Expression of conic section in polar coordinates

defn of conic section

$$\overline{OP} = e\overline{PD}$$

$e = 1$; parabola

$e < 1$; ellipse

$e > 1$; hyperbola

$$r = e\left(\overline{OD} - \overline{OQ}\right)$$

$$r = e(\overline{OD} - r\cos\theta)$$

$$r = e\overline{OD} - er\cos\theta$$

$e\overline{OD}$ constant, call it p

$$r = p - er\cos\theta$$

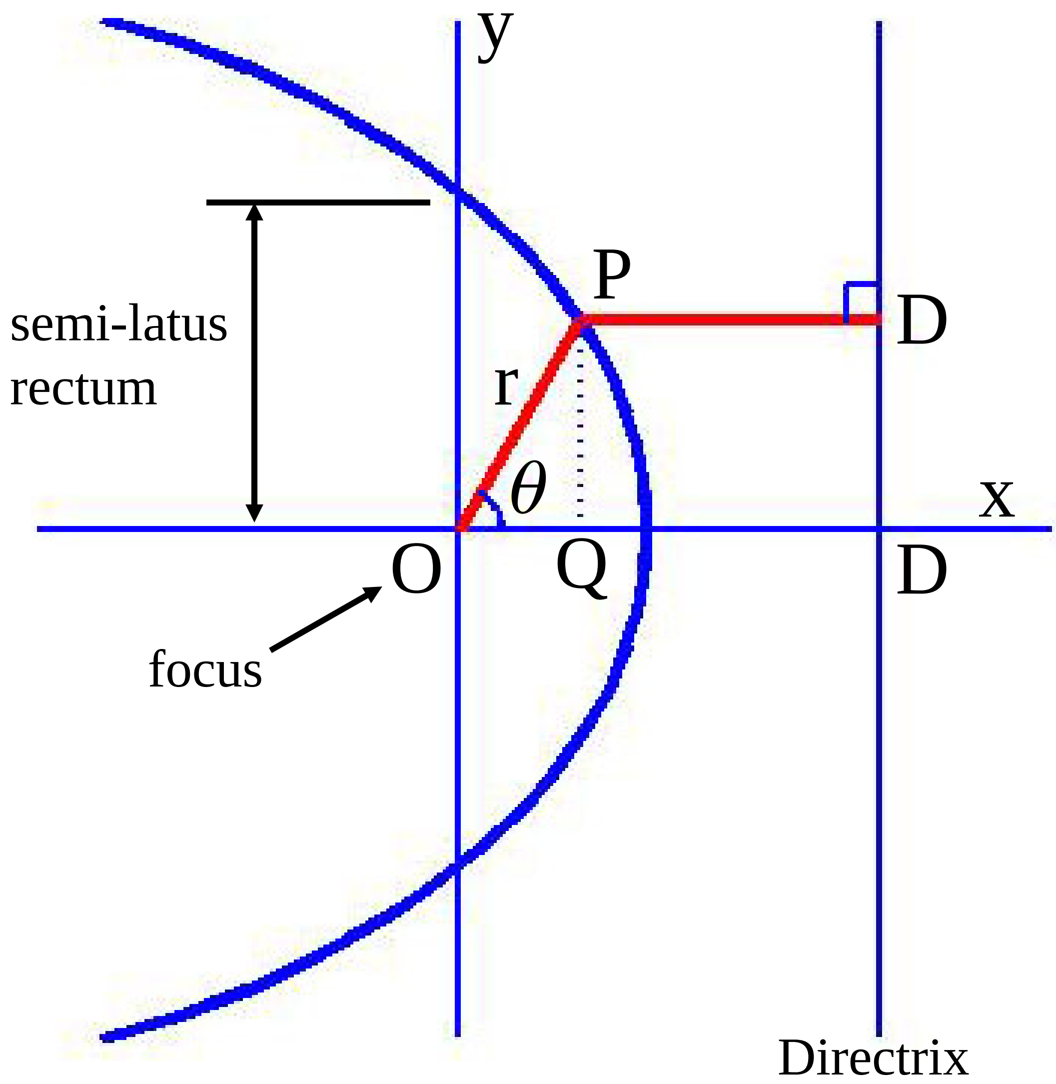
$$r + er\cos\theta = p$$

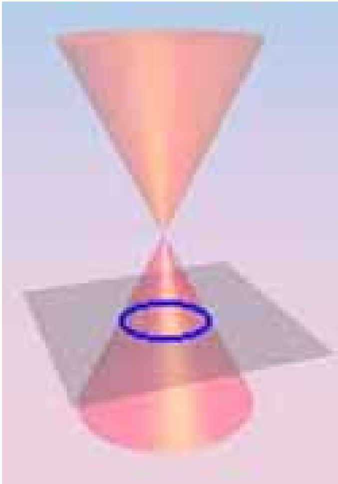
