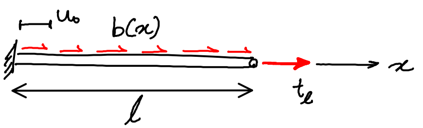
Ch 2: 1-D Problems

Ref: Hjelmstad Ch 5 & 6
Reddy Ch 2 & 3
Hughes Ch 1

Consider the 1-D problem:

Find $\sigma(x)$, $u(x)$:

$$Ⓢ \left[\begin{array}{lll} & \sigma' + b = 0 & \forall x \in (0,\ell) \\ \text{BC} & & \\ & u(0) = u_0 & \text{on } \Gamma_D \\ t(\ell) = & \sigma(\ell)(+1) = t_\ell & \text{on } \Gamma_N \end{array} \right.$$



$\sigma = \sigma(\epsilon)$

This is called the Strong Form Ⓢ of the governing differential equation (GDE).

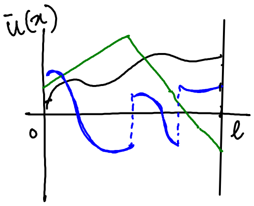
Method of Weighted Residuals:

$$\underset{\uparrow}{G(\sigma,\bar{u})} \equiv -\int_0^\ell \bar{u}\,(\sigma' + b)\,dx \;-\; \bar{u}(0)\underbrace{\left(t_0 + \sigma(0)\right)}_{t_0 = \sigma(0)(-1)} - \bar{u}(\ell)\underbrace{\left(t_\ell - \sigma(\ell)\right)}_{t_\ell = \sigma(\ell)(+1)}$$

functional (scalar)

If for some $\sigma(x)$:

$$G(\sigma,\bar{u}) = 0 \quad \forall\ \bar{u} \in \underset{\substack{\uparrow \\ \text{Function space}}}{V}(0,\ell)$$



Then ⇒

$$\begin{array}{ll} \sigma' + b = 0 & \forall x \text{ in } (0,\ell) \\ \sigma(\ell)(+1) = t_\ell & \text{at } x = \ell \\ \sigma(0)(-1) = t_0 & \text{at } x = 0 \end{array}$$

If $\bar{u} = (\sigma' + b)$: $G(\sigma,\bar{u}) = \int (\sigma' + b)^2 dx$

This is the fundamental theorem of Calculus of Variations.

This also poses a restriction $\bar{u} \in V(0,\ell)$:

$\bar{u}$ must be square integrable

ie $\underbrace{\int_0^\ell (\bar{u})^2 dx}_{L_2\text{-norm of } \bar{u}(x)}$ must exist (be finite)

Possible choices for function spaces:

- Square integrable : L_2 (or H^0)
- Sq. Int. upto 1st derivative : H^1
- Sq. Int upto m^{th} derivative : H^m

- Continuous functions : C^0
- Continuous upto 1st der. : C^1
- Continuous upto m^{th} der : C^m

Note: Dirac-delta $\delta(x - x_0)$ is not L_2.

Sunday, January 24, 2010
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Equivalence with Principle of Virtual Work: (PVW)

$$G(\sigma,\bar u) = -\int_0^\ell \bar u\,(\sigma'+b)\,dx - \bar u(\ell)\left(t_\ell - \sigma(\ell)\right) - \bar u(0)\left(\sigma(0) + t_0\right)$$

(Integration by parts)

$$G(\sigma,\bar u) = \int_0^\ell (\bar u'\sigma)\,dx - \underbrace{\left[\sigma\,\bar u\right]_0^\ell}_{} - \int_0^\ell \bar u\, b\,dx$$

$$- \cancel{\sigma(\ell)\,\bar u(\ell)} + \sigma(\cancel{0})\,\bar u(0)$$

$$+ \left(\cancel{\sigma(\ell)} - t_\ell\right)\bar u(\ell) - \left(\cancel{\sigma(0)} + t_0\right)\bar u(0)$$

because
$t_0 = \sigma(0)(-1)$
$t_\ell = \sigma(\ell)(+1)$
(Natural BCs)

$$\Rightarrow \boxed{G(\sigma,\bar u) = \underbrace{\int_0^\ell (\bar u'\sigma)\,dx}_{W_I} - \underbrace{\int_0^\ell \bar u\, b\,dx - t_\ell\,\bar u(\ell) - t_0\,\cancelto{0}{\bar u(0)}}_{W_E}}$$

Essential BC
$u(0) = u_0$

Note: Unknowns to solve for : $\sigma(x)$ & t_0
So further restrict $\boxed{\bar u(0) = 0 \quad \text{on } \Gamma_D}$

PVW

If for some $\sigma(x)$

$$G(\sigma,\bar u) = W_I - W_E = 0 \qquad \forall\ \bar u(x) \in H_0^1(0,\ell)$$

Then ← denotes $\bar u(0) = 0$

$$\Rightarrow \quad \sigma' + b = 0 \qquad \forall\ x \in (0,\ell)$$

$$\sigma(\ell) = t_\ell \quad \text{at } x = \ell$$

Note: It poses additional restriction on $\bar u \in H_0^1(0,\ell)$

Now Introduce $\sigma = \sigma(\epsilon)$ say: $\sigma = Cu'$

Define

$$\boxed{\tilde G(u,\bar u) \equiv \underbrace{\int_0^\ell \bar u'(Cu')\,dx}_{W_I} - \underbrace{\left[\int_0^\ell \bar u\, b\,dx + \bar u(\ell)\,t_\ell\right]}_{W_E}}$$

Problem statement:

Ⓦ Find $u(x) \in \left\{H^1(0,\ell) \text{ and } \boxed{u(0) = u_0} \text{ on } \Gamma_D : (x=0)\right\}$
such that

$$\tilde G(u,\bar u) = 0 \qquad \forall\ \bar u(x) \in H_0^1(0,\ell)$$

This is called the Weak form Ⓦ or integral form.

