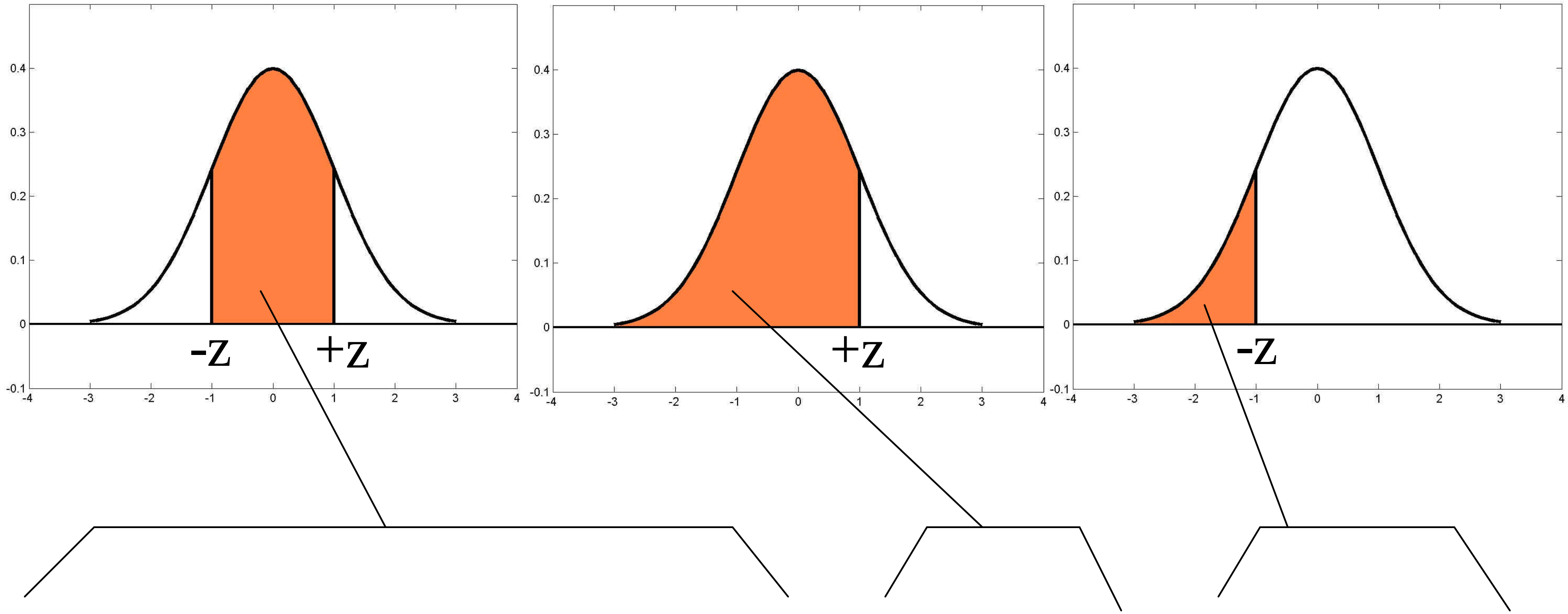
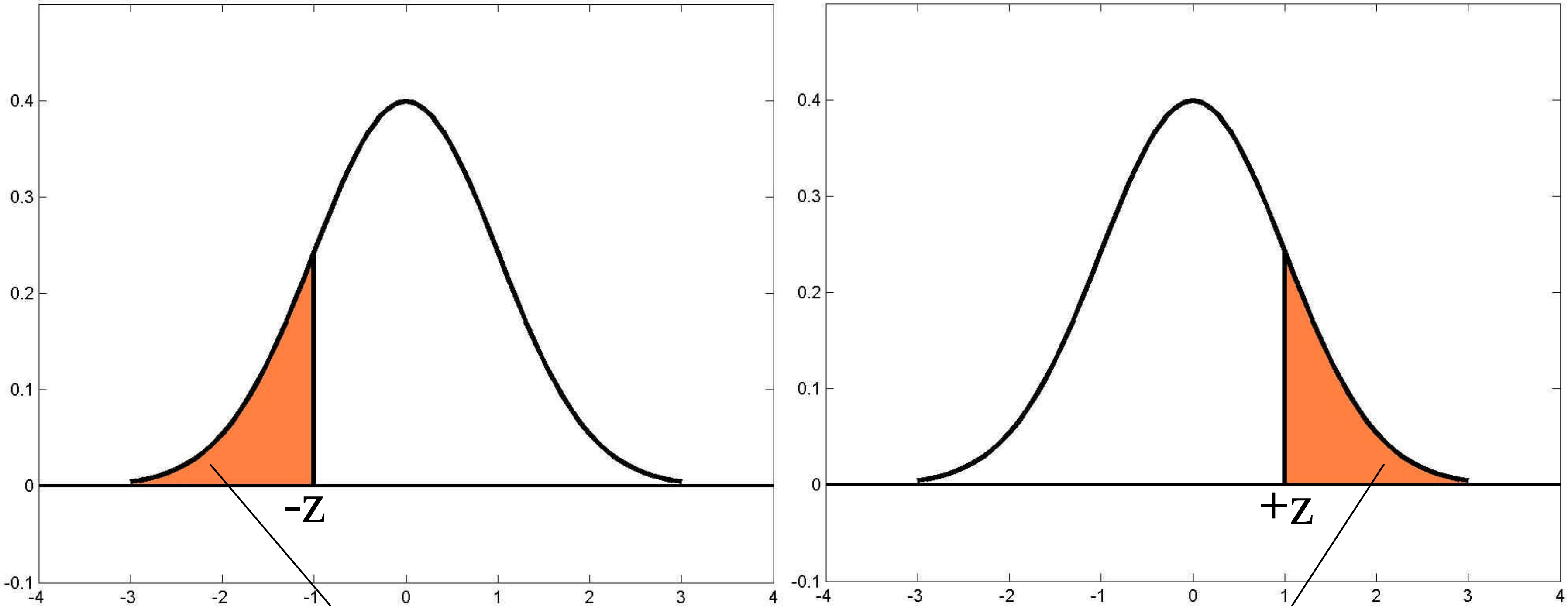
Statistical Confidence Statements



$$P\left(-z < RV < +z\right) = \Phi(z) - \Phi(-z)$$

Φ is the cumulative distribution function for standard normal distr.

By Symmetry



$$\Phi(-z) = 1 - \Phi(z)$$

Plug this into prior expression, then we can relate P and z for symmetric interval.

$$P = 2\Phi(z) - 1$$

Statistical Confidence Statements

Derive one dimensional confidence statement, s known with large d.f.

