Wednesday, April 14, 2010
7:39 AM

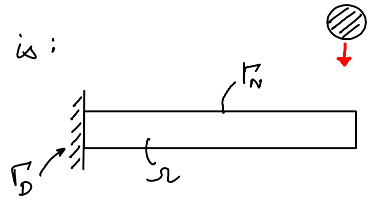
Dynamics

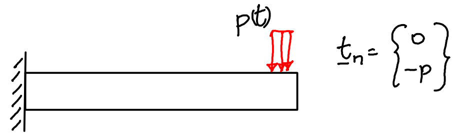
For dynamic problems the equation of motion is:

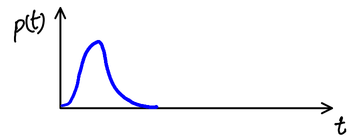
$$\operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \rho \underline{\ddot{u}} \qquad \text{in } \Omega$$

$$\underline{u}(\underline{x}, t) = u_0(\underline{x}, t) \qquad \text{on } \Gamma_D$$

$$\underline{t}_n(\underline{x}, t) = \underline{t}_0(\underline{x}, t) \qquad \text{on } \Gamma_N$$







Find $\underline{u}(\underline{x}, t)$

Weak form: Method of weighted residuals

$$G(\underline{u}, \underline{\bar{u}}) \equiv \int_\Omega \underline{\bar{u}} \left(\operatorname{div} \underset{\sim}{\sigma} + \underline{b} - \rho \underline{\ddot{u}} \right) d\Omega$$

Integration by parts:

$$G(\underline{u}, \underline{\bar{u}}) = - \left[\int_\Omega \left(\underline{\bar{u}} \cdot \rho \underline{\ddot{u}} + \nabla \underline{\bar{u}} : \underset{\sim}{\sigma} - \underline{\bar{u}} \cdot \underline{b} \right) d\Omega - \int_{\Gamma_N} \bar{u}\, \underline{t}_n \, d\Gamma \right]$$

Discretization + Finite Element Approximation:

$$\int_\Omega \cdot = \sum_{e=1}^{M} \int_{\Omega^e} \cdot$$

$$\underline{u}(\underline{x}, t) \approx \underset{\sim}{N}(\underline{x}) \; \underline{d}(t) \quad \Rightarrow \quad \underline{\ddot{u}} \approx \underset{\sim}{N} \, \underline{\ddot{d}}$$

$$\underline{\epsilon} \approx \underset{\sim}{B} \; \underline{d}$$

$$\underline{\sigma} \approx \underset{\sim}{D} \, \underset{\sim}{B} \; \underline{d}$$

$$\tilde{G}^h(\underline{d}, \underline{\bar{d}}) =$$

$$- \sum_{e=1}^{M} \underline{\bar{d}}^{e^T} \left[\underbrace{\left(\int_{\Omega^e} \rho \underset{\sim}{N}^T \underset{\sim}{N} \, d\Omega \right)}_{\underset{\sim}{M}^e} \underline{\ddot{d}}^e + \underbrace{\left(\int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D} \, \underset{\sim}{B} \, d\Omega \right)}_{\underset{\sim}{K}^e} \underline{d} - \underbrace{\left(\int_{\Omega^e} \underset{\sim}{N}^T \underline{b} \, d\Omega - \int_{\Gamma_N^e} \underset{\sim}{N}^T \underline{t}_n \, d\Gamma \right)}_{\underline{f}^e} \right] = 0$$

Assembly $\overset{M}{\underset{e=1}{\mathsf{A}}} \Rightarrow \tilde{G}^h(\underline{d}, \underline{\bar{d}}) = - \underline{\bar{d}}^{g^T} \left[\underset{\sim}{M}^g \underline{\ddot{d}}^g(t) + \underset{\sim}{K}^g \underline{d}^g(t) - \underline{f}^g(t) \right] = 0 \quad \forall \underline{\bar{d}}$

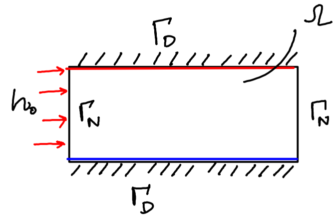
This gives us the <u>semi-discrete</u> equations of motion.
This is a 2^{nd} order <u>system</u> of <u>ordinary differential equations</u> (ODEs).