Note: Ⓢ $\Longleftrightarrow$ Ⓦ

Alternative Notation:

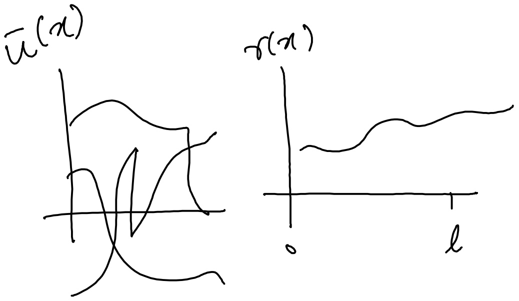
Bilinear forms: $a(\bar{u}, u) \equiv \int_0^\ell \bar{u}'(cu')\,dx \quad (= W_I) \quad (= B(\bar{u},u))$

$(\bar{u}, b) \equiv \int_0^\ell \bar{u}\, b\, dx$

Thus

$$\begin{aligned} \tilde{G}(u,\bar{u}) &= a(\bar{u},u) - \underbrace{(\bar{u},b) - \bar{u}(\ell)t_\ell} \\ &= B(\bar{u},u) - \ell(\bar{u},b) \\ &= W_I - W_E \end{aligned}$$

Aside: Gist of the "proof" of Fundamental theorem of calculus of variations
(see Lec 7 video for details)



$\bar{u} . r =$

$\downarrow$

$1 . r \rightarrow 0$

$2 \cdot r \rightarrow 0$

$\pi \cdot r \rightarrow 0$

$\frac{1}{\sqrt{3}}\ r \rightarrow 0$

$r . r \rightarrow 0$

if $r^2 = 0 \Rightarrow r = 0$

if

$\bar{u}(x) = r(x)$

$$\int_0^\ell [r(x)]^2 dx = 0$$

$\Rightarrow r(x) = 0$

$r(x) \equiv \sigma'(x) + b(x)$

$\forall\ \bar{u}(x) \quad G(\sigma,\bar{u}) = 0 \quad \Rightarrow \quad \sigma'(x) + b = 0$

Monday, January 25, 2010
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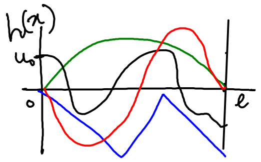
The Ritz Method

Ⓦ form of the problem is still infinite dimensional.

Introduce approximation:

(Assume a certain form of the solution)



$$u(x) \cong u^h(x) = \sum_{i=1}^{N} a_i\, h_i(x) \quad + h_0(x)$$ —

$$= \begin{bmatrix} a_1 & a_2 & \cdots & a_N \end{bmatrix} \begin{bmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_N(x) \end{bmatrix}$$

$h_1(x)$ —
$h_2(x)$ —
$h_N(x)$ —

$$u(x) = \underline{a}^T\, \underline{h}(x)$$

h : smooth enough
: complete

$\{h_i\}_{i=1:N} \subset H_0^1(0,\ell)$

(basis)
(shape)

<u>Note</u>: Essential BC $u(0) = u_0$ is satisfied by $h_0(x)$

Examples of $h_i(x)$:

- Polynomials $\left\{1, \frac{x}{\ell}, \left(\frac{x}{\ell}\right)^2, \cdots\right\}$
- Trigonometric $\left\{1, \sin\left(n\pi\frac{x}{\ell}\right), \cos\left(n\frac{\pi x}{\ell}\right)\right\}$ $n = 1, 2, 3 \ldots$
- Piecewise Polynomial (FE)

What about $\bar{u}(x)$?

<u>Galerkin Approximation</u>

$$\bar{u}(x) \cong \bar{u}^h(x) = \sum_{i=1}^{N} \bar{a}_i\, h_i(x)$$

$$= \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_N \end{bmatrix} \begin{bmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_N(x) \end{bmatrix}$$

$$\bar{u}(x) = \bar{\underline{a}}^T\, \underline{h}(x)$$

<u>Note</u> for <u>ALL</u> $\{\bar{a}_i\}$

Discretized Galerkin Form:

Ⓖ
Find $\underline{a} = \{a_1\ a_2\ \cdots\ a_N\}$

such that:

$$\tilde{G}^h(\underline{a},\bar{\underline{a}}) \equiv \int_0^\ell \underbrace{(\bar{\underline{a}}^T \underline{h})'}_{\bar{u}^{h\prime}}\, C\, \underbrace{(\underline{a}^T \underline{h} + h_0)'}_{u^{h\prime}}\, dx - \int_0^\ell \underbrace{(\bar{\underline{a}}^T \underline{h})}_{\bar{u}^h}\, b\, dx - \underbrace{(\bar{\underline{a}}^T \underline{h}(\ell))}_{\bar{u}^h(\ell)}\,(t_\ell)$$