$\hat{x}$ is a normally distributed random variable with mean m_x and standard deviation s_x. We want a P confidence interval for m_x.

$$\frac{\hat{x} - m_x}{s_x} \sim z \text{ (standard normal)}$$

$$P\left(-z < \frac{\hat{x} - m_x}{s_x} < +z \right) = \Phi(z) - \Phi(-z)$$

This says that given some boundary points we can determine the probability – we want to invert that: given a probability, what are the boundaries, i.e. the interval?

by symmetry, $\Phi(-z) = 1 - \Phi(z)$

$$\Phi(z) - \Phi(-z) = \Phi(z) - \left(1 - \Phi(z)\right) = 2\Phi(z) - 1$$

$$P\left(-z s_x < \hat{x} - m_x < +z s_x\right) = 2\Phi(z) - 1$$

$$P\left(-\hat{x} - z s_x < -m_x < -x + z s_x\right) = 2\Phi(z) - 1$$

$$P\left(\hat{x} + z s_x > m_x > \hat{x} - z s_x\right) = 2\Phi(z) - 1$$

1D Confidence Intervals

$$P\left(\hat{x} - z s_x < m_x < \hat{x} + z s_x\right) = 2\Phi(z) - 1$$

Procedure:

1. select P (i.e. 0.90)
2. solve $P = 2\Phi(z) - 1$ for z:

$$2\Phi(z) = P + 1$$

$$\Phi(z) = \frac{P+1}{2}$$ get z from this

3. with z, $\hat{x}$, and s_x construct interval

Useful MATLAB functions:

```
y = normpdf(x, mu, sigma)
p = normcdf(x, mu, sigma)
x = norminv(p, mu, sigma)
```

Multivariate Normal Random Variables

$$f(\mathbf{x}) = \left(\frac{1}{(2p)^{n/2}\sqrt{|\mathbf{S}|}} \right) \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{x})^\mathbf{T} \mathbf{S}^{-1} (\mathbf{x}-\boldsymbol{\mu}_\mathbf{x}) \right) \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

From Searle (Linear Models, corollary 2.2, p. 58):

$$\text{if } \mathbf{x} \sim N(0, \mathbf{S}_{\mathrm{xx}}) \text{ then } \mathbf{x}^\mathbf{T}\mathbf{A}\mathbf{x} \sim c_r^2 \text{ if } \mathbf{A}\mathbf{S}_{\mathrm{xx}} \text{ is idempotent with rank r}$$

For our problem,

$$\text{let } \mathbf{A} = \mathbf{S}_{\mathrm{xx}}^{-1},\ \mathbf{S}_{\mathrm{xx}}^{-1}\mathbf{S}_{\mathrm{xx}} = \mathbf{I}$$

$$\mathbf{A}\mathbf{S}_{\mathrm{xx}} = \mathbf{I} = \mathbf{I}^2 \Rightarrow \text{idempotent}$$

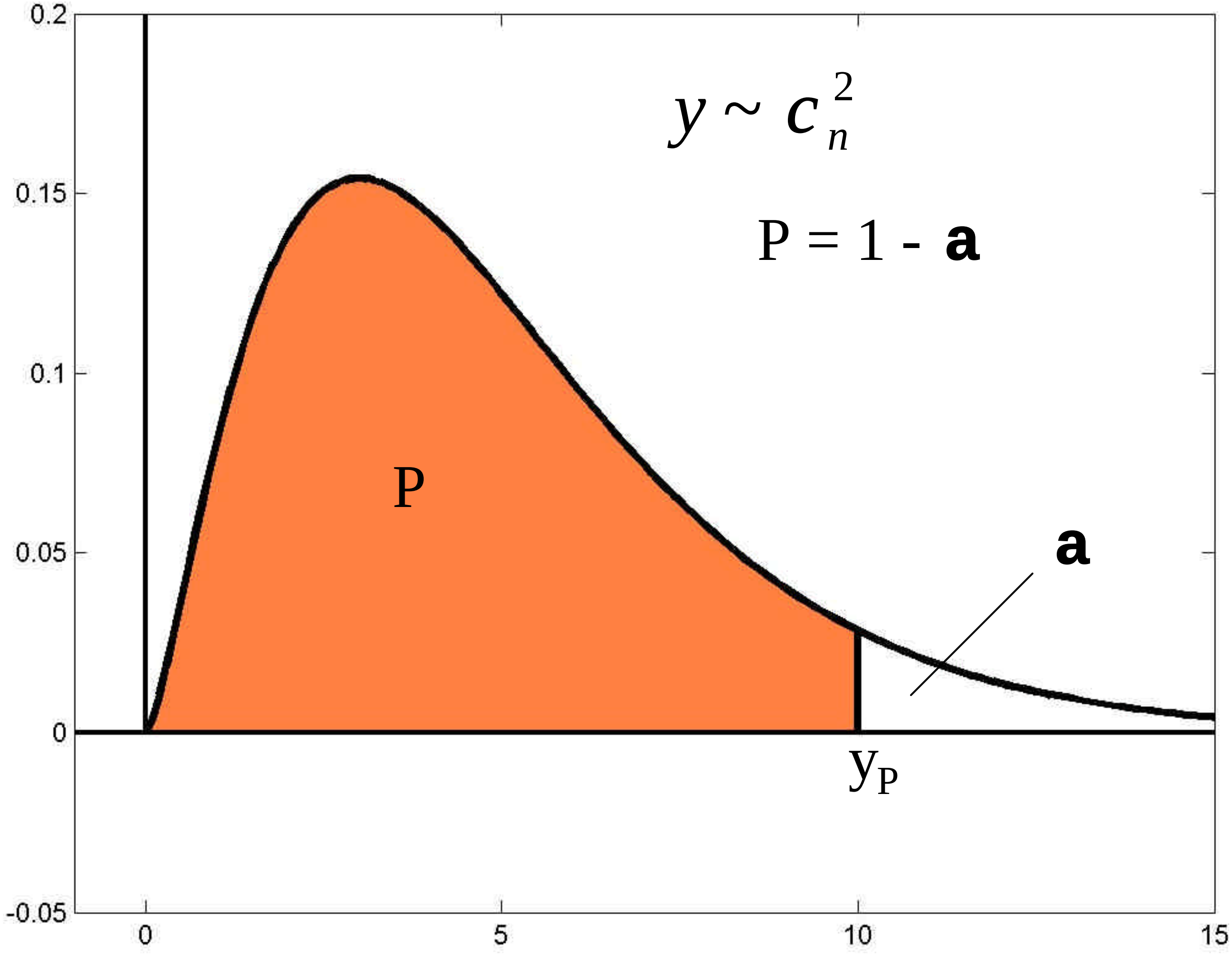
$$\text{rank} = \text{dimension}$$

Then, for random vector **x** with dimension n x 1, **y**, a scalar, is distributed as chi-squared

$$y = (\mathbf{x}-\boldsymbol{\mu}_\mathbf{x})^\mathbf{T} \mathbf{S}_{\mathrm{xx}}^{-1} (\mathbf{x}-\boldsymbol{\mu}_\mathbf{x}) \sim c_n^2$$

Given a value y_0 we can get $P(y < y_0)$. And we can easily invert. Given P, get a corresponding interval for y. More useful: given P & y-interval, what is the corresponding multidimensional *region* associated with **x** ? i.e what are all of the **x**'s that map into that interval of y's ?