Sometimes artificial <u>physical damping</u> is also included:

$$\underset{\sim}{M}^g \underline{\ddot{d}}^g + \underset{\sim}{C}^g \underline{\dot{d}}^g + \underset{\sim}{K}^g \underline{d}^g - \underline{f}^g = \underline{0}$$

where $\underset{\sim}{C} = \alpha \underset{\sim}{M} + \beta \underset{\sim}{K}$. This is called Raleigh Damping.

Transient Heat Conduction



Governing Equation :

$$\left.\begin{aligned} \rho c \dot{\theta} + \operatorname{div}(\underset{\sim}{q}) &= f \\ \underset{\sim}{q} &= -\underset{\sim}{K} \nabla\theta \end{aligned}\right] \text{ in } \Omega$$

BCs :

$$\theta(\underset{\sim}{x},t) = \theta_0(\underset{\sim}{x},t) \quad \text{on } \Gamma_D$$

$$\underset{\sim}{q}(\underset{\sim}{x},t)\cdot\underset{\sim}{n} = h_0(\underset{\sim}{x},t) \quad \text{on } \Gamma_N$$

Weak Form + Discretization + FE approximation

$$\Rightarrow \quad \tilde{G}^h(\underset{\sim}{d},\bar{\underset{\sim}{d}}) = -\bar{\underset{\sim}{d}}^{g^T}\left[\underset{\sim}{M}^g \dot{\underset{\sim}{d}}^g(t) + \underset{\sim}{K}^g \underset{\sim}{d}^g(t) - \underset{\sim}{f}^g \right] = 0 \quad \forall \bar{\underset{\sim}{d}}$$

where $\underset{\sim}{M}^g = \mathop{\mathsf{A}}_{e=1}^{M} \underset{\sim}{M}^e$ and $\underset{\sim}{M}^e = \int_{\Omega^e} \rho c\, \underset{\sim}{N}^T \underset{\sim}{N}\, d\Omega$

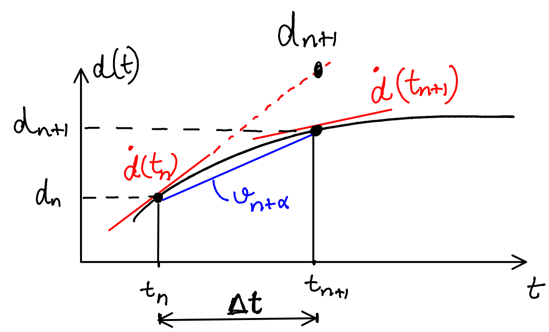
$\underset{\sim}{K}^g = \mathop{\mathsf{A}}_{e=1}^{M} \underset{\sim}{K}^e$ and $\underset{\sim}{K}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{K} \underset{\sim}{B}\, d\Omega$

Time Integration :

For 1st order systems :

$$d_n \approx d(t_n)$$

$$v_n \approx \dot{d}(t_n)$$



Euler's Method :

$$\boxed{\begin{aligned} \underset{-}{v}_{n+\alpha} &= (1-\alpha)\, \underset{-}{v}_n + \alpha\, \underset{-}{v}_{n+1} \\ \underset{-}{d}_{n+1} &= \underset{-}{d}_n + \Delta t\, \underset{-}{v}_{n+\alpha} \end{aligned}} \quad \begin{aligned} &— (1) \\ &— (2) \end{aligned}$$

$\alpha = 0 \Rightarrow$ Forward Euler (Explicit)

$\alpha = 1 \Rightarrow$ Backward Euler (Implicit)

$\alpha = \frac{1}{2} \Rightarrow$ Mid point rule
Trapezoidal "
Crank-Nicolson "
(Implicit)

Enforce governing equation at t_{n+1} :

$$\underset{\sim}{M}\, \underset{-}{v}_{n+1} + \underset{\sim}{K}\, \underset{-}{d}_{n+1} - \underset{-}{f}_{n+1} = \underset{-}{0}$$

Substitute (1) & (2) :-

$$\boxed{\left[\underset{\sim}{M} + \alpha\Delta t\, \underset{\sim}{K}\right] \underset{-}{v}_{n+1} = \left\{ \underset{-}{f}_{n+1} - \underset{\sim}{K}\left(\underset{-}{d}_n + \Delta t\,(1-\alpha)\, \underset{-}{v}_n \right) \right\}}$$

Solve for $\underset{-}{v}_{n+1}$ $\Rightarrow$ find $\underset{-}{d}_{n+1}$ from (1) & (2)

Repeat for next time step.

