1. $n = 5$ indirect observations

$n_0 = 3$ $\quad u = n_0 = 3$: parameters : a_0, a_1, a_2 $\qquad$ 1/6

$r = 2$ $\quad c = n$ condition equations, 1 per observation

$$\hat{y}_i = a_0 + a_1 x_i + a_2 x_i^2 = y_i + v_i = a_0 + a_1 x_i + a_2 x_i^2$$

$$1.74 + v_1 = a_0 + a_1(1) + a_2 \cdot (1), \quad v_1 = a_0 + a_1 + a_2 - 1.74$$
$$2.79 + v_2 = a_0 + a_1(2) + a_2(4), \quad v_2 = a_0 + 2a_1 + 4a_2 - 2.79$$
$$4.33 + v_3 = a_0 + a_1(3) + a_2(9), \quad v_3 = a_0 + 3a_1 + 9a_2 - 4.33$$
$$6.16 + v_4 = a_0 + a_1(4) + a_2(16), \quad v_4 = a_0 + 4a_1 + 16a_2 - 6.16$$
$$8.51 + v_5 = a_0 + a_1(5) + a_2(25), \quad v_5 = a_0 + 5a_1 + 25a_2 - 8.51$$

$$\phi = \sum v_i^2 = (a_0 + a_1 + a_2 - 1.74)^2 + (a_0 + 2a_1 + 4a_2 - 2.79)^2 + (a_0 + 3a_1 + 9a_2 - 4.33)^2$$
$$+ (a_0 + 4a_1 + 16a_2 - 6.16)^2 + (a_0 + 5a_1 + 25a_2 - 8.51)^2$$

$\partial\phi / \partial \text{parameters} = 0$

$$\frac{\partial\phi}{\partial a_0} = \cancel{2}(a_0 + a_1 + a_2 - 1.74) + \cancel{2}(a_0 + 2a_1 + 4a_2 - 2.79) + \cancel{2}(a_0 + 3a_1 + 9a_2 - 4.33)$$
$$+ \cancel{2}(a_0 + 4a_1 + 16a_2 - 6.16) + \cancel{2}(a_0 + 5a_1 + 25a_2 - 8.51) = 0$$

$$\frac{\partial\phi}{\partial a_1} = \cancel{2}(a_0 + a_1 + a_2 - 1.74) + \cancel{2}(a_0 + 2a_1 + 4a_2 - 2.79) \cdot 2 + \cancel{2}(a_0 + 3a_1 + 9a_2 - 4.33) \cdot 3$$
$$+ \cancel{2}(a_0 + 4a_1 + 16a_2 - 6.16) \cdot 4 + \cancel{2}(a_0 + 5a_1 + 25a_2 - 8.51) \cdot 5 = 0$$

$$\frac{\partial\phi}{\partial a_2} = \cancel{2}(a_0 + a_1 + a_2 - 1.74) + \cancel{2}(a_0 + 2a_1 + 4a_2 - 2.79) \cdot 4 + \cancel{2}(a_0 + 3a_1 + 9a_2 - 4.33) \cdot 9$$
$$+ \cancel{2}(a_0 + 4a_1 + 16a_2 - 6.16) \cdot 16 + \cancel{2}(a_0 + 5a_1 + 25a_2 - 8.51) \cdot 25 = 0$$

$$5a_0 + 15a_1 + 55a_2 = 23.53$$
$$15a_0 + 55a_1 + 225a_2 = 87.50$$
$$55a_0 + 225a_1 + 979a_2 = 363.18$$

$$\Delta = N^{-1} \cdot t = \begin{bmatrix} 1.0780 \\ 0.4524 \\ 0.2064 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 5 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix}}_{N} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{\Delta} = \underbrace{\begin{bmatrix} 23.53 \\ 87.50 \\ 363.18 \end{bmatrix}}_{t}$$

$$v = \begin{bmatrix} 0.00 \\ 0.02 \\ -0.04 \\ 0.03 \\ -0.01 \end{bmatrix} \qquad \hat{\ell} = \begin{bmatrix} 1.74 \\ 2.81 \\ 4.29 \\ 6.19 \\ 8.50 \end{bmatrix}$$

1.(a) do again w/ $\sigma_1, \sigma_2, \sigma_3 = 0.02$ and $\sigma_4, \sigma_5 = 0.04$

$\sigma_1^2 = .0004, \ \sigma_4^2 = .0016$ let $\sigma_0^2 = .0016$

Then $W_1, W_2, W_3 = \dfrac{\sigma_0^2}{\sigma_1^2} = \dfrac{.0016}{.0004} = 4$, $W_4, W_5 = \dfrac{\sigma_0^2}{\sigma_4^2} = \dfrac{.0016}{.0016} = 1$