Chi-squared Probability Density Function



$$P\left[y = \left(\mathbf{x}-\boldsymbol{\mu}_\mathbf{x}\right)^\mathbf{T} \mathbf{S}_{\mathbf{XX}}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}_\mathbf{x}\right) < y_P\right] = 1 - a$$

For a given value of P, get y, and get corresponding locus of **x**. That locus of **x** is what we will call a confidence region

$$P\left[(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{x}})^{\mathbf{T}}\mathbf{S}_{\mathrm{XX}}^{-\mathbf{1}}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{x}}) < c^{2}_{1-a,n}\right] = 1-a$$

Find **R** (orthogonal) which diagonalizes S_{XX} **,** i.e.

$$\mathbf{R}\mathbf{S}_{\mathrm{XX}}\mathbf{R}^{\mathrm{T}} = \mathbf{D}$$

$$P\left[(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{x}})^{\mathbf{T}}\mathbf{R}^{\mathrm{T}}\underbrace{\mathbf{R}\mathbf{S}_{\mathrm{XX}}^{-\mathbf{1}}\mathbf{R}^{\mathrm{T}}}_{\mathbf{M}}\mathbf{R}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{x}}) < c^{2}_{1-a,n}\right] = 1-a$$

$$\mathbf{R}\mathbf{S}_{\mathrm{XX}}^{-\mathbf{1}}\mathbf{R}^{\mathrm{T}} = \mathbf{M}$$

$$\mathbf{S}_{\mathrm{XX}}^{-\mathbf{1}} = \mathbf{R}^{\mathrm{T}}\mathbf{M}\mathbf{R}$$

$$\mathbf{S}_{\mathrm{XX}} = \left(\mathbf{R}^{\mathrm{T}}\mathbf{M}\mathbf{R}\right)^{-\mathbf{1}} = \mathbf{R}^{\mathrm{T}}\mathbf{M}^{-\mathbf{1}}\mathbf{R}$$

2D Confidence Regions

$$\underbrace{\mathbf{R}\mathbf{S}_{\mathbf{xx}}\mathbf{R}^{\mathbf{T}}}_{\mathbf{D},\ \text{diagonal (by defn. of } \mathbf{R})} = \mathbf{M}^{-1}$$

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},\quad \mathbf{M} = \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix}$$

d_i are eigenvalues of $\mathbf{S}_{\mathbf{xx}}$
rows of $\mathbf{R}$ are eigenvectors of $\mathbf{S}_{\mathbf{xx}}$

Now, assume n=2 (*) and let

$$\mathbf{w} = \mathbf{R}\left(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{x}}\right),\quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

(*) note: n can be anything.

2D Confidence Regions

rewrite the probability statement with substitutions

$$P\left(\mathbf{w}^{\mathbf{T}}\begin{bmatrix}\frac{1}{d_1} & 0\\ 0 & \frac{1}{d_2}\end{bmatrix}\mathbf{w} < c^2_{1-a,2}\right) = 1-a$$

$$P\left(\begin{bmatrix}w_1 & w_2\end{bmatrix}\begin{bmatrix}\frac{1}{d_1} & 0\\ 0 & \frac{1}{d_2}\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix} < c^2_{1-a,2}\right) = 1-a$$

$$P\left(\frac{w_1^2}{d_1} + \frac{w_2^2}{d_2} < c^2_{1-a,2}\right) = 1-a$$

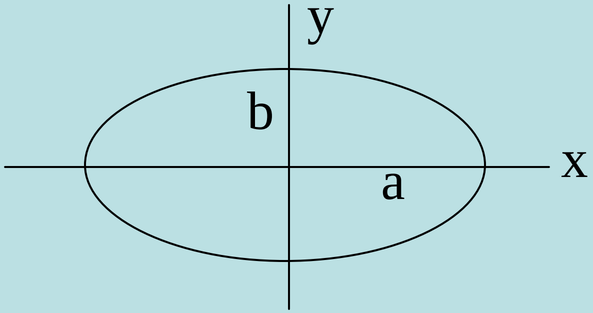
$$P\left(z_1^2 + z_2^2 < c^2_{1\text{-}a,2}\right) = 1-a$$

$$P\left(\frac{w_1^2}{d_1 c^2_{1-a,2}} + \frac{w_2^2}{d_2 c^2_{1-a,2}} < 1\right) = 1-a$$

Recall

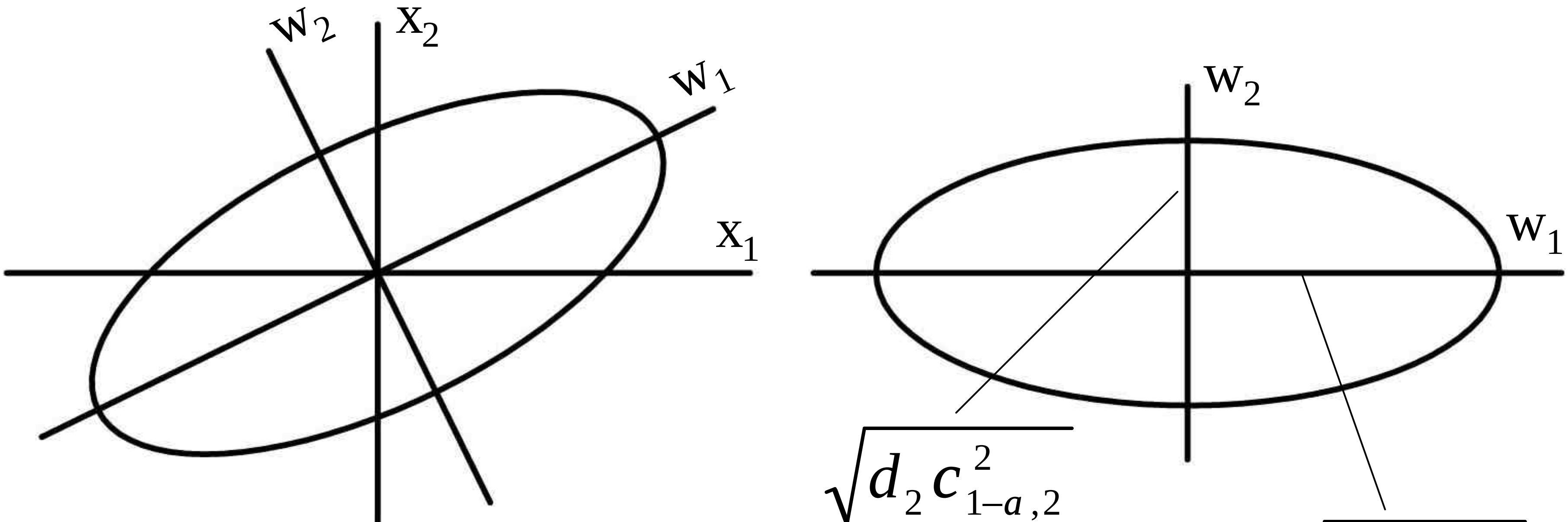
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Equation of ellipse



We recognize this expression as describing the interior of an ellipse with dimensions as shown on the next slide.

