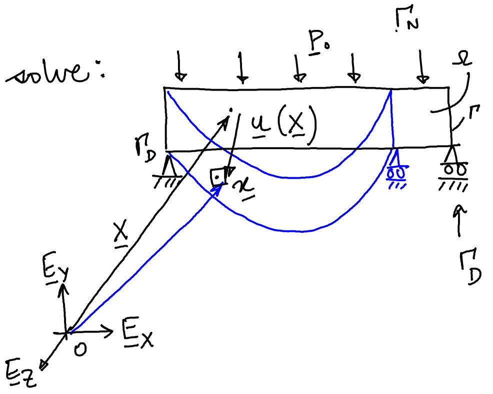
Friday, February 05, 2010
12:22 PM

Chapter 3: 2D & 3D Problems

Recall the problem we were trying to solve:



Given

- Structure (Domain) Geometry $\Omega, \Gamma_N, \Gamma_D$
- Loads $\underline{b}$: body force (self-weight)

  $\underline{p}_o$: surface tractions

Find

$$\underline{x} = \boxed{\underline{\phi}(\underline{X})}$$

such that

$$\text{div}\ \underset{\sim}{\sigma} + \underline{b} = \underline{0} \qquad \text{at all } \underline{x} \in \Omega_\phi$$

deformed configuration

i.e.

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \qquad \text{for } i = 1,2,3 \ (x,y,z)$$

Additional information

- Kinematics

$$\underline{x} = \underline{\phi}(\underline{X})$$

$$\underline{u}(\underline{X}) = \underline{x} - \underline{X}$$

Small strain

$$\Omega \approx \Omega_\phi$$

$$\underset{\sim}{\epsilon} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$$

Finite strain

$$\underset{\sim}{F} = \underset{\sim}{\nabla_X} \underline{\phi} = \underset{\sim}{I} + \nabla \underline{u}$$

$$\underset{\sim}{E} = \frac{1}{2}\left(\underset{\sim}{F}^T \underset{\sim}{F} - \underset{\sim}{I}\right) = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T + \nabla \underline{u}^T \nabla \underline{u}\right)$$

- Material Properties (at every $\underline{X}$)

$$\underset{\sim}{\sigma} = \underset{\approx}{C}\ \underset{\sim}{\epsilon}$$

$$\left(\sigma_{ij} = C_{ijkl}\, \epsilon_{kl}\right)$$

Hooke's Model: (Isotropic; Lin; Elastic)

$$\underset{\sim}{\sigma} = \lambda\left(\text{tr}\, \underset{\sim}{\epsilon}\right) \underset{\sim}{I} + 2\mu\, \underset{\sim}{\epsilon}$$

$$\left(\sigma_{ij} = \left(\lambda\, \epsilon_{kk}\right) \delta_{ij} + \left(2\mu\right) \epsilon_{ij}\right)$$

$$\psi(\underset{\sim}{F}) \qquad \hat{\psi}(\underset{\sim}{C}) \qquad \tilde{\psi}(\underset{\sim}{E})$$

$$\underset{\sim}{P} = \frac{\partial \psi}{\partial \underset{\sim}{F}} \qquad \underset{\sim}{S} = \frac{\partial \hat{\psi}}{\partial \underset{\sim}{E}}$$

$$\underset{\sim}{\sigma} = \frac{1}{J} \underset{\sim}{P}\, \underset{\sim}{F}^T \qquad \underset{\sim}{\sigma} = \frac{1}{J} \underset{\sim}{F}\, \underset{\sim}{S}\, \underset{\sim}{F}^T$$

$$\left(J = \det(\underset{\sim}{F})\right)$$

BCs :

$$\underline{\phi}(\underline{X}) = \underline{\phi}_D \qquad \text{on } \Gamma_D$$

$$\underset{\sim}{\sigma}\, \underline{n} = \underline{t}_N \ \left(= \underline{p}_o\right) \qquad \text{on } \Gamma_N$$

Wednesday, February 10, 2010
9:12 AM

"Elasticity" Matrix

Note

$$\underset{\sim}{\sigma} \rightsquigarrow \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad \& \quad \underset{\sim}{\epsilon} \rightsquigarrow \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

Using Voight Notation

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \hline \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{array} \right\} = \frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{ccc|ccc} (1-\nu) & \nu & \nu & & & \\ \nu & (1-\nu) & \nu & & 0 & \\ \nu & \nu & (1-\nu) & & & \\ \hline & & & \frac{1-2\nu}{2} & 0 & 0 \\ & 0 & & 0 & \frac{1-2\nu}{2} & 0 \\ & & & 0 & 0 & \frac{1-2\nu}{2} \end{array} \right] \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \hline 2\epsilon_{xy} \\ 2\epsilon_{yz} \\ 2\epsilon_{zx} \end{array} \right\} \begin{array}{l} \\ \\ \\ = \gamma_{xy} \\ = \gamma_{yz} \\ = \gamma_{zx} \end{array}$$

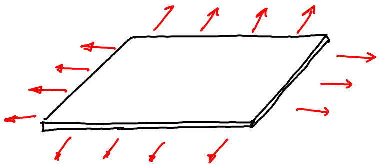
i.e. $\boxed{\underline{\sigma} = \underset{\sim}{D}\, \underline{\epsilon}}^{*}$ (* Careful)

2D Plane Problems

- Plane Stress

$\boxed{\sigma_{33} = 0}$; $\sigma_{13} = \sigma_{31} = 0$;

$\sigma_{23} = \sigma_{32} = 0$



Stress-strain relationship :

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{array} \right\}$$

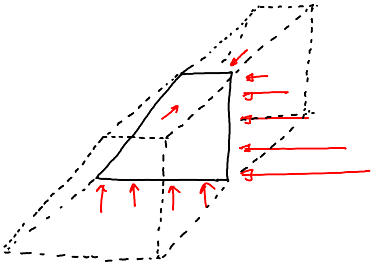
ie $\boxed{\underline{\sigma} = \underset{\sim}{D}_{p\sigma}\, \underline{\epsilon}}$

and $\epsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy})$

- Plane Strain

$\boxed{\epsilon_{33} = 0}$; $\epsilon_{13} = \epsilon_{31} = 0$;

$\epsilon_{23} = \epsilon_{23} = 0$



Stress-strain relationship :

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{array} \right\}$$

ie $\boxed{\underline{\sigma} = \underset{\sim}{D}_{p\epsilon}\, \underline{\epsilon}}$

Wednesday, February 10, 2010
9:12 AM

Strong Forms

{Ref: Reddy: Ch 8, 9, 11
Hughes: Ch 2
Z&T : Ch 2,3

- **Elasticity:**

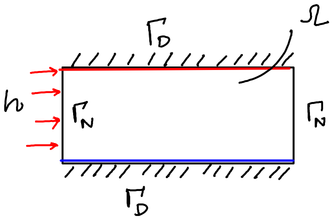
$$\operatorname{div} \underset{\sim}{\sigma} + \underline{b} = 0 \quad \text{in } \Omega_\phi$$

BC: $\underset{\sim}{\sigma}\,\underline{n} = \underline{t}_N$ on $\Gamma_{N\phi}$

$\underline{u} = \underline{u}_D$ on $\Gamma_{D\phi}$

Find $\underline{u}(\underline{x})$ (vector field) $\left\{\begin{matrix} u_x(X,Y,Z) \\ u_y(X,Y,Z) \\ u_z(X,Y,Z) \end{matrix}\right\}$

- **Heat conduction** Find $\theta(\underline{x})$ (scalar field)



$$\operatorname{div} \underline{q} = f \quad \text{in } \Omega$$

BC $\underline{q}\cdot\underline{n} = h$ on Γ_N

$\theta = \theta_D$ on Γ_D

Additional Info:

Temperature Gradient $\equiv \nabla\theta \rightsquigarrow \left\{\begin{matrix} \frac{\partial\theta}{\partial x_1} \\ \frac{\partial\theta}{\partial x_2} \end{matrix}\right\}$

$$= \frac{\partial\theta}{\partial x_j}\,\underline{e}_j$$

Material conductivities : $\underset{\sim}{K} \rightsquigarrow \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$

Fourier's Law: $\underline{q} = -\underset{\sim}{K}(\nabla\underline{\theta})$

Thus $\boxed{\operatorname{div}\left(\underset{\sim}{K}(\nabla\underline{\theta})\right) + f = 0}$ in Ω

ie $$\frac{\partial}{\partial x_i}\left[K_{ij}\left(\frac{\partial\theta}{\partial x_j}\right)\right] + f = 0$$

Equivalently: $(K_{ij}\,\theta_{,j})_{,i} + f = 0$ $\begin{bmatrix} x & x \\ x & x \end{bmatrix}\begin{bmatrix} x \\ x \end{bmatrix}$

$$\begin{bmatrix} \partial/\partial x_1 & \partial/\partial x_2 \end{bmatrix}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \end{bmatrix}\theta + f = 0$$

Wednesday, February 10, 2010
11:30 AM

Method of Weighted Residuals

Heat Conduction:

$$G(\theta,\bar\theta) \equiv \int_\Omega \bar\theta\left[\mathrm{div}\left(\underset{\sim}{k}(\underline{\nabla\theta})\right) + f\right] d\Omega$$

$$= \int_\Omega \bar\theta\left[\big(\overbrace{K_{ij}\,\theta_{,j}}^{q_i}\big)_{,i} + f\right] d\Omega$$

For integration by parts in 2-D & 3-D

Recall from 1-D: (Integration by parts / Product Rule)

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\int_0^\ell u\frac{dv}{dx}\,dx = \underbrace{\int_{[uv]_0}^{[uv]_\ell} d(uv)}_{[uv]_0^\ell} - \int_0^\ell \left(\frac{du}{dx}\right) v\,dx$$

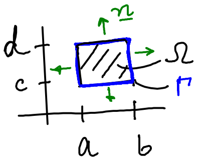
$\longrightarrow$ (because $\int_a^b \frac{df(x)}{dx}\,dx = f(b)-f(a)$)

Aside 1:

$$\int_\Omega (\ )\,d\Omega$$

$$\iint \quad \iiint$$

2D 3D



$$\int_c^d\left(\int_a^b (\ )\,dx\right)dy$$

Similarly:

$$\mathrm{div}(\bar\theta\,\underline{q}) = \underline{\nabla\bar\theta}\cdot\underline{q} + \bar\theta\,\mathrm{div}\,\underline{q} \quad \text{(product rule)}$$

i.e. $$(\bar\theta\, q_i)_{,i} = \bar\theta_{,i}\, q_i + \bar\theta\, q_{i,i} = \left(\frac{\partial\bar\theta}{\partial x_1}q_1 + \frac{\partial\bar\theta}{\partial x_2}q_2 + \frac{\partial\bar\theta}{\partial x_3}q_3\right) + \bar\theta\left(\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3}\right)$$

$$\Rightarrow \bar\theta\, q_{i,i} = (\bar\theta q_i)_{,i} - \bar\theta_{,i}\, q_i$$

Thus

$$G(\theta,\bar\theta) = -\int_\Omega(\bar\theta_{,i}\,K_{ij}\,\theta_{,j})\,d\Omega + \underbrace{\int_\Omega(\bar\theta q_i)_{,i}\,d\Omega}_{\oint_\Gamma(\bar\theta q_i)n_i\,d\Gamma} + \int_\Omega \bar\theta f\,d\Omega$$

$$\oint_\Gamma(\bar\theta q_i)n_i\,d\Gamma = \underbrace{\int_{\Gamma_D}\bar\theta q_i n_i\,d\Gamma}_{0} + \int_{\Gamma_N}\bar\theta q_i n_i\,d\Gamma$$

Aside 2:
(Divergence Theorem)

$$\int_\Omega \mathrm{div}(\underline{q})\,d\Omega = \oint_\Gamma(\underline{q}\cdot\underline{n})\,d\Gamma$$

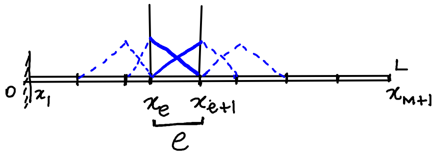
i.e. $$\int_\Omega q_{i,i}\,d\Omega = \oint_\Gamma q_i n_i\,d\Gamma$$

$$G(\theta,\bar\theta) = \underbrace{-\int_\Omega(\bar\theta_{,i}\,K_{ij}\,\theta_{,j})\,d\Omega}_{-W_I} + \underbrace{\int_\Omega\bar\theta f\,d\Omega + \int_{\Gamma_N}\bar\theta\, h\,d\Gamma}_{W_E}$$

Discretization

At this stage, we need to "discretize" our problem domain into smaller element domains.

Recall, in 1D we divided as:

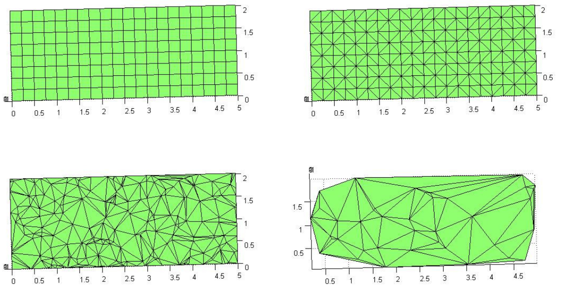


In 2-D & 3-D, this can be done in a variety of ways.

For example, consider the 2D domain shown below, along with four possible discretizations.

(i) Regular rectangular grid
(ii) Regular Triangular grid
(iii) Irregular Triangular grid
(iv) Arbitrary Domain Triangulation

(using Delaunay Triangulation)

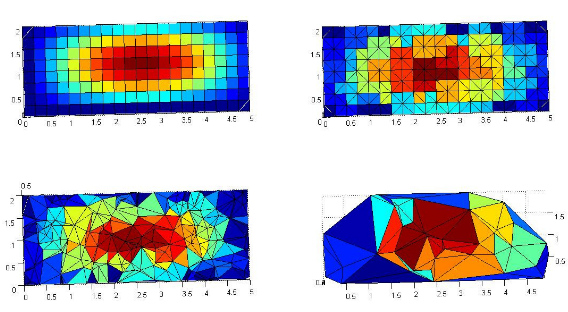


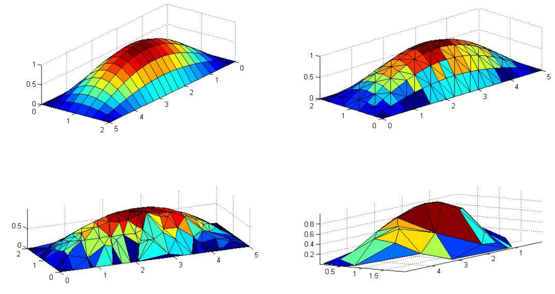
Having discretized our domain, we can write weak form as a sum of integrals.

$$\boxed{G(\theta,\bar{\theta}) = -\sum_{e=1}^{M} \int_{\Omega^e} \left(\bar{\theta}_{,i}\, K_{ij}\, \theta_{,j}\right) d\Omega + \sum_{e=1}^{M} \int_{\Omega^e} \bar{\theta} f \, d\Omega + \sum_{e=1}^{M} \int_{\Gamma_N^e} \bar{\theta} h \, d\Gamma}$$

Note: This is still "exact". We have only discretized our domain (we have not yet apporimated our solution).

Using these dicretizations, we can represent <u>some</u> function $f(x,y)$
In this case, (for illustration only) $f(x,y) = \sin\left(\frac{x\pi}{L}\right) * \sin\left(\frac{y\pi}{D}\right)$

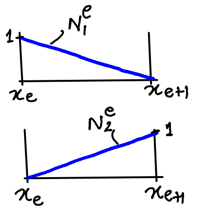




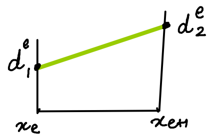
Clearly, the "quality" of the approximation depends upon the choice of the discretization.

Finite Element Approximation

Recall, in 1-D we used the following approximation

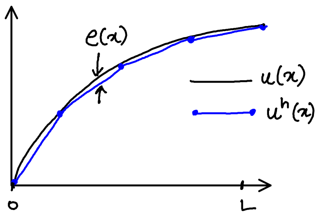


Using this, we could express our solution as:



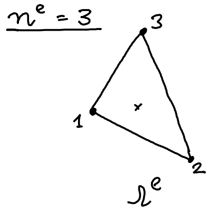
$$u(x) \approx u_e^h(x) = \underset{\sim}{N}\,\underline{d} = d_1^e\left(N_1^e(x)\right) + d_2^e\left(N_2^e(x)\right)$$

This, then allows us to express the global solution as an "assembly" of all elements.



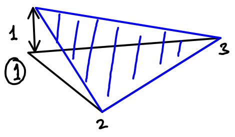
An analogous thing happens in 2D & 3D :

$$\theta(x,y) \approx \theta_e^h(x,y) = \sum_{\alpha=1}^{n^e} N_\alpha^e\, d_\alpha^e$$

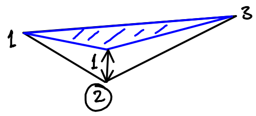


$$\theta_e^h(x,y) = \begin{bmatrix} N_1^e & N_2^e & N_3^e \end{bmatrix} \begin{bmatrix} d_1^e \\ d_2^e \\ d_3^e \end{bmatrix} = \underset{\sim}{N}\,\underline{d}$$

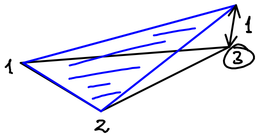
$$\bar{\theta}(x,y) \approx \bar{\theta}_e^h(x,y) = \underset{\sim}{N}\,\underline{\bar{d}}$$



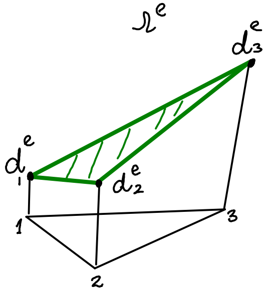
$N_1^e(x,y)$



$N_2^e(x,y)$



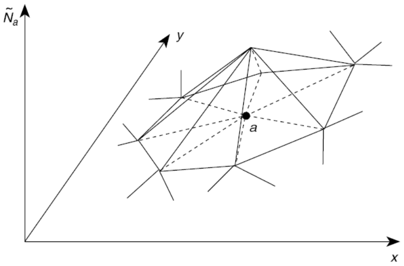
$N_3^e(x,y)$



$$\theta_e^h(x,y) = d_1^e\left(N_1^e(x,y)\right) + d_2^e\left(N_2^e(x,y)\right) + d_3^e\left(N_3^e(x,y)\right)$$

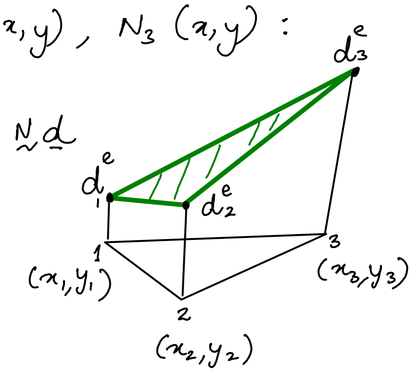
$$\nabla\underline{\theta}(x,y) \approx \nabla\underline{\theta}^h = \begin{Bmatrix} \dfrac{\partial\theta}{\partial x_1} \\ \dfrac{\partial\theta}{\partial x_2} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial N_1}{\partial x_1} & \dfrac{\partial N_2}{\partial x_1} & \dfrac{\partial N_3}{\partial x_1} \\ \dfrac{\partial N_1}{\partial x_2} & \dfrac{\partial N_2}{\partial x_2} & \dfrac{\partial N_3}{\partial x_2} \end{Bmatrix} \begin{Bmatrix} d_1^e \\ d_2^e \\ d_3^e \end{Bmatrix}$$

$$\nabla\underline{\theta}^h = \underset{\sim}{B}\,\underline{d}$$



Note : The "hat" shape functions of 2D finite elements are $H^1(\Omega)$

To obtain the actual functions $N_1(x,y)$, $N_2(x,y)$, $N_3(x,y)$:



$$\theta_e^h(x,y) = d_1^e\left(N_1^e(x,y)\right) + d_2^e\left(N_2^e(x,y)\right) + d_3^e\left(N_3^e(x,y)\right) = \underset{\sim}{N}\,\underline{d}$$

$$\theta_e^h(x,y) = a_1 + a_2 x + a_3 y \qquad \text{———} \circledast$$

$$= [a_1\ a_2\ a_3]\begin{bmatrix}1\\x\\y\end{bmatrix} = \underline{a}^T\,\underline{h}(x,y)$$

Thus we have three equations:

$$\begin{aligned} d_1^e &= a_1 + a_2x_1 + a_3y_1\\ d_2^e &= a_1 + a_2x_2 + a_3y_2\\ d_3^e &= a_1 + a_2x_3 + a_3y_3 \end{aligned}$$

solving for a_1, a_2, a_3:

$$\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix} = \begin{bmatrix}1&x_1&y_1\\1&x_2&y_2\\1&x_3&y_3\end{bmatrix}^{-1}\begin{Bmatrix}d_1^e\\d_2^e\\d_3^e\end{Bmatrix}$$

Substituting back into $\circledast$:

$$\theta_e^h(x,y) = \frac{1}{2\Delta}\Big[\underbrace{(A_1+B_1x+C_1y)}_{N_1(x,y)}d_1^e + \underbrace{(A_2+B_2x+C_2y)}_{N_2(x,y)}d_2^e + \underbrace{(A_3+B_3x+C_3y)}_{N_3(x,y)}d_3^e$$

where

$$\begin{array}{l|l|l} A_1 = x_2y_3 - x_3y_2 & A_2 = x_3y_1 - x_1y_3 & A_3 = x_1y_2 - x_2y_1\\ B_1 = y_2 - y_3 & B_2 = y_3 - y_1 & B_3 = y_1 - y_2\\ C_1 = x_3 - x_2 & C_2 = x_1 - x_3 & C_3 = x_2 - x_1 \end{array}$$

$$\boxed{N_\alpha(x,y) = \frac{1}{2\Delta}(A_\alpha + B_\alpha x + C_\alpha y)} \quad \alpha = 1,2,3 \qquad \text{where} \quad 2\Delta = \det\begin{vmatrix}1&x_1&y_1\\1&x_2&y_2\\1&x_3&y_3\end{vmatrix}$$

consider:

$$\sum_{\alpha=1}^{3} N_\alpha(x,y) = N_1(x,y) + N_2(x,y) + N_3(x,y)$$

$$= \frac{1}{2\Delta}\Big[\underbrace{(A_1+A_2+A_3)}_{2\Delta} + \underbrace{(B_1+B_2+B_3)}_{0}x + \underbrace{(C_1+C_2+C_3)}_{0}y\Big]$$

i.e.

$$\boxed{\sum N_\alpha(x,y) = 1}$$

Substituting the approximation $\theta_e^h(x,y) = \underset{\sim}{N}\, \underline{d}$

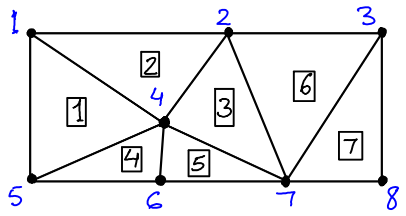
$$\nabla\underline{\theta}_e^h = \left[\theta_e^h(x,y)\right] = \underset{\sim}{B}\, \underline{d}$$

$$\tilde{G}^h(\underline{d},\bar{\underline{d}}) = -\sum_{e=1}^{M} \bar{\underline{d}}^{e^T} \underbrace{\int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{\kappa}\, \underset{\sim}{B}\, d\Omega}_{\underset{\sim}{K}^e}\ \underline{d}^e + \underbrace{\sum_{e=1}^{M} \bar{\underline{d}}^{e^T} \int_{\Omega^e} \underset{\sim}{N}^T f\, d\Omega + \sum_{e=1}^{M} \bar{\underline{d}}^{e^T} \int_{\Gamma_N} \underset{\sim}{N}^T h\, d\Gamma}_{\bar{\underline{d}}^{e^T} \underline{f}^e}$$

$$= -\overset{M}{\underset{e=1}{\mathsf{A}}}\ \bar{\underline{d}}^{e^T} \underset{\sim}{K}^e \underline{d}^e + \overset{M}{\underset{e=1}{\mathsf{A}}}\ \bar{\underline{d}}^{e^T} \underline{f} \qquad \text{(assembly)}$$

$$\boxed{\tilde{G}^h(\underline{d},\bar{\underline{d}}) = -\bar{\underline{d}}^{G^T}\left(\underset{\sim}{K}^G \underline{d}^G - \underline{f}^G\right) = \underline{0} \qquad \forall\ \bar{\underline{d}}^G}$$

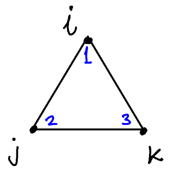
Assembly in 2D & 3D (sD)