$G^h(\underline{a},\bar{\underline{a}}) = 0$ FOR ALL $\bar{\underline{a}}$

This is called the discretized <u>Galerkin Form</u> Ⓖ

<u>Note</u>

$$Ⓢ \Longleftrightarrow Ⓦ \overset{\text{Approx}}{\approx} Ⓖ$$

Upon simplification:

$$\tilde{G}^h(\underline{a},\underline{\bar{a}}) = \underbrace{\underline{\bar{a}}^T\left[\int_0^{\ell} C\,\underline{h}'\,\underline{h}'^T\,dx\right]}_{\underline{\bar{a}}^T\,\underset{\sim}{K}}\underline{a} + \underline{\bar{a}}^T\left\{\int_0^{\ell}\underline{h}'\,C h_0'\,dx\right\}$$

$$\underbrace{-\,\underline{\bar{a}}^T\left\{\int_0^{\ell}\underline{h}\,b\,dx\right\} - \underline{\bar{a}}^T\left\{\underline{h}(\ell)\,t_{\ell}\right\}}_{-\underline{\bar{a}}^T\,\underline{f}}$$

ie. $\tilde{G}^h(\underline{a},\underline{\bar{a}}) = \underline{\bar{a}}^T\left(\underset{\sim}{K}\,\underline{a} - \underline{f}\right) = 0$

Note:

This equation would be satisfied FOR ALL $\underline{\bar{a}}^T$

if $\boxed{\underset{\sim}{K}\,\underline{a} = \underline{f}}$

Wednesday, January 27, 2010
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Steps for the Method of Weighted Residuals

1) GDE : multiply with $\bar{u}(x)$ → Integrate ⇒ $G(\sigma, \bar{u})$

2) Integrate $G(\sigma, \bar{u})$ by parts to balance the derivatives

3) Approximation $\bar{u}(x)$; $u(x)$: $\sum_{i=1}^{N} a_i \, h_i(x)$

4) Solution $\underline{a} = \underset{\sim}{K}^{-1} \underline{f}$ $\qquad \underset{\sim}{K} \underline{a} = \underline{f}$

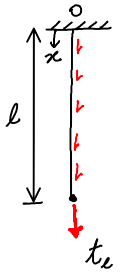
Example : Find $u(x)$ such that

$$(cu')' + b = 0 \qquad \text{on} \qquad x \in (0, \ell)$$

BC

$$u(0) = u_0 \qquad \text{at} \quad x = 0$$

$$\sigma(\ell) = (cu')(\ell) = t_\ell \qquad \text{at} \quad x = \ell$$



1
$b = \rho g$ 10
$c = 1$
$u_0 = 0$
$t_\ell = 10$

Soln:

① $$\tilde{G}(u, \bar{u}) \equiv -\int_0^\ell \bar{u}\left((cu')' + b\right) - \bar{u}(\ell)\, t_\ell$$

② $$= \underbrace{\int_0^\ell \bar{u}'(cu')\,dx}_{W_I} - \underbrace{\left(\int_0^\ell \bar{u}\, b\,dx + \bar{u}(\ell)\, t_\ell\right)}_{W_E}$$

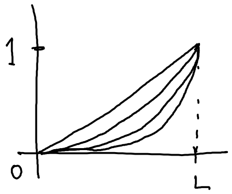
③ Approx: $$u(x) \cong u^h(x) = \sum_{i=1}^{N} a_i \, h_i(x)$$

$h_1(x) = x/L$

$h_2(x) = (x/L)^2$

$h_3(x) = (x/L)^3$

⋮



$$G(\underline{a}, \underline{\bar{a}}) = \underline{\bar{a}}^T \left(\underset{\sim}{K}\, \underline{a} - \underline{f}\right)$$

$$\underline{\bar{a}}^T = [\,\bar{a}_1 \;\; \bar{a}_2 \; \cdots \; \bar{a}_N\,]$$

$$\underline{a} = [\,a_1 \;\; a_2 \; -- \; a_N\,]$$

$$\underset{\sim}{K} = \int_0^\ell c \left\{ \begin{bmatrix} (1/L) \\ 2\,x/L^2 \\ 3\,x^2/L^3 \\ \vdots \\ N \dfrac{x^{N-1}}{L^N} \end{bmatrix} \begin{bmatrix} \left(1/L\right) & \left(\dfrac{2x}{L^2}\right) & \cdots & \left(N \dfrac{x^{N-1}}{L^N}\right) \end{bmatrix} \right\} dx$$

$$K_{ij} = c \int_0^{l} \left(i \frac{x^{i-1}}{L^i} \cdot j \frac{x^{j-1}}{L^j} \right) dx = \frac{i \cdot j}{L^{i+j}} \left[\frac{x^{(i+j-1)}}{(i+j-1)} \right]_0^L$$

$$= \frac{i \cdot j}{L(i+j-1)}$$

$$\underset{\sim}{K} = \frac{1}{L} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4/3 & 6/4 & 8/5 \\ 1 & 6/4 & 9/5 & 12/6 \\ 1 & 8/5 & 12/6 & 16/7 \end{bmatrix}$$

$$\underline{f} = \int_0^{l} \underline{h}\, b \, dx + \underline{h}(l)\, t_l + \int_0^{l} c \underline{h}' \cancelto{0}{h_0'} \, dx$$

$$= \int_0^{l} \begin{bmatrix} (x/L) \\ (x/L)^2 \\ \vdots \\ (x/L)^N \end{bmatrix} b \, dx + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} t_l$$

$$\underline{f} = \begin{bmatrix} \frac{100}{2} + 10 \\ \frac{100}{3} + 10 \\ \vdots \\ \frac{100}{N+1} + 10 \end{bmatrix}$$

