Axis/angle form of the rotation matrix, Bethel, 2005

In order to derive this matrix, review some preliminaries,

(a) columns of rotation matrix are components in the "to" system of unit basis vectors in the "from" system (recall direction cosines).

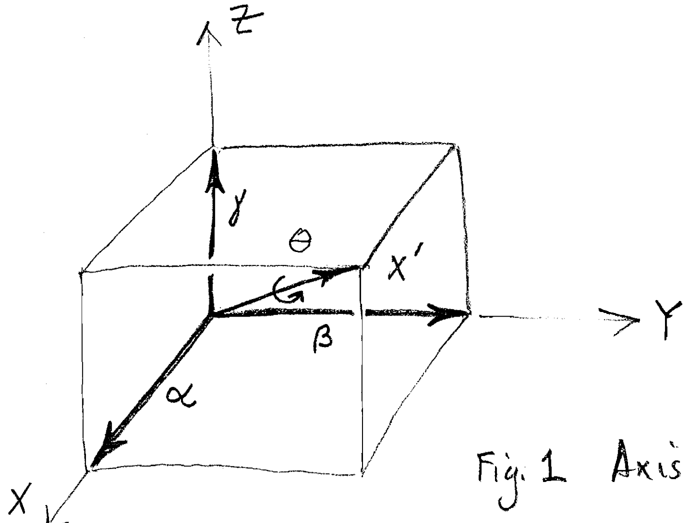
(b) rows of rotation matrix are components in the "from" system of unit basis vectors in the "to" system

(c) you may interpret a rotation in the usual way as a rotation of the axes, leaving the object in place. You may also interpret the same rotation as rotating the object (with the opposite sense) while leaving the axes in place.

(d) for an orthogonal matrix, the inner product of a column with itself or a row with itself is equal to 1. The inner product of a column with another column, or a row with another row is equal to Ø. (note inner product = dot product)

(e) An orthogonal matrix is equal to its cofactor matrix.

For Axis/angle rotation we want to rotate the X-axis of the original system into the desired axis direction, then rotate about that axis by angle θ, then rotate back to the original system. This is illustrated in the accompanying figure,



X' rotated position of X-axis, about which we rotate by θ

α, β, γ : direction cosines of this direction, referenced to original axes

Fig. 1 Axis/angle geometry

In order to specify the matrix which moves X into X', we invoke principle (b) above

$$M_1 = \begin{bmatrix} \alpha & \beta & \gamma \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \qquad (1)$$

elements of rows 2 and 3 are unknown, and unnecessary at this point. Having positioned the X' axis where we want it, now we must rotate about it by angle θ. Because we wish to rotate back into the original system later, it is convenient here to invoke principle (c), and to consider this step as leaving the axes fixed and rotating the object. Here one has a choice in the interpretation of the right-hand rule, or rotation sense designation. In photogrammetry, a positive rotation has traditionally meant rotating the <u>axes</u> consistent with the right hand rule. This yields the X elementary rotation matrix as,

$$M_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad (2)$$

Others designate a positive rotation as rotating the <u>object</u> consistent with the right-hand rule. By principle (c) this yields

$$M_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad (3)$$

To be consistent with Mikhail (2001), and Kanatani (1993) we adopt the second convention, for this case only, even though it is at odds with conventional photogrammetry. Therefore we adopt equation (3) as the second of our sequential rotations.

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

The third in the rotation sequence is to just reverse the first rotation and put the axes back in the original position. This is done with the transpose of the first rotation matrix,

$$M_3 = \begin{bmatrix} \alpha & m_{21} & m_{31} \\ \beta & m_{22} & m_{32} \\ \gamma & m_{23} & m_{33} \end{bmatrix} \tag{5}$$

As before, at this point, columns 2 and 3 are unknown and it will turn out that <u>any</u> valid entries here will work, and the final matrix will be invariant to the choices for these elements. The total, composite rotation matrix, thus becomes,

$$M_{axis/angle} = \begin{bmatrix} \alpha & m_{21} & m_{31} \\ \beta & m_{22} & m_{32} \\ \gamma & m_{23} & m_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$= M_3 M_2 M_1 \qquad (6)$$