Time Integration for Structural Dynamics

To integrate 2^{nd} order ODEs:

$$\underset{\sim}{M}\,\ddot{\underline{d}} + \underset{\sim}{C}\,\dot{\underline{d}} + \underset{\sim}{K}\,\underline{d} - \underline{f} = \underline{0}$$

We can recast them as:
(system of 1^{st} order ODEs)
(However, size is doubled)

$$\dot{\underline{d}} = \underline{v}$$
$$\underset{\sim}{M}\,\dot{\underline{v}} + \underset{\sim}{C}\,\underline{v} + \underset{\sim}{K}\,\underline{d} - \underline{f} = \underline{0}$$

Alternatively, use Newmark Family of Integrators:

Approximate $\underline{a}_n \approx \ddot{d}(t_n)$ (Parameters: γ, β)

$\underline{v}_n \approx \dot{d}(t_n)$

$\underline{d}_n \approx d(t_n)$

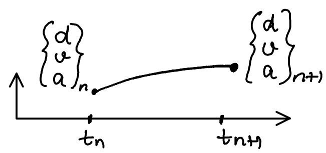
Method:

$$\boxed{\begin{aligned} \underline{v}_{n+1} &= \underline{v}_n + \Delta t\left[(1-\gamma)\,\underline{a}_n + \gamma\,\underline{a}_{n+1}\right] \\ \underline{d}_{n+1} &= \underline{d}_n + \Delta t\,\underline{v}_n + \Delta t^2\left[\left(\tfrac{1}{2}-\beta\right)\underline{a}_n + \beta\,\underline{a}_{n+1}\right] \end{aligned}}$$

— (1)

— (2)

Enforce equations of motion at t_{n+1}:



$$\boxed{\left[\underset{\sim}{M} + \gamma\Delta t\,\underset{\sim}{C} + \tfrac{1}{2}\beta\,\Delta t^2\,\underset{\sim}{K}\right]\underline{a}_{n+1} = \left\{\underline{f}_{n+1} - \left[\underline{v}_n + \Delta t(1-\gamma)\,\underline{a}_n\right]\underset{\sim}{C} - \left[\underline{d}_n + \Delta t\,\underline{v}_n + \Delta t^2\left(\tfrac{1}{2}-\beta\right)\underline{a}_n\right]\underset{\sim}{K}\right\}}$$

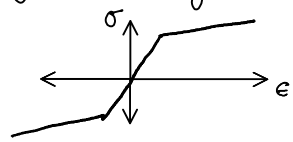
Solve for $\underline{a}_{n+1}$ and then $\underline{v}_{n+1}$ & $\underline{d}_{n+1}$ from (1) & (2).

Common Methods:

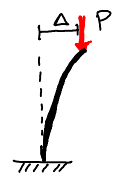
γ	β	Method	Type	Stability	Accuracy
$1/2$	0	Central-difference	Explicit	$\Delta t < \frac{2}{\omega_{max}}$	$O(\Delta t^2)$
$1/2$	$1/4$	Trapezoidal Rule (Const. Avg Accele)	Implicit	Unconditional	$O(\Delta t^2)$
$1/2$	$1/6$	Linear Avg. Acceleration	Implicit	$\Delta t < \frac{2\sqrt{3}}{\omega_{max}}$	$O(\Delta t^2)$

Non-Linear Problems

For structural / solid mechanics problems, nonlinearity arises from



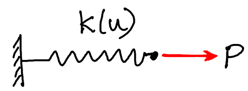
- Material → eg. Plasticity
- Geometric (P–Δ) → $\begin{cases} \underset{\sim}{F} = \dfrac{\partial x}{\partial X} \quad ; \quad \underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} \\ \underset{\sim}{E} = \frac{1}{2}\left(\underset{\sim}{C} - \underset{\sim}{I}\right) = \frac{1}{2}\left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T + \nabla \underset{\sim}{u}^T \nabla \underset{\sim}{u}\right) \end{cases}$



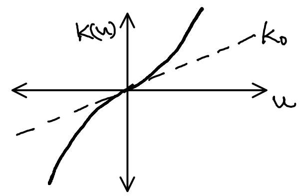
(or both)

Newton's Method for Numerical Solution of Non-linear Problems:

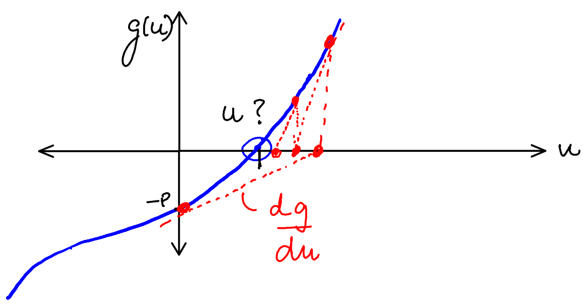
First, consider a Single-Degree-of-freedom (SDOF)



$$g(u) = k(u) - P = 0 \qquad (\text{find } u)$$



$$k(u) = k_o u + k_1 u^3$$



Steps: 1) Assume a solution $u = u^0$

2) Find $g(u)$

3) If $\quad g(u) \overset{*}{=} 0 \quad \rightarrow$ return u

Else

- Solve: $g(u + \Delta u) = 0$ for Δu

$$g(u+\Delta u) \approx g(u) + \underbrace{\left[\frac{dg}{du}\right]_u}_{\substack{K_T \\ (\text{Tangent})}} \Delta u = 0$$

$$[K_T]\,\Delta u = \{-g\} \qquad \left\{ K_T = \frac{dk}{du} = k_o + 3k_1 u^2 \right.$$

- $u = u + \Delta u$

End

4) Repeat 2 & 3.

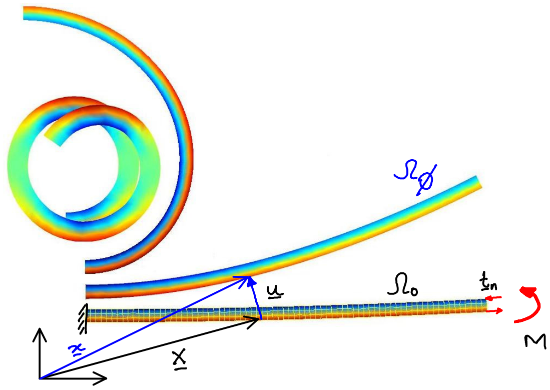
For System of Non-linear Equations

Strong form:

$$\underset{\frac{\partial}{\partial x}}{\operatorname{div}}\,\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \qquad \text{on } \Omega_\phi$$

or

$$\underset{\frac{\partial}{\partial X}}{\mathrm{DIV}}\,\mathbf{P} + \mathbf{B} = 0 \qquad \text{on } \Omega_0$$



Weak form:

$$G(\mathbf{u}, \bar{\mathbf{u}}) = \int_{\Omega_\phi} \bar{\mathbf{u}} \left(\operatorname{div} \boldsymbol{\sigma} + \mathbf{b} \right) d\Omega_\phi$$

$$= \int_{\Omega_0} \underbrace{\bar{\mathbf{u}} \left(\operatorname{div} \boldsymbol{\sigma} + \mathbf{b} \right) |J|}_{\bar{\mathbf{u}}\left(\mathrm{DIV}\,\mathbf{P} + \mathbf{B}\right)} \, d\Omega_0$$

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u} \; ; \quad |J| = \det(\mathbf{F})$$

$$\mathbf{P} = |J| \boldsymbol{\sigma} \mathbf{F}^{-T} \; ; \quad \mathbf{B} = \mathbf{b}\,|J|$$

$$\mathbf{S} = |J|\, \mathbf{F}^{-1} \boldsymbol{\sigma} \mathbf{F}^{-T}$$

Integration by parts:

$$G(\mathbf{u}, \bar{\mathbf{u}}) = -\left[\int_{\Omega_0} \underbrace{\left(\overset{\bar{\mathbf{F}}}{\nabla \bar{\mathbf{u}}} : \mathbf{P} \right)}_{\left(\bar{\mathbf{E}} \,:\, \mathbf{S}\right)} d\Omega_0 - \int_{\Omega_0} \bar{\mathbf{u}} \cdot \mathbf{B} \, d\Omega_0 - \int_{\Gamma_{N_0}} \bar{\mathbf{u}} \cdot \mathbf{T}_N \, d\Gamma_0 \right]$$