④ $\underline{a} = \underset{\sim}{K}^{-1} \underline{f}$

- Types of Weighted residuals

$$\tilde{G}(u,\bar{u}) = -\int_0^l \bar{u}\,\underbrace{\left((Cu')'+b\right)}_{r(x)\ \ (\text{residual})} dx \;-\; \bar{u}(l)\left(t_l - \sigma(l)\right)$$

Approximation

$$u(x) \cong u^h(x) = \underline{a}^T\,\underline{h}(x)$$

1) Galerkin (Bubnov-Galerkin) (same $\underline{h}(x)$)

$$\bar{u}(x) \cong \bar{u}^h(x) = \underline{\bar{a}}^T\left(\underline{h}(x)\right)$$

leads to

$$\tilde{G}(\underline{a},\underline{\bar{a}}) = \underline{\bar{a}}^T\left(\underset{\sim}{K}\,\underline{a} - \underline{f}\right) = 0$$

where $\underset{\sim}{K} = \int C\,\underline{h}'\,\underline{h}'^T dx$ (symmetric) $K_{ij} = K_{ji}$

2) Petrov-Galerkin

$$\bar{u}(x) \cong \bar{u}^h(x) = \underline{\bar{a}}^T\,\underline{\bar{h}}(x)$$

← Different $\underline{\bar{h}}(x)$

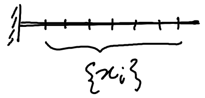
leads to $\tilde{G}^{PG} = \underline{\bar{a}}^T\left(\underset{\sim}{K}^{PG}\,\underline{a}^{PG} - \underline{f}^{PG}\right)$

$\underset{\sim}{K}^{PG} = \int C\,\underline{\bar{h}}'\,\underline{h}'^T dx$ (In general, non symmetric) $K^{PG}_{ij} \neq K^{PG}_{ji}$

3) Collocation:

Residual enforced =0 at a chosen Collection of points.

$$\bar{u}(x) = \delta(x - x_i)$$



Leads to:

$$G(u,\bar{u}) = \int_0^l \delta(x - x^i)\left[(C\underset{\underset{(\underline{a}^T\underline{h})}{\uparrow}}{u'})' + b\right] dx$$

$$\Rightarrow \left.\begin{array}{l} \left(C\,\underline{h}^{T'}\right)'\Big|_{x_1}\underline{a} = -b(x_1) \\ \left(C\,\underline{h}^{T'}\right)'\Big|_{x_2}\underline{a} = -b(x_2) \\ \quad\vdots \qquad\qquad\quad \vdots \\ \left(C\,\underline{h}^{T'}\right)'\Big|_{x_N}\underline{a} = -b(x_N) \end{array}\right\} \Rightarrow \underset{\sim}{K}^{co}\,\underline{a}^{co} = \underline{f}^{co}$$

4) Least squares method

Choose $\bar{u}(x) = (Cu')' + b = r(x)$

leads to $$\tilde{G}^{LS} = \int_0^\ell \overbrace{\left[\underset{\uparrow}{(Cu')'} + b\right]^2}^{[r(x)]^2} dx + \left[t_\ell - Cu'(\ell)\right]^2$$

Now substitute Approx: $u(x) = \underline{a}^T \underline{h}(x)$

i.e: $u'(x) = \underline{a}^T \underline{h}'(x)$

$$\Rightarrow \tilde{G}^{LS} = 2\int_0^\ell \left[\underline{a}^T (C\underline{h}')' + b\right]^2 dx + \left[t_\ell - \underline{a}^T(C\underline{h}'(\ell))\right]^2 = 0$$

Minimize $\tilde{G}^{LS}$ to get approximate solution:

$$\frac{\partial \tilde{G}^{LS}}{\partial \underline{a}} = \int_0^\ell (C\underline{h}')'\left[\underline{a}^T(C\underline{h}')' + b\right] dx + (C\underline{h}'(\ell))\left[t_\ell - \underline{a}^T(C\underline{h}'(\ell))\right]$$

Finally, we get $\underset{\sim}{K}^{LS}\, \underline{a}^{LS} = \underline{f}^{LS}$

where

$$\underset{\sim}{K}^{LS} = \int_0^\ell \begin{bmatrix} (Ch_1')' \\ (Ch_2')' \\ \vdots \\ (Ch_N')' \end{bmatrix} \left[(Ch_1')' \; (Ch_2')' \; \cdots \; (Ch_N')' \right] dx$$

(symmetric)

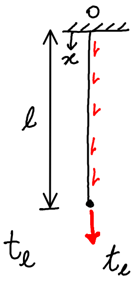
$$+ \begin{bmatrix} \vdots \\ C\,\underline{h}_i'(\ell) \\ \vdots \end{bmatrix} \left[\cdots \; C\underline{h}_i'(\ell) \; \cdots \right]$$