$$r\left(1 + e\cos\theta\right) = p$$

$$r = \frac{p}{1 + e\cos\theta} \quad [1]$$

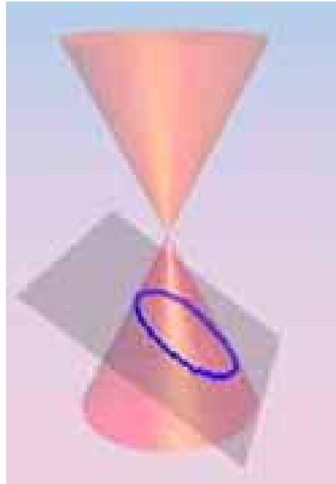
e : eccentricity

p : semi-latus rectum

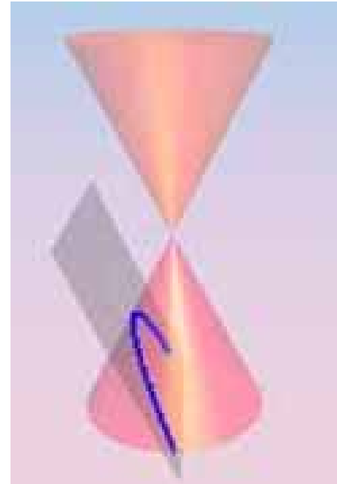




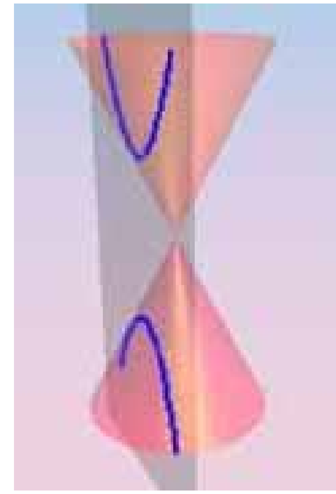
circle



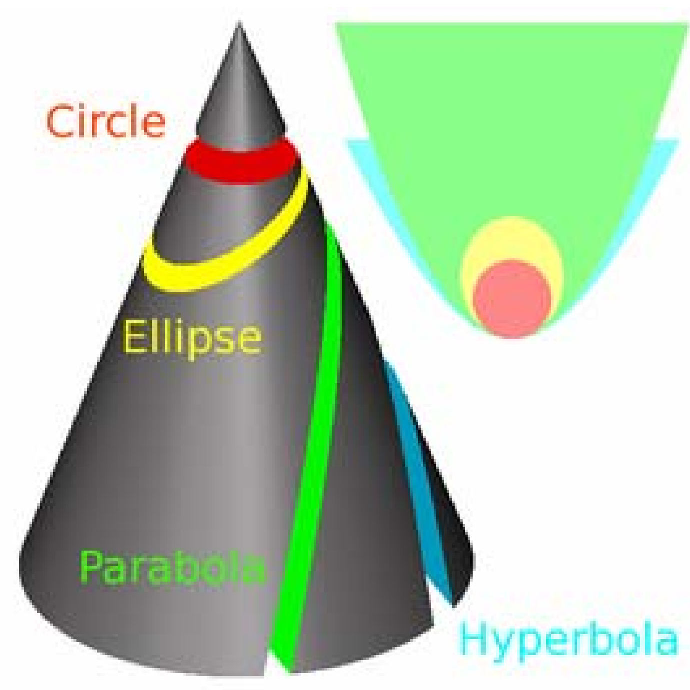
ellipse



parabola



hyperbola



Two Obscure Vector Identities

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \qquad [2]$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \qquad [3]$$

