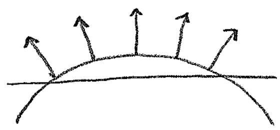

z: h ellipsoid height
H orthometric height
xy map projection

} not cartesian (may consider it cartesian over a small area)

$\Rightarrow$ we need a better model to do 3D surveying over a large area

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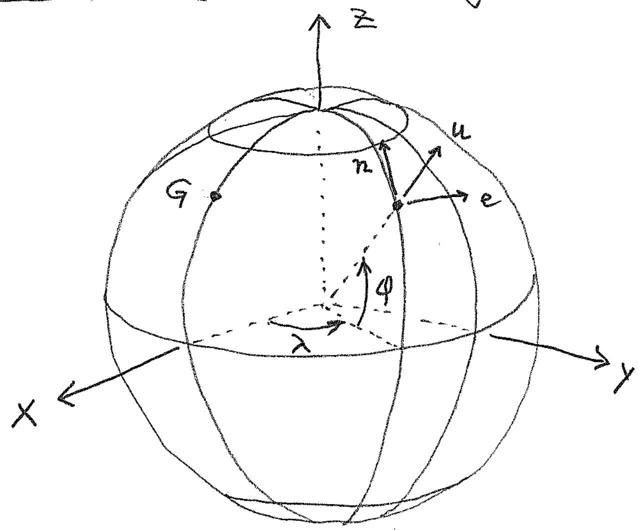
Adjustment of 3D geodetic surveying
16 oct 2019

Preliminaries: transform geocentric $\rightarrow$ local cartesian



rotate $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \rightarrow \begin{bmatrix} e \\ n \\ u \end{bmatrix}$

1. $M_z(\lambda + 90)$
2. $M_x(90 - \varphi)$

$$\begin{bmatrix} e \\ n \\ u \end{bmatrix} = M_x(90-\varphi)\, M_z(\lambda+90) \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix}$$

geocentric   local reference

$$M_x(90-\varphi)\, M_z(\lambda+90): \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(90-\varphi) & \sin(90-\varphi) \\ 0 & -\sin(90-\varphi) & \cos(90-\varphi) \end{bmatrix} \begin{bmatrix} \cos(\lambda+90) & \sin(90+\lambda) & 0 \\ -\sin(\lambda+90) & \cos(90+\lambda) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\varphi & \cos\varphi \\ 0 & -\cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\cos\lambda & -\sin\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}}_{\text{"to"}} = \underbrace{\begin{bmatrix} \square & \cdot & \cdot \\ \square & \cdot & \cdot \\ \square & \cdot & \cdot \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{"from"}}, \quad \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix} = \begin{bmatrix} \cdot & \square & \cdot \\ \cdot & \square & \cdot \\ \cdot & \square & \cdot \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{etc.}$$

columns of rotation matrix represent unit vectors of "from" system, expressed in coordinates of the "to" system.

$$\begin{bmatrix} e \\ n \\ u \end{bmatrix} = M \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}, \quad \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = M^T \begin{bmatrix} e \\ n \\ u \end{bmatrix}$$

we want unit vectors in e, n, u expressed in coordinates of X, Y, Z $\Rightarrow$ columns of M^T = rows of M

$$\hat{e} = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix}, \quad \hat{n} = \begin{bmatrix} -\sin\varphi\cos\lambda \\ -\sin\varphi\sin\lambda \\ \cos\varphi \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi \end{bmatrix}$$

$$\begin{pmatrix}\phi\\ \lambda\\ h\end{pmatrix} \rightleftarrows \begin{pmatrix}X\\ Y\\ Z\end{pmatrix},$$

ellipsoid	a	f
GRS80/NAD83	6378137 m	1/298.257222101
WGS84	6378137 m	1/298.257223563

$\phi\lambda h \to xyz$

$$X = (N+h)\cos\phi\cos\lambda$$
$$Y = (N+h)\cos\phi\sin\lambda$$
$$Z = (N(1-e^2)+h)\sin\phi$$

$$e^2 = 2f - f^2, \quad e = \sqrt{2f-f^2}$$

$$N = \frac{a}{\sqrt{1-e^2\sin^2\phi}}$$

$$f = \frac{a-b}{a}, \quad e^2 = \frac{a^2-b^2}{a^2}$$

$xyz \to \phi\lambda h$

$\lambda = \operatorname{atan}(Y/X)$ consider quadrant!

initial approximations:

$$\phi^\circ = \operatorname{atan}\left[\frac{Z}{\sqrt{X^2+Y^2}}\right], \quad h^\circ = 0$$

iterate:

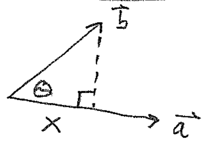
$$N = \frac{a}{\sqrt{1-e^2\sin^2\phi}}$$

$$\phi = \operatorname{atan}\left[\frac{Z}{\sqrt{X^2+Y^2}}\left(1-e^2\left(\frac{N}{N+h}\right)\right)^{-1}\right]$$

$$h = \frac{(X^2+Y^2)^{1/2}}{\cos\phi} - N$$

until no change in ϕ, h

projection of $\vec{b}$ onto $\vec{a}$

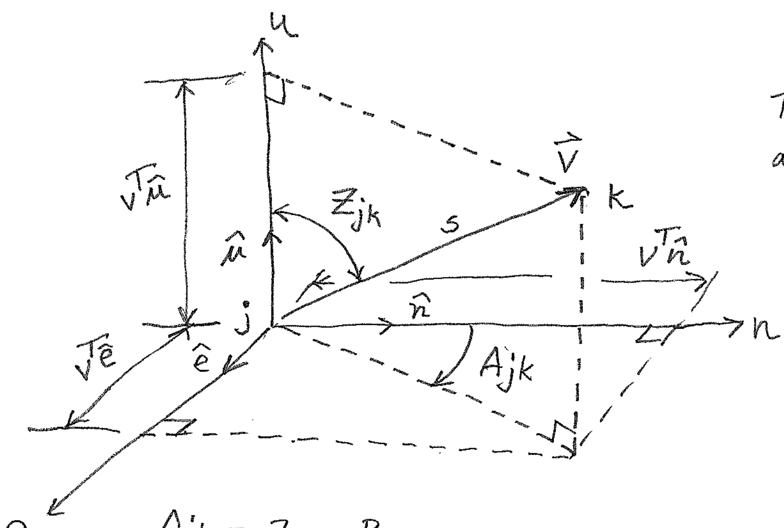


$$\left.\begin{array}{l}\vec{a}\cdot\vec{b} = |a||b|\cos\theta\\[4pt] \dfrac{x}{|b|} = \cos\theta\end{array}\right\} \quad \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{x}{|\vec{b}|}, \quad x = \vec{b}\cdot\hat{a}$$

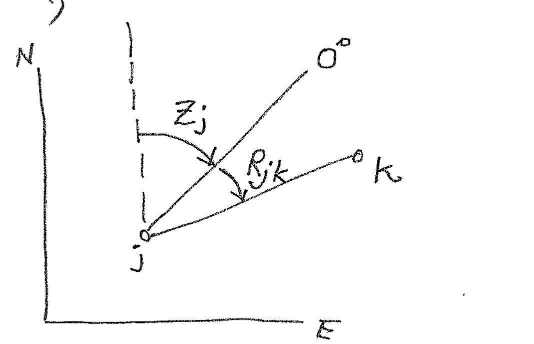
$x = b^T\hat{a}$

$\hat{a}$: unit vector

For 3D geodetic measurements, make a e,n,u local coordinate system at <u>each</u> instrument station. instrument station, j, and target point, k.



We handle the arbitrary location of the horizontal circle zero, by carrying an "orientation unknown", Z_j, see sketch,



$$A_{jk} = Z_j + R_{jk}$$

azimuth — orientation unknown — jk direction observation

$$A_{jk} = \operatorname{atan}\left(\frac{v^T\hat{e}}{v^T\hat{n}}\right)$$

(note: for the moment we ignore deflection of the vertical.)

condition equation for <u>jk-direction</u> $\boxed{F_{Az} = R_{jk} + Z_j - A_{jk} = 0}$

$$F_{Az} = R_{jk} + Z_j - \operatorname{atan}\left[\frac{-(x_k - x_j)\sin\lambda + (y_k - y_j)\cos\lambda}{-(x_k - x_j)\sin\varphi\cos\lambda - (y_k - y_j)\sin\varphi\sin\lambda + (z_k - z_j)\cos\varphi}\right]$$

$$\frac{\partial F_{Az}}{\partial x_j} = -\left[\frac{1}{1 + \left(\frac{v^Te}{v^Tn}\right)^2}\right] \cdot \frac{v^Tn\sin\lambda - v^Te\sin\varphi\cos\lambda}{(v^Tn)^2} \qquad v = \begin{bmatrix} x_k - x_j \\ y_k - y_j \\ z_k - z_j \end{bmatrix}$$