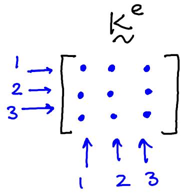


Total number of nodes : N
Total number of elements : M

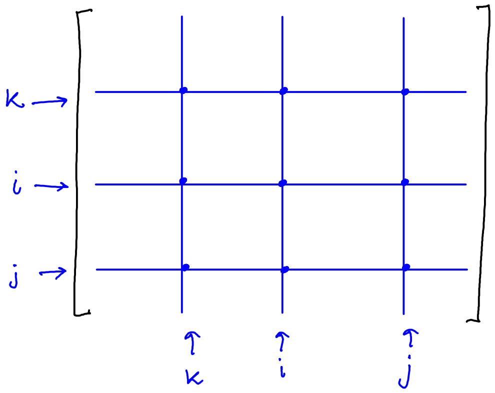
Degrees of freedom	Per Node	Total
· Scalar problem :	1	N

Thus, when we assemble:





$$\xrightarrow{\mathsf{A}} \quad \underset{\sim}{K}^G(\bullet,\bullet) = \underset{\sim}{K}^G(\bullet,\bullet) + \underset{\sim}{K}^e$$

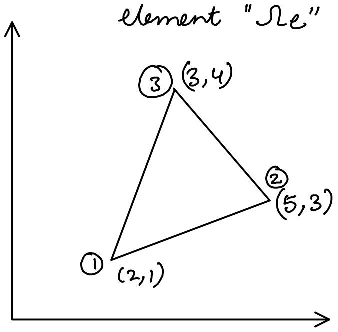


global element dofs : i, j, k

Example :

For the element "Ωe", find :

- shape functions, $\underset{\sim}{N}^e$
- derivatives $\underset{\sim}{B}^e$
- "stiffness" matrix $\underset{\sim}{K}^e$ (say $\underset{\sim}{k} = \kappa \underset{\sim}{I}$)



$$\Theta_e^h(x,y) = a_1 + a_2 x + a_3 y$$

$$\begin{aligned} d_1^e &= a_1 + a_2(2) + a_3(1) \\ d_2^e &= a_1 + a_2(5) + a_3(3) \\ d_3^e &= a_1 + a_2(3) + a_3(4) \end{aligned}$$

Solving
$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 3 \\ 1 & 3 & 4 \end{bmatrix}^{-1} \begin{Bmatrix} d_1^e \\ d_2^e \\ d_3^e \end{Bmatrix} = \frac{1}{7} \begin{bmatrix} 11 & -5 & 1 \\ -1 & 3 & -2 \\ -2 & -1 & 3 \end{bmatrix} \begin{Bmatrix} d_1^e \\ d_2^e \\ d_3^e \end{Bmatrix}$$

Substituting back in

$$\begin{aligned} \Theta_h^e(x,y) &= a_1 + a_2 x + a_3 y \\ &= \frac{1}{7}\left[(11 d_1^e - 5 d_2^e + 1 d_3^e) + (-d_1^e + 3 d_2^e - 2 d_3^e)x + (-2d_1^e - d_2^e + 3 d_3^e)\, y\right] \end{aligned}$$

Rearranging $= \frac{1}{7}\left[\underbrace{(11 - x - 2y)}_{N_1^e(x,y)} d_1^e + \underbrace{(-5 + 3x - y)}_{N_2^e(x,y)} d_2^e + \underbrace{(1 - 2x + 3y)}_{N_3^e(x,y)} d_3^e \right]$

Shape functions : $\underset{\sim}{N}^e = \left[N_1^e(x,y) \quad N_2^e(x,y) \quad N_3^e(x,y) \right]$

Derivatives : $\underset{\sim}{B}^e = \begin{bmatrix} N_{1,1}^e & N_{2,1}^e & N_{3,1}^e \\ N_{1,2}^e & N_{2,2}^e & N_{3,2}^e \end{bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ -2 & -1 & 3 \end{bmatrix}$

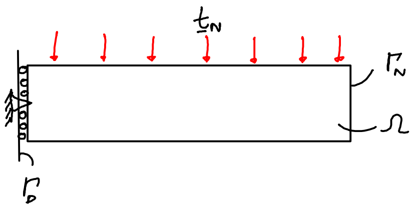
Element stiffness matrix :

$$\underset{\sim}{K}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{\kappa}\, \underset{\sim}{B}\, d\Omega = \kappa \int_{\Omega^e} \begin{bmatrix} -1 & -2 \\ 3 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 \\ -2 & -1 & 3 \end{bmatrix} d\Omega$$

$$= \kappa \begin{bmatrix} 5 & -1 & -4 \\ & 10 & -9 \\ \text{Sym} & & 13 \end{bmatrix} \underbrace{\int_{\Omega^e} d\Omega}_{A^e} = \kappa A \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}$$

Friday, February 12, 2010
11:00 AM

Elasticity:



$$G(\underline{u}, \bar{\underline{u}}) \equiv \int_\Omega \bar{\underline{u}} \cdot (\operatorname{div} \underset{\sim}{\sigma} + \underline{b})\, d\Omega$$

$$= \int_\Omega \bar{u}_i \left[\underbrace{(\sigma_{ij,j})}_{} + b_i\right] d\Omega$$

Consider

$$(\bar{u}_i\ \sigma_{ij})_{,j} = \bar{u}_{i,j}\ \sigma_{ij} + \bar{u}_i\ \sigma_{ij,j}$$

$$\bar{u}_i\, \sigma_{ij,j} = (\bar{u}_i\, \sigma_{ij})_{,j} - \bar{u}_{i,j}\, \sigma_{ij}$$

Thus

$$G(\underline{u}, \bar{\underline{u}}) = -\int_\Omega \underbrace{\bar{u}_{i,j}\ \sigma_{ij}}_{\bar{\underset{\sim}{\epsilon}} : \underset{\sim}{\sigma}}\ d\Omega + \int_\Omega \underbrace{\bar{u}_i\, b_i}_{\bar{\underline{u}} \cdot \underline{b}}\ d\Omega + \oint_\Gamma (\bar{u}_i\ \sigma_{ij}) n_j\, d\Gamma$$

$$\int_{\Gamma_D} \underset{0}{\cancel{\bar{u}_i}}\ t_{D_i}\ d\Gamma + \int_{\Gamma_N} \underbrace{\bar{u}_i\ \underline{t}_{N_i}}_{\bar{\underline{u}} \cdot \underline{t}_N}$$

Also Note:

$$\bar{u}_{i,j} = \underbrace{\tfrac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i})}_{\text{sym}\, \nabla \bar{\underset{\sim}{u}}\ \ (= \bar{\underset{\sim}{\epsilon}})} + \underbrace{\tfrac{1}{2}(\bar{u}_{i,j} - \bar{u}_{j,i})}_{\text{skew}\ \nabla \bar{\underset{\sim}{u}}}$$

$$\bar{u}_{i,j}\, \sigma_{ij} = \bar{\epsilon}_{ij}\ \sigma_{ij}$$

$$= \bar{\underset{\sim}{\epsilon}} : \underset{\sim}{\sigma} = \begin{bmatrix} x & x \\ x & x \end{bmatrix} : \begin{bmatrix} x & x \\ x & x \end{bmatrix} = \bar{\epsilon}_{11}\, \sigma_{11} + \underbrace{\bar{\epsilon}_{12}\, \sigma_{12} + \bar{\epsilon}_{21}\, \sigma_{21}}_{\downarrow} + \bar{\epsilon}_{22}\, \sigma_{22}$$

In the Voight Notation:

$$= \bar{\underline{\epsilon}} \cdot \underline{\sigma} = \bar{\underline{\epsilon}}^T \underline{\sigma} = [\bar{\epsilon}_{xx}\ \ \bar{\epsilon}_{yy}\ \ \bar{\gamma}_{xy}] \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

$$= \bar{\underline{\epsilon}}^T \underset{\sim}{D}_{P*}\ \underline{\epsilon}$$

Thus,

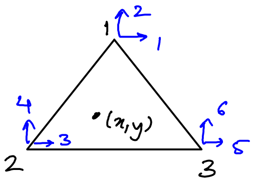
$$\Rightarrow \boxed{G(\underline{u}, \bar{\underline{u}}) = -\int_\Omega (\bar{\underline{\epsilon}}^T \underset{\sim}{D}\, \underline{\epsilon})\, d\Omega + \int_\Omega \bar{\underline{u}}^T \underline{b}\, d\Omega + \int_{\Gamma_N} \bar{\underline{u}}^T \underline{t}_N\, d\Gamma}$$

$$\underbrace{\qquad\qquad\qquad}_{-W_I} \qquad \underbrace{\qquad\qquad\qquad\qquad\qquad}_{W_E}$$

Discretize

$$G(\underline{u}, \bar{\underline{u}}) = \sum_{e=1}^{M} \left[-\int_{\Omega^e} (\bar{\underline{\epsilon}}^T \underset{\sim}{D}\, \underline{\epsilon})\, d\Omega + \int_{\Omega^e} \bar{\underline{u}}^T \underline{b}\, d\Omega + \int_{\Gamma_N^e} \bar{\underline{u}}^T \underline{t}_N\, d\Gamma \right]$$

Now introduce Approximation within every element Ω^e:



$$\boxed{\underline{u}(x,y) = \underset{\sim}{N}^e\,\underline{d}^e}$$

$$\begin{Bmatrix} u_1(x,y) \\ u_2(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} d^e_1 \\ d^e_2 \\ d^e_3 \\ d^e_4 \\ d^e_5 \\ d^e_6 \end{bmatrix}$$

Galerkin

$$\Rightarrow \quad \bar{\underline{u}}(x,y) = \underset{\sim}{N}^e\,\bar{\underline{d}}^e$$

$$\boxed{\underline{\epsilon}(x,y) = \underset{\sim}{B}^e\,\underline{d}^e}$$

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} N_{1,1} & 0 & N_{2,1} & 0 & N_{3,1} & 0 \\ 0 & N_{1,2} & 0 & N_{2,2} & 0 & N_{3,2} \\ N_{1,2} & N_{1,1} & N_{2,2} & N_{2,1} & N_{3,2} & N_{3,1} \end{bmatrix} \begin{bmatrix} d^e_1 \\ d^e_2 \\ d^e_3 \\ d^e_4 \\ d^e_5 \\ d^e_6 \end{bmatrix}$$

$$\boxed{\underline{\sigma}(x,y) = \underset{\sim}{D}\,\underset{\sim}{B}^e\,\underline{d}^e}$$

Thus the Final Weak form:

$$\tilde{G}^h(\underline{d},\bar{\underline{d}}) = -\sum_{e=1}^{M}\left[\bar{\underline{d}}^{e^T}\underbrace{\int_{\Omega^e}(\underset{\sim}{B}^T\underset{\sim}{D}\,\underset{\sim}{B})\,d\Omega}_{\underset{\sim}{K}^e}\;\underline{d}^e - \bar{\underline{d}}^{e^T}\underbrace{\left(\int_{\Omega^e}\underset{\sim}{N}^T\underline{b}\,d\Omega + \int_{\Gamma^e_N}\underset{\sim}{N}^T\underline{t}_N\,d\Gamma\right)}_{\underline{f}^e}\right]$$

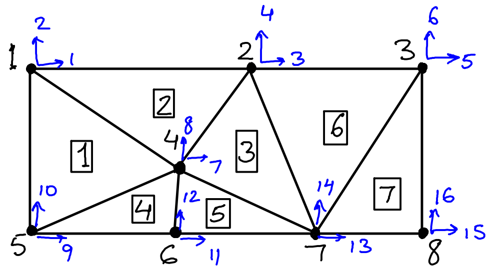
$$\Rightarrow \quad \boxed{\tilde{G}^h(\underline{d},\bar{\underline{d}}) = -\,\bar{\underline{d}}^{G^T}\left(\underset{\sim}{K}^G\,\underline{d}^G - \underline{f}^G\right)}$$

where

$$\underset{\sim}{K}^G = \mathop{\mathbf{A}}_{e=1}^{M}\underset{\sim}{K}^e \quad ; \quad \underline{f}^G = \mathop{\mathbf{A}}_{e=1}^{M}\underline{f}^e$$

Assembly in 2D & 3D (sD)

Total number of nodes : N
Total number of elements : M

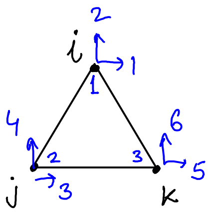


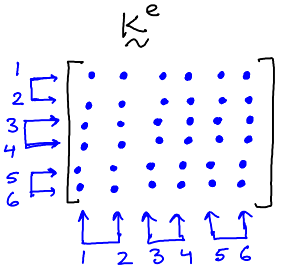
Degrees of freedom	Per Node	Total
· Scalar problem :	1	N
· Vector problem :	s	sN

Global Degrees of freedom corresponding to node "i" :

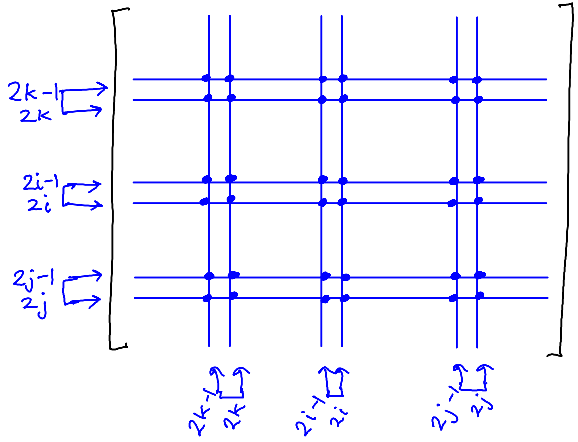
$$s * (i-1) + 1 \quad ; \quad s * (i-1) + 2 \quad \dots \quad s * (i-1) + s$$

For example





$$\xrightarrow{\mathbf{A}} \quad \underset{\sim}{K}^G(\bullet,\bullet) = \underset{\sim}{K}^G(\bullet,\bullet) + \underset{\sim}{K}^e$$



global element dofs :

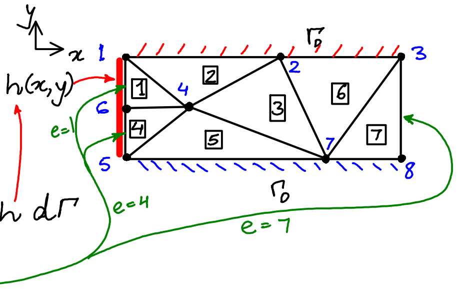
$2i-1, 2i \quad , \quad 2j-1, 2j \quad , \quad 2k-1, 2k$

Integrals for the "f" vector

Recall,

- Heat Conduction

$$\underset{\sim}{f}^{G} = \mathop{\mathbf{A}}_{e=1}^{M} \underset{\sim}{f}^{e}$$

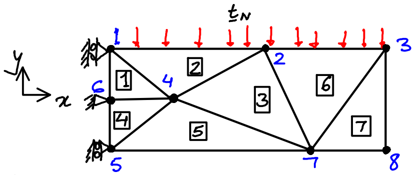


where
$$\underset{(3\times1)}{\underset{\sim}{f}^{e}} = \int_{\Omega^e} \underset{\sim}{N}^T f \, d\Omega + \int_{\boxed{\Gamma_N^e}} \underset{\sim}{N}^T h \, d\Gamma$$

$$= \int_{\Omega^e} \begin{bmatrix} N_1^e(x,y) \\ N_2^e(x,y) \\ N_3^e(x,y) \end{bmatrix} f(x,y) \, d\Omega + \int_{\Gamma_N^e} \begin{bmatrix} N_1^e(x,y) \\ N_2^e(x,y) \\ N_3^e(x,y) \end{bmatrix} h(x,y) \, d\Gamma$$

- 2D Elasticity

$$\underset{\sim}{f}^{G} = \mathop{\mathbf{A}}_{e=1}^{M} \underset{\sim}{f}^{e}$$



where
$$\underset{(6\times1)}{\underset{\sim}{f}^{e}} = \int_{\Omega^e} \underset{\sim}{N}^T \underset{\sim}{b} \, d\Omega + \int_{\Gamma_N^e} \underset{\sim}{N}^T \underset{\sim}{t}_N \, d\Gamma$$

$$= \int_{\Omega^e} \left[\begin{array}{cc} N_1 & 0 \\ 0 & N_1 \\ \hdashline N_2 & 0 \\ 0 & N_2 \\ \hdashline N_3 & 0 \\ 0 & N_3 \end{array}\right] \begin{Bmatrix} b_x(x,y) \\ b_y(x,y) \end{Bmatrix} d\Omega + \int_{\Gamma_N^e} \left[\begin{array}{cc} N_1 & 0 \\ 0 & N_1 \\ \hdashline N_2 & 0 \\ 0 & N_2 \\ \hdashline N_3 & 0 \\ 0 & N_3 \end{array}\right] \begin{Bmatrix} t_{N_x}(x,y) \\ t_{N_y}(x,y) \end{Bmatrix} d\Gamma$$

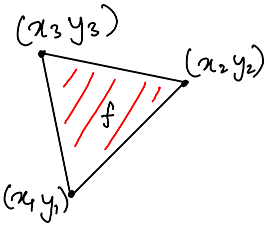
These integrals will, in general, need to be evaluated numerically.
However, for Linear 3-node triangles ($\Omega^e = \Delta$),
and when $f(x,y)$ (or $\underset{\sim}{b}(x,y)$) are constants f_0 (or $\underset{\sim}{b}_0$)
and when $h(x,y)$ (or $\underset{\sim}{t}_N(x,y)$) are constants h_0 (or $\underset{\sim}{t}_{N_0}$)

Then, we can find these integrals exactly for a Δ.
This will help us save computational time when we implement it on a computer (MATLAB).

Note: If you have a non-constant "f" function, you can still assume that f is constant within one Δ. Then as you refine your mesh, your solution will converge.

Friday, February 19, 2010
12:25 PM

Lets first consider the domain term:



$$\int_{\Omega^e} N_\alpha f_0 \, d\Omega = f_0 \int_\Delta N_\alpha(x,y)\, d\Delta$$

$$= \frac{f_0}{2\Delta}\int_\Delta \left(A_\alpha + B_\alpha x + C_\alpha y\right) d\Delta$$

$$= \frac{f_0}{2\Delta}\left[A_\alpha \underbrace{\left(\int_\Delta d\Delta\right)}_{\Delta} + B_\alpha \underbrace{\left(\int_\Delta x\, d\Delta\right)}_{x_c \Delta} + C_\alpha \underbrace{\left(\int_\Delta y\, d\Delta\right)}_{y_c\Delta}\right]$$

Thus,

$$\int_{\Omega^e} N_\alpha(x,y) f_0\, d\Omega = \frac{f_0}{2}\left[A_\alpha + B_\alpha\left(\frac{x_1+x_2+x_3}{3}\right) + C_\alpha\left(\frac{y_1+y_2+y_3}{3}\right)\right]$$

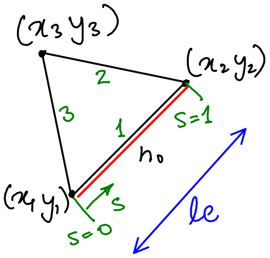
$$= \frac{f_0}{2}\left[\left(\frac{A_\alpha+B_\alpha x_1 + C_\alpha y_1}{3}\right) + \left(\frac{A_\alpha + B_\alpha x_2 + C_\alpha y_2}{3}\right) + \left(\frac{A_\alpha + B_\alpha x_3 + C_\alpha y_3}{3}\right)\right]$$

$$= \frac{f_0}{2}\left[\frac{1}{3}(2\Delta)\underbrace{\left(\frac{A_\alpha + B_\alpha x_\alpha + C_\alpha y_\alpha}{2\Delta}\right)}_{1} + 0 + 0\right] \qquad (\text{no sum on } \alpha)$$

$$\Rightarrow \boxed{\int_{\Omega^e} N_\alpha(x,y) f_0\, d\Omega = \frac{f_0\Delta}{3}}$$

when "f" can be assumed constant.

Now consider the boundary term



$$\int_{\Gamma_N^e} N_\alpha(x,y)\, h_0\, d\Gamma^e = h_0 \sum_{\substack{\Gamma_N^e \\ \text{edges}}}\left[\int_0^1 N_\alpha(x(s),y(s))\, ds\right]$$

We can express for the edge 1-2:

$$\left.\begin{aligned} x(s) &= (1-s)\,x_1 + s\,x_2 \\ y(s) &= (1-s)\,y_1 + s\,y_2 \end{aligned}\right]$$

Note $s=0 \Rightarrow (x,y) = (x_1\ y_1)$ $(l=0)$

$s=1 \Rightarrow (x,y) = (x_2\ y_2)$ $(l=l_e)$

$l(s) = s\,l_e \Rightarrow \frac{dl}{ds} = l_e$

Thus $N_\alpha(x,y) = \frac{1}{2\Delta}\left(A_\alpha + B_\alpha x + C_\alpha y\right)$

$$\Rightarrow \tilde{N}_\alpha(s) = \frac{1}{2\Delta}\left(A_\alpha + B_\alpha x_1 + C_\alpha y_1\right) + \frac{1}{2\Delta}\left[B_\alpha(x_2 - x_1) + C_\alpha(y_2 - y_1)\right] s$$

$$h_0\int_{\Gamma^e} N_\alpha(x,y)\, d\Gamma = h_0\int_0^{l_e} N_\alpha(x,y)\, dl = h_0\int_0^1 \tilde{N}_\alpha(s)\overbrace{\left(\frac{dl}{ds}\right)}^{l_e} ds$$

Sunday, February 21, 2010
10:56 AM

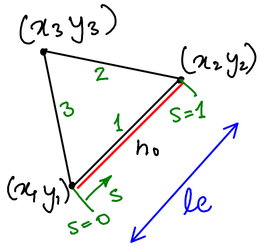
$$h_\circ \int_{\Gamma^e} N_\alpha(x,y)\, d\Gamma = h_0 \int_0^{l_e} N_\alpha(x,y)\, dl = h_\circ \int_0^1 \tilde{N}_\alpha(s) \overbrace{\left(\frac{dl}{ds}\right)}^{l_e} ds$$

$$= \frac{h_0 l_e}{2\Delta}\left[\left(A_\alpha + B_\alpha x_1 + C_\alpha y_1\right)\underbrace{\int_0^1 ds}_{1} + \left[B_\alpha(x_2 - x_1) + C_\alpha(y_2 - y_1)\right]\underbrace{\int_0^1 s\, ds}_{1/2}\right]$$

$$= \frac{h_0 \ l_e}{2\Delta}\left[A_\alpha + B_\alpha\left(\frac{x_1 + x_2}{2}\right) + C_\alpha\left(\frac{y_1 + y_2}{2}\right)\right]$$

$$= \frac{h_0 \ l_e}{2\Delta}\left[\left(\frac{A_\alpha + B_\alpha x_1 + C_\alpha y_1}{2}\right) + \left(\frac{A_\alpha + B_\alpha x_2 + C_\alpha y_2}{2}\right)\right]$$

$$= \frac{h_0 l_e}{2}\left[\underbrace{\left(\frac{A_\alpha + B_\alpha x_\alpha + C_\alpha y_\alpha}{2\Delta}\right)}_{1} + 0\right] \quad (\text{no sum on } \alpha)$$

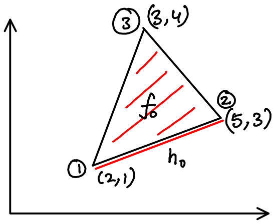


In general for edge i-j :

$$\boxed{\int_{\Gamma_N^e(i\text{-}j)} N_\alpha(x,y)\, h_0\, d\Gamma = \frac{h_0 l_e}{2}}$$

when "h" can be assumed constant.