2D Confidence Region



$$\sqrt{d_2 c^2_{1-a,2}}$$

$$\sqrt{d_1 c^2_{1-a,2}}$$

$1-a$ confidence region for $\left(m_x, m_y\right)$

$$\mathbf{w} = \mathbf{R}\left(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{x}}\right),\ \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

3D Confidence Regions

$$\underbrace{\mathbf{R}\mathbf{S}_{\mathbf{xx}}\mathbf{R}^{\mathbf{T}}}_{\mathbf{D},\ \text{diagonal (by defn. of } \mathbf{R})} = \mathbf{M}^{-1}$$

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{bmatrix}$$

d_i are eigenvalues of $\mathbf{S}_{\mathbf{xx}}$

rows of $\mathbf{R}$ are eigenvectors of $\mathbf{S}_{\mathbf{xx}}$

Now, assume n=3 (*) and let

$$\mathbf{w} = \mathbf{R}\left(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{x}}\right), \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

(*) note: n can be anything.

3D Confidence Regions

$$P\left(\mathbf{w}^{\mathbf{T}} \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix} \mathbf{w} < c^2_{1-a,3} \right) = 1-a$$

$$P\left(\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} < c^2_{1-a,3} \right) = 1-a$$

$$P\left(\frac{w_1^2}{d_1} + \frac{w_2^2}{d_2} + \frac{w_3^2}{d_3} < c^2_{1-a,3} \right) = 1-a$$

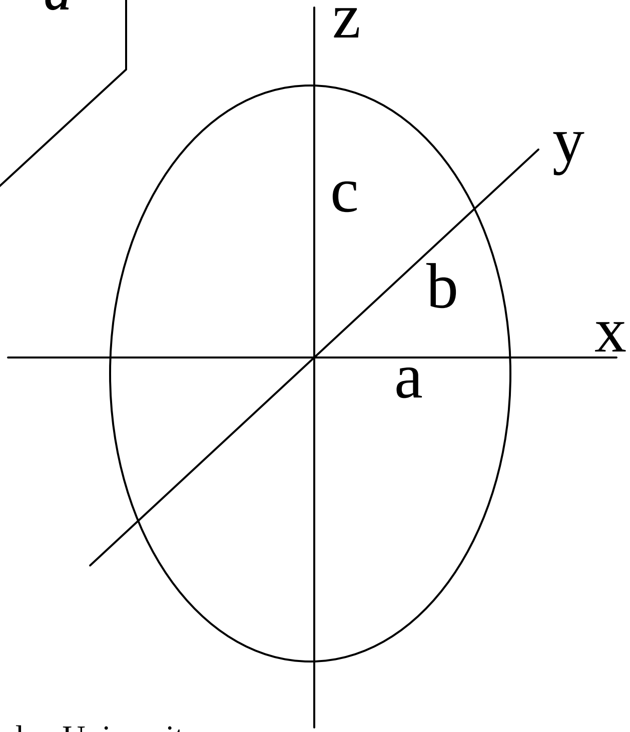
$$P\left(z_1^2 + z_2^2 + z_3^2 < c^2_{1\text{-}a,3} \right) = 1-a$$

$$P\left(\frac{w_1^2}{d_1 c^2_{1-a,3}} + \frac{w_2^2}{d_2 c^2_{1-a,3}} + \frac{w_3^2}{d_3 c^2_{1-a,3}} < 1 \right) = 1-a$$

Recall

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Equation of ellipsoid



Error Propagation for a Pass Point

$$\mathbf{S}_{\mathbf{p45}} = \begin{bmatrix} 0.00115403 & -0.00000036 & 0.00004374 \\ -0.00000036 & 0.00123980 & -0.00023149 \\ 0.00004374 & -0.00023149 & 0.00421319 \end{bmatrix}$$

$$s_X = 0.034m$$
$$s_Y = 0.035m$$
$$s_Z = 0.065m$$

P = 90%

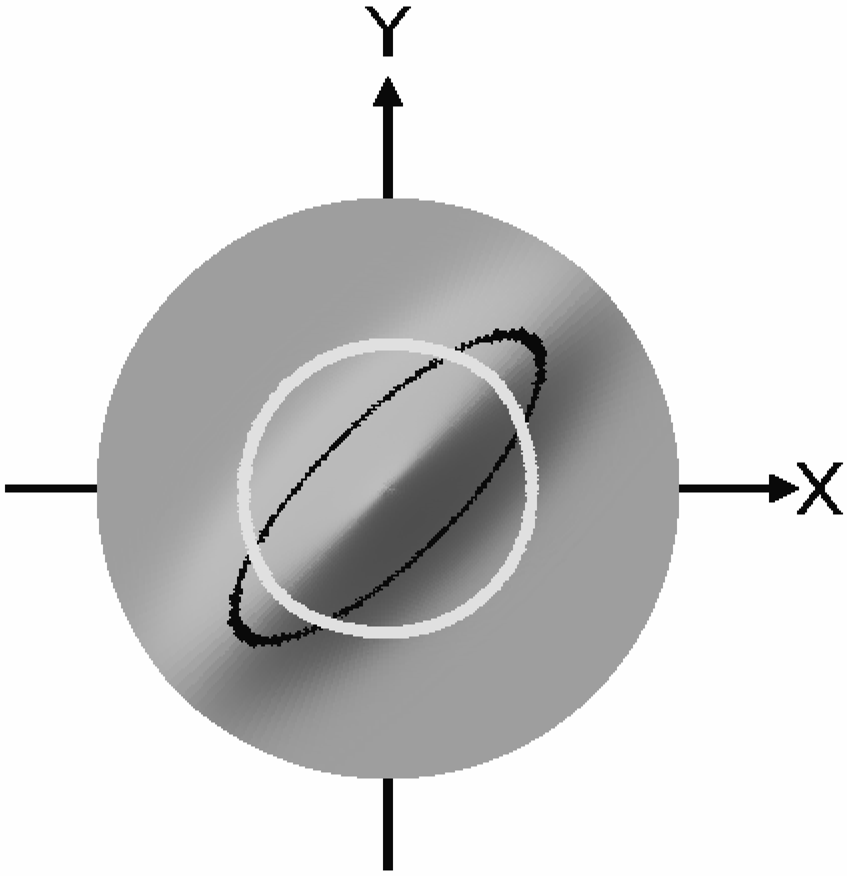
$$a = 0.10$$

Useful MATLAB function:

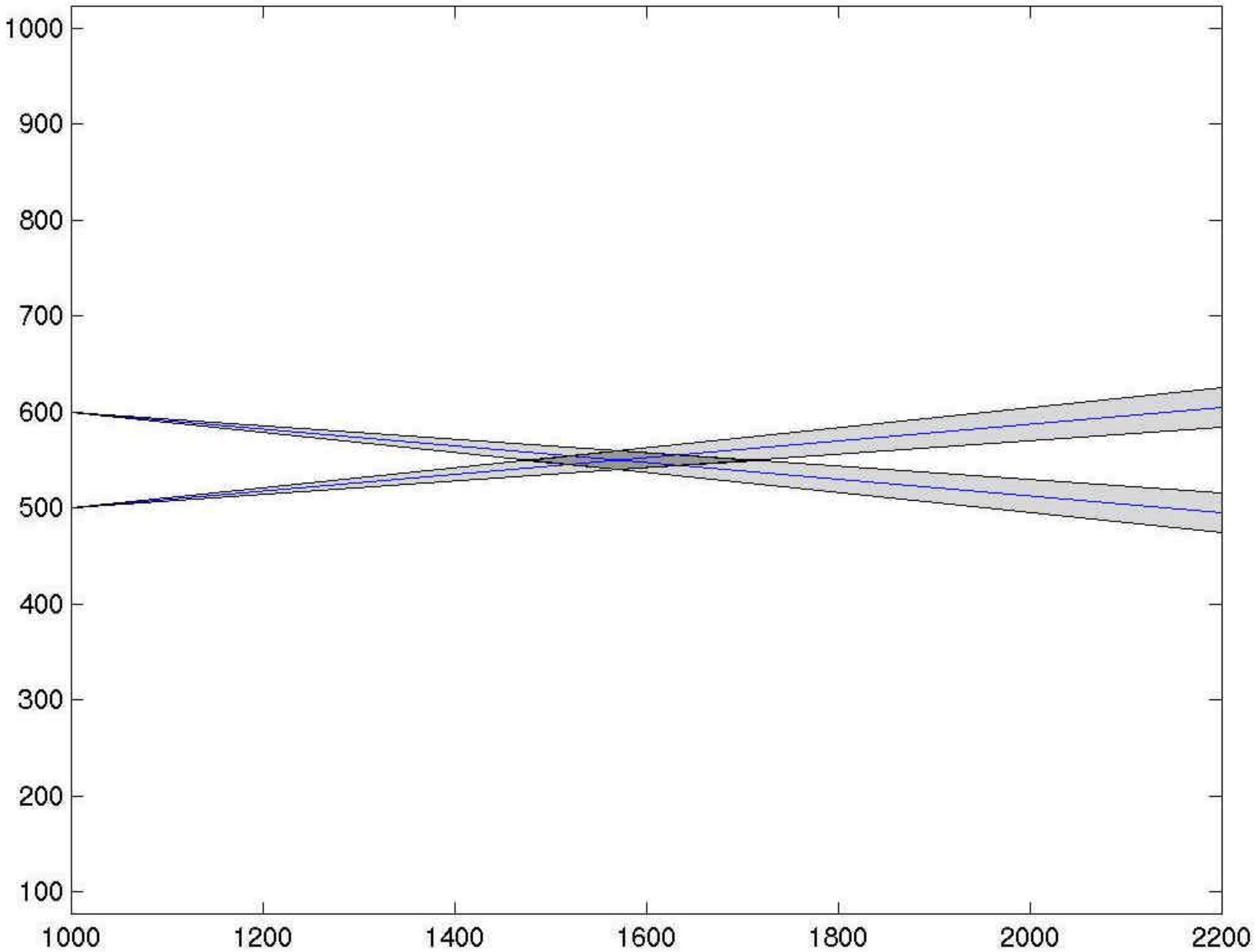
x=chi2inv(p,v)

	90% 1D	90% 2D	90% 3D
X	0.056	0.073	0.085
Y	0.058	0.076	0.088
Z	0.107	0.139	0.162
factor	$z_{1-0.10/2}$	$\sqrt{c^2_{0.90,2}}$	$\sqrt{c^2_{0.90,3}}$
factor	1.645	2.146	2.500