1-D Finite Element Basis

Example : Find $u(x)$ such that

$$\text{(S)} \left[\begin{array}{ll} (Cu')' + b = 0 & \text{on } x \in (0,\ell) \\ \underline{BC} \quad u(0) = u_0 & \text{at } x = 0 \\ \sigma(\ell) = (Cu')(\ell) = t_\ell & \text{at } x = \ell \end{array} \right.$$

i.e. $$G(u,\bar{u}) = \int_0 \bar{u}'(Cu')\,dx - \int_0 \bar{u}\,b\,dx - \bar{u}(\ell)\,t_\ell$$



$b = \rho g$ ($\rho = 1$, $g = 10$)
$C = 1$
$u_0 = 0$
$t_\ell = 10$

Find $u(x) \in \{ H^1(0,\ell) \text{ and } u(0) = u_0 \}$

$$\text{(W)} \left[\; G(u,\bar{u}) = 0 \quad \forall\ \bar{u} \in H^1_0(0,\ell) \right.$$

$\bar{u}(0) = 0$

Approximation

$$u(x) = \sum_{i=1}^{N} a_i h_i(x) + \cancelto{0}{a_0} h_0(x)$$

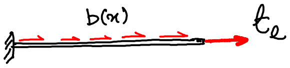
$$= [a_1\ a_2\ \cdots\ a_N] \begin{bmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_N(x) \end{bmatrix}$$

$$\boxed{u(x) = \underline{a}^T \underline{h}(x)}$$

$$\boxed{\bar{u}(x) = \bar{\underline{a}}^T \underline{h}(x)} \quad \text{(Galerkin)}$$

$$\Rightarrow \quad G^h(a,\bar{a}) = \underline{\bar{a}}^T\left(\underset{\sim}{K}\, \underline{a} - \underline{f} \right)$$

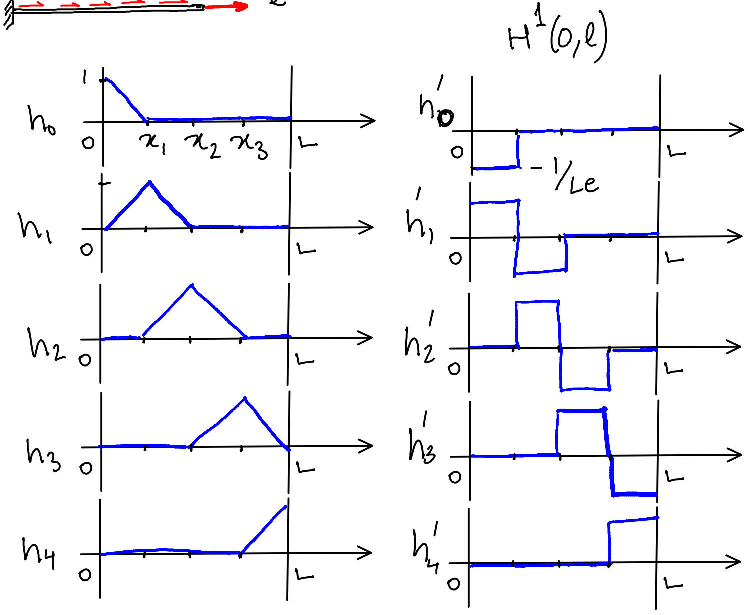
1-D Finite Element Basis Functions



$$L_e = \frac{L}{N} \quad (N=4)$$

$$h_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & : [x_{i-1}, x_i] \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & : [x_i, x_{i+1}] \end{cases} \qquad x \in$$

$$h_i'(x) = \begin{cases} 1/\ell_e & : [x_{i-1}, x_i] \\ -1/\ell_e & : [x_i, x_{i+1}] \end{cases}$$



$$\underset{\sim}{K} = \int_0^l c\, \underline{h}'\, \underline{h}'^T\, dx$$

$$= \int_0^l c \underbrace{\left\{ \begin{bmatrix} h_1' \\ h_2' \\ \vdots \\ h_4' \end{bmatrix} \begin{bmatrix} h_1' & h_2' & h_3' & h_4' \end{bmatrix} \right\}}\, dx \qquad \begin{bmatrix} h_1'h_1' & h_1'h_2' & 0 & 0 \\ & h_2'h_2' & h_2'h_3' & 0 \\ \text{sym} & & h_3'h_3' & h_3'h_4' \\ & & & h_4'h_4' \end{bmatrix}$$

$$\left.\begin{aligned} K_{ii} &= \int_{x_{i-1}}^{x_i} c\left(\frac{1}{l_e}\right)\left(\frac{1}{l_e}\right) dx = \frac{c\,[x]_{x_{i-1}}^{x_i}}{l_e^2} = \frac{c}{l_e} \\ &+ \int_{x_i}^{x_{i+1}} c\left(\frac{-1}{l_e}\right)\left(\frac{-1}{l_e}\right) dx = \frac{c}{l_e} \end{aligned}\right] \quad \frac{2c}{l_e}$$

(no summation)

$$K_{i,i+1} = \int_{x_i}^{x_{i+1}} c\left(\frac{1}{l_e}\right)\left(\frac{-1}{l_e}\right) dx = -\frac{c}{l_e}$$

$$\underset{\sim}{K} = \frac{c}{l_e}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\underline{f} = \int_0^l \underline{h}\, b\, dx + \underline{h}(l)\, t_l = \left(\int_0^l \underline{h}\, dx\right) b + \underline{h}(l)\, t_l$$

$$\downarrow$$

$$\begin{bmatrix} l_e \\ l_e \\ l_e \\ l_e/2 \end{bmatrix} b + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} t_e = \begin{bmatrix} l_e b \\ l_e b \\ l_e b \\ \frac{l_e b}{2} + t_l \end{bmatrix}$$