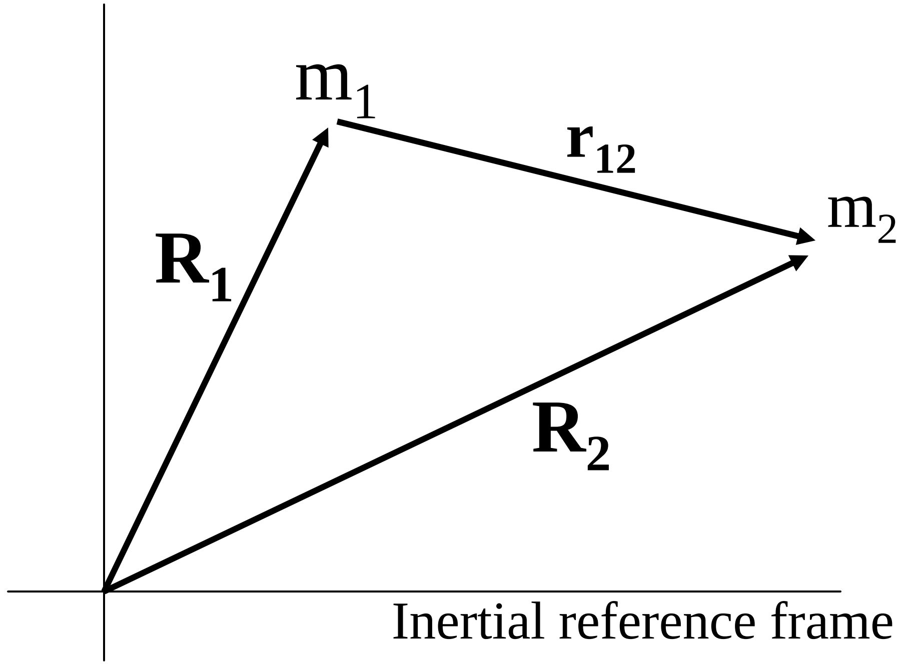
You can prove these by expanding using definitions of cross product and inner / dot product, yielding equal expressions

Newton, 2nd law of motion

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\mathbf{a} \qquad [4]$$

Newton, law of gravitation

$$\mathbf{F} = \frac{Gm_1 m_2}{r^2}\left(\frac{\mathbf{r}}{r}\right) \qquad [5]$$



For the 2-body problem,

$$\mathbf{F_1} = \frac{Gm_1 m_2}{r_{12}^3}\mathbf{r_{12}} = m_1\left[\frac{Gm_2}{r_{12}^3}\mathbf{r_{12}}\right] = m_1\ddot{\mathbf{R}}_\mathbf{1} \qquad [6]$$

$$\mathbf{F}_2 = \frac{Gm_1 m_2}{r_{12}^3}\mathbf{r_{21}} = m_2\left[\frac{Gm_1}{r_{12}^3}\mathbf{r_{21}}\right] = m_2\ddot{\mathbf{R}}_\mathbf{2} \qquad [7]$$

Next take the difference between the accelerations at 2 and 1 from equations 6 and 7.

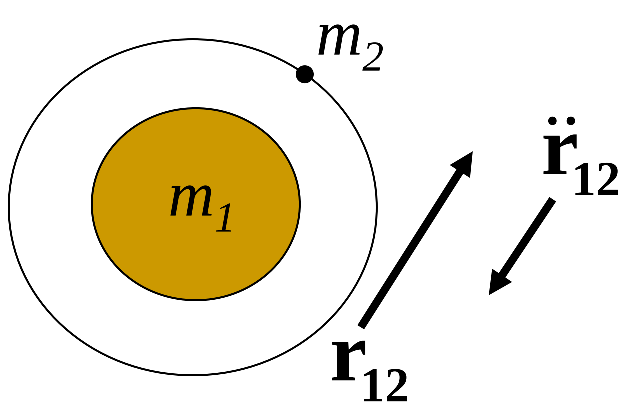
$$\ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1 = \frac{Gm_1}{r_{12}^3}\mathbf{r}_{21} - \frac{Gm_2}{r_{12}^3}\mathbf{r}_{12} \qquad [8]$$