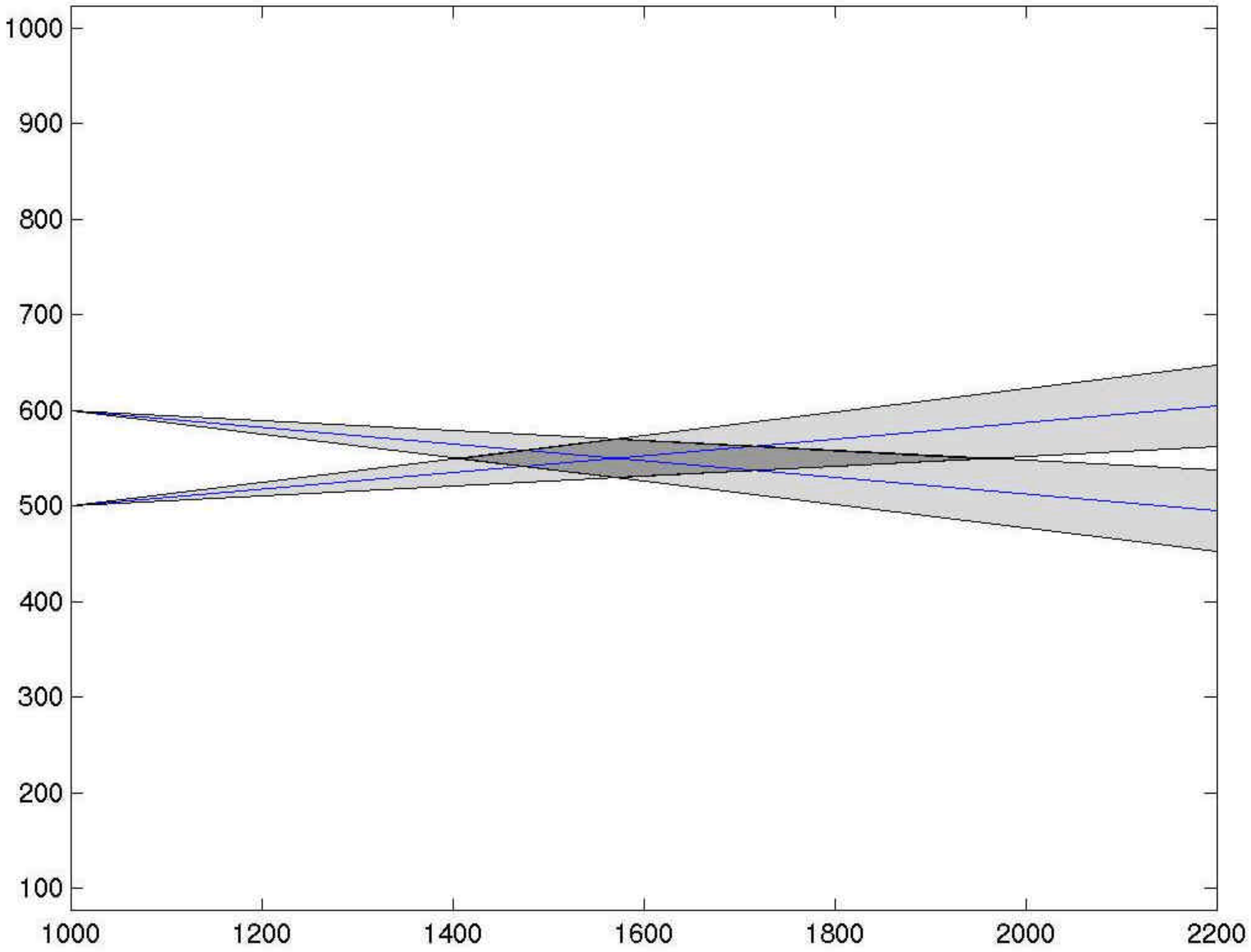
Error Region Tutorial



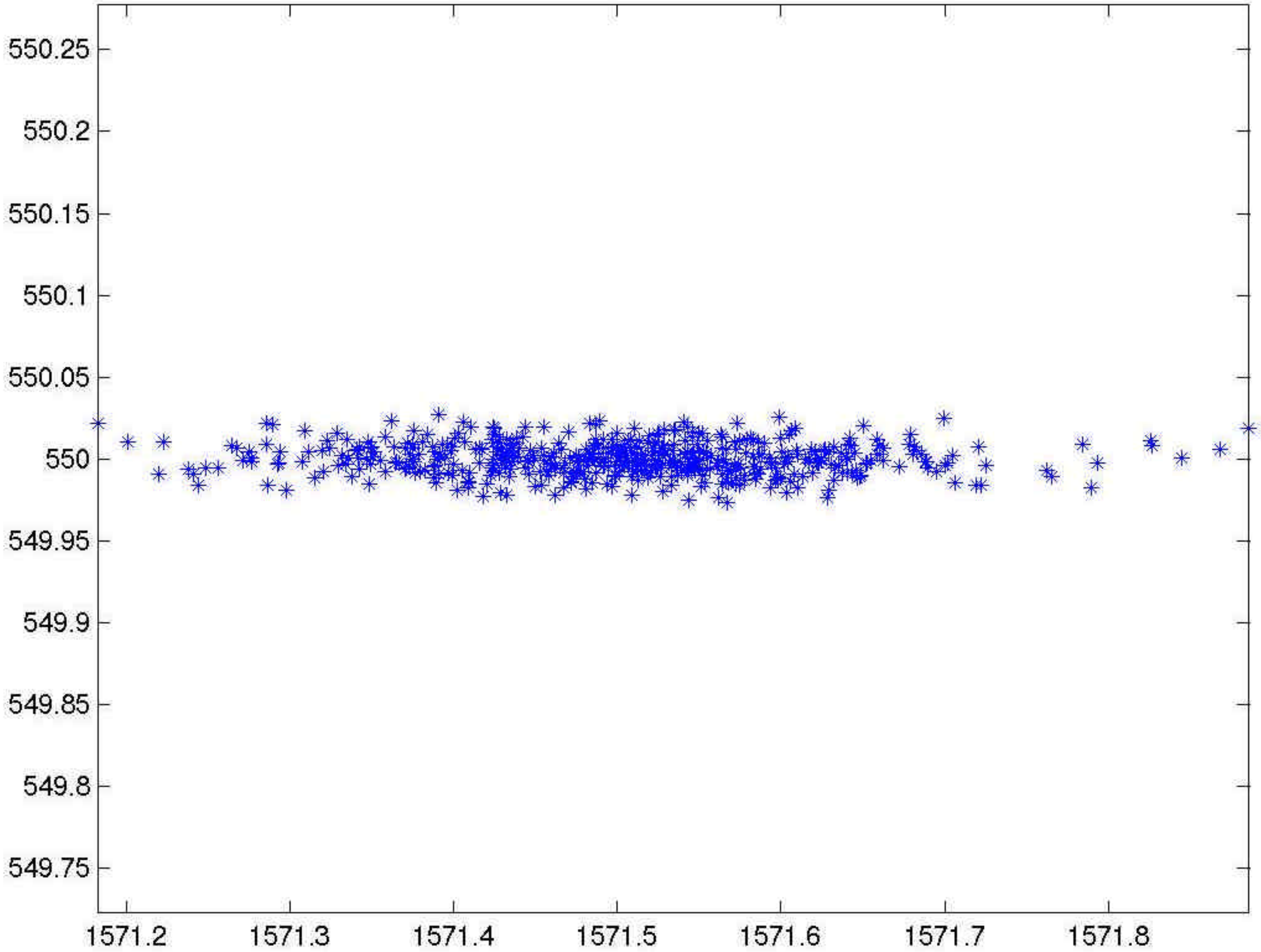
Propagate Angular Uncertainty into an Uncertainty Region around the Intersection Point