Discretization + FE approximation: $\mathbf{u} = \mathbf{N}(\mathbf{X})\, \mathbf{d}$; $\mathbf{E} = \overset{*}{\mathbf{B}}(\mathbf{X})\, \mathbf{d}$

$$\tilde{G}^h(\mathbf{d}, \bar{\mathbf{d}}) = -\sum_{e=1}^{M} \bar{\mathbf{d}}^{eT} \left[\underbrace{\int_{\Omega_0^e} \mathbf{B}^T \underset{\text{Voight}}{\mathbf{S}} \, d\Omega_0}_{} - \underbrace{\int_{\Omega_0^e} \mathbf{N}^T \mathbf{B} \, d\Omega_0 - \int_{\Gamma_{N_0}} \mathbf{N}^T \mathbf{T}_N \, d\Gamma_0}_{} \right]$$

$$\tilde{G}^h(\mathbf{d}, \bar{\mathbf{d}}) = -\bar{\mathbf{d}}^{gT} \left(\mathbf{f}^{int}(\mathbf{d}^g) - \mathbf{f}^{ext}(\mathbf{d}^g) \right) = 0 \qquad \forall\, \bar{\mathbf{d}}^g$$

Thus to solve for $\mathbf{d}^g$:

$$\boxed{\mathbf{g}(\mathbf{d}^g) = \mathbf{f}^{int}(\mathbf{d}^g) - \mathbf{f}^{ext}(\mathbf{d}^g) = 0}$$

For Newton's method we need "tangent":

$$\mathbf{K}_T^g = \frac{\partial\, \mathbf{g}(\mathbf{d})}{\partial \mathbf{d}} = \mathop{\mathrm{A}}_{e=1}^{M} \mathbf{K}_T$$

$$\mathbf{K}_T = \int_{\Omega_0^e} \mathbf{B}^T \overset{\mathbf{D}}{\left(\frac{\partial \mathbf{S}}{\partial \mathbf{E}} \right)} \mathbf{B} \, d\Omega_0 + \int_{\Omega_0^e} \frac{\partial \mathbf{N}^T}{\partial \mathbf{X}} \mathbf{S} \frac{\partial \mathbf{N}}{\partial \mathbf{X}} \, d\Omega_0 + \int_{\Omega_0^e} \mathbf{N}^T \left(\frac{\partial \mathbf{B}}{\partial \mathbf{d}} \right) d\Omega_0 + \int_{\Gamma_N^e} \mathbf{N}^T \left(\frac{\partial \mathbf{T}_N}{\partial \mathbf{d}} \right) d\Gamma_0$$

$$\mathbf{K}_T = \underset{\text{Material}}{\mathbf{K}_M} + \underset{\text{Geometric}}{\mathbf{K}_G} + \underset{\text{Load}}{\mathbf{K}_L}$$

Newton's Method:

Load Steps

1) Assume a solution $\underline{d}_i^g = \underline{d}_0^g$

2) Calculate $\underline{g}(\underline{d}_i^g)$

3) If $\|\underline{g}(\underline{d}_i^g)\| < tol \rightarrow$ return $\underline{d}_i^g$

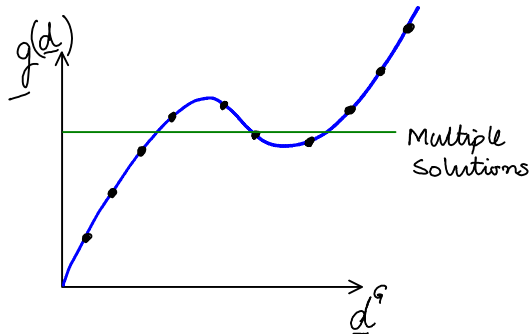
Else

- solve: $[\underset{\sim}{K}_T^g]\,\Delta\underline{d} = -\underline{g}(\underline{d}_i^g)$
- $\underline{d}_{i+1}^g = \underline{d}_i^g + \Delta\underline{d}$

End

4) Repeat 2 & 3

Repeat 1) thru 4).



In Class Assignment:

$$K(u) = K_0 u + K_1 u^3 \quad ; \quad K_0 = 1 \quad K_1 = 1$$

$$g(u) = K(u) - P = 0 \quad ; \quad P = 1$$

$$\Rightarrow u = ?$$

(i) Assumption $u^0 = \frac{P}{K_0} = \frac{1}{1} = 1$

$$\Rightarrow g(u^0) = 1 + 1 - 1 = 1$$

Find u^1 ?

$$K_T = \frac{\partial g}{\partial u} = K_0 + 3K_1 u^2 = 4$$

$$\Rightarrow K_T \Delta u = -g \Rightarrow \Delta u = \frac{-1}{4}$$

$$u^1 = u^0 + \Delta u = 1 - 1/4 = 3/4$$