Example



Consider

$f = f_0$ (constant) on Δ

$h = h_0$ (constant) on 1-2:

$$N_\alpha(x,y) = \frac{1}{2\Delta}\left(A_\alpha + B_\alpha x + C_\alpha y\right)$$

$$\begin{array}{l|l|l}
A_1 = x_2 y_3 - x_3 y_2 & A_2 = x_3 y_1 - x_1 y_3 & A_3 = x_1 y_2 - x_2 y_1 \\
B_1 = y_2 - y_3 & B_2 = y_3 - y_1 & B_3 = y_1 - y_2 \\
C_1 = x_3 - x_2 & C_2 = x_1 - x_3 & C_3 = x_2 - x_1
\end{array}$$

$$\frac{1}{7}\left[\underbrace{(11 - x - 2y)}_{N_1^e(x,y)} d_1^e + \underbrace{(-5 + 3x - y)}_{N_2^e(x,y)} d_2^e + \underbrace{(1 - 2x + 3y)}_{N_3^e(x,y)} d_3^e\right]$$

Thus

$$\int_{\Omega^e} N_\alpha(x,y)\, f_0\, d\Omega = \frac{f_0}{2}\left[A_\alpha + B_\alpha\left(\frac{x_1+x_2+x_3}{3}\right) + C_\alpha\left(\frac{y_1+y_2+y_3}{3}\right)\right]$$

Note: $2\Delta = A_1 + A_2 + A_3 = 7 \Rightarrow \Delta = 3.5$ $\quad x_c = \frac{10}{3} = 3.333$ $\quad y_c = \frac{8}{3} = 2.667$

$$\int_{\Omega^e} N_1(x,y)\, f_0\, d\Omega = \frac{f_0}{2}\left[11 + (-1)\left(\frac{10}{3}\right) + (-2)\left(\frac{8}{3}\right)\right] = f_0\left(\tfrac{1}{3}\right)\times 3.5 \quad ✓$$

$$\int_{\Omega^e} N_2(x,y)\, f_0\, d\Omega = \frac{f_0}{2}\left[-5 + 3\left(\frac{10}{3}\right) + (-1)\left(\tfrac{8}{3}\right)\right] = f_0\left(\tfrac{1}{3}\right)\times 3.5 \quad ✓$$

$$\int_{\Omega^e} N_3(x,y)\, f_0\, d\Omega = \frac{f_0}{2}\left[1 + (-2)\left(\frac{10}{3}\right) + 3\left(\tfrac{8}{3}\right)\right] = f_0\left(\tfrac{1}{3}\right)\times 3.5 \quad ✓$$

And

$$\sqrt{(5-2)^2 + (3-1)^2} = \sqrt{13}$$

$$\int_{\Gamma_N^e} N_\alpha h_0\, d\Gamma = h_0 \int_0^1 N_\alpha\left(x(s), y(s)\right) l_e\, ds \qquad (\text{for } \alpha = 1, 2, 3)$$

$$= \frac{h_0 l_e}{2\Delta}\left[A_\alpha + B_\alpha \int_0^1 \overbrace{\left[(1-s)x_1 + s\, x_2\right]}\, ds + C_\alpha \int_0^1 \overbrace{\left[(1-s)y_1 + s\, y_2\right]}\, ds\right]$$

So
$$\int_{\Gamma_N^e} N_1 h_0\, d\Gamma = \frac{h_0 l_e}{2\Delta}\left[A_1 + B_1\left(\frac{x_1+x_2}{2}\right) + C_1\left(\frac{y_1+y_2}{2}\right)\right]$$

$$= \frac{h_0 l_e}{7}\left[11 + (-1)\left(\frac{2+5}{2}\right) + (-2)\left(\frac{1+3}{2}\right)\right] = h_0 \frac{\overset{\sqrt{13}}{l_{12}}}{2} \quad ✓$$

$$\int_{\Gamma_N^e} N_2 h_0\, d\Gamma = \frac{h_0 l_e}{7}\left[A_2 + B_2\left(\frac{x_1+x_2}{2}\right) + C_1\left(\frac{y_1+y_2}{2}\right)\right]$$

$$= \frac{h_0}{7}\left[-5 + 3\left(\frac{2+5}{2}\right) + (-1)\left(\frac{1+3}{2}\right)\right] = h_0 \frac{\overset{\sqrt{13}}{l_{12}}}{2} \quad ✓$$

$$\int_{\Gamma_N^e} N_3 h_0\, d\Gamma = 0$$

Thus
$$\boxed{\underline{f}^e = \begin{bmatrix} \Delta/3\, f_0 + \frac{l_{12}}{2} h_0 \\ \Delta/3\, f_0 + \frac{l_{12}}{2} h_0 \\ \Delta/3\, f_0 + 0 \end{bmatrix}}$$

Friday, February 19, 2010
1:21 PM

Integrals for "$\underset{\sim}{K}$" stiffness Matrix

Recall,

- Heat Conduction:

$$\underset{\sim}{K}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{k}\, \underset{\sim}{B}\, d\Omega$$

$$N_\alpha(x,y) = \frac{1}{2\Delta}\left(A_\alpha + B_\alpha x + C_\alpha y\right)$$

$$\left.\begin{aligned} \Rightarrow N_{\alpha,1} &= \frac{1}{2\Delta} B_\alpha \\ N_{\alpha,2} &= \frac{1}{2\Delta} C_\alpha \end{aligned}\right\} \text{constants.}$$

$$\underset{\sim}{N} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

$$\underset{\sim}{B} = \left.\left[\begin{array}{c:c:c} N_{1,1} & N_{2,1} & N_{3,1} \\ N_{1,2} & N_{2,2} & N_{3,2} \end{array}\right]\right\} \text{constant}$$

$$\boxed{\underset{\sim}{K}^e = \underbrace{\int_\Delta d\Omega}_{\Delta} \left(\frac{1}{2\Delta}\right)^2 \begin{bmatrix} B_1 & C_1 \\ B_2 & C_2 \\ B_3 & C_3 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}} \quad \text{if } \underset{\sim}{k} \text{ is constant within } \Delta.$$

- Elasticity:

$$\underset{\sim}{K}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, d\Omega$$

<u>N_α : same</u>

$$\underset{\sim}{B} = \left[\begin{array}{cc:cc:cc} N_{1,1} & 0 & N_{2,1} & 0 & N_{3,1} & 0 \\ 0 & N_{1,2} & 0 & N_{2,2} & 0 & N_{3,2} \\ N_{1,2} & N_{1,1} & N_{2,2} & N_{2,1} & N_{3,2} & N_{3,1} \end{array}\right]$$

$$\boxed{\underset{\sim}{K}^e = \underbrace{\int_\Delta d\Omega}_{\Delta} \left(\frac{1}{2\Delta}\right)^2 \left[\begin{array}{ccc} B_1 & 0 & C_1 \\ 0 & C_1 & B_1 \\ \hdashline B_2 & 0 & C_2 \\ 0 & C_2 & B_2 \\ \hdashline B_3 & 0 & C_3 \\ 0 & C_3 & B_3 \end{array}\right] \begin{bmatrix} & & \\ & \underset{\sim}{D} & \\ & & \end{bmatrix} \left[\begin{array}{cc:cc:cc} B_1 & 0 & B_2 & 0 & B_3 & 0 \\ 0 & C_1 & 0 & C_2 & 0 & C_3 \\ C_1 & B_1 & C_2 & B_2 & C_3 & B_3 \end{array}\right] \quad \text{if } \underset{\sim}{D} \text{ is constant in } \Delta.}$$

<u>Note</u>: $\underset{\sim}{D}$ is the elasticity matrix for Plane stress or Plane strain. (Ch3-pg 2)

Wednesday, February 24, 2010
7:26 AM

Outline of Steps for Complete Problem Solution

(i) Strong Form (Governing Differential Equations)

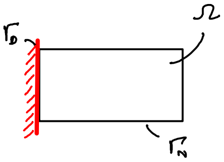
- Heat Conduction

  {unknown : $\theta(x,y)$}

$$\left.\begin{aligned} \operatorname{div} \underline{q} &= f \\ \underline{q} &= -\underset{\sim}{\kappa}\nabla\theta \end{aligned}\right] \text{ on } \Omega$$

$$\theta = \theta_0 \,\Big]\text{ on } \Gamma_D$$

$$\underline{q}\cdot\underline{n} = h \,\Big]\text{ on } \Gamma_N$$



- 2D Elasticity

  {unknown $\underline{u}(x,y)$ $\begin{Bmatrix} u_1(x,y) \\ u_2(x,y) \end{Bmatrix}$}

$$\left.\begin{aligned} \operatorname{div} \underset{\sim}{\sigma} + \underline{b} &= \underline{0} \\ \underset{\sim}{\sigma} &= \underset{\approx}{C}\,\underset{\sim}{\epsilon} \end{aligned}\right] \text{ on } \Omega$$

(Plane Stress/Strain)

$\underline{\sigma} = \underset{\sim}{D}\,\underline{\epsilon}$ (Voight)

$$\underline{u} = \underline{u}_0 \,\Big]\text{ on } \Gamma_D$$

$$\underset{\sim}{\sigma}\,\underline{n} = \underline{t}_N \,\Big]\text{ on } \Gamma_N$$

(ii) Weak form : Method of Weighted residuals + Integration by parts :

- Heat Conduction

$$G(\theta,\bar{\theta}) = -\int_\Omega (\underline{\nabla\bar{\theta}})\cdot\underset{\sim}{\kappa}(\underline{\nabla\theta})\, d\Omega + \int_\Omega \bar{\theta} f \, d\Omega + \int_{\Gamma_N} \bar{\theta} h \, d\Gamma$$

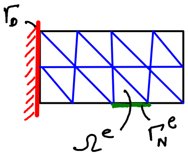
- 2D Elasticity

$$G(\underline{u},\underline{\bar{u}}) = -\int_\Omega \underbrace{\underset{\sim}{\bar{\epsilon}}\cdot(\underset{\approx}{C}\,\underset{\sim}{\epsilon})}_{\downarrow\text{Voight}} d\Omega + \int_\Omega \underline{\bar{u}}\cdot\underline{b}\, d\Omega + \int_{\Gamma_N} \underline{\bar{u}}\cdot\underline{t}_N \, d\Gamma$$

$$= -\int_\Omega \overbrace{\underline{\bar{\epsilon}}^T \underset{\sim}{D}\,\underline{\epsilon}}\, d\Omega + \int_\Omega \underline{\bar{u}}^T \underline{b}\, d\Omega + \int_{\Gamma_N} \underline{\bar{u}}^T \underline{t}_N \, d\Gamma$$

(iii) Discretization :-

$$\int_\Omega (\cdot)\, d\Omega = \sum_{e=1}^{M} \int_{\Omega^e} (\cdot)\, d\Omega \;; \qquad \int_{\Gamma_N} (\cdot)\, d\Gamma = \sum_{e=1}^{M} \int_{\Gamma_N^e} (\cdot)\, d\Gamma$$



(iv) Finite element Approximation :

- Heat Conduction :

$$\theta_e(x) \approx \theta_e^h(x) = \underset{\sim}{N}\,\underline{d} = \begin{bmatrix} N_1^e & N_2^e & N_3^e \end{bmatrix} \begin{bmatrix} d_1^e \\ d_2^e \\ d_3^e \end{bmatrix}$$

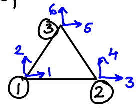
← Nodal Temperatures

$$\underline{\nabla\theta_e}(x) \approx \nabla\theta_e^h(x) = \underset{\sim}{B}\,\underline{d} = \begin{bmatrix} N_{1,1}^e & N_{2,1}^e & N_{3,1}^e \\ N_{1,2}^e & N_{2,2}^e & N_{3,2}^e \end{bmatrix} \begin{bmatrix} d_1^e \\ d_2^e \\ d_3^e \end{bmatrix}$$

- 2D Elasticity :

$$\underline{u}_e(x) \approx u_e^h(x) = \underset{\sim}{N}\,\underline{d} = \left[\begin{array}{cc:cc:cc} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e \end{array}\right] \left[\begin{array}{c} d_1^e \\ d_2^e \\ \hdashline d_3^e \\ d_4^e \\ \hdashline d_5^e \\ d_6^e \end{array}\right]$$

← Nodal Displacements



$$\underline{\epsilon}_e(x) \approx \underline{\epsilon}_e^h(x) = \underset{\sim}{B}\,\underline{d}$$

$$\downarrow$$

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \left[\begin{array}{cc:cc:cc} N_{1,1}^e & 0 & N_{2,1}^e & 0 & N_{3,1}^e & 0 \\ 0 & N_{1,2}^e & 0 & N_{2,2}^e & 0 & N_{3,2}^e \\ N_{1,2}^e & N_{1,1}^e & N_{2,2}^e & N_{2,1}^e & N_{3,2}^e & N_{3,1}^e \end{array}\right] \underline{d}$$

(v) Calculate Element Matrices & Vectors for all elements.

- Heat Conduction :

$$\underset{(3\times3)}{\underset{\sim}{K}^e} = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{\kappa}\, \underset{\sim}{B}\, d\Omega \qquad \underset{(3\times1)}{\underline{f}^e} = \int_{\Omega^e} \underset{\sim}{N}^T f\, d\Omega + \int_{\Gamma_N^e} \underset{\sim}{N}^T h\, d\Gamma$$

- 2D Elasticity :

$$\underset{(6\times6)}{\underset{\sim}{K}^e} = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, d\Omega \qquad \underset{(6\times1)}{\underline{f}^e} = \int_{\Omega^e} \underset{\sim}{N}^T \underline{b}\, d\Omega + \int_{\Gamma_N^e} \underset{\sim}{N}^T \underline{t}_N\, d\Gamma$$

(vi) Assemble

$$\underset{\sim}{K}^G = \mathop{\mathrm{A}}_{e=1}^{M} \underset{\sim}{K}^e \quad ; \quad \underline{f}^G = \mathop{\mathrm{A}}_{e=1}^{M} \underline{f}^e$$

$$\Rightarrow G(\cdot,\cdot) \approx \tilde{G}^h(\underline{d}, \bar{\underline{d}}) = \boxed{-\bar{\underline{d}}^T\left(\underset{\sim}{K}^G \underline{d}^G - \underline{f}^G\right) = 0} \qquad \forall\, \bar{\underline{d}}$$

(vii) Solve , enforcing Boundary Conditions :

$$\begin{bmatrix} K_{ff} & K_{fs} \\ K_{sf} & K_{ss} \end{bmatrix} \begin{Bmatrix} \overset{?}{d_f} \\ \underset{\checkmark}{d_s} \end{Bmatrix} = \begin{Bmatrix} \overset{\checkmark}{f_f} \\ \underset{?}{f_s} \end{Bmatrix}$$

(viii) Post computation (Plot)

- Temperature Value at all nodes and temperature field over all elements (MATLAB : patch( ))
- Displaced shape using new locations and stress distribution over all elements (MATLAB : patch( ))

Note:

For Δ element, derivatives are constant over the entire element.

Thus

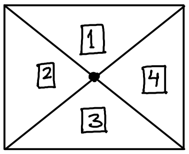
Strain : $\underline{\epsilon}(x,y)$ = constant = $\underset{\sim}{B}\, \underline{d}$

Stress : $\underline{\sigma}(x,y)$ = constant = $\underset{\sim}{D}\, \underline{\epsilon} = \underset{\sim}{D}\, \underset{\sim}{B}\, \underline{d}$

For this reason, the 3-node triangle is also called $\boxed{\text{CST}}$ element.

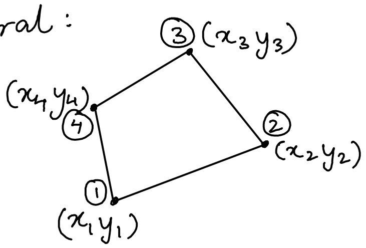
CST : Constant stress/strain Triangle.

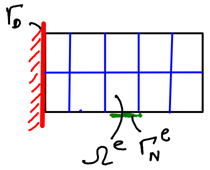
We can also "average" the stresses/Strains at a node from neighboring elements.



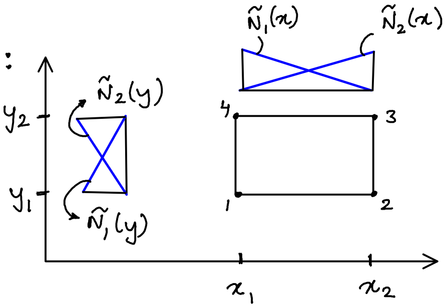
Q4 element : 4-node Quadrilateral

In the discretization step, one can also choose quadrilaterals. The quadrilaterals can be general:





However, lets first consider pure rectangles: The finite element approximation can be obtained by multiplying the 1-D shape functions in "x" and "y".



$$N_1^e(x,y) = \tilde{N}_1(x) * \tilde{N}_1(y)$$

$$N_2^e(x,y) = \tilde{N}_2(x) * \tilde{N}_1(y)$$

$$N_3^e(x,y) = \tilde{N}_2(x) * \tilde{N}_2(y)$$

$$N_4^e(x,y) = \tilde{N}_1(x) * \tilde{N}_2(y)$$

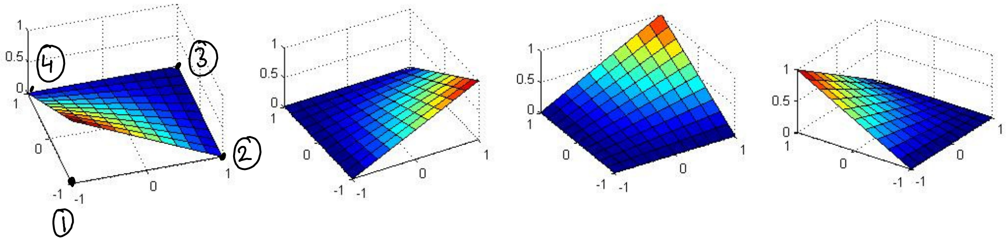
$$\tilde{N}_1(x) = \frac{x-x_2}{x_1-x_2}$$

$$\tilde{N}_2(x) = \frac{x-x_1}{x_2-x_1}$$

$$\tilde{N}_1(y) = \frac{y-y_2}{y_1-y_2}$$

$$\tilde{N}_2(y) = \frac{y-y_1}{y_2-y_1}$$

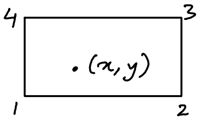
The two dimensional shape functions look like:



Note:

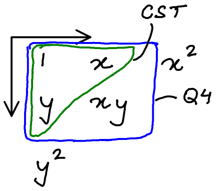
- $$\sum_{\alpha=1}^{4} N_\alpha(x,y) = \tilde{N}_1(x)\underbrace{\left[\tilde{N}_1(y)+\tilde{N}_2(y)\right]}_{1} + \tilde{N}_2(x)\underbrace{\left[\tilde{N}_1(y)+\tilde{N}_2(y)\right]}_{1}$$

  $$= \underbrace{\tilde{N}_1(x)+\tilde{N}_2(x)}_{1} \Rightarrow \boxed{\sum_{\alpha=1}^{4} N_\alpha(x,y) = 1}$$



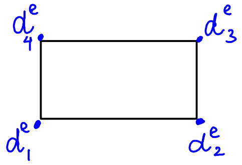
- These shape functions are not linear.

  $$N_\alpha(x,y) = a_0 + a_1(x) + a_2(y) + a_3(xy)$$



Using these shape functions, the unknown variable can be approximated as usual:

- Heat conduction

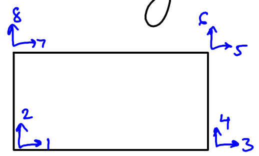


$$\theta_e(x,y) \approx \theta_e^h(x) = \underset{\sim}{N}\,\underline{d} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} d_1^e \\ d_2^e \\ d_3^e \\ d_4^e \end{bmatrix}$$

$$\underline{\nabla}\theta_e(x,y) \approx \underline{\nabla}\theta_e^h(x) = \underset{\sim}{B}\,\underline{d}$$

$$\underset{\sim}{B} = \begin{bmatrix} N_{1,1} & N_{2,1} & N_{3,1} & N_{4,1} \\ N_{1,2} & N_{2,2} & N_{3,2} & N_{4,2} \end{bmatrix}$$

- 2D Elasticity



$$\underline{u}_e(x,y) \approx \underline{u}_e^h(x,y) = \underset{\sim}{N}\,\underline{d} = \left[\begin{array}{cc|cc|cc|cc} N_1 & & N_2 & & N_3 & & N_4 & \\ & N_1 & & N_2 & & N_3 & & N_4 \end{array}\right] \begin{bmatrix} d_1^e \\ d_2^e \\ d_3^e \\ d_4^e \\ d_5^e \\ d_6^e \\ d_7^e \\ d_8 \end{bmatrix}$$

$$\underline{\epsilon}_e(x,y) \approx \underline{\epsilon}_e^h(x,y) = \underset{\sim}{B}\,\underline{d}$$

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \left[\begin{array}{cc|cc|cc|cc} N_{1,1} & & N_{2,1} & & N_{3,1} & & N_{4,1} & \\ & N_{1,2} & & N_{2,2} & & N_{3,2} & & N_{4,2} \\ N_{1,2} & N_{1,1} & N_{2,2} & N_{2,1} & N_{3,2} & N_{3,1} & N_{4,2} & N_{4,1} \end{array}\right] \underline{d}$$

Element Integrals

- Heat Conduction

$$\underset{(4\times4)}{\underset{\sim}{K}^e} = \int_{▨} \underset{\sim}{B}^T \underset{\sim}{\kappa}\, \underset{\sim}{B}\, dx dy \quad ; \quad \underset{(4\times1)}{\underline{f}^e} = \int_{▨} \underset{\sim}{N}^T f\, dx dy + \int_{\square} \underset{\sim}{N}^T h\, dl$$

- 2D Elasticity

$$\underset{(8\times8)}{\underset{\sim}{K}^e} = \int_{▨} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, dx dy \quad ; \quad \underset{(8\times1)}{\underline{f}^e} = \int_{▨} \underset{\sim}{N}^T \underline{b}\, dx dy + \int_{\square} \underset{\sim}{N}^T \underline{t}_N\, dl$$

Note: The derivatives of $N_\alpha(x,y)$ will not be constant. These element integrals are generally evaluated numerically.

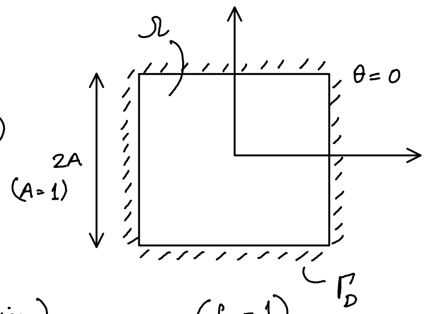
Finally assemble and solve as usual.

$$\underset{\sim}{K}^g = \overset{M}{\underset{e=1}{\mathbf{A}}} \underset{\sim}{K}^e \quad ; \quad \underline{f}^g = \overset{M}{\underset{e=1}{\mathbf{A}}} \underline{f}^e \quad \Rightarrow \quad \underline{\bar{d}}^T\left(\underset{\sim}{K}^g \underline{d} - \underline{f}\right) = 0 \quad \forall \underline{\bar{d}}$$

Example

Heat Conduction :

$$\left.\begin{array}{l}\operatorname{div}\,\underline{q} = f_0 \\ \underline{q} = -\underset{\sim}{K}(\nabla\theta)\end{array}\right] \text{ in } \Omega \text{ (square)}$$



(say $\underset{\sim}{K} = \underset{\sim}{I}$) $\Rightarrow \operatorname{div}(\nabla\theta) + f_0 = 0$

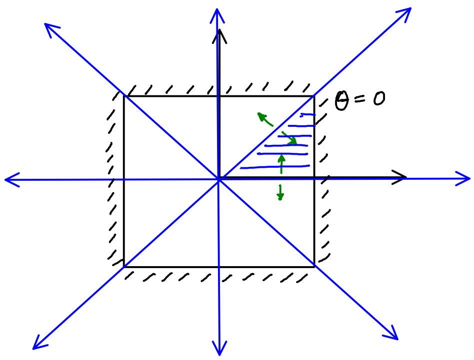
$\Rightarrow \boxed{\nabla^2\theta + f_0 = 0}$ (Poisson Equation) $(f_0 = 1)$

$$\nabla^2\theta \equiv \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} \quad \text{(Laplacian)}$$

BC : $\theta = 0$ on Γ_D

$\Gamma_N = \emptyset$

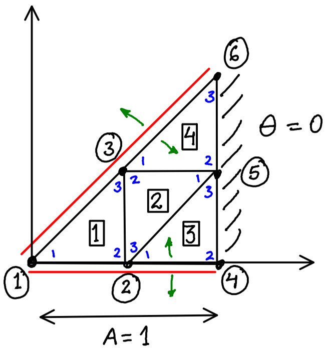
Note: $\boxed{\text{Symmetry}}$ of the problem allows us to reduce the problem domain to $\frac{1}{8}$ of the original size.