(same condition equations as before)

$$\phi = \sum w_i v_i^2 = 4V_1^2 + 4V_2^2 + 4V_3^2 + V_4^2 + V_5^2$$

$$= 4(a_0 + a_1 + a_2 - 1.74)^2 + 4(a_0 + 2a_1 + 4a_2 - 2.79)^2 + 4(a_0 + 3a_1 + 9a_2 - 4.33)^2$$
$$+ (a_0 + 4a_1 + 16a_2 - 6.16)^2 + (a_0 + 5a_1 + 25a_2 - 8.51)^2$$

$\partial\phi / \partial \text{parameters} = 0$

$$\frac{\partial\phi}{\partial a_0} = \cancel{2}\cdot 4(a_0 + a_1 + a_2 - 1.74) + \cancel{2}\cdot 4(a_0 + 2a_1 + 4a_2 - 2.79) + \cancel{2}\cdot 4(a_0 + 3a_1 + 9a_2 - 4.33)$$
$$+ \cancel{2}(a_0 + 4a_1 + 16a_2 - 6.16) + \cancel{2}(a_0 + 5a_1 + 25a_2 - 8.51) = 0$$

$$\frac{\partial\phi}{\partial a_1} = \cancel{2}\cdot 4(a_0 + a_1 + a_2 - 1.74) + \cancel{2}\cdot 4(a_0 + 2a_1 + 4a_2 - 2.79)\cdot 2 + \cancel{2}\cdot 4(a_0 + 3a_1 + 9a_2 - 4.33)\cdot 3$$
$$+ \cancel{2}(a_0 + 4a_1 + 16a_2 - 6.16)\cdot 4 + \cancel{2}(a_0 + 5a_1 + 25a_2 - 8.51)\cdot 5 = 0$$

$$\frac{\partial\phi}{\partial a_2} = \cancel{2}\cdot 4(a_0 + a_1 + a_2 - 1.74) + \cancel{2}\cdot 4(a_0 + 2a_1 + 4a_2 - 2.79)\cdot 4 + \cancel{2}\cdot 4(a_0 + 3a_1 + 9a_2 - 4.33)\cdot 9$$
$$+ \cancel{2}(a_0 + 4a_1 + 16a_2 - 6.16)\cdot 16 + \cancel{2}(a_0 + 5a_1 + 25a_2 - 8.51)\cdot 25 = 0$$

$$14a_0 + 33a_1 + 97a_2 = 50.11$$

$$33a_0 + 97a_1 + 333a_2 = 148.43$$

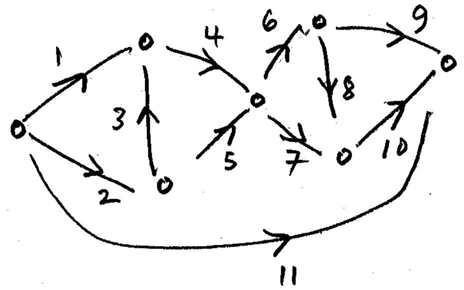
$$97a_0 + 333a_1 + 1273a_2 = 518.79$$

$$\underbrace{\begin{bmatrix} 14 & 33 & 97 \\ 33 & 97 & 333 \\ 97 & 333 & 1273 \end{bmatrix}}_{N} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{\Delta} = \underbrace{\begin{bmatrix} 50.11 \\ 148.43 \\ 518.79 \end{bmatrix}}_{t}$$

$$\Delta = N^{-1}t = \begin{bmatrix} 1.0578 \\ 0.4705 \\ 0.2038 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.01 \\ 0.02 \\ -0.03 \\ 0.04 \\ 0.00 \end{bmatrix} \qquad \hat{l} = \begin{bmatrix} 1.73 \\ 2.81 \\ 4.30 \\ 6.20 \\ 8.51 \end{bmatrix}$$

2.



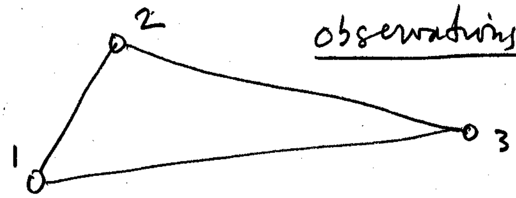
$n = 4$
$n_0 = 6$
$r = 5$

3/6

3.