To put this in a form convenient to use, let's multiply the 3 matrices and simplify the expressions using orthogonal matrix identities. Multiplying yields,

$$\begin{bmatrix} \alpha & m_{21} & m_{31} \\ \beta & m_{22} & m_{32} \\ \gamma & m_{23} & m_{33} \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma \\ \cos\theta\, m_{21} - \sin\theta\, m_{31} & \cos\theta\, m_{22} - \sin\theta\, m_{32} & \cos\theta\, m_{23} - \sin\theta\, m_{33} \\ \sin\theta\, m_{21} + \cos\theta\, m_{31} & \sin\theta\, m_{22} + \cos\theta\, m_{32} & \sin\theta\, m_{23} + \cos\theta\, m_{33} \end{bmatrix}$$

$$= M_3 (M_2 M_1) \qquad (7)$$

$$M =$$

$$\begin{bmatrix}
\alpha^2 + \cos\theta\, m_{21}^2 - \sin\theta\, m_{21} m_{31} + \sin\theta\, m_{31} m_{21} + \cos\theta\, m_{31}^2 & \alpha\beta + \cos\theta\, m_{21} m_{22} - \sin\theta\, m_{21} m_{32} + \sin\theta\, m_{31} m_{22} + \cos\theta\, m_{31} m_{32} & \alpha\gamma + \cos\theta\, m_{21} m_{23} - \sin\theta\, m_{21} m_{33} + \sin\theta\, m_{31} m_{23} + \cos\theta\, m_{31} m_{33} \\
\alpha\beta + \cos\theta\, m_{22} m_{21} - \sin\theta\, m_{22} m_{31} + \sin\theta\, m_{32} m_{21} + \cos\theta\, m_{32} m_{31} & \beta^2 + \cos\theta\, m_{22}^2 - \sin\theta\, m_{22} m_{32} + \sin\theta\, m_{32} m_{22} + \cos\theta\, m_{32}^2 & \beta\gamma + \cos\theta\, m_{22} m_{23} - \sin\theta\, m_{22} m_{33} + \sin\theta\, m_{32} m_{23} + \cos\theta\, m_{32} m_{33} \\
\gamma\alpha + \cos\theta\, m_{23} m_{21} - \sin\theta\, m_{23} m_{31} + \sin\theta\, m_{33} m_{21} + \cos\theta\, m_{33} m_{31} & \gamma\beta + \cos\, m_{23} m_{22} - \sin\theta\, m_{23} m_{32} + \sin\theta\, m_{33} m_{22} + \cos\theta\, m_{33} m_{32} & \gamma^2 + \cos\theta\, m_{23}^2 - \sin\theta\, m_{23} m_{33} + \sin\theta\, m_{33} m_{23} + \cos\theta\, m_{33}^2
\end{bmatrix} \qquad (8)$$

Notice that each of the diagonal elements has a pair of terms which cancel. Now group the remaining terms,

$$M =$$

$$\begin{bmatrix}
\alpha^2 + \cos\theta\,(m_{21}^2 + m_{31}^2) & \alpha\beta + \cos\,(\underline{m_{21}m_{22} + m_{31}m_{32}}) + \sin\theta\,(m_{31}m_{22} - m_{21}m_{32}) & \alpha\gamma + \cos\theta\,(\underline{m_{21}m_{23} + m_{31}m_{33}}) + \sin\theta\,(m_{31}m_{23} - m_{21}m_{33}) \\
\alpha\beta + \cos\theta\,(\underline{m_{22}m_{21} + m_{32}m_{31}}) + \sin\theta\,(m_{32}m_{21} - m_{22}m_{31}) & \beta^2 + \cos\theta\,(m_{22}^2 + m_{32}^2) & \beta\gamma + \cos\theta\,(\underline{m_{22}m_{23} + m_{32}m_{33}}) + \sin\theta\,(m_{32}m_{23} - m_{22}m_{33}) \\
\gamma\alpha + \cos\theta\,(\underline{m_{23}m_{21} + m_{33}m_{31}}) + \sin\theta\,(m_{33}m_{21} - m_{23}m_{31}) & \gamma\beta + \cos\theta\,(\underline{m_{23}m_{22} + m_{33}m_{32}}) + \sin\theta\,(m_{33}m_{22} - m_{23}m_{32}) & \gamma^2 + \cos\theta\,(m_{23}^2 + m_{33}^2)
\end{bmatrix} \tag{9}$$

We simplify 3 groups of expressions above, by the following methods,

- - - - - - - - - - : use inner product of columns of matrix (1) with themselves (= 1)