Symmetry Conditions : Problem domain,
+ boundary conditions,
+ Loads

New problem:

$$\nabla^2\theta + f_0 = 0 \quad \text{in } (\Omega/8)$$



Boundary conditions :

$\boxed{\theta = 0}$ on edge 4-5-6

$\frac{\partial\theta}{\partial y} = 0$ on edge 1-2-4

$\frac{\partial\theta}{\partial n} = 0$ on edge 1-3-6

Note:

$$h = \underline{q}\cdot\underline{n} = -\underset{\sim}{K}\underbrace{(\nabla\underline{\theta})\cdot\underline{n}}_{\frac{\partial\theta}{\partial n}} = -\begin{bmatrix}\frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y}\end{bmatrix}\cdot\begin{bmatrix}n_x \\ n_y\end{bmatrix}$$

For edge 1-2-4: $\begin{Bmatrix}n_x \\ n_y\end{Bmatrix} = \begin{Bmatrix}0 \\ -1\end{Bmatrix} \Rightarrow \frac{\partial\theta}{\partial y} = \boxed{h = 0}$

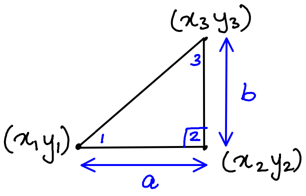
For edge 1-3-6: $\begin{Bmatrix}n_x \\ n_y\end{Bmatrix} = \begin{Bmatrix}-1/\sqrt{2} \\ 1/\sqrt{2}\end{Bmatrix} \Rightarrow \frac{1}{\sqrt{2}}\left(-\frac{\partial\theta}{\partial x} + \frac{\partial\theta}{\partial y}\right) = \boxed{h = 0}$

Recall : Weak form:

$$G(\theta,\bar{\theta}) = \int_\Omega (\nabla\bar{\theta})\,\underset{\sim}{K}\,(\nabla\theta)\,d\Omega - \int_\Omega \bar{\theta} f\,d\Omega - \int_{\Gamma_N} \bar{\theta} h\,d\Gamma$$

Friday, February 26, 2010
8:44 AM

Discretized & Approximated (Galerkin) form:



- For each element : $\theta_e^h(x,y) = \underset{\sim}{N}\,\underline{d}$

$$\nabla \underline{\theta}_e^h(x,y) = \underset{\sim}{B}\,\underline{d}$$

$$\underset{\sim}{K}^{el} = \int_{\Delta} \underset{\sim}{B}^T \underset{\sim}{\kappa}\, \underset{\sim}{B}\, d\Delta \quad ; \quad \underline{f}^{el} = \int_{\Delta} \underset{\sim}{N}^T f\, d\Delta + \underbrace{\int_{\Delta} \underset{\sim}{N}^T h\, dl}_{0}$$

$$\underset{\sim}{K}^G = \mathop{\Large\mathsf{A}}_{e=1}^{M} \underset{\sim}{K}^{el} \quad ; \quad \underline{f}^G = \mathop{\Large\mathsf{A}}_{e=1}^{M} \underline{f}^{el}$$

$$\Rightarrow \tilde{G}^h(\underline{d}, \underline{\bar{d}}) = \boxed{\underline{\bar{d}}^{G^T}\left(\underset{\sim}{K}^G \underline{d}^G - \underline{f}^G\right) = 0} \quad \text{for } \forall\, \underline{\bar{d}}^G$$

$$\underset{\sim}{\kappa} = \underset{\sim}{I}$$

$$\Rightarrow \underset{\sim}{K}^{el} = \frac{1}{2ab}\begin{bmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ & -a^2 & a^2 \end{bmatrix}$$

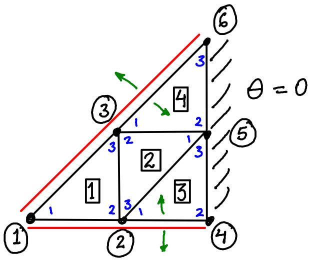
$$\underline{f}^{el} = \frac{f_0 ab}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

For the present problem $a = b = A/2 = 1/2$

$$\underset{\sim}{K}^{el} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad ; \quad \underline{f}^{el} = \frac{f_0}{24}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad el = 1, 2, 3, 4.$$

Assembly:

$$\underset{\sim}{K}^G = \frac{1}{2}\begin{array}{cc|cccccc} & & \multicolumn{3}{c}{f} & \multicolumn{3}{c}{s} \\ & & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline f & 1 & 1 & -1 & & & & \\ & 2 & -1 & 4 & -2 & -1 & & \\ & 3 & & -2 & 4 & 0 & -2 & \\ s & 4 & & -1 & 0 & 2 & -1 & \\ & 5 & & & -2 & -1 & 4 & -1 \\ & 6 & & & & & -1 & 1 \end{array} \quad ; \quad \underline{f}^G = \frac{1}{24}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{Bmatrix} (f_0 = 1) \\ 1 \\ 3 \\ 3 \\ 1 + Q_4 \\ 3 + Q_5 \\ 1 + Q_6 \end{Bmatrix}$$



Boundary Conditions

$$\Rightarrow \left[\begin{array}{c|c} \underset{\sim}{K}_{ff}^G & \underset{\sim}{K}_{fs}^G \\ \hline \underset{\sim}{K}_{sf}^G & \underset{\sim}{K}_{ss}^G \end{array}\right]\begin{bmatrix} \underline{d}_f^G \\ \underbrace{\underline{d}_s^G}_{0} \end{bmatrix} = \begin{bmatrix} \underline{f}_f^G \\ \underline{f}_s^G \end{bmatrix} \quad \Rightarrow \quad [K_{ff}^G]\{d_f^G\} = \{\underline{f}_f^G\}$$

$$\Rightarrow \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1/24 \\ 1/8 \\ 1/8 \end{Bmatrix} \quad \Rightarrow \quad \boxed{\underline{d}_f^G = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0.3125 \\ 0.2292 \\ 0.1771 \end{Bmatrix}}$$

Post computation:

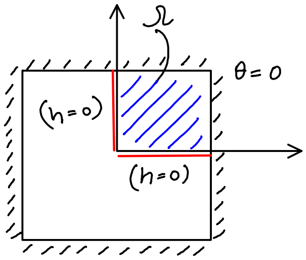
- Heat flux on Γ_D (reactions)

$$\begin{Bmatrix} Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = -\begin{Bmatrix} 1/24 \\ 1/8 \\ 1/24 \end{Bmatrix} + [\underset{\sim}{K}_{sf}^G]\{\underline{d}_f^G\} = \begin{Bmatrix} -0.1979 \\ -0.3021 \\ -0.0417 \end{Bmatrix}$$

- Temperature gradient in each element (Strains / stresses) $\quad (\nabla\underline{\theta})_e = \underset{\sim}{B}^e \underline{d}^e$

Rectangular Elements:

We can only use 1/4 symmetry in this case.

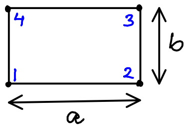


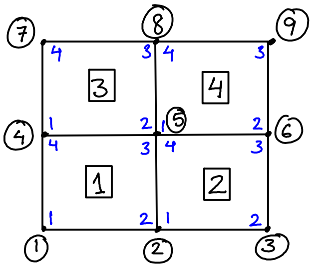
- Same weak form.
- Discretization & Approximation

$$\theta_e^h(x,y) = \underset{\sim}{N}\,\underline{d} = [N_1 \;\; N_2 \;\; N_3 \;\; N_4]\,\underline{d}$$

$$\nabla\theta_e^h(x,y) = \underset{\sim}{B}\,\underline{d} = \begin{bmatrix} N_{1,1} & N_{2,1} & N_{3,1} & N_{4,1} \\ N_{1,2} & N_{2,2} & N_{3,2} & N_{4,2} \end{bmatrix}\underline{d}$$

For a rectangular element of sides "a" & "b"





$$N_1(x,y) = \tilde{N}_1(x) * \tilde{N}_1(y) = \frac{(x-a)(y-b)}{ab}$$

$$N_2(x,y) = \tilde{N}_2(x) * \tilde{N}_1(y) = \frac{x(y-b)}{(-ab)}$$

$$N_3(x,y) = \tilde{N}_2(x) * \tilde{N}_2(y) = \frac{xy}{ab}$$

$$N_4(x,y) = \tilde{N}_1(x) * \tilde{N}_2(y) = \frac{(x-a)y}{(-ab)}$$

$$N_{1,1} = \frac{(y-b)}{ab} \;;\quad N_{2,1} = \frac{(y-b)}{(-ab)} \;;\quad N_{3,1} = \frac{y}{ab} \;;\quad N_{4,1} = \frac{y}{(-ab)}$$

$$N_{1,2} = \frac{(x-a)}{ab} \;;\quad N_{2,2} = \frac{x}{(-ab)} \;;\quad N_{3,2} = \frac{x}{ab} \;;\quad N_{4,2} = \frac{(x-a)}{(-ab)}$$

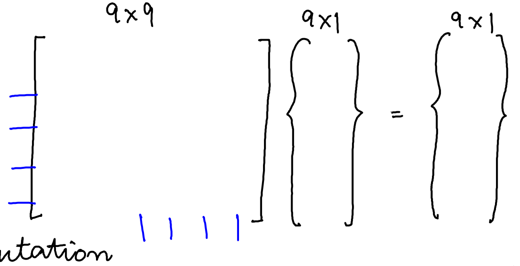
$$\underset{\sim}{B} = \begin{bmatrix} N_{1,1} & N_{2,1} & N_{3,1} & N_{4,1} \\ N_{1,2} & N_{2,2} & N_{3,2} & N_{4,2} \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{K}^{el} = \int_{\blacksquare} \underset{\sim}{B}^T \underset{\sim}{k}\, \underset{\sim}{B}\, d\Box = \int_0^a\left(\int_0^b \underset{\sim}{B}^T \underset{\sim}{k}\, \underset{\sim}{B}\, dy\right)dx = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ & 4 & -1 & -2 \\ & & 4 & -1 \\ \text{sym} & & & 4 \end{bmatrix}$$

$$\underline{f}^{el} = \int_{\blacksquare} \underset{\sim}{N}^T f_0\, d\Box + \int_{\Box} \underset{\sim}{N}^T \cancelto{0}{h}\, dl = \frac{f_0}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Global system of equations:

$$\underset{\sim}{K}^g\, \underline{d}^g = \underline{f}^g$$



$$\Rightarrow \left[K^G_{ff}\right]\left\{d^G_f\right\} = \left\{\underline{f}^G_f\right\}$$

Post computation

- Heat fluxes on Γ_D (reactions)
- Temperature gradients in Ω (strains/stresses)

2D Plane Stress Problem

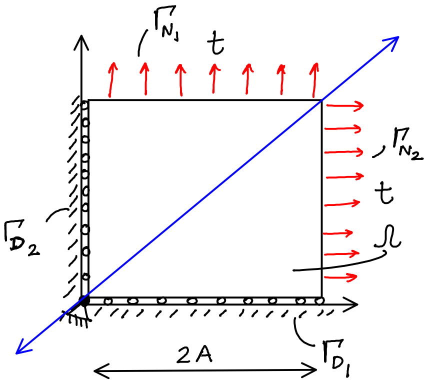
Strong Form (GDE):

$$\text{div}\ \underset{\sim}{\sigma} + \underline{b} = \underline{0} \quad \text{in } \Omega$$

Boundary Conditions:

$$u_2 = 0 \quad \text{on } \Gamma_{D_1}$$

$$u_1 = 0 \quad \text{on } \Gamma_{D_2}$$



(A = 1)
(t = 1)
E, ν given

i.e.

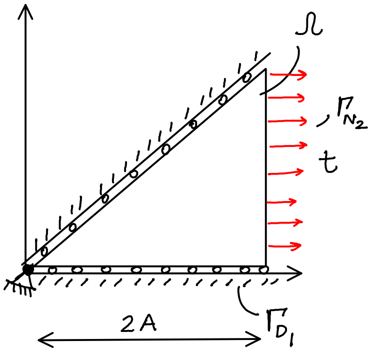
$$\underline{u} = \begin{Bmatrix} ? \\ 0 \end{Bmatrix} \qquad \underline{t}_N = \begin{Bmatrix} 0 \\ ? \end{Bmatrix} \qquad \text{on } \Gamma_{D_1}$$

$$\underline{u} = \begin{Bmatrix} 0 \\ ? \end{Bmatrix} \qquad \underline{t}_N = \begin{Bmatrix} ? \\ 0 \end{Bmatrix} \qquad \text{on } \Gamma_{D_2}$$

$$\underline{t}_N = \begin{Bmatrix} 0 \\ t \end{Bmatrix} \qquad \text{on } \Gamma_{N_1}$$

$$\underline{t}_N = \begin{Bmatrix} t \\ 0 \end{Bmatrix} \qquad \text{on } \Gamma_{N_2}$$

Symmetry reduces the problem to ½

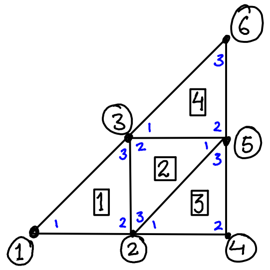


Using four 3-node (CST) Triangular elements:

$$\underline{u}(x,y) = \begin{Bmatrix} u_1(x,y) \\ u_2(x,y) \end{Bmatrix} = \underset{\sim}{N}\ \underline{d}$$

$$= \left[\begin{array}{cc|cc|cc} N_1 & & N_2 & & N_3 & \\ & N_1 & & N_2 & & N_3 \end{array}\right] \begin{bmatrix} d_1^e \\ d_2^e \\ \hdashline d_3^e \\ d_4^e \\ \hdashline d_5^e \\ d_6^e \end{bmatrix}$$

$$\underline{\epsilon}(x,y) = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \underset{\sim}{B}\ \underline{d}$$



$$\underset{\sim}{K}^{el} = \int_{\blacktriangle} \underset{\sim}{B}^T \underset{\sim}{D}\ \underset{\sim}{B}\ dA \quad ; \quad \underline{f}^{el} = \int_{\blacktriangle} \underset{\sim}{N}^T \underline{b}\ dA + \int_{\Delta} \underset{\sim}{N}^T \underline{t}_N\ dl$$

$$\underset{\sim}{K}^G = \mathop{\mathbf{A}}_{e=1}^{M} \underset{\sim}{K}^{el} \quad ; \quad \underline{f}^g = \mathop{\mathbf{A}}_{e=1}^{M} \underline{f}^{el}$$

Saturday, February 27, 2010
10:37 AM

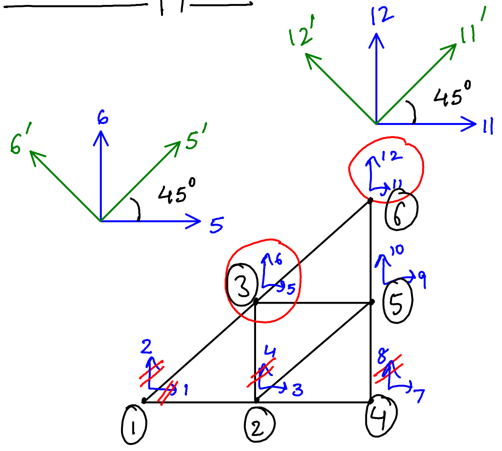
Enforcing Boundary conditions on an inclined support:

$$\underset{\sim}{K}^G \underline{d}^G = \underline{f}^G$$

(12×12)

Transform specific dofs:

$$\left[\begin{array}{c} \\ \\ \\ \\ \bullet \\ \bullet \\ \\ \\ \\ \\ \bullet \\ \bullet \end{array}\right]\begin{array}{l} 1\\ 2\\ 3\\ 4\\ 5'\\ 6'\\ 7\\ 8\\ 9\\ 10\\ 11'\\ 12' \end{array} = \underbrace{\begin{bmatrix} 1 &&&&&&&&&&& \\ & 1 &&&&&&&&&& \\ && 1 &&&&&&&&& \\ &&& 1 &&&&&&&& \\ &&&& c & s &&&&&& \\ &&&& -s & c &&&&&& \\ &&&&&& 1 &&&&& \\ &&&&&&& 1 &&&& \\ &&&&&&&& 1 &&& \\ &&&&&&&&& 1 && \\ &&&&&&&&&& c & s \\ &&&&&&&&&& -s & c \end{bmatrix}}_{\underset{\sim}{T}} \left[\begin{array}{c} \\ \\ \\ \\ \bullet \\ \bullet \\ \\ \\ \\ \\ \bullet \\ \bullet \end{array}\right]\begin{array}{l} 1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10\\ 11\\ 12 \end{array}$$



Note:

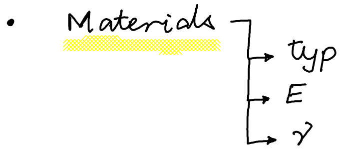
$$\begin{Bmatrix} d_5' \\ d_6' \end{Bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} d_5 \\ d_6 \end{Bmatrix}$$

$$\underbrace{\left(\underset{\sim}{T}\,\underset{\sim}{K}^G\,\underset{\sim}{T}^T\right)}\underbrace{\left(\underset{\sim}{T}\,\underline{d}^G\right)} = \underbrace{\left(\underset{\sim}{T}\,\underline{f}^G\right)}$$

$$\left[\underset{\sim}{K}^{G'}\right]\left\{\underline{d}^{G'}\right\} = \left\{\underline{f}^{G'}\right\}$$ → Now Enforce BCs & solve.

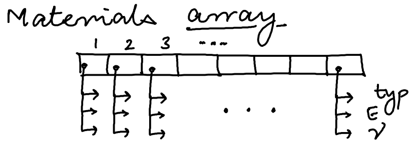
2D Finite element Code Structure

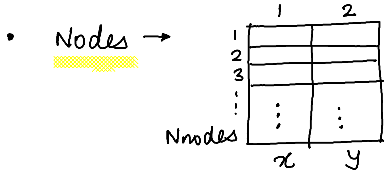
Data structure

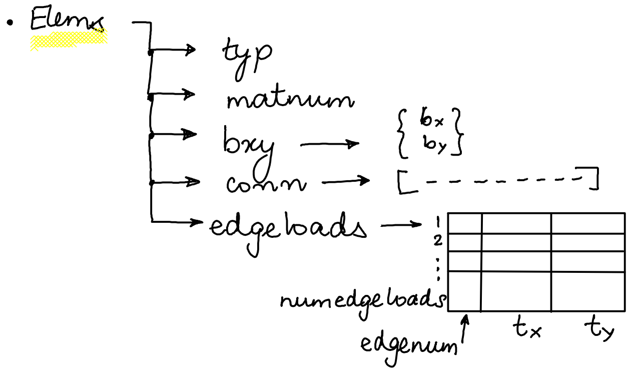


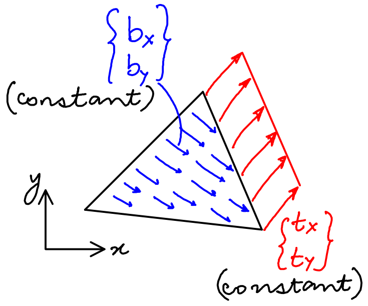
("Struct" in MATLAB)

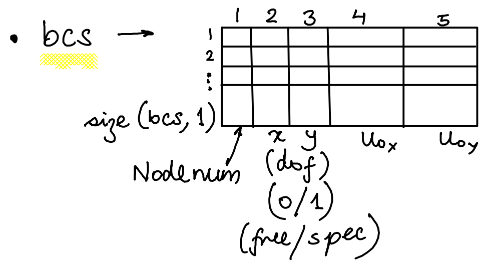
Materials array

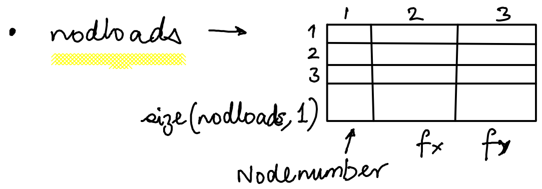












Overall Code Flow:

(1) Input

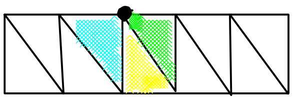
(2) Loop over elements
- Obtain $\underset{\sim}{K}^{el}$, $\underline{f}^{el}$

(3) Assemble in $\underset{\sim}{K}^{g}$, $\underline{f}^{g}$

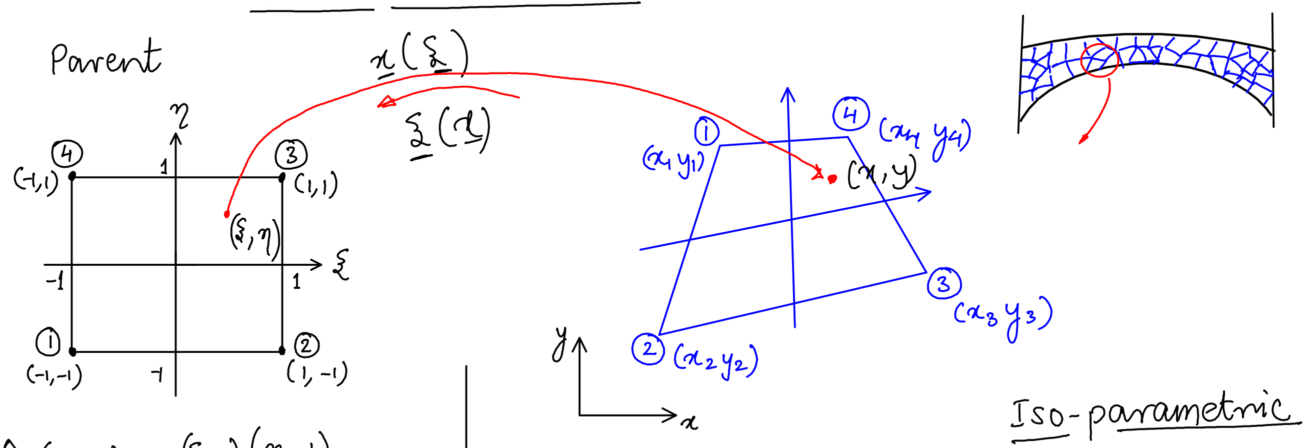
(4) Enforce BCs (doffree, dofspec)

(5) Solve for $\{\underline{d}^{g}_{f}\}$; $\{\underline{f}^{g}_{s}\}$

(6) Plot Results
- Displacements
- Stresses (for all elements) (Unaveraged/Averaged)



General Q4 Element



Iso-parametric Map.

$$\hat{N}_1(\xi,\eta) = \frac{(\xi-1)(\eta-1)}{(-1-1)(-1-1)}$$

$$\hat{N}_2(\xi,\eta) = \frac{(\xi+1)(\eta-1)}{(1+1)(-1-1)}$$

$$\hat{N}_3(\xi,\eta) = \frac{(\xi+1)(\eta+1)}{(1+1)(1+1)}$$

$$\hat{N}_4(\xi,\eta) = \frac{(\xi-1)(\eta+1)}{(-1-1)(1+1)}$$

$N_\alpha(x,y) = ?$

Choose:

$$\boxed{\underline{x}(\underline{\xi}) = \sum_{\alpha=1}^{4} \hat{N}_\alpha(\underline{\xi})\, \underline{x}_\alpha}$$

ie $\underline{x} = \underset{\sim}{\hat{N}}_\alpha\, \underline{x}_\alpha$

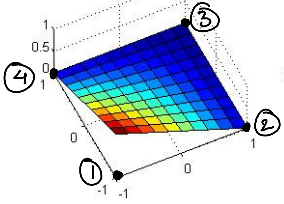
$$\begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = \left[\begin{array}{cc|cc|cc|cc} \hat{N}_1 & & \hat{N}_2 & & \hat{N}_3 & & \hat{N}_4 & \\ & \hat{N}_1 & & \hat{N}_2 & & \hat{N}_3 & & \hat{N}_4 \end{array}\right] \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

In general,

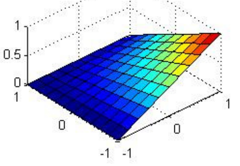
$$\boxed{\hat{N}_\alpha(\xi,\eta) = \frac{1}{4}(1+\xi_\alpha \xi)(1+\eta_\alpha \eta) = N_\alpha(x,y)}$$

where $(\xi_\alpha, \eta_\alpha)$ are coordinates $(\pm 1, \pm 1)$

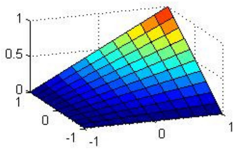
$\hat{N}_1(\xi,\eta)$



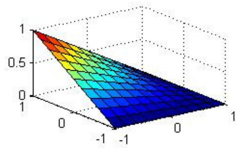
$\hat{N}_2(\xi,\eta)$



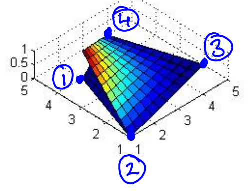
$\hat{N}_3(\xi,\eta)$



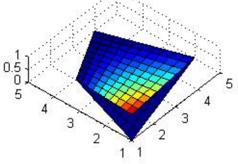
$\hat{N}_4(\xi,\eta)$



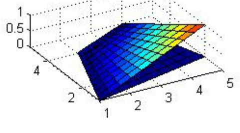
$N_1(x,y)$



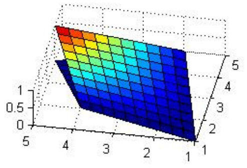
$N_2(x,y)$



$N_3(x,y)$

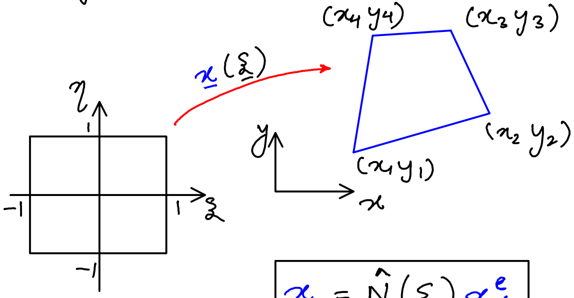


$N_4(x,y)$



Mapped Q4 finite element formulation

FE Approximation:



$$\boxed{\underline{x} = \underset{\sim}{\hat{N}}(\underline{\xi})\,\underline{x}^e_\alpha}$$