Taking the difference in acceleration and changing sign of vector

$$\ddot{\mathbf{r}}_{12} = -\frac{Gm_1}{r_{12}^3}\mathbf{r}_{12} - \frac{Gm_2}{r_{12}^3}\mathbf{r}_{12} \qquad [9]$$

Now all vectors are *relative*, rearrange

$$\ddot{\mathbf{r}}_{12} = -\frac{G(m_1 + m_2)}{r_{12}^3}\mathbf{r}_{12} \qquad [10]$$



But, $m_1 >> m_2$ for artificial earth satellite

$$G(m_1 + m_2) \approx Gm_1 = \mu \qquad [11]$$

Dropping the subscripts, acceleration of satellite in earth gravity field

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \qquad [12]$$

This is 2-body equation of relative motion

Constants and Notation

$$m_1 = M_{earth} = 5.974 \times 10^{24}\, kg$$

$$m_2 = m_{sat}$$

$$G = 6.67259 \times 10^{-11}\, m^3 kg^{-1} s^{-2}$$

Product is known to greater precision than individual factors,

$$\mu = 398600.4405 km^3 s^{-2}$$

$$\mu = 3.986004405 \times 10^{14}\, m^3 s^{-2}$$

The 2-body relative equation of motion (12) can be manipulated to yield a trajectory equation, we will show it to be a conic section

The angular momentum vector, **h**

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} \qquad [13]$$

Show the time derivative of **h** is zero, therefore **h** is constant

$$\frac{d}{dt}\mathbf{h} = \dot{\mathbf{h}} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} \qquad [14]$$

Cross product of parallel vectors is zero, substitute from 12

$$\dot{\mathbf{h}} = \mathbf{0} - \frac{\mu}{r^3}\mathbf{r} \times \mathbf{r} \qquad [15]$$

again, cross product of parallel vectors,

$$\dot{\mathbf{h}} = \mathbf{0} \qquad [16]$$

Constant **h** implies that relative motion takes place in a plane, with **h** normal to that plane

Find the time derivative of unit vector in the direction of **r**,

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{r\dot{\mathbf{r}} - \dot{r}\mathbf{r}}{r^2} \qquad [17]$$

quotient rule

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$$

But,

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \frac{d}{dt}\left(r^2\right) \qquad [18]$$

$$2\dot{\mathbf{r}}\cdot\mathbf{r} = 2r\dot{r} \qquad [19]$$

$$\dot{\mathbf{r}}\cdot\mathbf{r} = r\dot{r} \qquad [20]$$

$$\dot{r} = \frac{\dot{\mathbf{r}}\cdot\mathbf{r}}{r} \qquad [21]$$

Plug this result into equation 17

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{1}{r^3}\left(r^2\dot{\mathbf{r}} - (\dot{\mathbf{r}}\cdot\mathbf{r})\mathbf{r}\right) \qquad [22]$$

Now consider $\frac{d}{dt}\left(\dot{\mathbf{r}} \times \mathbf{h}\right)$

$$\frac{d}{dt}\left(\dot{\mathbf{r}} \times \mathbf{h}\right) = \ddot{\mathbf{r}} \times \mathbf{h} + \dot{\mathbf{r}} \times \dot{\mathbf{h}} \qquad [23]$$

second term above is zero since $\dot{\mathbf{h}}$ is zero, substitute from eqn. 13

$$= \ddot{\mathbf{r}} \times \left(\mathbf{r} \times \dot{\mathbf{r}}\right) \qquad [24]$$

Substitute expression for $\ddot{\mathbf{r}}$ from equation 12

$$= -\frac{\mu}{r^3}\mathbf{r} \times \left(\mathbf{r} \times \dot{\mathbf{r}}\right) \qquad [25]$$

Evaluate this triple cross product by vector identity in equation 2

$$= -\frac{\mu}{r^3}\left(\left(\mathbf{r} \cdot \dot{\mathbf{r}}\right)\mathbf{r} - r^2\dot{\mathbf{r}}\right) \qquad [26]$$