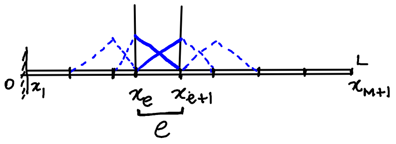
$$\boxed{\underline{a} = \underset{\sim}{K}^{-1}\underline{f}}$$

1-D Finite Element Implementation

When implementing Finite Elements on a computer, it is more convenient to express all quantities in an "element"-wise fashion.

Recall:

$$G(u,\bar{u}) = \int_0^l \bar{u}' C u' \, dx - \int_0^l \bar{u}\, b\, dx - \bar{u}(l)\, t_e$$



This integral may be written as a sum over "elements":

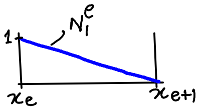
$$G(u,\bar{u}) = \sum_{e=1}^{M} \left[\int_{x_e}^{x_{e+1}} \bar{u}' \, C u' \, dx \right] - \sum_{e=1}^{M} \left[\int_{x_e}^{x_{e+1}} \bar{u}\, b\, dx \right] - \bar{u}(l)\, t_e$$

Approximating $u(x)$ and $\bar{u}(x)$ within element "e" as:

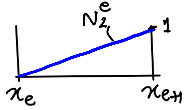
$$u(x) \approx u_e^h(x) = N_1^e(x)\, d_1^e + N_2^e(x)\, d_2^e$$

$$= \left[N_1^e(x) \;\vdots\; N_2^e(x) \right] \begin{bmatrix} d_1^e \\ \hdashline d_2^e \end{bmatrix} \qquad \text{i.e.} \quad \boxed{u_e^h(x) = \underset{\sim}{N}\, \underline{d}}$$

where $$N_1^e(x) = h_e(x)\Big|_{(x_e < x < x_{e+1})} = \left(\frac{x_{e+1} - x}{x_{e+1} - x_e} \right)$$



and $$N_2^e(x) = h_{e+1}(x)\Big|_{(x_e < x < x_{e+1})} = \left(\frac{x - x_e}{x_{e+1} - x_e} \right)$$



Similarly

$$\bar{u}(x) \approx \bar{u}_e^h(x) = \left[N_1^e(x) \;\vdots\; N_2^e(x) \right] \begin{bmatrix} \bar{d}_1^e \\ \hdashline \bar{d}_2^e \end{bmatrix}$$

Using this approximation:

$$u'(x) \approx \frac{d\, u_e^h}{dx} = \left[\frac{dN_1^e}{dx} \;\vdots\; \frac{dN_2^e}{dx} \right] \begin{bmatrix} d_1^e \\ \hdashline d_2^e \end{bmatrix} \qquad \text{i.e.} \quad \boxed{\varepsilon_e^h(x) = \underset{\sim}{B}\, \underline{d}}$$

and

$$\bar{u}'(x) \approx \frac{d\, \bar{u}_e^h}{dx} = \left[\frac{dN_1^e}{dx} \;\vdots\; \frac{dN_2^e}{dx} \right] \begin{bmatrix} \bar{d}_1^e \\ \hdashline \bar{d}_2^e \end{bmatrix}$$

Substituting the boxed equations into the weak form:

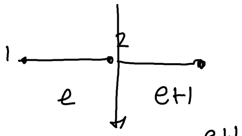
$$G(u,\bar{u}) = \sum_{e=1}^{M}\left[\int_{x_e}^{x_{e+1}} \bar{u}'\, C u'\, dx\right] - \sum_{e=1}^{M}\left[\int_{x_e}^{x_{e+1}} \bar{u}\, b\, dx\right] - \bar{u}(l) t_\ell$$

$$G(u,\bar{u}) \approx \tilde{G}^h\left(\{\underline{d}\}_{e=1}^{M}, \{\bar{\underline{d}}\}_{e=1}^{M}\right)$$

$$= \sum_{e=1}^{M} \bar{\underline{d}}^{e^T} \underbrace{\left[\int_{x_e}^{x_{e+1}} [\underset{\sim}{B}^T C \underset{\sim}{B}]\, dx\right]}_{\boxed{\underset{\sim}{K}^e = \int_{x_e}^{x_{e+1}} \begin{bmatrix} N_1^{e'} \\ N_2^{e'} \end{bmatrix} C \begin{bmatrix} N_1^{e'} & N_2^{e'} \end{bmatrix} dx}} \underline{d}^e - \sum_{e=1}^{M} \bar{\underline{d}}^{e^T} \underbrace{\left[\int_{x_e}^{x_{e+1}} \underset{\sim}{N}^T b\, dx\right]}_{\boxed{\underline{f}^e = \int_{x_e}^{x_{e+1}} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} b\, dx}} - \bar{u}(l)\, t_\ell$$

