3^2 \longrightarrow \text{minimum}$$

objective function

there are an infinite number of sets of $V_i$ which satisfy the condition equation. to select <u>one</u> <u>unique</u> set of $V_i$ we impose a minimization constraint




$$n = 6$$
$$n_0 = 3$$
$$\overline{\phantom{n_0 = 3}}$$
$$r = 3$$

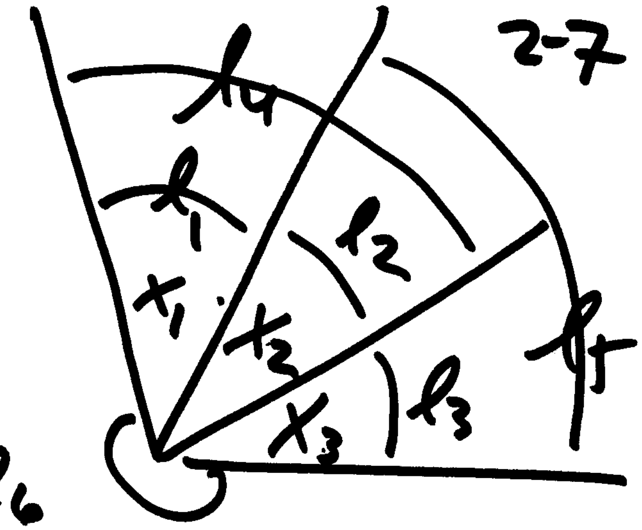
<u>observations only</u>, $c = r = 3$

$$\hat{\ell}_1 + \hat{\ell}_2 + \hat{\ell}_3 + \hat{\ell}_6 = 360^\circ$$

$$\hat{\ell}_1 + \hat{\ell}_2 = \hat{\ell}_4$$

$$\hat{\ell}_2 + \hat{\ell}_3 = \hat{\ell}_5$$

$$\hat{l}_1+\hat{l}_2+\hat{l}_3+\hat{l}_6=360^\circ$$
$$\hat{l}_3+\hat{l}_4+\hat{l}_6=360^\circ$$
$$\hat{l}_1+\hat{l}_5+\hat{l}_6=360^\circ$$




3 parameters : $x_1, x_2, x_3$

$n=6,\ \underline{n_0=3},\ r=3$

select $n_0=3$ parameters

ind. obs. $\Rightarrow$ $n$ conditions

$$l_1+v_1=x_1$$
$$l_2+v_2=x_2$$
$$l_3+v_3=x_3$$
$$l_4+v_4=x_1+x_2$$
$$l_5+v_5=x_2+x_3$$
$$l_6+v_6=360^\circ-x_1-x_2-x_3$$

# indirect observations (longhand method)

1. analyze prob. $n, n_0, r$
2. choose $n_0$ parameters (must define model)
3. write $n$ cond. equations, of form

$$l_i + V_i = f(x)$$

$$v_i = \underline{f(x) - l_i}$$

4. $\Phi = V_1^2 + V_2^2 + \cdots + V_n^2$, $\Phi = \sum_{i=1}^{n} V_n^2$
5. differentiate $\Phi$ w.r.t. parameters, $= 0$
6. solve $n_0 \times n_0$ system, for parameter
7. substitute into cond. equ., solve for $V_i$

8. $l_i + v_i = \hat{l}_i$ adjusted obs.

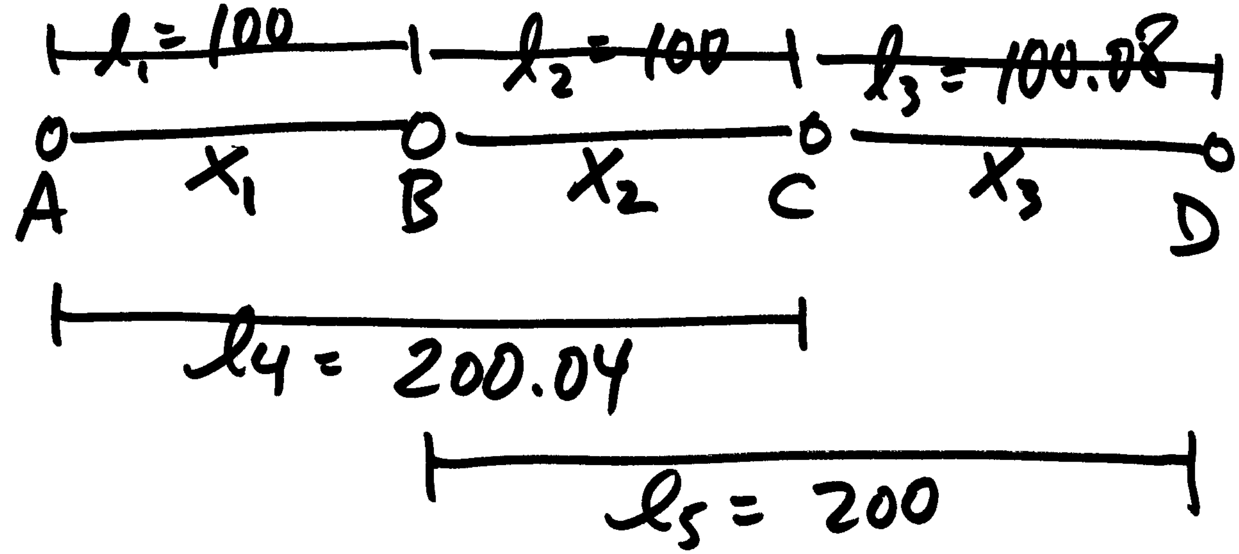
9. implicit: are the corrections **reasonable**?

# Ex 3-4

$n = 5$

$n_0 = 3$

$r = 2$




$l_1 + v_1 = X_1 \quad , \quad v_1 = X_1 - 100$

$l_2 + v_2 = X_2 \quad , \quad v_2 = X_2 - 100$

$l_3 + v_3 = X_3 \quad , \quad v_3 = X_3 - 100.08$

$l_4 + v_4 = X_1 + X_2 , \quad v_4 = X_1 + X_2 - 200.04$

$l_5 + v_5 = X_2 + X_3 , \quad v_5 = X_2 + X_3 - 200$

$$\Phi = V_1^2 + V_2^2 + V_3^2 + V_4^2 + V_5^2$$

$$\Phi = (x_1 - 100)^2 + (x_2 - 100)^2 + (x_3 - 100.08)^2 + (x_1 + x_2 - 200.04)^2 + (x_2 + x_3 - 200)^2$$

$$\frac{\partial \Phi}{\partial x_1} = \cancel{2}(x_1 - 100) + \cancel{2}(x_1 + x_2 - 200.04) = 0$$

$$\frac{\partial \Phi}{\partial x_2} = \cancel{2}(x_2 - 100) + \cancel{2}(x_1 + x_2 - 200.04) + \cancel{2}(x_2 + x_3 - 200) = 0$$

$$\frac{\partial \Phi}{\partial x_3} = \cancel{2}(x_3 - 100.08) + \cancel{2}(x_2 + x_3 - 200) = 0$$

$$2x_1 + x_2 = 300.04$$

$$x_1 + 3x_2 + x_3 = 500.04$$

$$x_2 + 2x_3 = 300.08$$

**Normal Equations**

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 300.04 \\ 500.04 \\ 300.08 \end{bmatrix}$$

Symmetric

$$Nx = t$$

$$x = N^{-1}t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 100.025 \\ 99.990 \\ 100.045 \end{bmatrix}$$