- Heat conduction

$$\theta^h_e(x,y) = \underset{\sim}{N}(x,y)\,\underline{d}^e$$

i.e. $\theta^h_e(\underline{x}(\underline{\xi})) = \underset{\sim}{\hat{N}}(\xi,\eta)\,\underline{d}^e$

$$= \begin{bmatrix}\hat{N}_1 & \hat{N}_2 & \hat{N}_3 & \hat{N}_4\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\end{bmatrix}$$

- Elasticity:

$$\underline{u}^h_e(x,y) = \underset{\sim}{N}(x,y)\,\underline{d}^e$$

$$= \underset{\sim}{\hat{N}}(\xi,\eta)\,\underline{d}^e$$

$$\begin{Bmatrix}u_1(x(\xi,\eta),y(\xi,\eta))\\ u_2(x(\xi,\eta),y(\xi,\eta))\end{Bmatrix} = \left[\begin{array}{cc:cc:cc:cc}\hat{N}_1 & & \hat{N}_2 & & \hat{N}_3 & & \hat{N}_4 & \\ & \hat{N}_1 & & \hat{N}_2 & & \hat{N}_3 & & \hat{N}_4\end{array}\right]\left\{\begin{array}{c}d^e_1\\ d^e_2\\ \hdashline d^e_3\\ d^e_4\\ \hdashline d^e_5\\ d^e_6\\ \hdashline d^e_7\\ d^e_8\end{array}\right\}$$

Calculation of Gradients:

- Heat Conduction : $(\nabla\underline{\theta}) = \underset{\sim}{B}\,\underline{d}^e = \begin{bmatrix}N_{1,1} & N_{2,1} & N_{3,1} & N_{4,1}\\ N_{1,2} & N_{2,2} & N_{3,2} & N_{4,2}\end{bmatrix}\underline{d}^e$
- Elasticity: $\underline{\epsilon} = \underset{\sim}{B}\,\underline{d}^e = \left[\begin{array}{cc:cc:cc:cc}N_{1,1} & & N_{2,1} & & N_{3,1} & & N_{4,1} & \\ & N_{1,2} & & N_{2,2} & & N_{3,2} & & N_{4,2}\\ N_{1,2} & N_{1,1} & N_{2,2} & N_{2,1} & N_{3,2} & N_{3,1} & N_{4,2} & N_{4,1}\end{array}\right]\underline{d}^e$

In both cases $\underset{\sim}{B}$ matrices involve $\dfrac{\partial N_\alpha(x,y)}{\partial x}$ and $\dfrac{\partial N_\alpha(x,y)}{\partial y}$

To calculate $\dfrac{\partial N_\alpha}{\partial x}$ or $\dfrac{\partial N_\alpha}{\partial y}$, use chain rule (not $\dfrac{\partial \hat{N}_\alpha}{\partial \xi}$ or $\dfrac{\partial \hat{N}_\alpha}{\partial \eta}$)

$$\hat{N}_\alpha(\xi,\eta) = N_\alpha(x,y)$$

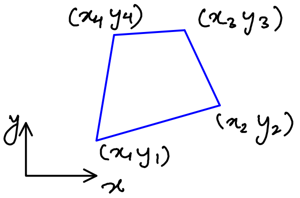
$$\left.\begin{aligned}\frac{\partial \hat{N}_\alpha}{\partial \xi} &= \frac{\partial N_\alpha}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial N_\alpha}{\partial y}\cdot\frac{\partial y}{\partial \xi}\\ \frac{\partial \hat{N}_\alpha}{\partial \eta} &= \frac{\partial N_\alpha}{\partial x}\,\frac{\partial x}{\partial \eta} + \frac{\partial N_\alpha}{\partial y}\cdot\frac{\partial y}{\partial \eta}\end{aligned}\right\} \Rightarrow \begin{Bmatrix}\dfrac{\partial \hat{N}_\alpha}{\partial \xi}\\ \dfrac{\partial \hat{N}_\alpha}{\partial \eta}\end{Bmatrix} = \underbrace{\begin{bmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi}\\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta}\end{bmatrix}}_{\underset{\sim}{J}^T}\begin{Bmatrix}\dfrac{\partial N_\alpha}{\partial x}\\ \dfrac{\partial N_\alpha}{\partial y}\end{Bmatrix}$$

Monday, March 01, 2010
12:17 PM

Using the iso-parametric map:

$$x = \sum_{\alpha=1}^{4} \hat{N}_\alpha x_\alpha = \hat{N}_1 x_1 + \hat{N}_2 x_2 + \hat{N}_3 x_3 + \hat{N}_4 x_4$$

$$y = \sum_{\alpha=1}^{4} \hat{N}_\alpha y_\alpha = \hat{N}_1 y_1 + \hat{N}_2 y_2 + \hat{N}_3 y_3 + \hat{N}_4 y_4$$



So,

$$\frac{\partial x}{\partial \xi} = \sum_{\alpha=1}^{4} \frac{\partial \hat{N}_\alpha}{\partial \xi} x_\alpha \quad ; \quad \frac{\partial x}{\partial \eta} = \sum_{\alpha=1}^{4} \frac{\partial \hat{N}_\alpha}{\partial \eta} x_\alpha$$

$$\frac{\partial y}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial \hat{N}_\alpha}{\partial \xi} y_\alpha \quad ; \quad \frac{\partial y}{\partial \eta} = \sum_{\alpha=1}^{4} \frac{\partial \hat{N}_\alpha}{\partial \eta} y_\alpha$$

These terms can be arranged in matrix called the <u>Jacobian</u> of the map.

$$\underset{\sim}{J} \equiv \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \Rightarrow \begin{Bmatrix} \frac{\partial N_\alpha}{\partial x} \\ \frac{\partial N_\alpha}{\partial y} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1}}_{[\underset{\sim}{J}^{-T}]} \begin{Bmatrix} \frac{\partial \hat{N}_\alpha}{\partial \xi} \\ \frac{\partial \hat{N}_\alpha}{\partial \eta} \end{Bmatrix}$$

So Finally, to calculate $\frac{\partial N_\alpha}{\partial x}$ & $\frac{\partial N_\alpha}{\partial y}$ (at a <u>particular</u> (x,y) or (ξ,η))

(i) Calculate

$$\left.\begin{aligned} \frac{\partial \hat{N}_\alpha}{\partial \xi} &= \frac{1}{4} \xi_\alpha (1 + \eta_\alpha \eta) \\ \frac{\partial \hat{N}_\alpha}{\partial \eta} &= \frac{1}{4} (1 + \xi_\alpha \xi) \eta_\alpha \end{aligned}\right\} \text{for } \alpha = 1,2,3,4$$

(ii) Calculate $[\underset{\sim}{J}]$

(iii) Calculate

$$\begin{Bmatrix} N_{\alpha,1} \\ N_{\alpha,2} \end{Bmatrix} = [\underset{\sim}{J}]^{-T} \begin{Bmatrix} N_{\alpha,\xi} \\ N_{\alpha,\eta} \end{Bmatrix}$$

Now we can calculate $\underset{\sim}{B}$ matrices.

Finally, Element Matrices :-

- Heat Conduction

$$\underset{\sim}{K}^{el} = \int_{\blacksquare} \underset{\sim}{B}^T \underset{\sim}{\kappa} \, \underset{\sim}{B} \, d\blacksquare \quad ; \quad \underline{f}^{el} = \int_{\blacksquare} \underset{\sim}{N}^T f \, d\blacksquare + \int_{\square} \underset{\sim}{N}^T h \, dl$$

- Elasticity

$$\underset{\sim}{K}^{el} = \int_{\blacksquare} \underset{\sim}{B}^T \underset{\sim}{D} \, \underset{\sim}{B} \, d\blacksquare \quad ; \quad \underline{f}^{el} = \int_{\blacksquare} \underset{\sim}{N}^T \underline{b} \, d\blacksquare + \int_{\square} \underset{\sim}{N}^T t_N \, dl$$

Assembly:

$$\underset{\sim}{K}^{g} = \overset{M}{\underset{e=1}{\mathbf{A}}} \underset{\sim}{K}^{el} \quad ; \quad \underline{f}^{g} = \overset{M}{\underset{e=1}{\mathbf{A}}} \underline{f}^{el}$$

Solve:

$$\tilde{G}^h(\underline{d}, \bar{\underline{d}}) = -\bar{\underline{d}}^T \left(\underset{\sim}{K}^g \underline{d}^g - \underline{f}^g \right) = 0 \qquad \forall \bar{\underline{d}}$$

(with Boundary Conditions).

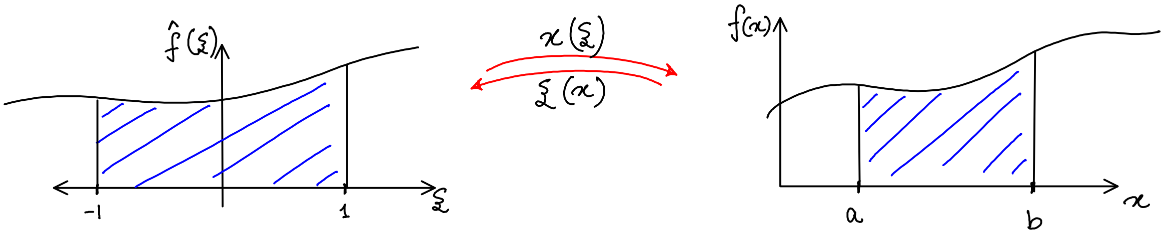
Tuesday, March 02, 2010
6:22 PM

Numerical Integration

In order to calculate the domain and boundary integrals:

$$\int_{\blacksquare} (\cdot)\, d\blacksquare \qquad \text{and} \qquad \int_{\square} (\cdot)\, d\ell$$

Consider the 1-D integral : $\int_a^b f(x)\, dx$



$$x(\xi) = \hat{N}_1(\xi)\, a + \hat{N}_2(\xi)\, b$$

$$x(\xi) = \frac{(\xi-1)}{(-2)}\, a + \frac{(\xi+1)}{2}\, b$$

Using this "change of variables" ; or "iso-parametric transformation"

$$\frac{dx}{d\xi} = \frac{d\hat{N}_1}{d\xi} \cdot a + \frac{d\hat{N}_2}{d\xi}\, b = -\frac{a}{2} + \frac{b}{2} \Rightarrow dx = \underbrace{\frac{2}{(b-a)}}_{J}\, d\xi$$

Thus
$$\int_a^b f(x)\, dx = \int_{-1}^{1} \hat{f}(\xi)\, J\, d\xi$$

To evaluate $\int_{-1}^{1} g(\xi)\, d\xi$:

we can use a variety of different numerical integration schemes or (Quadrature methods)

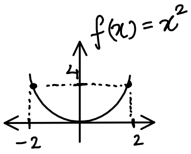
Note:
$$\hat{f}(\xi) = f(x(\xi))$$

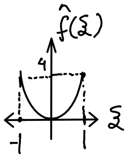
eg $f(x) = x^2$ $\quad -2 < x < 2$ $(a < x < b)$

$x(\xi) = 2\xi$

$\hat{f}(\xi) = f(x(\xi)) = (2\xi)^2$

$= 4\xi^2$





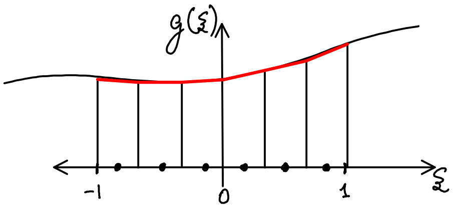
$$\left.\begin{matrix} \hat{f}(\cdot) = (\cdot)^2 \\ f(\cdot) = 4(\cdot)^2 \end{matrix}\right\} \text{are different}$$

For example

(i) Trapezoidal Rule

Divide into "N" parts

If equal points : $\left(\Delta\ell = \frac{2}{N}\right)$



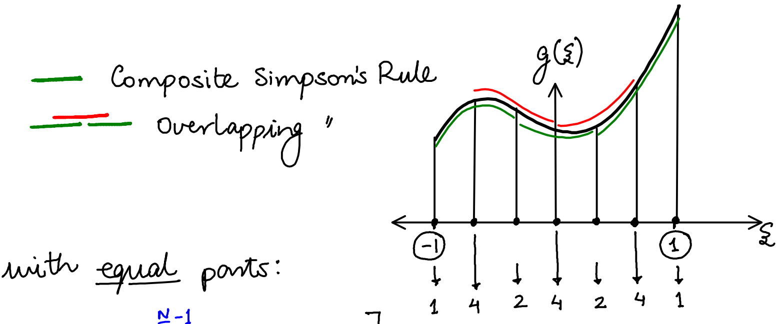
$$\int_{-1}^{1} g(\xi)\, d\xi \approx \sum_{i=1}^{N} g(\xi_i)\, \Delta\ell \qquad \left(\text{where } \xi_i = -1 + \frac{(2i-1)}{2}\Delta\ell\right)$$

or
$$\int_{-1}^{1} g(\xi)\, d\xi \approx \frac{\Delta\ell}{2}\left[g(-1) + 2\sum_{j=1}^{N-1} g(\xi_j) + g(1)\right] \qquad \left(\text{where } \xi_j = -1 + j\Delta\ell\right)$$

Tuesday, March 02, 2010
7:25 PM

(ii) Simpson's Rule:

Divide into "N" parts $\left(\Delta l = \frac{2}{N}\right)$
("N" must be even.)



For composite Simpson's Rule with equal parts:

$$\int_{-1}^{1} g(\xi)\,d\xi \approx \frac{\Delta l}{3}\left[g(-1) + 4\sum_{i=1}^{N/2} g(\xi_i) + 2\sum_{j=1}^{\frac{N}{2}-1} g(\xi_j) + g(1)\right]$$

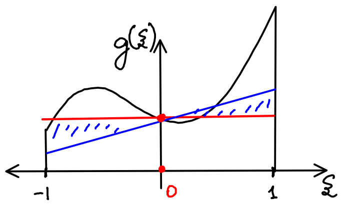
$\xi_i = -1 + (2i-1)\Delta l$
$\xi_j = -1 + 2j\,\Delta l$

When the quadrature points ξ_i (or ξ_j) are pre-determined whether in Trapezoidal or Simpson's rule and polynomials are used to "fit" the function values at $g(\xi_i)$ (or $g(\xi_j)$), then these families of methods are called Newton-Cotes formulas.

In general, these formulas are of the type: $\boxed{\int_{-1}^{1} g(\xi)\,d\xi \approx \sum_{i=1}^{n} g(\xi_i)\,w_i}$

(iii) Gauss Quadrature

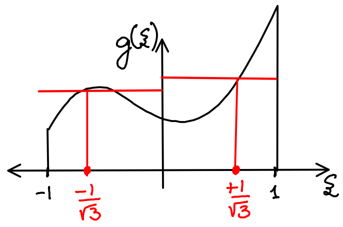
Instead of pre-determining the locations ξ_i, if we determine them so as to minimize the error in the integral.

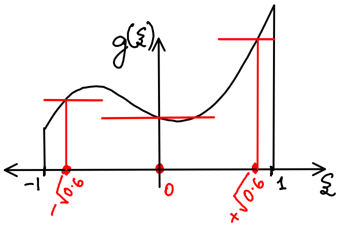


The following table (Ref. Z&T) gives some Gassian Quadrature formulas:

Table 5.2 Gaussian quadrature abscissae and weights for $\int_{-1}^{1} f(x)\mathrm{d}x = \sum_{j=1}^{n} f(\xi_j)\,w_j$.

$\pm\xi_j$		w_j
	$n = 1$	
0		2.000 000 000 000 000
	$n = 2$	
$1/\sqrt{3}$		1.000 000 000 000 000
	$n = 3$	
$\sqrt{0.6}$		5/9
0.000 000 000 000 000		8/9
	$n = 4$	
0.861 136 311 594 053		0.347 854 845 137 454
0.339 981 043 584 856		0.652 145 154 862 546





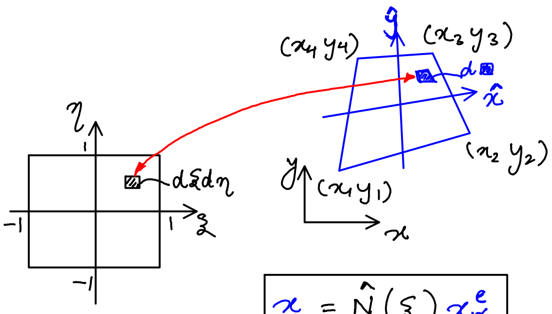
Note: An "n" point Gauss Quadrature formula is able to integrate $g(\xi)$ exactly upto "2n-1" polynomial terms.

Integration of the weak form for Q4 elements

Recall:

$$\underset{\sim}{K}^{el} = \int_{\text{▨}} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, d\text{▨}$$

$$\underset{\sim}{f}^{el} = \int_{\text{▨}} \underset{\sim}{N}^T \underset{\sim}{b}\, d\text{▨} + \int_{\square} \underset{\sim}{N}^T \underset{\sim}{t}_N\, dl$$



$$\boxed{\underset{\sim}{x} = \hat{\underset{\sim}{N}}(\underset{\sim}{\xi})\, \underset{\sim}{x}^e_\alpha}$$

- Consider the domain integrals first:

$$I_{\blacksquare} = \int_{\text{▨}} f(\underset{\sim}{x}(\underset{\sim}{\xi}))\, d\text{▨}$$

$$= \int_{\hat{y}_1}^{\hat{y}_2} \int_{\hat{x}_1}^{\hat{x}_2} f(\underset{\sim}{x}(\underset{\sim}{\xi}))\, d\hat{x}\, d\hat{y}$$

$$= \int_{-1}^{1} \int_{-1}^{1} f(\underset{\sim}{x}(\underset{\sim}{\xi})) \underbrace{\left[\frac{d\hat{x}}{d\xi} \cdot \frac{d\hat{y}}{d\eta}\right]} \cdot (d\xi\, d\eta)$$

One can show:

$$|\underset{\sim}{J}| = \det \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

Thus

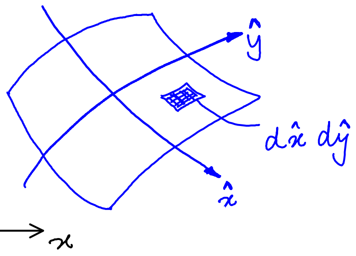
$$I_{\blacksquare} = \int_{-1}^{1} \left[\underbrace{\int_{-1}^{1} f(\underset{\sim}{x}(\underset{\sim}{\xi})) |\underset{\sim}{J}(\underset{\sim}{\xi})|\, d\xi}_{g(\xi)} \right] d\eta$$

Note:

- d▨ is an elemental area in CURVILINEAR COORDINATES $(\hat{x}, \hat{y})$
- It is not dxdy $(d\text{▨} \neq dx\, dy)$

$$d\text{▨} = d\hat{x}\, d\hat{y}$$

- In general:



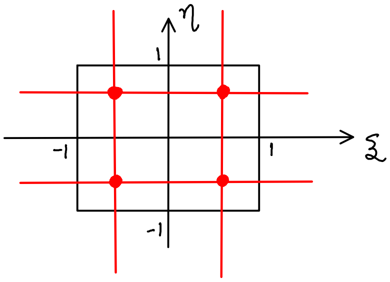
- In 1-D $|J| = \frac{dl}{dL}$

In 2-D $|\underset{\sim}{J}| = \frac{da}{dA}$

In 3-D $|\underset{\sim}{J}| = \frac{dv}{dV}$

$$\Rightarrow I_{\blacksquare} \approx \int_{-1}^{1} \left[\sum_{i=1}^{n_1} f(\underset{\sim}{x}(\xi_i, \eta)) |\underset{\sim}{J}(\xi_i, \eta)| w_i \right] d\eta$$

$$\Rightarrow \boxed{I_{\blacksquare} \approx \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} f(\underset{\sim}{x}(\xi_i, \eta_j)) |\underset{\sim}{J}(\xi_i, \eta_j)| w_i w_j}$$



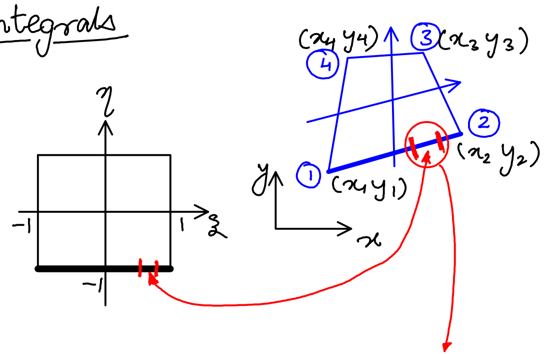
$n_1 = 2$
$n_2 = 2$

2×2 Gauss

- Now Lets look at the boundary integrals

$$I_{\square} = \int_{\square} f(\underset{\sim}{x}(\underset{\sim}{\xi}))\, dl$$

$$= \int_{1-2} (\cdot)\, dl + \int_{2-3} (\cdot)\, dl + \int_{3-4} (\cdot)\, dl + \int_{4-1} (\cdot)\, dl$$



- For edges 1-2 and 3-4 (η = constant)

$$dl = \left[\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2}\,\right] (d\xi)$$

- For edges 2-3 and 4-1 (ξ = const)

$$dl = \left[\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2}\,\right] (d\eta)$$

Thus

$$I_{1-2} \approx \sum_{i=1}^{n_1} f(\underset{\sim}{x}(\xi_i, \overset{\eta}{-1}))\left[\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2}\,\right] w_i$$

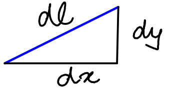
$$I_{2-3} \approx \sum_{i=1}^{n_2} f(\underset{\sim}{x}(\overset{\xi}{1}, \eta_i))\left[\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2}\,\right] w_i$$

Similarly

I_{3-4} = - - - -

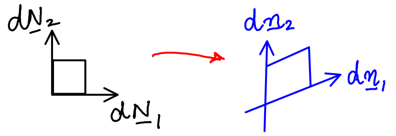
I_{4-1} = - - - - -

Note:



- $dl = \sqrt{dx^2 + dy^2}$
- In 3-D boundary is "da"

This is mapped with the "Piola's Area transformation" (or Nanson's formula)



$d\underline{A} = d\underline{N}_1 \times d\underline{N}_2$

$d\underline{a} = d\underline{n}_1 \times d\underline{n}_2$

$d\underline{a} = (\underset{\sim}{J}\, d\underline{N}_1) \times (\underset{\sim}{J} \times d\underline{N}_2)$

$\underset{\sim}{J}^T d\underline{a} = \det|\underset{\sim}{J}|\, d\underline{A}$

$$\Rightarrow \boxed{d\underline{a} = \det|\underset{\sim}{J}|\; \underset{\sim}{J}^{-T} d\underline{A}}$$

Finally Element Matrices

- Heat Conduction

$$\underset{\sim}{K}^{el} = \int_{▨} \underset{\sim}{B}^T \underset{\sim}{\kappa} \underset{\sim}{B} \, d▨ \approx \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left[\left(\underset{\sim}{B}^T \underset{\sim}{\kappa} \underset{\sim}{B} \right) |\underset{\sim}{J}| \right]_{(\xi_i, \eta_j)} w_i w_j$$

$$\underline{f}^{el} = \int_{▨} \underset{\sim}{N}^T f \, d▨ \approx \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left[\left(\underset{\sim}{N}^T f \right) |\underset{\sim}{J}| \right]_{(\xi_i, \eta_j)} w_i w_j$$

$$+ \int_{□} \underset{\sim}{N}^T h \, d□ \qquad + \sum_{edges} \left[\sum_{i=1}^{n} \left[\left(\underset{\sim}{N}^T h \right) \sqrt{(\cdot)^2 + (\cdot)^2} \right]_{(\xi, \eta)} w_i \right]$$

- 2D Elasticity :

$$\underset{\sim}{K}^{el} = \int_{▨} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, d▨ \approx \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left[\left(\underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \right) |\underset{\sim}{J}| \right]_{(\xi_i, \eta_j)} w_i w_j$$

$$\underline{f}^{el} = \int_{▨} \underset{\sim}{N}^T \underline{b} \, d▨ \approx \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left[\left(\underset{\sim}{N}^T \underline{b} \right) |\underset{\sim}{J}| \right]_{(\xi_i, \eta_j)} w_i w_j$$

$$+ \int_{□} \underset{\sim}{N}^T \underline{t}_N \, d□ \qquad + \sum_{edges} \left[\sum_{i=1}^{n} \left[\left(\underset{\sim}{N}^T \underline{t}_N \right) \sqrt{(\cdot)^2 + (\cdot)^2} \right]_{(\xi, \eta)} w_i \right]$$

- Assemble:

$$\underset{\sim}{K}^{g} = \mathop{\mathrm{A}}_{e=1}^{M} \underset{\sim}{K}^{el} \qquad ; \qquad \underline{f}^{g} = \mathop{\mathrm{A}}_{e=1}^{M} \underline{f}^{el}$$

- Enforce BCs & Solve.

$$[\underset{\sim}{K}_{ff}] \{\underline{d}_f\} = \{\underline{f}_f\} - [\underset{\sim}{K}_{fs}] \{\underline{f}_s\}$$

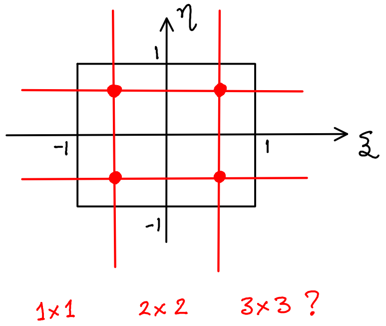
- Post-computation :
  - Plot deformed shape
  - Calculate stresses in each element at each <u>Gauss Point</u>

Appropriate Order of Quadrature

Recall

$$\underset{\sim}{K}^{el} = \int_{\blacksquare} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, d\blacksquare = \int_{-1}^{1}\int_{-1}^{1} \left(\underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\right)\Big|_{(\xi,\eta)} J(\xi,\eta)\, d\xi\, d\eta$$

$$\approx \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left[\left(\underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\right)\Big|_{(\xi_i,\eta_j)} \left|\underset{\sim}{J}(\xi_i,\eta_j)\right| \right] w_i w_j$$



How many points (ξ_i,η_j) should we choose?