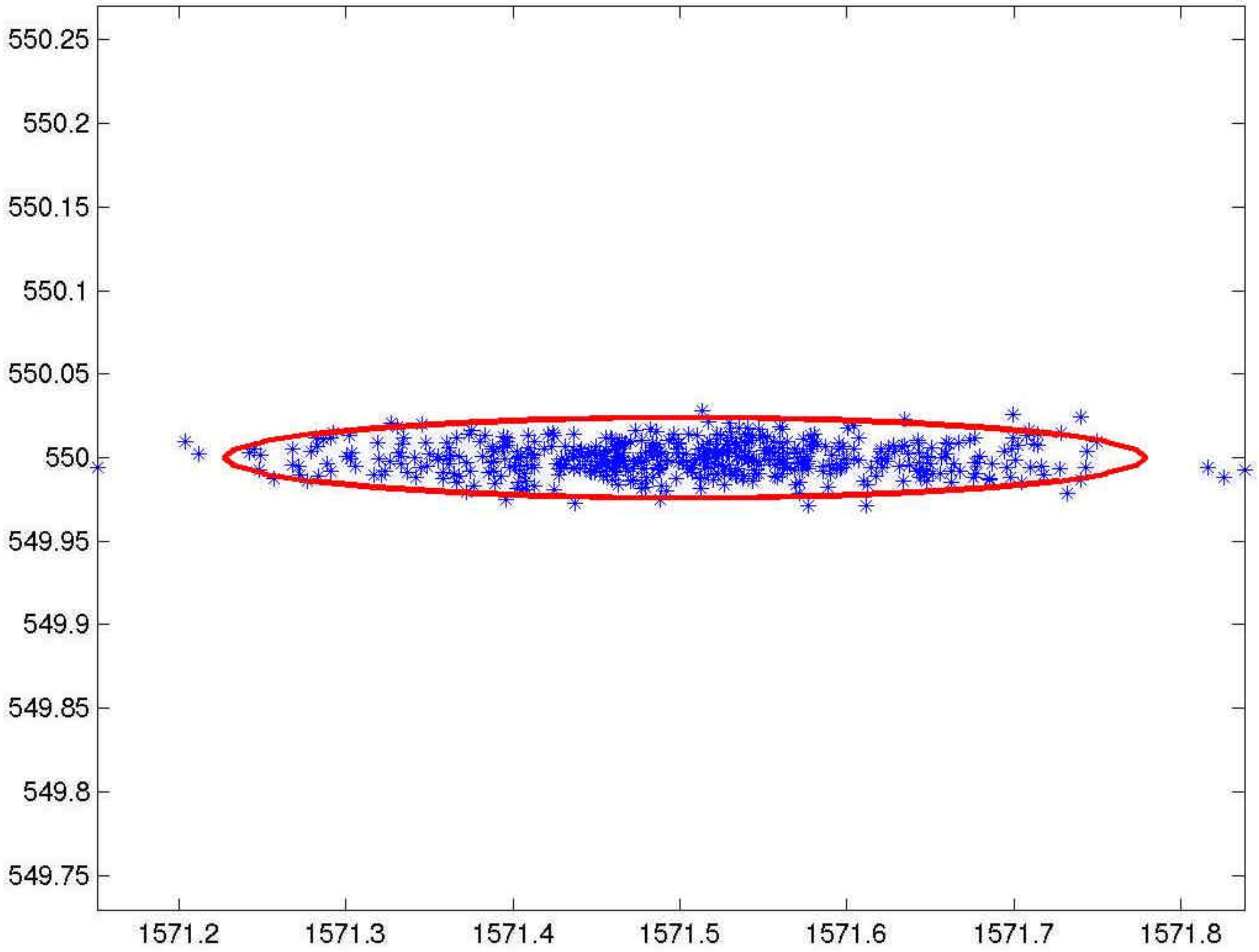
Increase the Angular Uncertainty



Monte Carlo Simulation Yields an Equivalent Picture of the Uncertainty Region as a Scatter Diagram

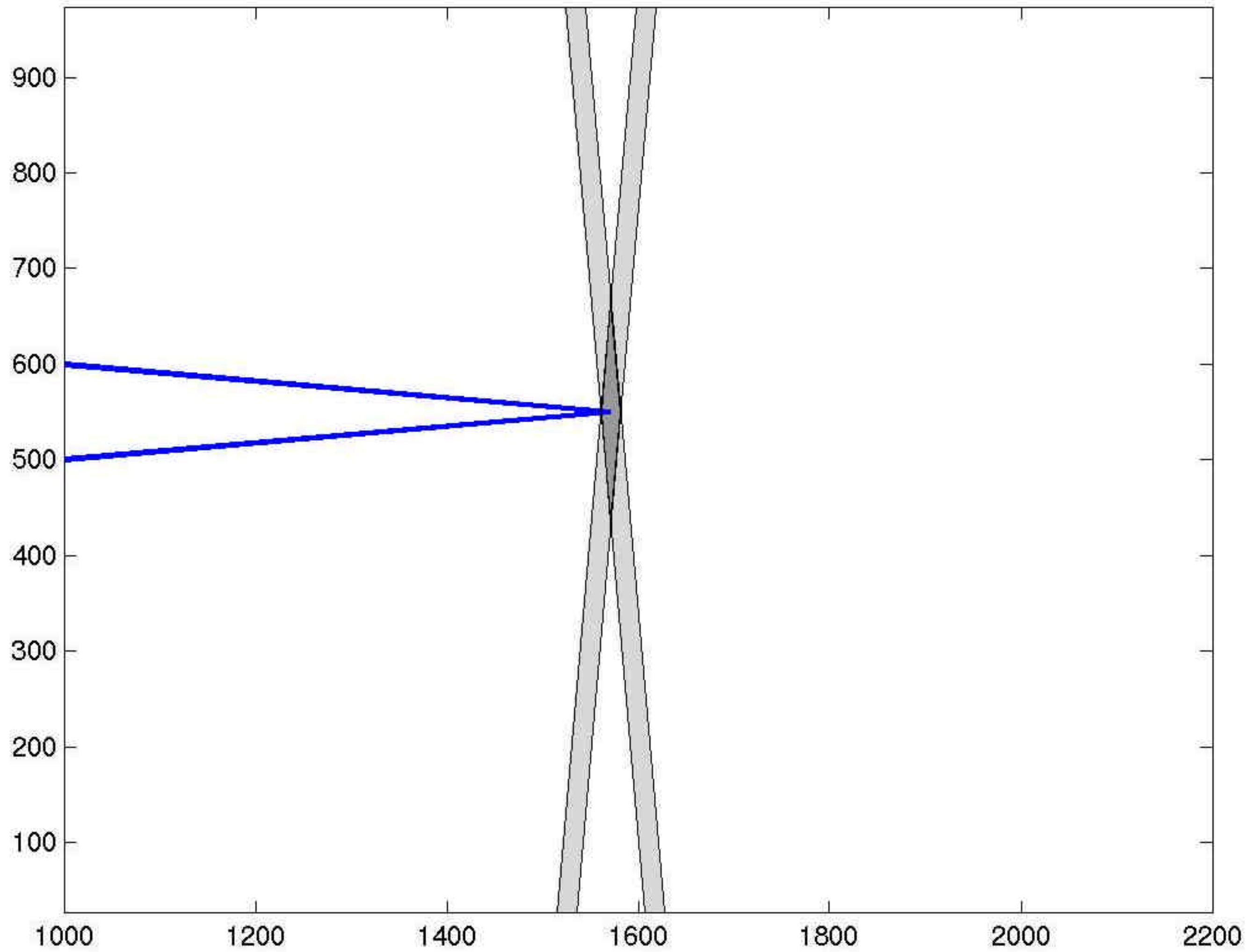


From the Least Squares Equations We Can Compute a Confidence Region (Error Ellipse)