$dx_{12} = 5.2$
$dx_{13} = 24.7$
$dx_{23} = 19.9$
$dy_{12} = 12.1$
$dy_{13} = 4.4$
$dy_{32} = 7.9$

$n = 6$
$n_0 = 4$
$r = 2$



observations only w/ L.M.

write r = 2 condition equations

$$\hat{d}x_{12} + \hat{d}x_{23} = \hat{d}x_{13}$$
$$\hat{d}y_{12} = \hat{d}y_{13} + \hat{d}y_{32}$$

$$\left.\begin{array}{l} dx_{12} + Vx_{12} + dx_{23} + Vx_{23} - dx_{13} - Vx_{13} = 0 \\ dy_{12} + Vy_{12} - dy_{13} - Vy_{13} - dy_{32} - Vy_{32} = 0 \end{array}\right\}$$

$$Vx_{12} + Vx_{23} - Vx_{13} = -dx_{12} - dx_{23} + dx_{13}$$
$$Vy_{12} - Vy_{13} - Vy_{32} = -dy_{12} + dy_{13} + dy_{32}$$

$$Vx_{12} + Vx_{23} - Vx_{13} = -0.4$$
$$Vy_{12} - Vy_{13} - Vy_{32} = 0.2$$

$$\phi' = Vx_{12}^2 + Vx_{13}^2 + Vx_{23}^2 + Vy_{12}^2 + Vy_{13}^2 + Vy_{32}^2 + 2k_1(Vx_{12} + Vx_{23} - Vx_{13} + 0.4)$$
$$+ 2k_2(Vy_{12} - Vy_{13} - Vy_{32} - 0.2)$$

$$\partial\phi'/\partial Vx_{12} = \cancel{2}Vx_{12} - \cancel{2}k_1 = 0$$
$$\partial\phi'/\partial Vx_{13} = \cancel{2}Vx_{13} + \cancel{2}k_1 = 0$$
$$\partial\phi'/\partial Vx_{23} = \cancel{2}Vx_{23} - \cancel{2}k_1 = 0$$
$$\partial\phi'/\partial Vy_{12} = \cancel{2}Vy_{12} - \cancel{2}k_2 = 0$$
$$\partial\phi'/\partial Vy_{13} = \cancel{2}Vy_{13} + \cancel{2}k_2 = 0$$
$$\partial\phi'/\partial Vy_{32} = \cancel{2}Vy_{32} + \cancel{2}k_2 = 0$$

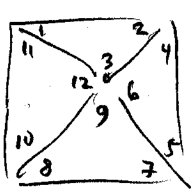
$$\partial\phi'/\partial k_1 = \cancel{2}(Vx_{12} + Vx_{23} - Vx_{13} + 0.4) = 0$$
$$\partial\phi'/\partial k_2 = \cancel{2}(Vy_{12} - Vy_{13} - Vy_{32} - 0.2) = 0$$

$$
\begin{array}{cccccccc}
V_{x_{12}} & V_{x_{13}} & V_{x_{23}} & V_{y_{12}} & V_{y_{13}} & V_{y_{32}} & k_1 & k_2
\end{array}
$$

$$
\left[\begin{array}{cccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
\hline
1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & 0 & 0
\end{array}\right]
\begin{bmatrix} V_{x_{12}} \\ V_{x_{13}} \\ V_{x_{23}} \\ V_{y_{12}} \\ V_{y_{13}} \\ V_{y_{32}} \\ k_1 \\ k_2 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.4 \\ 0.2 \end{bmatrix}
$$

$$
\begin{bmatrix} V_{x12} \\ V_{x13} \\ V_{x23} \\ V_{y12} \\ V_{y13} \\ V_{y32} \\ K_1 \\ K_2 \end{bmatrix}
=
\begin{bmatrix} -0.13 \\ 0.13 \\ -0.13 \\ 0.07 \\ -0.07 \\ -0.07 \\ 0.13 \\ -0.07 \end{bmatrix}
\qquad
\begin{bmatrix} \hat{d}x_{12} = \\ \hat{d}x_{13} = \\ \hat{d}x_{23} = \\ dy_{12} = \\ dy_{13} = \\ dy_{32} = \end{bmatrix}
\begin{bmatrix} 5.1 \\ 24.8 \\ 19.8 \\ 12.2 \\ 4.3 \\ 7.8 \end{bmatrix}
$$

4.



$n = 12$
$n_0 = 6$
$r = 6$

$\hat{\ell}_1 + \hat{\ell}_2 + \hat{\ell}_3 = 180^\circ$

$\hat{\ell}_4 + \hat{\ell}_5 + \hat{\ell}_6 = 180^\circ$

$\hat{\ell}_7 + \hat{\ell}_8 + \hat{\ell}_9 = 180^\circ$

$\hat{\ell}_{10} + \hat{\ell}_{11} + \hat{\ell}_{12} = 180^\circ$

$\hat{\ell}_3 + \hat{\ell}_6 + \hat{\ell}_9 + \hat{\ell}_{12} = 360^\circ$

$\hat{\ell}_1 + \hat{\ell}_2 + \hat{\ell}_4 + \hat{\ell}_5 + \hat{\ell}_7 + \hat{\ell}_8 + \hat{\ell}_{10} + \hat{\ell}_{11} = 360^\circ$