For Q4 element, (with constant $\underset{\sim}{D}$)

- $\underset{\sim}{B}$ matrix is linear in ξ and in η
  (so $\underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}$ will be quadratic in ξ & η.)
- $\left|\underset{\sim}{J}(\xi,\eta)\right| = \det(\underset{\sim}{J})$ is linear in ξ and in η
  (so the integrand is cubic in ξ & η) i.e. $(\cdot)\xi^3 * \eta^3$
- Thus order of quadrature required for Full integration is $\boxed{2\times 2}$. ⊛

However, full integration is NOT required for convergence. The minimum requirement for convergence is that the weak form converge in the limit of mesh refinement.

i.e. $\tilde{G}^h(\underline{d},\bar{\underline{d}}) \longrightarrow G(\underline{u},\bar{\underline{u}})$ as "h" $\longrightarrow 0$

It can be shown that this condition is satisfied

$\boxed{\text{if an element is able to reproduce the state of constant stress}}$

in the limit as $h \to 0$.

This requirement forms the basis of "testing" new elements with a test called the "Patch" test.

Note:

- The CST element satisfies this requirement automatically.
- For the general iso-parametrically mapped elements:
  (Q4 or higher)

  The constant stress (strain) requirement is equivalent to having a constant $\underset{\sim}{B}$ matrix. Thus if your integration rule can integrate the weak form for a constant $\underset{\sim}{B}$ matrix, then convergence will be achieved for the element.

Friday, March 05, 2010
9:08 AM

i.e. $$\tilde{G}^h(\underline{d}^g, \bar{\underline{d}}^g) = \bar{\underline{d}}^T \left[\left[\overset{M}{\underset{e=1}{\mathbf{A}}} \underset{\sim}{K}^{el} \right] \underline{d}^g - \underline{f}^g \right] \rightarrow G(\underline{u}, \bar{\underline{u}}) \quad \text{as } h \rightarrow 0$$

i.e. $$\sum_{e=1}^{M} \bar{\underline{d}}^{e^T} \left[\int_{\blacksquare} (\underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B}) \, d\blacksquare \right] \underline{d}^e \rightarrow \int_{\Omega} \bar{\underline{\epsilon}} \underset{\sim}{D} \underline{\epsilon} \, d\Omega \quad \begin{pmatrix} \text{for all} \\ \text{elements} \end{pmatrix} \text{ as } h \rightarrow 0$$

If $\underset{\sim}{B}$ is constant within each element then,

$$\int_{\blacksquare} (\underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B}) \, d\blacksquare = (\underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B}) \underbrace{\int_{\blacksquare} d\blacksquare}_{\text{Area}}$$

i.e. All we need to integrate "exactly" is the area of all elements. This specifies the <u>minimum</u> order of Quadrature required for Q4 (and higher order) elements.

So, $$\int_{\blacksquare} d\blacksquare = \int_{-1}^{1} \int_{-1}^{1} |\underset{\sim}{J}(\xi, \eta)| \, d\xi \, d\eta$$

This integral needs to be evaluated <u>exactly</u>. (Not $\underset{\sim}{K}^{el}$)

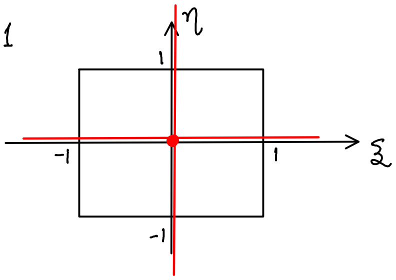
For Q4 elements, $J(\xi, \eta)$ is <u>linear</u> in ξ and η.

Thus the minimum order quadrature required is

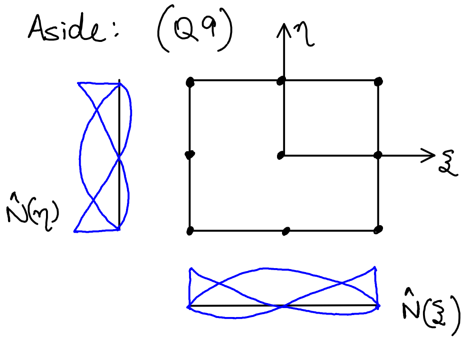
1×1 Gauss i.e. $\xi_i = 0$; $\eta_j = 0$ $\quad i = j = 1$

$\omega_i = 2$; $\omega_j = 2$

This is called <u>Reduced</u> integration.



Aside: (Q9)



$N(\xi, \eta) \rightarrow$ quadratic : $\xi^2 \eta^2$

$$\Rightarrow \underset{\sim}{B}(\xi, \eta) \rightarrow \begin{cases} N_{\alpha,\xi} \rightarrow \xi \eta^2 \\ N_{\alpha,\eta} \rightarrow \xi^2 \eta \end{cases}$$

Thus $\underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \rightarrow \xi^3 \eta^3$
(for constant $\underset{\sim}{D}$)

Also $$|\underset{\sim}{J}(\xi, \eta)| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} \rightarrow \begin{vmatrix} \xi \eta^2 & \xi^2 \eta \\ \xi \eta^2 & \xi^2 \eta \end{vmatrix} \rightarrow \xi^3 \eta^3$$

$\Rightarrow$ Full 4×4 ⊛

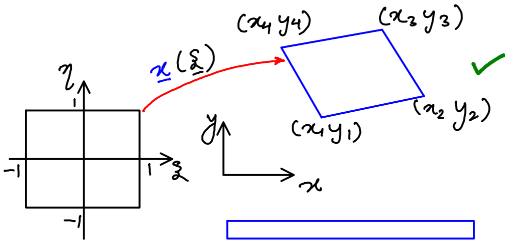
Reduced 2×2 ✓

Note on "Full" Integration:

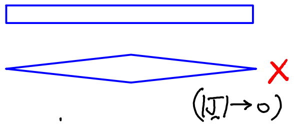
- The quadrature rules for "Full" integration, developed in the previous section are approximate. i.e. "Full" integration does not mean that the element integrals are integrated "exactly".

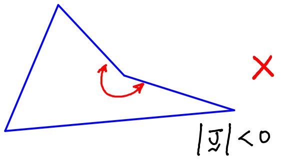
- These rules assume that the distortion due to iso-parametric mapping from the parent element to the actual element is almost "uniform". i.e. $|\underset{\sim}{J}(\xi,\eta)| \to$ constant

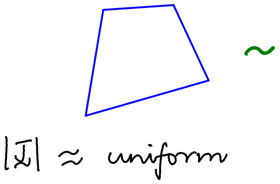
- This is ensured, when the parent element is mapped to a parallelogram.



- This restriction on iso-parametric distortion places restrictions on actual element quality.







- In practice, all FE meshes have elements that are distorted. When $|\underset{\sim}{J}|$ is non-uniform, the terms in $\underset{\sim}{B}$ matrix cannot be expressed as polynomials of ξ, η.

$$\underset{\sim}{B} \to \begin{Bmatrix} N_{\alpha,x} \\ N_{\alpha,y} \end{Bmatrix} \to \underset{\sim}{J}^{-T} \begin{Bmatrix} \hat{N}_{\alpha,\xi} \\ \hat{N}_{\alpha,\eta} \end{Bmatrix}$$

where

$$\underset{\sim}{J}^{-1} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} = \frac{1}{|\underset{\sim}{J}|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \qquad (J^{-1})_{Ii} = \frac{1}{J_{iI}}$$

Thus for General Q4 elements:

$$\underset{\sim}{K}^{el} = \int_{-1}^{1}\int_{-1}^{1} \underbrace{\underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}}_{\frac{1}{|\underset{\sim}{J}|^2}\begin{bmatrix} \sim\xi & \sim\xi \\ \sim\eta & \sim\eta \end{bmatrix}^2} |\underset{\sim}{J}|\, d\xi\, d\eta \;\propto\; \int_{-1}^{1}\int_{-1}^{1} \frac{\sim O(\xi^2,\eta^2)}{|\underset{\sim}{J}|}\, d\xi\, d\eta$$

Note: $|\underset{\sim}{J}| \propto O(\xi,\eta)$

The integrand is a Rational function of polynomials. Gauss quadrature cannot integrate Rational functions exactly. (and we don't need to).

Thus, strictly speaking, there is no "Full" integration rule that integrates the weak form exactly.

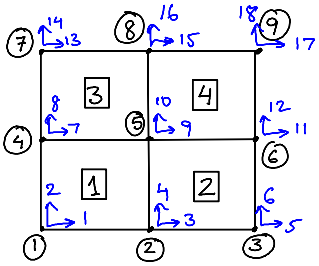
Post - Computation

After solving $[\underset{\sim}{K}^G]\{d^G\} = \{\underset{\sim}{f}^G\}$ (with boundary conditions)

we obtain $\{d^G\}$: xy-displacements at each node

Using these $\{d^G\}$ we can calculate the element dofs $\{d^e\}$ for any element. This process is called restriction. (Reverse of assembly).

- Plotting displacements:
  Ideally we should use $\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{N}\, d^e$
  (i.e. the actual shape functions)
  to calculate & plot displacements within each element



- Plotting stresses : $\underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{B}\,d^e$

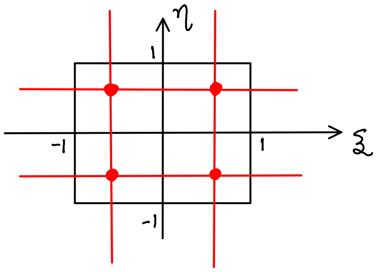
i.e.

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix} = \underset{\sim}{D}\underbrace{\left[\begin{array}{c|cc|c} \cdots & N_{\alpha,1} & 0 & \cdots\\ \cdots & 0 & N_{\alpha,2} & \cdots \\ & N_{\alpha,2} & N_{\alpha,1} & \end{array}\right]}_{\underset{\sim}{B}} \begin{Bmatrix}\vdots\\ d_1^\alpha\\ d_2^\alpha\\ \vdots\end{Bmatrix}$$

Note:

- Previously for CST Δs $\underset{\sim}{B}$ was constant and its computation was not expensive.
- However, for Iso-parametric elements (such as Q4), $\underset{\sim}{B}$ matrix in general is not constant and needs to be computed again for plotting stresses.

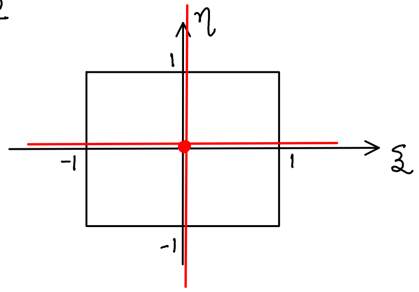
(This can be quite expensive, so you may choose to store the $\underset{\sim}{B}$ matrix for each element at every Gauss integration point. This would a huge amount of storage, but can turn out to be faster.)



- We could use Reduced Integration (ie. 1×1)

Consequences of Reduced Integration

$$\underset{\sim}{K}^{el} \approx \left[\left(\underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\right) \left|\underset{\sim}{J}\right| \right]_{(\xi_i = 0,\, \eta_j = 0)} * \overset{\overbrace{\omega_i \omega_j}}{4}$$



★ Less Computation => fast ↑

For Q4 elements only 1-Gauss point is used for numerical integration of the weak form.

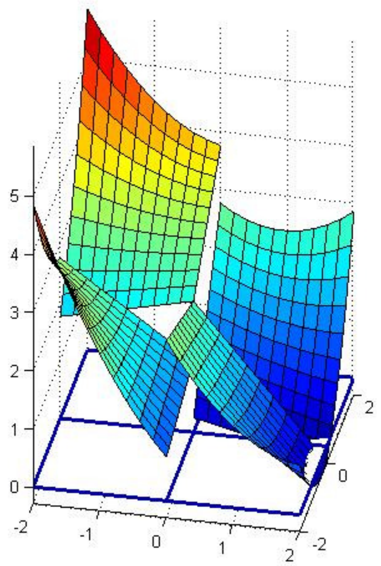
★ Super-convergence of stress ↑

It turns out that the "optimal" locations for calculating the stresses in post-computation are the Gauss point locations of 1-order less than what is required for full integration.

These are called Barlow points.

eg. Q4 : Full integration (2×2)

1-order less (1×1) i.e. Reduced integration.

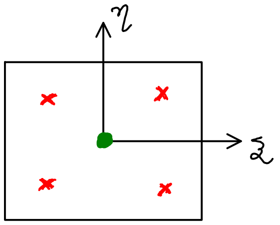


Stresses obtained using

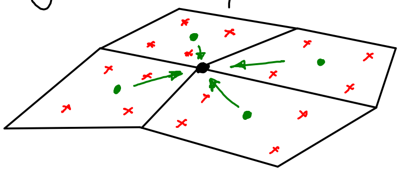
$$\underset{\sim}{\sigma}(x,y) = \underset{\sim}{D}\,\underset{\sim}{B}\,\underset{\sim}{d}^e$$

are very poor at the nodes.

They are much better at the Barlow points.

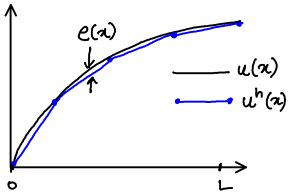


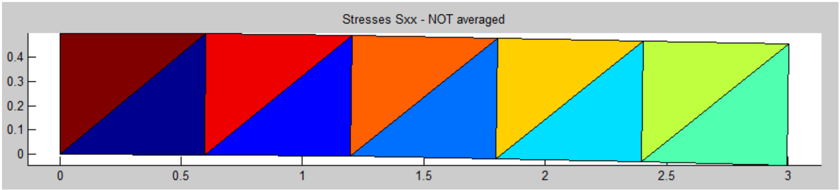
Stress averaging should be done using Barlow pts.



★ "Softer" (more accurate response) ↑

A finite element solution is usually "stiffer" than the actual continuum problem.





If we use full integration to integrate the $\underset{\sim}{K}^{el}$ exactly, then does not help.

If we use reduced integration to approximately under-integrate the $\underset{\sim}{K}^{el}$ on purpose, then the two effects tend to negate each other and we get better results.

★ Mesh instabilities occur ↓

$\underset{\sim}{K}^{el}$ (8×8) ⇒ 8 Eigenvalues
8 Eigenvectors

Full integration: too-stiff.

Reduced integration: too-soft (sometimes even unstable)

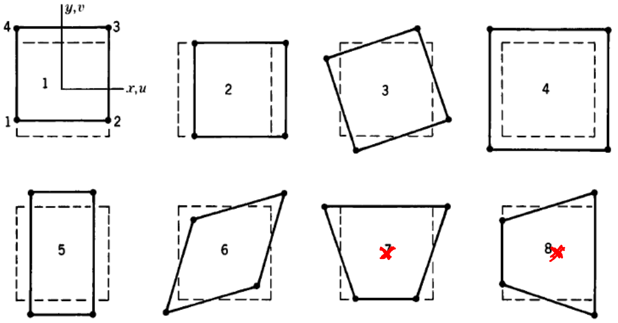


Figure 6.12-1. Independent displacement modes of a bilinear element.

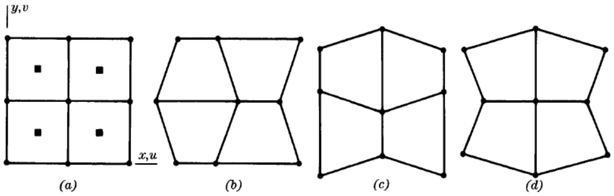
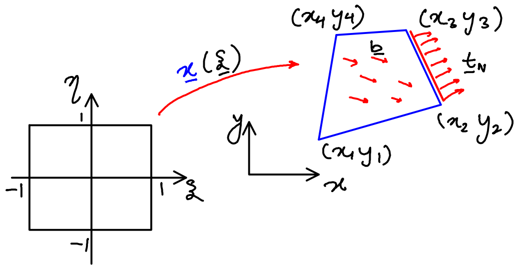


Figure 6.12-2. (*a*) Mesh of four bilinear elements, showing Gauss points of an order 1 rule in each element (squares). (*b,c,d*) Possible mechanisms ("hourglass" modes).

Computer Implementation of General Q4 element

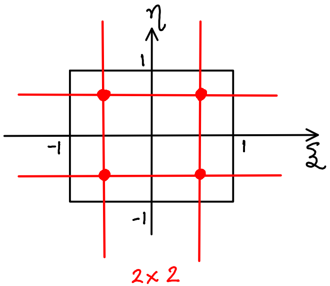


Given

- Actual co-ordinates $(x_\alpha\ y_\alpha)$ $\alpha=1,2,3,4$
- Material Properties:
  - Heat Conduction $\underset{\sim}{k}$
  - 2D Elasticity $\underset{\sim}{D}$
- Domain term: - Heat Conduction: Body heat source: f
  - 2D Elasticity: Body Force: $\underline{b}$
- Boundary term: (Γ_N) - Heat conduction: Edge heat source: h
  - 2D Elasticity: Edge Traction: $\underline{t}_N$

Output:

$\underset{\sim}{K}^{el}$; $\underline{f}^{el}$



Steps:

(i) Determine integration rule (ξ_i, η_j) & $w_i w_j$

(ii) Loop over number of integration points $(n_i * n_j)$

(a) Calculate

$\hat{N}_\alpha(\xi_i, \eta_j)$ & $\begin{Bmatrix} \hat{N}_{\alpha,\xi} \\ \hat{N}_{\alpha,\eta} \end{Bmatrix} (\xi_i, \eta_j)$ $(\alpha = 1,2,3,4)$ $\quad \underline{x} = \underset{\sim}{\hat{N}}\, x^e_\alpha$

(b) Calculate

$$\underset{\sim}{J}(\xi_i, \eta_j) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{\alpha=1}^{4} (\hat{N}_{\alpha,\xi}\, x_\alpha) & \sum_{\alpha=1}^{4} (\hat{N}_{\alpha,\eta}\, x_\alpha) \\ \sum_{\alpha=1}^{4} (\hat{N}_{\alpha,\xi}\, y_\alpha) & \sum_{\alpha=1}^{4} (\hat{N}_{\alpha,\eta}\, y_\alpha) \end{bmatrix}$$

(c) Calculate $|\underset{\sim}{J}|$ & $\underset{\sim}{J}^{-1}(\xi_i, \eta_j)$

(d) Calculate $\begin{Bmatrix} N_{\alpha,x} \\ N_{\alpha,y} \end{Bmatrix} = \underset{\sim}{J}^{-T} \begin{Bmatrix} \hat{N}_{\alpha,\xi} \\ \hat{N}_{\alpha,\eta} \end{Bmatrix}$ at $(\xi_i\, \eta_j)$

(e) Construct $\underset{\sim}{N}$ matrix $= \left[\begin{array}{cc|cc|cc|cc} \hat{N}_1 & & \hat{N}_2 & & \hat{N}_3 & & \hat{N}_4 & \\ & \hat{N}_1 & & \hat{N}_2 & & \hat{N}_3 & & \hat{N}_4 \end{array}\right]$ at $(\xi_i\, \eta_j)$

$\underset{\sim}{B}$ matrix $= \left[\begin{array}{c|cc|c} & N_{\alpha,x} & 0 & \\ \cdots\cdots & 0 & N_{\alpha,y} & \cdots\cdots \\ & N_{\alpha,y} & N_{\alpha,x} & \end{array}\right]$ at $(\xi_i\, \eta_j)$

(2D elasticity)

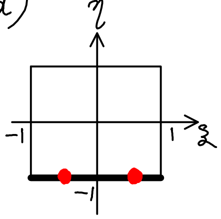
(f) Add to Element Matrices:

- $$\underset{(8\times 8)}{\underset{\sim}{K}^{el}} = \underset{\sim}{K}^{el} + \left[\underset{\sim}{B}^{T}\,\underset{\sim}{D}\,\underset{\sim}{B}\,|\underset{\sim}{J}|\right]\Big|_{(\xi_i,\eta_j)} \omega_i\,\omega_j$$

- $$\underset{(8\times 1)}{\underline{f}^{el}} = \underline{f}^{el} + \left\{\underset{\sim}{N}^{T}\,\underline{b}\,|\underset{\sim}{J}|\right\}\Big|_{(\xi_i,\eta_j)} \omega_i\,\omega_j$$

(iii) Loop over all boundary (Γ_N : 1-2, 2-3, 3-4, 4-1) (if needed)

(a) Calculate $\hat{N}_\alpha$, $\begin{Bmatrix} \hat{N}_{\alpha,\xi} \\ \hat{N}_{\alpha,\eta} \end{Bmatrix}$ at $(\xi_i, -1)$ $\quad \alpha = 1,2,3,4$



(b) Calculate $$[\underset{\sim}{J}] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

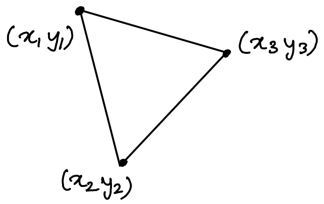
(c) Calculate "length factor" : $$|J_\ell| \equiv \left[\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2}\,\right]$$

(d) Construct $\underset{\sim}{N}$ matrix $$= \left[\begin{array}{cc|cc|cc|cc} \hat{N}_1 & & \hat{N}_2 & & \hat{N}_3 & & \hat{N}_4 & \\ & \hat{N}_1 & & \hat{N}_2 & & \hat{N}_3 & & \hat{N}_4 \end{array}\right] \quad \text{at } (\xi_i, -1)$$

(e) Add
$$\underset{(8\times 1)}{\underline{f}^{el}} = \underline{f}^{el} + \left\{\underset{\sim}{N}^{T}\,\underline{t}_N\,|J_\ell|\right\} \omega_i$$

(iv) Return $\underset{\sim}{K}^{el}$; $\underline{f}^{el}$

Higher Order Elements : Triangular



- Triangular Elements

  Polynomial approximation :

  Pascal's Triangle

1

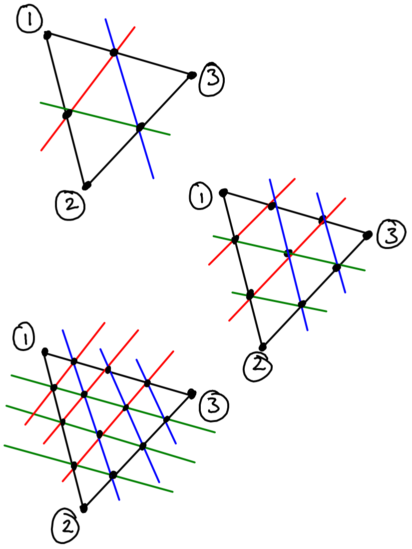
$x \quad y$ → CST (Linear)

$x^2 \quad xy \quad y^2$ → LST (Quadratic)

$x^3 \quad x^2y \quad xy^2 \quad y^3$ → T-10 (Cubic)

$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$ → T-15 (Quartic)