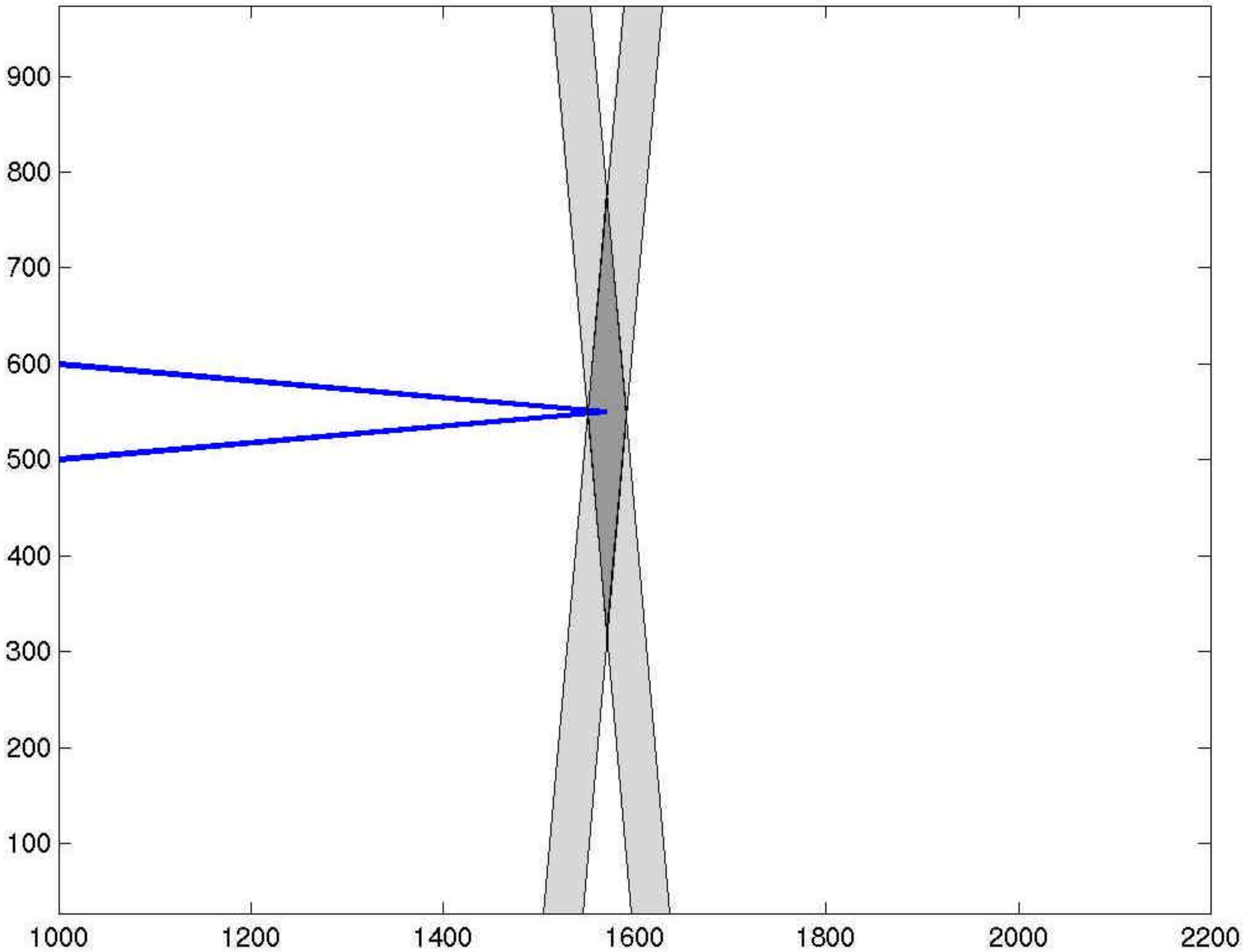


Notice that if we assume an uncertainty this can be plotted *without making any actual observations.* Thus we have a design mode where we can predict confidence ellipses based on assumed observation error and network geometry.

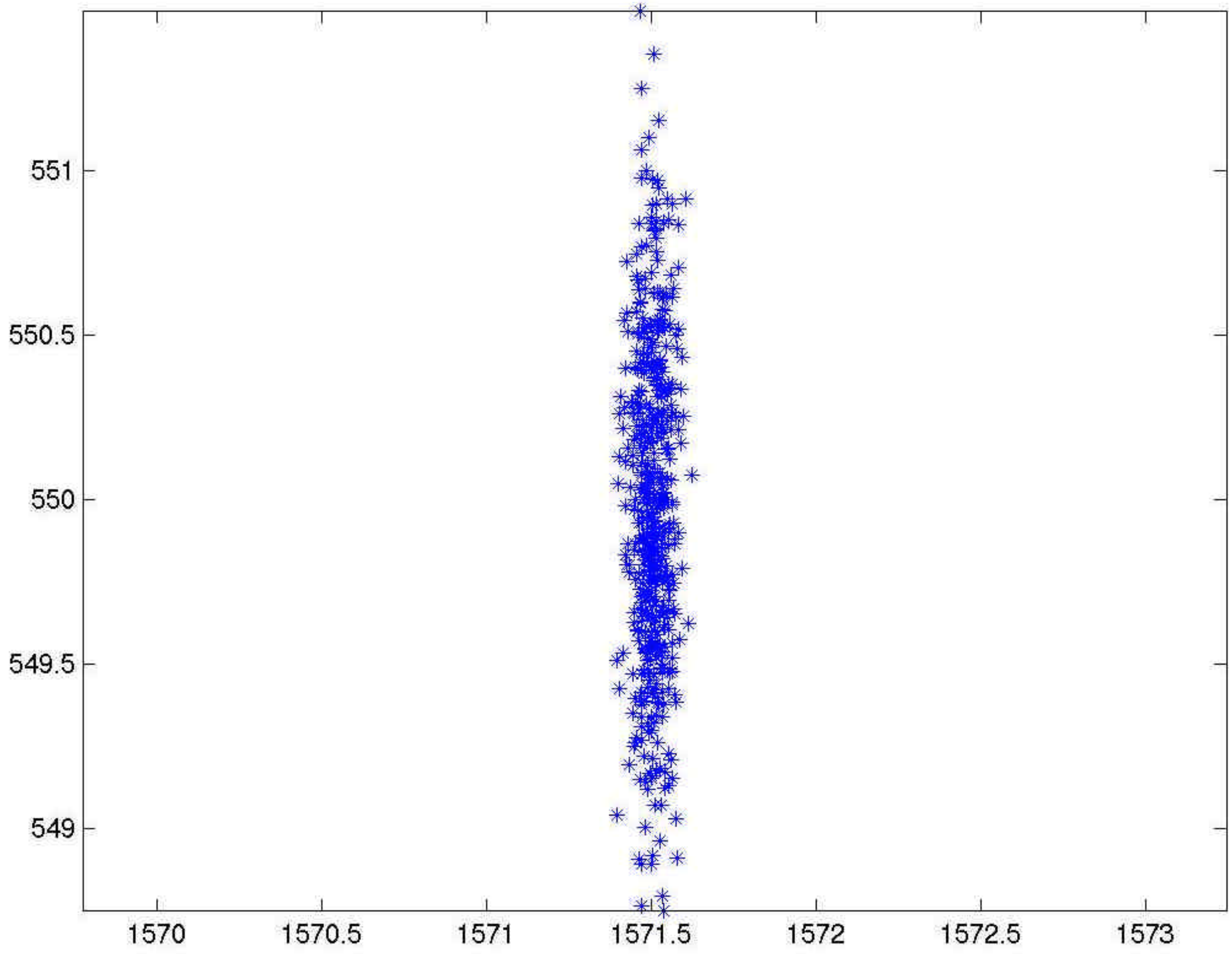
Same Idea – Let's Look at Uncertainty in Distance and Its Effect on the Uncertainty in the Position



Increase the Uncertainty in the Distance



Monte Carlo Simulation on the Same Geometric Model yields a Scatter Diagram



Compute the Corresponding Confidence Ellipse from the Mathematical Model