In the expanded form:

$$\tilde{G}^h\left(\{\underline{d}\}_{e=1}^{M}, \{\bar{\underline{d}}\}_{e=1}^{M}\right) = \begin{bmatrix} \bar{d}_1^1 & \bar{d}_2^1 \end{bmatrix} \left\{ \begin{bmatrix} K_{11}^1 & K_{12}^1 \\ K_{21}^1 & K_{22}^1 \end{bmatrix} \begin{bmatrix} d_1^1 \\ d_2^1 \end{bmatrix} - \begin{bmatrix} f_1^1 \\ f_2^1 \end{bmatrix} \right\}$$



$$d_2^e = d_1^{e+1}$$

$$+ \begin{bmatrix} \bar{d}_1^2 & \bar{d}_2^2 \end{bmatrix} \left\{ \begin{bmatrix} K_{11}^2 & K_{12}^2 \\ K_{21}^2 & K_{22}^2 \end{bmatrix} \begin{bmatrix} d_1^2 \\ d_{22}^2 \end{bmatrix} - \begin{bmatrix} f_1^2 \\ f_2^2 \end{bmatrix} \right\}$$

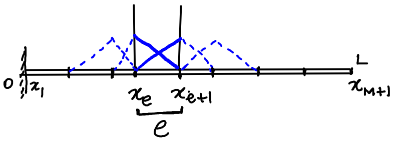
$$+ \qquad \vdots$$

$$+ \begin{bmatrix} \bar{d}_1^M & \bar{d}_2^M \end{bmatrix} \left\{ \begin{bmatrix} K_{11}^M & K_{12}^M \\ K_{21}^M & K_{22}^M \end{bmatrix} \begin{bmatrix} d_1^M \\ d_{22}^M \end{bmatrix} - \begin{bmatrix} f_1^M \\ f_2^M \end{bmatrix} \right\} - \underbrace{\bar{u}(l)}_{\downarrow\; d^{M+1}} \underbrace{t_\ell}_{\downarrow}$$

$$\tilde{G}^h\left(\{\underline{d}\}_{e=1}^{M}, \{\bar{\underline{d}}\}_{e=1}^{M}\right) = 0$$

Note:

$$d_1^G = d_1^1$$
$$d_2^G = d_2^1 = d_1^2$$
$$d_3^G = d_2^2 = d_1^3$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$d_M^G = d_2^{M-1} = d_1^M$$
$$d_{M+1}^G = d_2^M$$



This means that "Global" equation can be $\boxed{\text{ASSEMBLED}}$ by taking these terms common:-

ie.

$$\tilde{G}^h\left(\{\underline{d}\}_{e=1}^{M}, \{\bar{\underline{d}}\}_{e=1}^{M}\right)$$

$$= \left\{\bar{d}_1^G \; \bar{d}_2^G \; \cdots \; \bar{d}_M^G \; \bar{d}_{M+1}^G\right\} \begin{bmatrix} K^1_{11} & \overset{\underset{\sim}{K}^1}{K^1_{12}} & & & & 0 \\ K^1_{21} & (K^1_{22}+K^2_{11}) & \overset{\underset{\sim}{K}^2}{K^2_{12}} & & & \\ & K^2_{21} & (K^2_{22}+K^3_{11}) & \underset{\sim}{K}^3 & & \\ & & & \ddots & & \\ & & & & \overset{\underset{\sim}{K}^{M-1}}{(K^{M-1}_{22}+K^M_{11})} & \overset{\underset{\sim}{K}^M}{K^M_{12}} \\ 0 & & & & K^M_{21} & K^M_{22} \end{bmatrix} \begin{bmatrix} d_1^G \\ d_2^G \\ d_3^G \\ \vdots \\ d_{M-1}^G \\ d_M^G \end{bmatrix}$$

$$- \left\{\bar{d}_1^G \; \bar{d}_2^G \; \cdots \; \bar{d}_M^G \; \bar{d}_{M+1}^G\right\} \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ \vdots \\ f_2^{M-1} + f_1^M \\ f_2^M \end{Bmatrix} + t_\ell$$

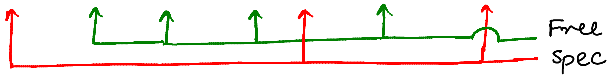
Finally $\boxed{\tilde{G}^h(\underline{d}^G, \bar{\underline{d}}^G) = \bar{\underline{d}}^G\left[\underset{\sim}{K}^G \underline{d}^G - \underline{f}^G\right] = 0}$

Boundary Conditions:

Note that our "N" functions do NOT satisfy essential boundary conditions.

To enforce BCs, we divide our "dofs" into "free" & "specified"

Say $\underline{d}^{G^T} = \left\{ d_1^G \;\; d_2^G \;\; d_3^G \;\; \cdots\cdots \;\; d_M^G \;\; d_{M+1}^G \right\}$



Thus rearrange so that:

unknown

$$\begin{bmatrix} \underset{\sim}{K}_{ff}^G & \underset{\sim}{K}_{fs}^G \\ \underset{\sim}{K}_{sf}^G & \underset{\sim}{K}_{ss}^G \end{bmatrix} \begin{bmatrix} \underline{d}_f^G \\ \underline{d}_s^G \end{bmatrix} = \begin{bmatrix} \underline{f}_f^G \\ \underline{f}_s^G \end{bmatrix}$$

given

$\Rightarrow$ Solve $\boxed{[\underset{\sim}{K}_{ff}^G]\{\underline{d}_f^G\} = \{\underline{f}_f^G\} - [\underset{\sim}{K}_{fs}^G]\{\underline{d}_s^G\}}$ for $\underline{d}_f^G$

and

$$\underline{f}_s^G = [\underset{\sim}{K}_{sf}^G]\{\underline{d}_f^G\} + [\underset{\sim}{K}_{ss}^G]\{\underline{d}_s^G\}$$ (support reactions)

Postprocessing:

Using $\underline{d}^G$ calculate stresses in each element.