⋮



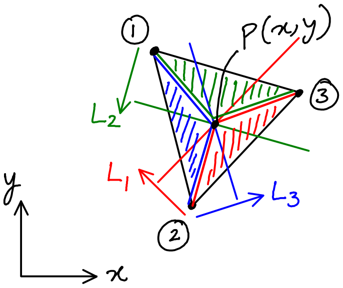
Shape functions are usually expressed in terms of Area Co-ordinates :

$$L_1 = \frac{\text{Area } P23}{\Delta}$$

$$L_2 = \frac{\text{Area } P31}{\Delta}$$

$$L_3 = \frac{\text{Area } P12}{\Delta}$$

Note : $\boxed{L_1 + L_2 + L_3 = 1}$

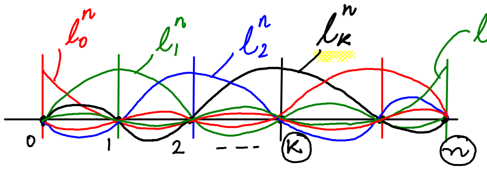


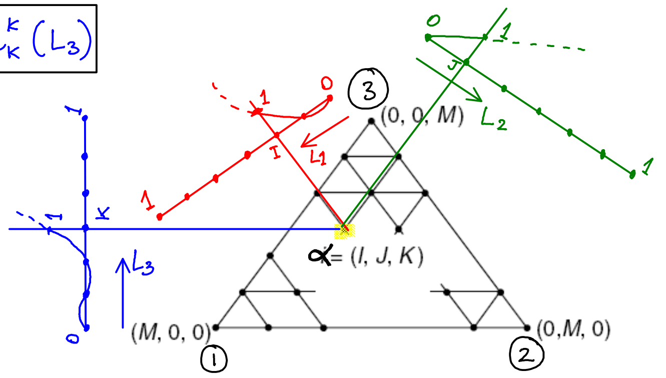
In terms of Area co-ordinates, one can express shape functions as Lagrange polynomials in each L_1 L_2 L_3 :

$$\boxed{N_\alpha(L_1, L_2, L_3) = \ell_I^I(L_1) * \ell_J^J(L_2) * \ell_K^K(L_3)}$$

↳ (I, J, K)

where Lagrange polynomials:





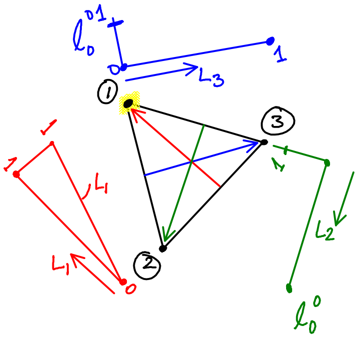
$$l_k^n(\xi) = \frac{(\xi - \xi_0)(\xi - \xi_1)\cdots(\xi - \xi_{k-1})(\xi - \xi_{k+1})\cdots(\xi - \xi_n)}{(\xi_k - \xi_0)(\xi_k - \xi_1)\cdots(\xi_k - \xi_{k-1})(\xi_k - \xi_{k+1})\cdots(\xi_k - \xi_n)} = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{\xi - \xi_i}{\xi_k - \xi_i}$$

Note : $\ell_0^0(\xi) = 1$

For example :

- CST (T3):

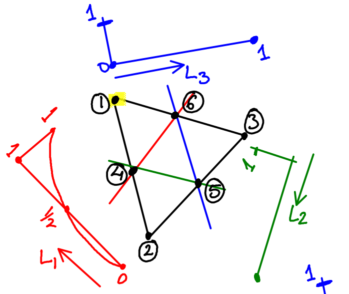
$$N_1(L_1,L_2,L_3) = L_1$$
$$N_2(L_1,L_2,L_3) = L_2$$
$$N_3(L_1,L_2,L_3) = L_3$$



- LST (T6) :

Corner Nodes:

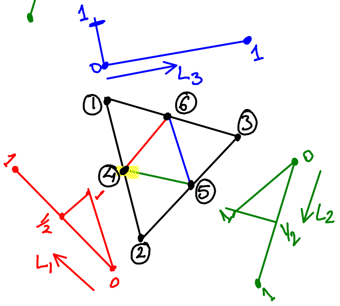
$$N_1(L_1,L_2,L_3) = \frac{(L_1-\frac{1}{2})(L_1-0)}{(1-\frac{1}{2})(1-0)} \cdot 1 \cdot 1$$



$$\Rightarrow \quad \left.\begin{array}{l} N_1 = (2L_1-1)L_1 \\ N_2 = (2L_2-1)L_2 \\ N_3 = (2L_3-1)L_3 \end{array}\right\} \text{Corner Nodes}$$

Mid-side Nodes:

$$N_4(L_1,L_2,L_3) = \frac{(L_1-0)}{(\frac{1}{2}-0)}\frac{(L_2-0)}{(\frac{1}{2}-0)} \cdot 1$$



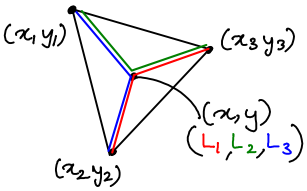
$$\Rightarrow \quad N_4 = 4L_1L_2$$
$$N_5 = 4L_2L_3$$
$$N_6 = 4L_3L_1$$

Using these shape functions, one would have to compute the derivatives and element integrals.

The following identities will be helpful:

- $L_1 + L_2 + L_3 = 1$

$$x = L_1x_1 + L_2x_2 + L_3x_3$$
$$y = L_1y_1 + L_2y_2 + L_3y_3$$



- $L_1 = N_1 = \frac{1}{2\Delta}(A_1 + B_1x + C_1y)$

$$\Delta = \frac{1}{2}\det\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

For derivatives : $\frac{\partial L_1}{\partial x} = \frac{B_1}{2\Delta}$; $\frac{\partial L_1}{\partial y} = \frac{C_1}{2\Delta}$

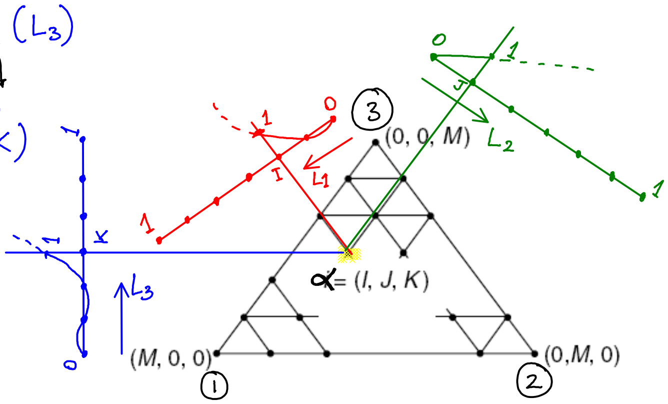
Similary L_2 & L_3

- $N_\alpha(L_1, L_2, L_3) = \ell_I^I(L_1) * \ell_J^J(L_2) * \ell_K^K(L_3)$

polynomial order: $\quad \downarrow 2\ (I) \quad \downarrow 1\ (J) \quad \downarrow 3\ (K)$

$\alpha = (I, J, K)$

where $I + J + K = \boxed{M}$



- Derivatives

$$\left.\begin{aligned}\frac{\partial N_\alpha}{\partial x} &= \frac{\partial N_\alpha}{\partial L_1}\cdot\frac{\partial L_1}{\partial x} + \frac{\partial N_\alpha}{\partial L_2}\cdot\frac{\partial L_2}{\partial x} + \frac{\partial N_\alpha}{\partial L_3}\cdot\frac{\partial L_3}{\partial x}\\ \frac{\partial N_\alpha}{\partial y} &= \frac{\partial N_\alpha}{\partial L_1}\,\frac{\partial L_1}{\partial y} + \frac{\partial N_\alpha}{\partial L_2}\cdot\frac{\partial L_2}{\partial y} + \frac{\partial N_\alpha}{\partial L_3}\cdot\frac{\partial L_3}{\partial y}\end{aligned}\right\}$$

can be explicitly obtained for Triangles with <u>straight</u> edges.

- Element Integrals:

$$\underset{\sim}{K}^{el} = \int_{\blacktriangle} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, d\blacktriangle \quad ; \quad \underline{f}^{el} = \int_{\blacktriangle} \underset{\sim}{N}^T \underline{b}\, d\blacktriangle + \int_{\Delta} \underset{\sim}{N}^T \underline{t}_N\, d\Delta$$

For straight edged triangles, all the quantities will be <u>polynomials</u> of L_1, L_2, L_3.

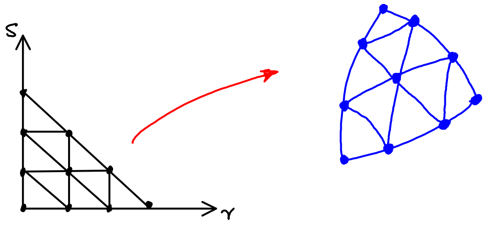
The following formulas are <u>exact</u>: (for straight edged Δs)

$$\boxed{\int_{\blacktriangle} L_1^i\, L_2^j\, L_3^k \,.\, d\blacktriangle = \frac{i!\ j!\ k!}{(i+j+k+2)!}\, 2\Delta}$$

$$\boxed{\int_{\Delta(1\text{-}2)} L_1^i\, L_2^j\, d\ell = \frac{i!\ j!}{(i+j+1)!}\,(\ell_{1\text{-}2})}$$

For triangles with <u>curved</u> edges, the elements have to be mapped to a "parent" triangular element and use numerical integration.

Ref: $\left(\begin{array}{l}\text{Reddy § 9.3.4}\\ \text{Hughes App. 3.1}\end{array}\right)$ for details



$L_1 \leftrightarrow r \quad \leftrightarrow \xi$

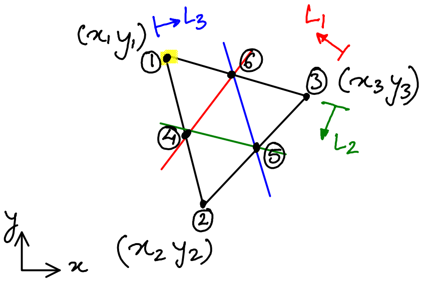
$L_2 \leftrightarrow s \quad \leftrightarrow \eta$

$L_3 \leftrightarrow t = 1 - r - s$

Friday, March 26, 2010
8:31 AM

HW-6 Hint

For a T6 triangle, given coordinates



$$\left.\begin{aligned} N_1 &= (2L_1-1)L_1 \\ N_2 &= (2L_2-1)L_2 \\ N_3 &= (2L_3-1)L_3 \end{aligned}\right\} \text{Corner Nodes}$$

$$\left.\begin{aligned} N_4 &= 4L_1L_2 \\ N_5 &= 4L_2L_3 \\ N_6 &= 4L_3L_1 \end{aligned}\right\} \text{Mid-side Nodes}$$

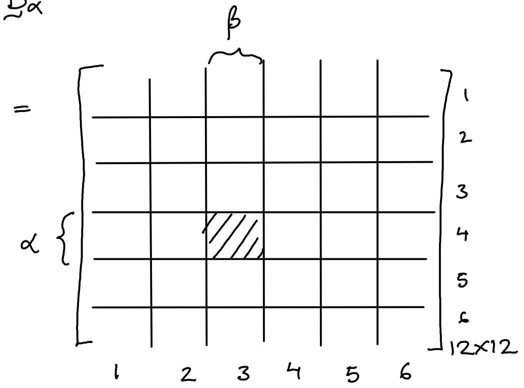
where
$$\begin{aligned} L_1 &= N_1^{T3} = \frac{1}{2\Delta}(A_1+B_1x+C_1y) \\ L_2 &= N_2^{T3} = \frac{1}{2\Delta}(A_2+B_2x+C_2y) \\ L_3 &= N_3^{T3} = \frac{1}{2\Delta}(A_3+B_3x+C_3y) \end{aligned}$$

$$\underset{\sim}{B} = \left[\begin{matrix} \text{------} & \underbrace{\left|\begin{matrix} N_{\alpha,x} & \\ & N_{\alpha,y} \\ N_{\alpha,y} & N_{\alpha,x} \end{matrix}\right|}_{\underset{\sim}{B}_\alpha} & \text{------} \end{matrix}\right]_{3\times12}$$

$$N_{\alpha,x} = \sum_{i=1}^{3} \frac{\partial N_\alpha}{\partial L_i} \cdot \boxed{\frac{\partial L_i}{\partial x}} = \frac{B_i}{2\Delta}$$

$$N_{\alpha,y} = \sum_{i=1}^{3} \frac{\partial N_\alpha}{\partial L_i} \cdot \boxed{\frac{\partial L_i}{\partial y}} = \frac{C_i}{2\Delta}$$

$$\underset{\sim}{K} = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, d\Omega =$$



$$\underset{\sim}{K}_{\alpha\beta} = \int_\Omega \underset{\sim}{B}_\alpha^T \underset{\sim}{D}\, \underset{\sim}{B}_\beta\, d\Omega$$

$$\underset{2\times2}{\underset{\sim}{K}_{\alpha\beta}} = \int_\Omega \begin{bmatrix} N_{\alpha,x} & & N_{\alpha,y} \\ & N_{\alpha,y} & N_{\alpha,x} \end{bmatrix} \underset{\underset{\sim}{D}}{\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}} \begin{bmatrix} N_{\beta,x} & \\ & N_{\beta,y} \\ N_{\beta,y} & N_{\beta,x} \end{bmatrix} d\Omega$$

$$\underset{\sim}{K}_{\alpha\beta} = \int_\Omega \begin{bmatrix} \vdots & \vdots \end{bmatrix} d\Omega$$ where each term is at most quadratic in $L_1 L_2 L_3$:

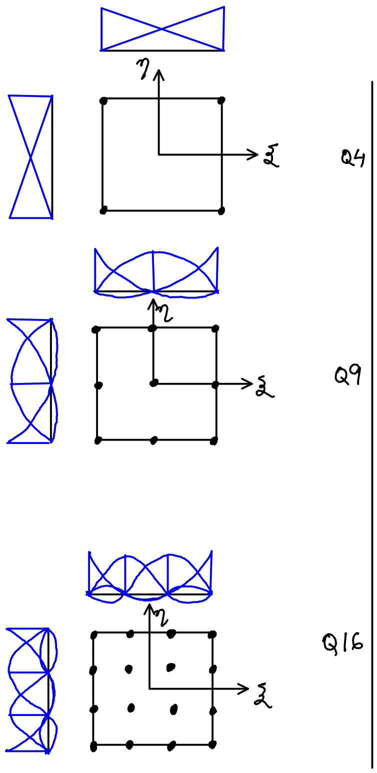
$$L_1^i\, L_2^j\, L_3^k \longrightarrow i+j+k \le 2$$

$$\text{(say)} \quad (\bullet) = \int_\Omega \left(C_1 + C_2L_1 + C_3L_2 + C_4L_1^2 + C_5L_1L_2 + C_6L_2^2 \right) d\Omega$$

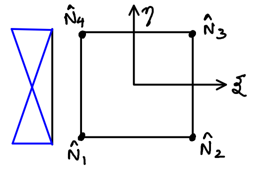
Higher Order Elements : Rectangular

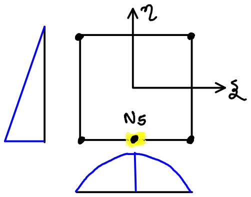
Lagrange family

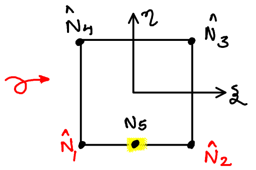
Serendipity Family



Q4

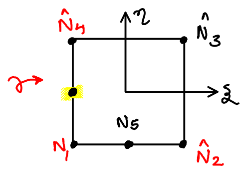






$\hat{N}_1 = \hat{N}_1 - \frac{1}{2} N_5$

$\hat{N}_2 = \hat{N}_2 - \frac{1}{2} N_5$



$N_1 = \hat{N}_1 - \frac{1}{2} N_5 - \frac{1}{2} N_8$

$N_4 = \hat{N}_4 - \frac{1}{2} N_8$

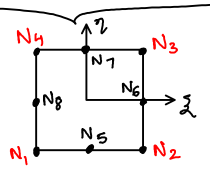
Q8

$N_1 = \hat{N}_1 - \frac{1}{2} N_5 - \frac{1}{2} N_8$

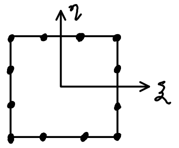
$N_2 = \hat{N}_2 - \frac{1}{2} N_5 - \frac{1}{2} N_6$

$N_3 = \hat{N}_3 - \frac{1}{2} N_6 - \frac{1}{2} N_7$

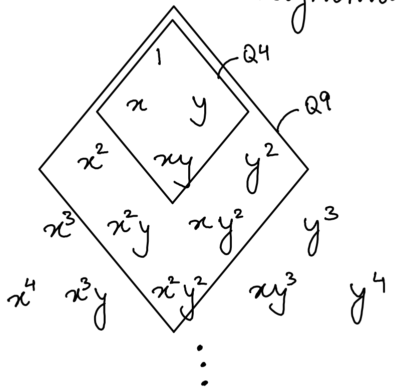
$N_4 = \hat{N}_4 - \frac{1}{2} N_7 - \frac{1}{2} N_8$

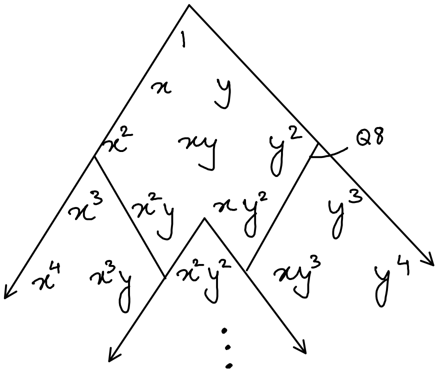


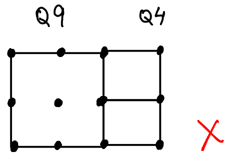
Q12

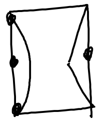


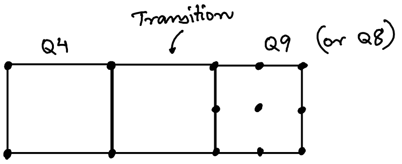
Polynomial Approximation :

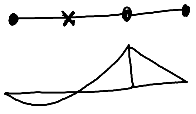


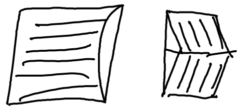


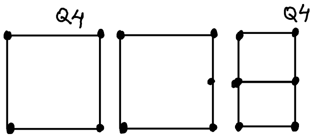


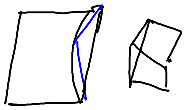










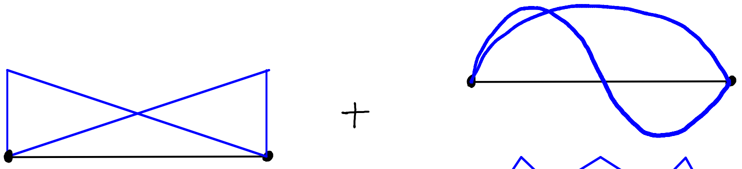


Higher Order Hierarchical Elements

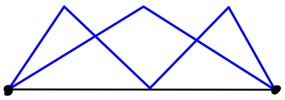
Hierarchical shape functions are generated by retaining the original "Standard" finite element shape functions and simply including more bubble functions.

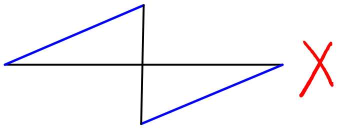
For example:

- 1-D

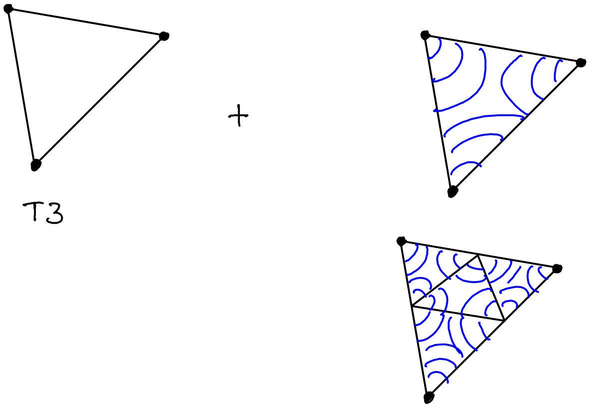


$$u(x) = \sum_{\alpha=1}^{2} N_\alpha d_\alpha^e + \sum_{\beta=1}^{n} \hat{N}_\beta c_\beta$$



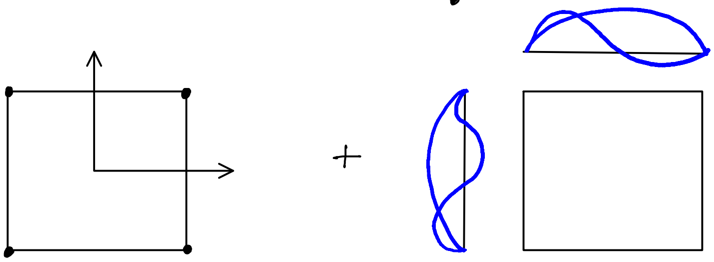


- 2-D Triangles



T3

- 2D Rectangles:



Q4

Note:

- The "Base" shape functions guarantee convergence
- Bubbles can be condensed out statically.
- $\sum N_\alpha = 1$ holds only for the "Base" functions.
- Kronecker delta $\delta_{ab} = \begin{cases} 1 & a=b \\ 0 & a \neq b \end{cases}$ is still maintained

Incompatible Bubbles : Variational Crime

General Q4 elements are simple and convenient to implement, however, they usually give poor results on coarse meshes in bending dominated problems.

To obtain "good" results with coarse meshes a hierarchical element with 2 non-conforming "bubbles" was developed.

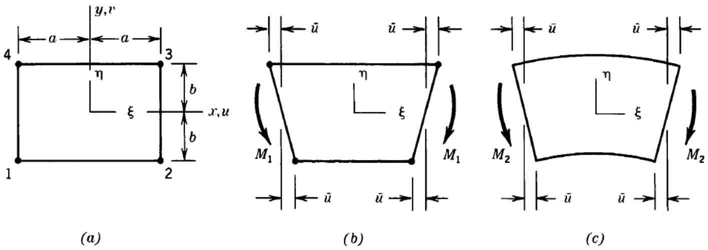
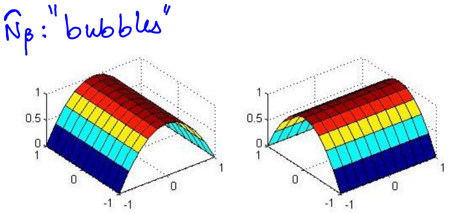
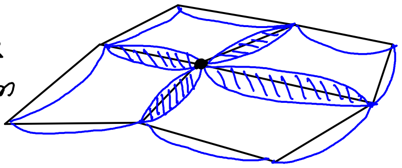


Figure 8.3-1. *(a)* A rectangular bilinear element. *(b)* The bilinear element deformed by bending moment M_1. *(c)* Correct deformed geometry for pure bending under bending moment M_2.



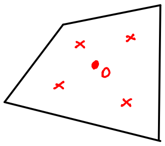
- This Q6 element is not compatible at the element boundaries, so is not in $H^1(\Omega)$. Discontinuities (Gaps / Overlaps) may occur.



- This element reproduces the constant strain condition only for rectangular / parallelogram configurations of the underlying Q4.

- For a general Q6, with non-constant iso-parametric distortion, it fails the constant strain (patch test).

- To "trick" it into passing the patch test :



$$\underset{\sim}{B} \longrightarrow \begin{Bmatrix} \hat{N}_{\beta,1} \\ \hat{N}_{\beta,2} \end{Bmatrix} = \frac{|J_0|\, J_0^{-T}}{|J(\xi,\eta)|} \begin{Bmatrix} \hat{N}_{\beta,\xi} \\ \hat{N}_{\beta,\eta} \end{Bmatrix}$$

- Famous Quote:

INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 29, 1595–1638 (1990)

A CLASS OF MIXED ASSUMED STRAIN METHODS AND THE METHOD OF INCOMPATIBLE MODES*

'*. . . two wrongs do make a right in California*' G. STRANG (1973)
'*. . . two rights make a right even in California*' R. L. TAYLOR (1989)

J. C. SIMO† AND M. S. RIFAI‡

Division of Applied Mechanics, Department of Mechanical Engineering, Stanford University, Stanford, CA 94305, U.S.A.

Convergence & Error Estimates

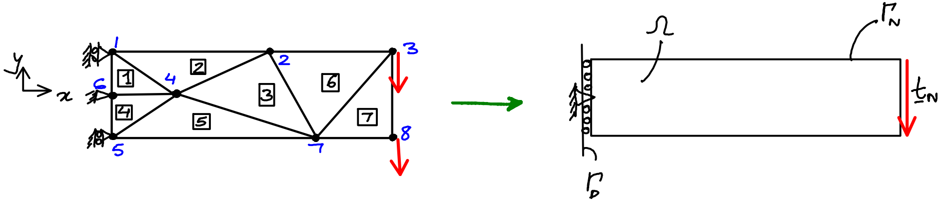
Convergence comprises of two conditions:

$$\boxed{\text{Convergence} = \text{Consistency} + \text{Stability}}$$

① Consistency: As $h \to 0$,

$$\tilde{G}^h(\underline{d},\underline{\bar{d}}) = G^h(\underline{u}^h,\underline{\bar{u}}^h) \longrightarrow G(\underline{u},\underline{\bar{u}})$$

i.e.