$v_1 + v_2 + v_3 = 180 - \ell_1 - \ell_2 - \ell_3$

$v_4 + v_5 + v_6 = 180 - \ell_4 - \ell_5 + \ell_6$

$v_7 + v_8 + v_9 = 180 - \ell_7 - \ell_8 - \ell_9$

$v_{10} + v_{11} + v_{12} = 180 - \ell_{10} - \ell_{11} - \ell_{12}$

$v_3 + v_6 + v_9 + v_{12} = 360 - \ell_3 - \ell_6 - \ell_9 - \ell_{12}$

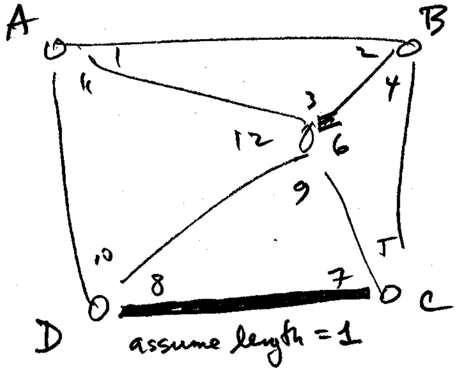
$v_1 + v_2 + v_4 + v_5 + v_7 + v_8 + v_{10} + v_{11} = 360 - \ell_1 - \ell_2 - \ell_4 - \ell_5 - \ell_7 - \ell_8 - \ell_{10} - \ell_{11}$?

$$
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12
\end{array}
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

oops - it's rank $\underline{\underline{5}}$

cannot get 6 independent equations with "angles only"!
must use "side condition"; idea:



1. assume length = 1
2. use law of sines to compute successively EC, then EB, then AB
3. use law of sines to compute successively ED, AE, then AB
4. expressions from 2 & 3 must be equal. That's is the "mystery" 6th cond. eqn. but it is non linear!

$$\frac{\sin(\theta_9)}{1} = \frac{\sin(\theta_8)}{EC}, \quad EC = \frac{\sin(\theta_8)}{\sin(\theta_9)}$$

$$\frac{\sin(\theta_4)}{EC} = \frac{\sin(\theta_5)}{EB}, \quad EB = \frac{\sin(\theta_5)}{\sin(\theta_4)} \cdot EC$$

$$EB = \frac{\sin\theta_5}{\sin\theta_4} \cdot \frac{\sin(\theta_8)}{\sin(\theta_9)}$$

$$\frac{\sin\theta_1}{EB} = \frac{\sin(\theta_3)}{AB}, \quad AB = \frac{\sin(\theta_3)}{\sin(\theta_1)} \cdot EB$$

$$\boxed{AB = \frac{\sin\theta_5}{\sin\theta_4} \cdot \frac{\sin\theta_8}{\sin\theta_9} \cdot \frac{\sin\theta_3}{\sin\theta_1}}$$

$$\frac{1}{\sin\theta_9} = \frac{ED}{\sin\theta_7} \quad , \quad ED = \frac{\sin\theta_7}{\sin\theta_9}$$

$$\frac{ED}{\sin\theta_{11}} = \frac{AE}{\sin\theta_{10}} \quad , \quad AE = \frac{\sin\theta_{10}}{\sin\theta_{11}} \cdot ED$$

$$AE = \frac{\sin\theta_{10}}{\sin\theta_{11}} \cdot \frac{\sin\theta_7}{\sin\theta_9}$$

$$\frac{AE}{\sin\theta_2} = \frac{AB}{\sin\theta_3} \quad , \quad AB = \frac{\sin\theta_3}{\sin\theta_2} \cdot AE$$

$$\boxed{AB = \frac{\sin\theta_3}{\sin\theta_2} \cdot \frac{\sin\theta_{10}}{\sin\theta_{11}} \cdot \frac{\sin\theta_7}{\sin\theta_9}}$$

condition equation

$$\frac{\sin\theta_5}{\sin\theta_4} \cdot \frac{\sin\theta_8}{\cancel{\sin\theta_9}} \cdot \frac{\cancel{\sin\theta_3}}{\sin\theta_1} = \frac{\cancel{\sin\theta_3}}{\sin\theta_2} \cdot \frac{\sin\theta_{10}}{\sin\theta_{11}} \cdot \frac{\sin\theta_7}{\cancel{\sin\theta_9}}$$

$$\boxed{\frac{\sin\theta_5}{\sin\theta_4} \cdot \frac{\sin\theta_8}{\sin\theta_1} = \frac{\sin\theta_7}{\sin\theta_2} \cdot \frac{\sin\theta_{10}}{\sin\theta_{11}}}$$

but non linear