———————— : use inner product of columns of matrix (1) with other columns (= 0)

~~~~~~~~~~ : use the fact that an element of an orthogonal matrix equals its cofactor, from matrix (1)
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Recall,
$$M_1 = \begin{bmatrix} \alpha & \beta & \gamma \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \qquad (1)$$

from (9),

$(1,1)$: $\alpha^2 + m_{21}^2 + m_{31}^2 = 1$, $m_{21}^2 + m_{31}^2 = 1 - \alpha^2$

$(2,1)$: $\alpha\beta + m_{21}m_{22} + m_{31}m_{32} = 0$, $m_{21}m_{22} + m_{31}m_{32} = -\alpha\beta$

$m_{21}m_{32} - m_{31}m_{22} = \gamma$

$(3,1)$: $\alpha\gamma + m_{21}m_{23} + m_{31}m_{33} = 0$, $m_{21}m_{23} + m_{31}m_{33} = -\alpha\gamma$

$m_{21}m_{33} - m_{31}m_{23} = -\beta$

$(1,2)$: $m_{21}m_{22} + m_{31}m_{32} = -\alpha\beta$, see $(2,1)$

$m_{31}m_{22} - m_{21}m_{32} = -\gamma$, see $(2,1)$ (10)

$(2,2)$: $\beta^2 + m_{22}^2 + m_{32}^2 = 1$, $m_{22}^2 + m_{32}^2 = 1 - \beta^2$

$(3,2)$: $\beta\gamma + m_{22}m_{23} + m_{32}m_{33} = 0$, $m_{22}m_{23} + m_{32}m_{33} = -\beta\gamma$

$m_{33}m_{22} - m_{23}m_{32} = \alpha$

$(1,3)$: $m_{21}m_{23} + m_{31}m_{33} = -\alpha\gamma$, see $(3,1)$

$m_{31}m_{23} - m_{21}m_{33} = \beta$, see $(3,1)$

$(2,3)$: $m_{22}m_{23} + m_{32}m_{33} = -\beta\gamma$, see $(3,2)$

$m_{32}m_{23} - m_{22}m_{33} = -\alpha$

$(3,3)$ $\gamma^2 + m_{23}^2 + m_{33}^2 = 1$, $m_{23}^2 + m_{33}^2 = 1 - \gamma^2$

Substituting back into equation (9) we get,

$$\begin{bmatrix} \alpha^2 + \cos\theta(1-\alpha^2) & \alpha\beta + \cos\theta(-\alpha\beta) + \sin\theta(-\gamma) & \alpha\gamma + \cos\theta(-\alpha\gamma) + \sin\theta(\beta) \\ \alpha\beta + \cos\theta(-\alpha\beta) + \sin\theta(\gamma) & \beta^2 + \cos\theta(1-\beta^2) & \beta\gamma + \cos\theta(-\beta\gamma) + \sin\theta(-\alpha) \\ \alpha\gamma + \cos\theta(-\alpha\gamma) + \sin\theta(-\beta) & \beta\gamma + \cos\theta(-\beta\gamma) + \sin\theta(\alpha) & \gamma^2 + \cos\theta(1-\gamma^2) \end{bmatrix} \quad (11)$$

Rearranging,

$$\begin{bmatrix} \alpha^2(1-\cos\theta) + \cos\theta & \alpha\beta(1-\cos\theta) - \gamma\sin\theta & \alpha\gamma(1-\cos\theta) + \beta\sin\theta \\ \alpha\beta(1-\cos\theta) + \gamma\sin\theta & \beta^2(1-\cos\theta) + \cos\theta & \beta\gamma(1-\cos\theta) - \alpha\sin\theta \\ \alpha\gamma(1-\cos\theta) - \beta\sin\theta & \beta\gamma(1-\cos\theta) + \alpha\sin\theta & \gamma^2(1-\cos\theta) + \cos\theta \end{bmatrix} =$$

$$M_{axis/angle} \quad (12)$$

agrees with Mikhail p.449, Kanatani p.102. Note all of the m_{ij} disappear, showing invariance with respect to their values.

References

Kanatani, Kenichi, 1993, Geometric Computation for Machine Vision, Oxford Science Publications, Clarendon Press

Mikhail, et al, 2001, Introduction to Modern Photogrammetry, John Wiley & Sons