Note:

$$G(\underline{u},\underline{\bar{u}}) = a(\underline{u},\underline{\bar{u}}) - (f,\underline{\bar{u}}) = 0 \quad \forall \; \underline{\bar{u}} \in H_0^1(\Omega)$$

$$G^h(\underline{u}^h,\underline{\bar{u}}^h) = a^h(\underline{u}^h,\underline{\bar{u}}^h) - (f,\underline{\bar{u}}^h) = 0 \quad \forall \; \underline{\bar{u}}^h \in FE_0^1(\Omega)$$

If a method is consistent then $a^h(\cdot,\underline{\bar{u}}^h) = a(\cdot,\underline{\bar{u}}^h)$ within $FE_0^1(\Omega)$.

So, restricting the space to $FE_0^1(\Omega)$ and subtracting:

$$\boxed{a(\underline{u}^h - \underline{u},\underline{\bar{u}}^h) = 0 \qquad \forall \; \underline{\bar{u}}^h \in FE_0^1(\Omega)}$$

This is called the error equation.

Note:

- "Error" $\underline{e}(\underline{x}) = \underline{u}^h(\underline{x}) - \underline{u}(\underline{x})$
- Thus $a(\underline{e},\underline{\bar{u}}^h) = 0 \quad \forall \; \underline{\bar{u}}^h \in FE_0^1(\Omega)$

  i.e. error is always "orthogonal" to the FE space.
  i.e. FE solution is the best possible solution in $FE_0^1(\Omega)$.
- However $a(\underline{e},\underline{\bar{u}}) \neq 0 \quad \forall \; \underline{\bar{u}} \in H_0^1(\Omega)$

In particular, $a(\underline{e},\underline{e})$ is called the "energy norm" of the error.

(Recall, $\Pi(\underline{u})$ is the energy functional corresponding to $G(\underline{u},\underline{\bar{u}})$.)

$$\Pi(\underline{e}) \propto \frac{1}{2} a(\underline{e},\underline{e}) = \frac{1}{2}\int_\Omega \underline{\varepsilon}(\underline{e})^T \underset{\sim}{D}\, \underline{\varepsilon}(\underline{e})\, d\Omega \qquad \left\{ \begin{array}{l} \text{strain-energy} \\ \text{of the error} \end{array} \right\}$$

It can be shown that

$$\Pi(\underline{e}) = \Pi(\underline{u}^h) - \Pi(\underline{u}) \qquad \left\{ \begin{array}{l} \text{Energy of the error} \\ = \text{Error in the energy} \end{array} \right\}$$

It can be shown, that if complete polynomials of order "p" are used in the FE approximation, then

$$\Pi(\underline{e}) = C_1 h^{2(p+1-m)}$$

where C_1 is a constant of proportionality, "h" is the element-size, and "m" is the order of derivatives in the strains (here $m=1$).

Thus for plane strain problems:

$$\Pi(\underline{e}) = C_1 h^{2p} \qquad \text{i.e. } O(h^{2p})$$

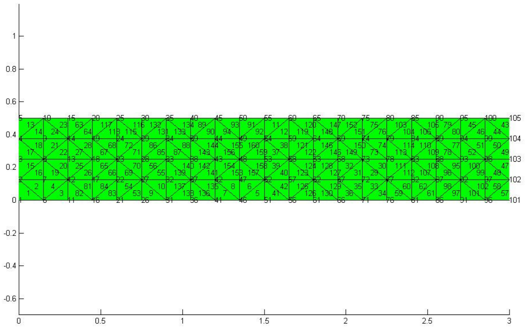
Other error measures:

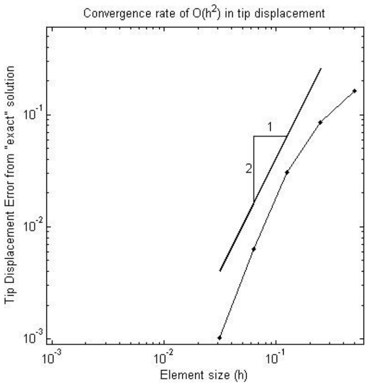
$$\|\underline{e}\| = \|\underline{u}^h - \underline{u}\| = c_2 h^{p+1} \qquad \text{i.e. } O(h^{p+1})$$

$$\|\nabla\underline{e}\| = \|\underline{\epsilon}^h - \underline{\epsilon}\| = c_3 h^p \qquad \text{ie } O(h^p)$$

In general, a method is said to be consistent of order "q" if $\|\underline{e}\| = \|\underline{u}^h - \underline{u}\| = c h^q$ with $q > 0$

eg. Recall





② Stability:

This refers to the solvability of the final FE equation.

$$\underline{d}^{g^T}(\underset{\sim}{K}^G \underline{d}^g - \underline{f}^g) = 0 \qquad \kappa(\underset{\sim}{K}) = \frac{\max(\lambda)}{\min(\lambda)}$$

If $\underset{\sim}{K}^G$ (after BCs) has zero (or close to zero) eigen-values, then this means that there are zero-energy modes and the computed solution may have large errors. (e.g. hourglass modes in Q4 with reduced integration).

Stabilized Methods

- Ad-hoc stabilization : $\underset{\sim}{\hat{K}}^G = \underset{\sim}{K}^G + \alpha \underset{\sim}{I}$ (not consistent)
- Robust stabilization methods require "functional analysis".

Wednesday, March 24, 2010
10:31 AM

Patch Test

Recall, the criteria for convergence of a finite element formulation:

- Continuity / Compatibility.
  The shape-functions must be square-integrable : $H^1(\Omega)$
- Completeness
  shape functions must be complete upto polynomial order "m" (m^{th} derivatives).
  In 2D elasticity, $m=1$, so atleast linear polynomials are required ie. $\{1, x, y\}$.

In addition we use the "patch test" to ensure convergence of a new element:

i.e. $\tilde{G}^h(\underline{d}, \bar{\underline{d}}) \longrightarrow G(\underline{u}, \bar{\underline{u}})$ as $h \to 0$

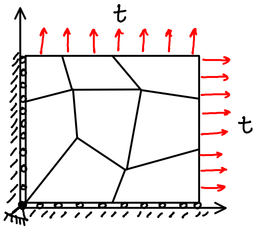
It can be shown that this condition is met if the element can reproduce a state of constant strain imposed on it.

The Patch test checks this ability of an element.

Note:

- The elements we have discussed already pass the patch test.
- This test is used for "new" elements / shape functions.

Idea:



- Consider a domain with an arbitrary patch of the "new" elements
- Apply a state of constant stress and see the stresses within each element are constant also.
- The exact solution may be

$$\underline{u}(x,y) = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{Bmatrix} a_1 + b_1 x + c_1 y \\ a_2 + b_2 x + c_2 y \end{Bmatrix} \quad ⊛$$

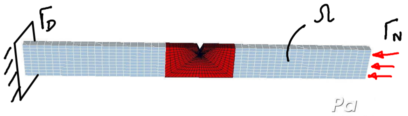
$$\Rightarrow \quad \underline{\epsilon}(x,y) = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} b_1 \\ c_2 \\ c_1 + b_2 \end{Bmatrix} \quad \longleftarrow \text{constant}$$

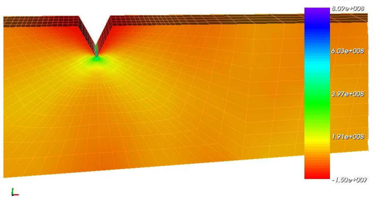
- Calculate the "nodal" displacements at the boundary from ⊛ and solve a pure "boundary" problem.
- Verify that the exact solution is produced as close as possible to the numerical precision of the computer.

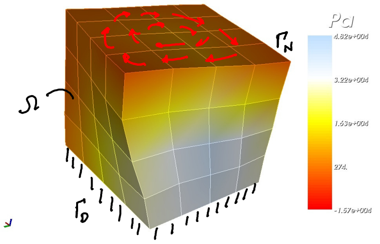
For details, refer Z&T (Ch 9) : 3 types patch tests.

Tuesday, March 23, 2010
4:42 PM

Extensions to 3D







The same process can be generalized to 3D.

GDE: $\quad \operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \underline{0} \quad$ over Ω

BC: $\quad \underset{\sim}{\sigma}\, \underset{\sim}{n} = t_N \quad$ on Γ_N

$\qquad \underline{u} = u_0 \quad$ on Γ_D

Strain-displacement

$$\underset{\sim}{\epsilon} = \tfrac{1}{2}\left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T\right)$$

Material:

$$\underset{\sim}{\sigma}(\underset{\sim}{\epsilon}) = \lambda\, tr(\underset{\sim}{\epsilon})\, \underset{\sim}{I} + 2\mu\, \underset{\sim}{\epsilon}$$

(or equivalently) $\quad \underset{\sim}{\sigma} = \underset{\approx}{C}\, \underset{\sim}{\epsilon}$

Weak form:

$$G(\underline{u}, \bar{\underline{u}}) = \int_\Omega \bar{\underline{u}} \cdot (\operatorname{div} \underset{\sim}{\sigma} + \underline{b})\, d\Omega = 0 \qquad \forall\, \bar{u} \in H_0^1(\Omega)$$

Integration by parts (using Divergence theorem):

$$\text{recall} \begin{cases} \operatorname{div}(\underset{\sim}{\sigma}^T \bar{\underline{u}}) = \bar{\underline{u}} \cdot \operatorname{div}(\underset{\sim}{\sigma}) + \boxed{\underset{\sim}{\sigma} : \nabla \bar{\underset{\sim}{u}}} = \underset{\sim}{\sigma} : \bar{\underset{\sim}{\epsilon}} \\ (\sigma_{ij}\, \bar{u}_i)_{,j} = \sigma_{ij,j}\, \bar{u}_i + \sigma_{ij}\, \bar{u}_{i,j} \end{cases}$$

$$G(\underline{u}, \bar{\underline{u}}) = \underbrace{\int_\Omega \operatorname{div}(\underset{\sim}{\sigma}^T \bar{\underline{u}})\, d\Omega}_{\downarrow \text{ Div. Th.}} - \int_\Omega \bar{\underset{\sim}{\epsilon}} : \underset{\sim}{\sigma}\, d\Omega + \int_\Omega \bar{\underline{u}} \cdot \underline{b}\, d\Omega$$

$$= \overbrace{\int_\Gamma (\underset{\sim}{\sigma}^T \bar{\underline{u}}) \cdot \underline{n}\, d\Gamma} - \int_\Omega \bar{\underset{\sim}{\epsilon}} : \underset{\sim}{\sigma}\, d\Omega + \int_\Omega \bar{\underline{u}} \cdot \underline{b}\, d\Omega$$

$\Gamma = \Gamma_D \cup \Gamma_N$ and $\bar{\underline{u}} = 0$ on Γ_D and $(\underset{\sim}{\sigma}^T \bar{\underline{u}}) \cdot \underline{n} = (\underset{\sim}{\sigma}\, \underline{n}) \cdot \bar{\underline{u}} = \bar{\underline{u}} \cdot t_N$

$$\Rightarrow \quad G(\underline{u}, \bar{\underline{u}}) = -\int_\Omega \bar{\underset{\sim}{\epsilon}} : \underset{\sim}{\sigma}\, d\Omega + \int_\Omega \bar{\underline{u}} \cdot \underline{b}\, d\Omega + \int_{\Gamma_N} \bar{\underline{u}} \cdot t_N\, d\Gamma$$

Using Voight Notation:

Displacement vector (same)

$$\underline{u} = \begin{Bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{Bmatrix}$$

Strain Tensor $\quad$ Strain "Vector"

$$\underset{\sim}{\epsilon} \rightarrow \underline{\epsilon} = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \qquad \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

Stress Tensor $\quad$ Stress Vector

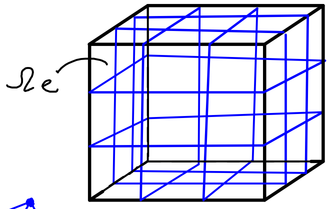
$$\underset{\sim}{\sigma} \rightarrow \underline{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} \qquad \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$\lfloor \epsilon_{zx} \quad \epsilon_{zy} \quad \sigma_{zz} \rfloor \qquad \lfloor \gamma_{xz} \rfloor \qquad \lfloor \sigma_{zx} \quad \sigma_{zy} \quad \sigma_{zz} \rfloor \qquad \lfloor \sigma_{xz} \rfloor$

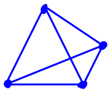
$$G(\underline{u}, \underline{\bar{u}}) = -\int_{\Omega} \underline{\bar{\epsilon}}^T \underset{\sim}{D}\, \underline{\epsilon}\, d\Omega + \int_{\Omega} \underline{\bar{u}}^T \underline{b}\, d\Omega + \int_{\Gamma} \underline{\bar{u}}^T \underline{t}_N\, d\Gamma$$

Discretization :

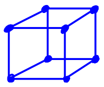
$$G(\underline{u}, \underline{\bar{u}}) = -\sum_{e=1}^{M} \left[\int_{\Omega^e} \left(\underline{\bar{\epsilon}}^T \underset{\sim}{D}\, \underline{\epsilon} - \underline{u}^T \underline{b} \right) d\Omega - \int_{\Gamma_N^e} \underline{\bar{u}}^T \underline{t}_N\, d\Gamma \right]$$



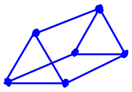
Element types :



Tetrahedron



Hexahedron



Prisms

FE approximation :

$$\underline{u}^e(x,y,z) = \underset{\sim}{N}\, \underline{d}^e = \left[\begin{array}{ccc:ccc:c:ccc} N_1 & & & N_2 & & & & N_n & & \\ & N_1 & & & N_2 & & \cdots\cdots & & N_n & \\ & & N_1 & & & N_2 & & & & N_n \end{array} \right] \left\{ \begin{array}{c} d_x^1 \\ d_y^1 \\ d_z^1 \\ \hdashline d_x^2 \\ d_y^2 \\ d_z^3 \\ \hdashline \vdots \\ d_x^n \\ d_y^n \\ d_z^n \end{array} \right\}$$

$$\underline{\epsilon}^e(x,y,z) = \underset{\sim}{B}\, \underline{d}^e$$

$$\underset{\sim}{B} = \left[\begin{array}{c:ccc:c} & N_{\alpha,x} & 0 & 0 & \\ & 0 & N_{\alpha,y} & 0 & \\ \cdots\cdots & 0 & 0 & N_{\alpha,z} & \cdots\cdots \\ & N_{\alpha,y} & N_{\alpha,x} & 0 & \\ & 0 & N_{\alpha,z} & N_{\alpha,y} & \\ & N_{\alpha,z} & 0 & N_{\alpha,x} & \end{array} \right]$$

$$\underline{\sigma}(x,y,z) = \underset{\sim}{D}\, \underset{\sim}{B}\, \underline{d}^e$$

Substituting FE approximation and Integrating

$$\tilde{G}^h(\underline{d}^G, \underline{\bar{d}}^G) = -\underline{\bar{d}}^{G^T} \underbrace{\left(\underset{\sim}{K}^G\, \underline{d}^G - \underline{f}^G \right)}_{\text{Solve enforcing BCs.}} = 0 \qquad \forall\ \underline{\bar{d}}^G$$

Postprocessing :

- Plot displaced shape
- Plot averaged stresses.

Same error measures hold : Displacement : $O(h^{p+1})$

Strains/Stresses : $O(h^p)$

Stored Strain Energy : $O(h^{2p})$

Variational Principles in 2D & 3D

Recall,

$$\begin{tikzcd} \Pi(\underline{u}) \arrow[r, bend right, "\text{Minimize}"'] & G(\underline{u},\underline{\bar{u}}) \arrow[l, bend right, "\text{Vainberg}"'] \arrow[r, bend right, "\begin{array}{c}\text{Int. by parts}\\(\text{Unbalance})\end{array}"'] & \begin{array}{c}\mathrm{div}\,\underset{\sim}{\sigma}+\underline{b}=\underline{0}\\(\text{Euler})\end{array} \arrow[l, bend right, "\text{PVW/MWR}"'] \end{tikzcd}$$

For 2D/3D Elasticity :

$$\Pi(\underline{u}) \equiv \int_{\Omega} \underset{(\epsilon_{ij}\,\sigma_{ij})}{\frac{1}{2}\,\underset{\sim}{\epsilon}:\underset{\sim}{\sigma}}\, d\Omega - \int_{\Omega} \underline{u}\cdot\underline{b}\, d\Omega - \int_{\Gamma_N} \underline{u}\cdot\underline{t}_N\, d\Gamma$$

Minimize using directional (Gateaux) derivative :

$$D\Pi(\underline{u})\cdot\underline{\bar{u}} = \left[\frac{d}{de}\Pi(\underline{u}+e\underline{\bar{u}})\right]\Bigg|_{e=0}$$

$$= \frac{d}{d\varepsilon}\left\{\int_{\Omega} \frac{1}{2}\left[\frac{1}{2}(\nabla\underset{\sim}{u}+\nabla\underset{\sim}{u}^T)+e\,\frac{1}{2}(\nabla\underset{\sim}{\bar{u}}+\nabla\underset{\sim}{\bar{u}}^T)\right]:\underset{\approx}{C}\left[\frac{1}{2}(\nabla\underset{\sim}{u}+\nabla\underset{\sim}{u}^T)+e\,\frac{1}{2}(\nabla\underset{\sim}{\bar{u}}+\nabla\underset{\sim}{\bar{u}}^T)\right]d\Omega\right.$$

$$\left. - \int_{\Omega}(\underline{u}+e\underline{\bar{u}})\cdot\underline{b}\, d\Omega - \int_{\Omega}(\underline{u}+e\underline{\bar{u}})\cdot\underline{t}_N\, d\Gamma\right\}\Bigg|_{e=0}$$

$$= \int_{\Omega}\underbrace{\left[\frac{1}{2}(\nabla\underset{\sim}{\bar{u}}+\nabla\underset{\sim}{\bar{u}}^T)\right]}_{\underset{\sim}{\bar{\epsilon}}}:\underset{\approx}{C}\underbrace{\left[\frac{1}{2}(\nabla\underset{\sim}{u}+\nabla\underset{\sim}{u}^T)\right]}_{\underset{\sim}{\epsilon}}d\Omega - \int_{\Omega}\underline{\bar{u}}\cdot\underline{b}\, d\Omega - \int_{\Gamma_N}\underline{\bar{u}}\cdot\underline{t}_N\, d\Gamma$$

$$D\Pi(\underline{u})\cdot\underline{\bar{u}} = G(\underline{u},\underline{\bar{u}})$$

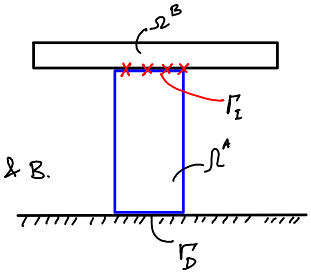
Tuesday, March 23, 2010
4:42 PM

Constraints

Variational Methods can be used to enforce constraints on the problem.

Constraints can be written as :

$$\underset{\sim}{c}(\underset{\sim}{u}) = 0$$



$$\underset{\sim}{K}^G \underline{d}^G = f^G$$

$$\downarrow$$

$$\begin{Bmatrix} - \\ \sim \\ - \\ \sim \end{Bmatrix}$$

For example :

(i) Say Γ_I must remain "connected" between A & B.

$$\underset{\sim}{c}(\underset{\sim}{u}) \equiv \underbrace{\underset{\sim}{u}^A(x,y) - u^B(x,y) = 0}_{} \qquad \text{on } \Gamma_I$$

Note : • This is a linear displacement-based constraint.

- This can be enforced simply by "assembling" the elements from Ω_A & Ω_B correctly.
- In general displacement Boundary conditions are also constraints.

(ii) Incompressible materials $(\nu \to 0.5)$

$$\underset{\sim}{c}(\underset{\sim}{u}) \equiv \underbrace{\det(\underset{\sim}{F}) = 0}_{} \qquad \text{on } \Omega$$

This constraint is usually enforced with "mixed" methods.

$$\underset{\sim}{\sigma} = \lambda\, tr(\underset{\sim}{\epsilon})\, \underset{\sim}{I} + 2\mu\, \underset{\sim}{\epsilon}$$

$$= \underbrace{\left(\lambda + \tfrac{2}{3}\mu\right)}_{K \text{ (Bulk Modulus)}} tr(\underset{\sim}{\epsilon})\, \underset{\sim}{I} + 2\mu\, \underset{\sim}{\epsilon}^{DEV}$$

$$\begin{cases} \underset{\sim}{\epsilon} = \underset{\sim}{\epsilon}^{VOL} + \epsilon^{DEV} \\ \underset{\sim}{\epsilon} = \underbrace{\tfrac{1}{3}\, tr(\underset{\sim}{\epsilon})\, \underset{\sim}{I}}_{\underset{\sim}{\epsilon}^{VOL}} + \underbrace{\left(\underset{\sim}{I} - \tfrac{1}{3}\, tr(\underset{\sim}{\epsilon})\, \underset{\sim}{I}\right) \underset{\sim}{\epsilon}}_{\underset{\sim}{\epsilon}^{DEV}} \end{cases}$$

$$\underset{\sim}{\sigma} = -\underset{\hookrightarrow \text{(pressure)}}{p}\, \underset{\sim}{I} + 2\mu\, \underset{\sim}{\epsilon}^{DEV}$$

The modified "mixed u-p" Variational form is given by :

$$\Pi(\underset{\sim}{u}, p) \equiv \int_\Omega \bar{\underline{\epsilon}}^T \underset{\sim}{D}^{DEV} \underline{\epsilon}\, d\Omega + \int_\Omega \bar{\underline{\epsilon}}^T \begin{bmatrix} p \\ p \\ p \\ 0 \\ 0 \\ 0 \end{bmatrix} d\Omega - \int_\Omega \bar{\underline{u}}^T \underline{b}\, d\Omega - \int_{\Gamma_N} \bar{\underline{u}}^T \underline{t}_N\, d\Gamma$$

$$+ \int_\Omega \bar{p} \left[p - K\, tr(\underset{\sim}{\epsilon}) \right] d\Omega = 0 \qquad \forall \begin{Bmatrix} \bar{u} \\ \bar{p} \end{Bmatrix}$$

(Ref. Z&T : Ch10, 11 for details)

$$\begin{bmatrix} K^G_{uu} & K^G_{up} \\ K^G_{pu} & K^G_{pp} \end{bmatrix} \begin{Bmatrix} \underline{d} \\ p \end{Bmatrix} = \begin{Bmatrix} f_u \\ f_p \end{Bmatrix}$$

Monday, March 29, 2010
11:33 AM

Augmented Lagrangian approaches for constraints:

(i) Lagrange Multiplier:

$$\tilde{\Pi}(\underline{u},\underline{\lambda}) = \Pi(\underline{u}) + \underline{\lambda}\cdot\underline{c}(\underline{u})$$

↳ Lagrange Multiplier

$$D\tilde{\Pi}(\underline{u},\underline{\lambda})\cdot\bar{u} = G(\underline{u},\bar{\underline{u}}) + \underline{\lambda}^T\cdot\left[D\,c(\underline{u})\cdot\bar{u}\right] = 0$$

$$D\tilde{\Pi}(\underline{u},\underline{\lambda})\cdot\bar{\underline{\lambda}} = \underline{c}(\underline{u}) = 0$$

$$D\tilde{\Pi}(\underline{u},\underline{\lambda})\cdot\begin{Bmatrix}\bar{\underline{u}}\\ \bar{\underline{\lambda}}\end{Bmatrix} = \left\{\bar{\underline{d}}^{G^T}\ \ \underline{\lambda}^T\right\}\cdot\left\{\begin{bmatrix}\underset{\sim}{K}^G & \underset{\sim}{C}^T\\ \underset{\sim}{C} & 0\end{bmatrix}\begin{Bmatrix}\underline{d}^G\\ \underline{\lambda}\end{Bmatrix} - \begin{Bmatrix}\underline{f}^G\\ \underline{0}\end{Bmatrix}\right\} = 0$$

(Note: BB conditions for "Mixed" methods)

(ii) Penalty Methods:

$$\tilde{\Pi}(\underline{u}) \equiv \Pi(\underline{u}) + \frac{1}{2}\alpha\left(\underline{c}(\underline{u})\right)^2$$

↳ Large penalty parameter

Minimization:

$$D\tilde{\Pi}(\underline{u})\cdot\bar{\underline{u}} = G(\underline{u},\bar{\underline{u}}) + \alpha\ \underbrace{\underline{c}(\underline{u}).D\,\underline{c}(\underline{u})\cdot\bar{\underline{u}}}_{\text{Usually "inconsistent"}} \quad \forall\,\bar{\underline{u}}$$

Leads to:

$$\bar{\underline{d}}^{G^T}\left[\left(\underset{\sim}{K}^G + \underset{\uparrow}{\underset{\sim}{K}^*}\right)\underline{d}^G - \left(\underline{f}^G + \underset{\uparrow}{\underline{f}^*}\right)\right] = 0 \quad \forall$$

Penalty contribution