$$\frac{\partial F_{Az}}{\partial y_j} = -\left[\frac{1}{1 + \left(\frac{v^Te}{v^Tn}\right)^2}\right] \cdot \frac{-v^Tn\cos\lambda - v^Te\sin\varphi\sin\lambda}{(v^Tn)^2}$$

$$\frac{\partial F_{Az}}{\partial z_j} = -\left[\frac{1}{1 + \left(\frac{v^Te}{v^Tn}\right)^2}\right] \cdot \frac{v^Te\cos\phi}{(v^Tn)^2}$$

notation confusion: 3 z's
Z_j : orientation unknown
z_j : z-coordinate
Z_{jk} : zenith angle

$$\frac{\partial F_{Az}}{\partial x_k} = -\frac{\partial F_{Az}}{\partial x_j}, \quad \frac{\partial F_{Az}}{\partial y_k} = -\frac{\partial F_{Az}}{\partial y_j}, \quad \frac{\partial F_{Az}}{\partial z_k} = \frac{\partial F_{Az}}{\partial z_j}$$

$$\frac{\partial F_{Az}}{\partial Z_j} = 1$$

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condition equation for <u>zenith angle</u> $\quad \cos Z_{jk} = \frac{v^Tu}{s}$

$$Z_{jk} = \operatorname{acos}\left(\frac{v^Tu}{s}\right) = \operatorname{acos}\left[\frac{(x_k - x_j)\cos\varphi\cos\lambda + (y_k - y_j)\cos\varphi\sin\lambda + (z_k - z_j)\sin\varphi}{\left[(x_k - x_j)^2 + (y_k - y_j)^2 + (z_k - z_j)^2\right]^{1/2}}\right]$$

$$s = \sqrt{v^Tv}, \qquad \frac{d}{dx}\operatorname{acos} u = \frac{-1}{(1 - u^2)^{1/2}} \cdot \frac{du}{dx}$$

$$\boxed{F_Z = Z_{jk} - \operatorname{acos}\left(\frac{v^Tu}{s}\right) = 0}$$

$$\frac{\partial F_z}{\partial x_j} = \frac{1}{\left[1-\left(\frac{v^T u}{s}\right)^2\right]^{1/2}} \cdot \frac{-s\cos\varphi\cos\lambda + \frac{v^T u}{s}(x_k - x_j)}{s^2}$$

$$\frac{\partial F_z}{\partial y_j} = \frac{1}{\left[1-\left(\frac{v^T u}{s}\right)^2\right]^{1/2}} \cdot \frac{-s\cos\varphi\sin\lambda + \frac{v^T u}{s}(y_k - y_j)}{s^2}$$

$$\frac{\partial F_z}{\partial z_j} = \frac{1}{\left[1-\left(\frac{v^T u}{s}\right)^2\right]^{1/2}} \cdot \frac{-s\sin\varphi + \frac{v^T u}{s}(z_k - z_j)}{s^2}$$

$$\frac{\partial F_z}{\partial x_k} = -\frac{\partial F_z}{\partial x_j}, \quad \frac{\partial F_z}{\partial y_k} = -\frac{\partial F_z}{\partial y_j}, \quad \frac{\partial F_z}{\partial z_k} = -\frac{\partial F_z}{\partial z_j}$$

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condition equation for slope distance / slant distance:

$$s_{jk} = (v^T v)^{1/2} = \left[(x_k - x_j)^2 + (y_k - y_j)^2 + (z_k - z_j)^2\right]^{1/2}$$

$$\boxed{F_d = s_{jk} - \left[(x_k - x_j)^2 + (y_k - y_j)^2 + (z_k - z_j)^2\right]^{1/2} = 0}$$

$$\frac{\partial F_d}{\partial x_j} = \frac{x_k - x_j}{s}$$

$$\frac{\partial F_d}{\partial y_j} = \frac{y_k - y_j}{s}$$

$$\frac{\partial F_d}{\partial z_j} = \frac{z_k - z_j}{s}$$

$$\frac{\partial F_d}{\partial x_k} = -\frac{\partial F_d}{\partial x_j}, \quad \frac{\partial F_d}{\partial y_k} = -\frac{\partial F_d}{\partial y_j}, \quad \frac{\partial F_d}{\partial z_k} = -\frac{\partial F_d}{\partial z_j}$$

References:
1. Ghilani & Wolf, Adjustment Computations..., 4th Ed., Wiley 2006
2. Strang & Borre, Linear Algebra, Geodesy, + GPS, Wellesley, 1997
3. Leick, GPS Satellite Surveying, 3rd Ed., Wiley, 2004