Move the minus sign inside parenthesis

$$= \frac{\mu}{r^3}\left(r^2\dot{\mathbf{r}} - \left(\mathbf{r} \cdot \dot{\mathbf{r}}\right)\mathbf{r}\right) \qquad [27]$$

This looks like the right side of equation 22, replace by left side

$$\frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \mu \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \qquad [28]$$

Therefore,

$$\frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) - \mu \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \mathbf{0} \qquad [29]$$

$$\frac{d}{dt}\left(\dot{\mathbf{r}} \times \mathbf{h} - \mu \frac{\mathbf{r}}{r}\right) = \mathbf{0} \qquad [30]$$

The quantity in the parentheses,

$$\mathbf{A} = \dot{\mathbf{r}} \times \mathbf{h} - \mu \frac{\mathbf{r}}{r} \qquad [31]$$

is the Runge-Lenz vector, and it is constant since its time derivative is zero

Now consider $\mathbf{A} \cdot \mathbf{r}$

$$\mathbf{A} \cdot \mathbf{r} = Ar\cos\theta \qquad [32]$$

Also, from equation 31,

$$\mathbf{A} \cdot \mathbf{r} = \left(\dot{\mathbf{r}} \times \mathbf{h} - \mu \frac{\mathbf{r}}{r} \right) \cdot \mathbf{r} \qquad [33]$$

Substitute definition of **h** from equation 13, and simplify

$$= \left(\dot{\mathbf{r}} \times \left(\mathbf{r} \times \dot{\mathbf{r}} \right) - \mu \frac{\mathbf{r}}{r} \right) \cdot \mathbf{r} \qquad [34]$$

$$= \mathbf{r} \cdot \left(\dot{\mathbf{r}} \times \left(\mathbf{r} \times \dot{\mathbf{r}} \right) \right) - \left(\mathbf{r} \cdot \mathbf{r} \right) \frac{\mu}{r} \qquad [35]$$

$$= \mathbf{r} \cdot \left(\dot{\mathbf{r}} \times \left(\mathbf{r} \times \dot{\mathbf{r}} \right) \right) - \mu r \qquad [36]$$

Now use vector identity in equation 1 to replace the triple cross product

$$= \mathbf{r} \cdot \left(\left(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \right) \mathbf{r} - \left(\dot{\mathbf{r}} \cdot \mathbf{r} \right) \dot{\mathbf{r}} \right) - \mu r \qquad [37]$$

$$\mathbf{A}\cdot\mathbf{r} = (\mathbf{r}\cdot\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}) - (\mathbf{r}\cdot\dot{\mathbf{r}})(\dot{\mathbf{r}}\cdot\mathbf{r}) - \mu r \qquad [38]$$

Now use vector identity, equation 3, on the first two terms above, and then simplify,

$$= (\mathbf{r}\times\dot{\mathbf{r}})\cdot(\mathbf{r}\times\dot{\mathbf{r}}) - \mu r \qquad [39]$$

$$= \mathbf{h}\cdot\mathbf{h} - \mu r \qquad [40]$$

$$= h^2 - \mu r \qquad [41]$$

Replace left side with definition of dot product from equation 32

$$Ar\cos\theta = h^2 - \mu r \qquad [42]$$

Solve for *r*,

$$Ar\cos\theta + \mu r = h^2 \qquad [43]$$

$$r(A\cos\theta + \mu r) = h^2 \qquad [44]$$

$$r = \frac{h^2}{\mu + A\cos\theta} \qquad [45]$$

Divide numerator and denominator by μ

$$r = \frac{h^2/\mu}{1+(A/\mu)\cos\theta} \qquad [46]$$

let $p = h^2/\mu$ and $e = A/\mu$

$$r = \frac{p}{1+e\cos\theta} \qquad [47]$$

Referring back to equation 1, this is the equation of a conic section in polar coordinates. e is the eccentricity and p is the semi-latus rectum. θ is the true anomaly, often represented as f or ν. For a complete description of a particular satellite orbit we must specify the conic section, as here, plus the orientation of the orbit plane, and the orientation of the curve in the plane, and the relationship of time and the orbiting body. These parameters may be expressed as state vectors or kepler elements.

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