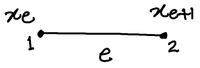
Wednesday, February 03, 2010
10:16 AM

• Properties of the Stiffness Matrix $\underset{\sim}{K}$

Recall:

$$\underset{\sim}{K}^G = \overset{M}{\underset{e=1}{\mathbf{A}}}\ \underset{\sim}{K}^e \qquad \text{where} \qquad \underset{\sim}{K}^e = \int_{x_e}^{x_{e+1}} \underbrace{\underset{\sim}{B}^T C\, \underset{\sim}{B}}_{\downarrow} dx$$

$$C \begin{bmatrix} N_1^{e\prime} \\ N_2^{e\prime} \end{bmatrix} [N_1^{e\prime} \mid N_2^{e\prime}]$$



In order to solve:

unknown

$$\left[\begin{array}{c|c} \underset{\sim}{K}_{ff}^G & \underset{\sim}{K}_{fs}^G \\ \hline \underset{\sim}{K}_{sf}^G & \underset{\sim}{K}_{ss}^G \end{array}\right] \begin{bmatrix} \underline{d}_f^G \\ \underline{d}_s^G \end{bmatrix} = \begin{bmatrix} \underline{f}_f^G \\ \underline{f}_s^G \end{bmatrix}$$

given

i.e.

$$\boxed{[\underset{\sim}{K}_{ff}^G]\{\underline{d}_f^G\} = \{\underline{f}_f^G\}_{\text{eff.}}} \qquad \text{where} \quad \{\underline{f}_f^G\}_{\text{eff}} = \{\underline{f}_f^G\} - [\underset{\sim}{K}_{fs}^G]\{\underline{d}_s^G\}$$

Can we always solve for $\{\underline{d}_f^G\}$?

- Eigenvalues will help us decide.
- Recall when solving $\underset{\sim}{A}\,\underline{x} = \underline{b}$; $(\underset{\sim}{K}\,\underline{d} = \underline{f})$

Eigenvalues & Eigenvectors of $\underset{\sim}{A}$; $(\underset{\sim}{K})$

are $\underset{\sim}{A}\,\underline{v} = \lambda\,\underline{v}$; $\underset{\sim}{K}\,\underline{v} = \lambda\,\underline{v}$

ie. $\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$; $\det(\underset{\sim}{K} - \lambda \underset{\sim}{I})$

Properties of $\underset{\sim}{K}$:

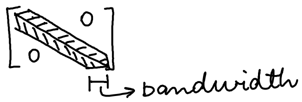
- Symmetric

$\underset{\sim}{K}^G = \underset{\sim}{K}^{G^T}$; $(\underset{\sim}{K}^e = \underset{\sim}{K}^{e^T})$

$\underset{\sim}{K}_{ff}^G = \underset{\sim}{K}_{ff}^{G^T}$

⇒ • Eigenvalues are Real.
• Solvers for symmetric matrices can be used (more efficient).

- Banded :



⇒ • Fast sparse solvers

- Positive Definite

A matrix $\underset{\sim}{A}$ is said to positive-definite

iff

- $\underline{c}^T \underset{\sim}{A}\,\underline{c} \geqslant 0$ for all $\underline{c}$ vectors
- $\underline{c}^T \underset{\sim}{A}\,\underline{c} = 0 \ \boxed{\Rightarrow}\ \underline{c} = 0$

⇒ • EigenVal are Positive $\lambda > 0$
• solvers

Note:

- $\underset{\sim}{K}^e$ and $\underset{\sim}{K}^G$ are semi-positive definite $(\lambda \geqslant 0)$

ie. there are some $\lambda_i = 0$ (Rigid body modes)

Wednesday, February 03, 2010
10:56 AM

e.g.

Solve:

$$\frac{CA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -f \\ f \end{bmatrix}$$

Eigenvalues & Eigenvectors

$$\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$(1-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 2\lambda = 0$$

$$\Rightarrow \lambda_1 = 0; \qquad \lambda_2 = 2\left(\frac{CA}{l_e}\right)$$

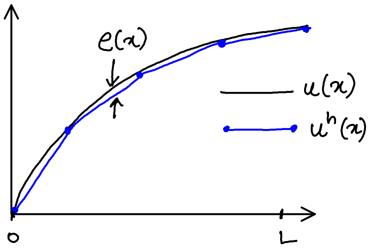
$$\underline{v} = \begin{bmatrix} a \\ a \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} +a \\ -a \end{bmatrix}$$

Rigid Body Displacement ↗ Deformation ↗ mode

- $[K^G]_{ff}$ is positive definite $(\lambda > 0)$
(because Rigid body modes are ruled out by BCs). ⇒ • Unique Solution

Convergence and Accuracy of the FE solution

- $u^h(x_e) = u(x_e)$: Exact at nodes (only in 1-D)



- Pointwise error in the solution:

$$e(x) \equiv u^h(x) - u(x)$$

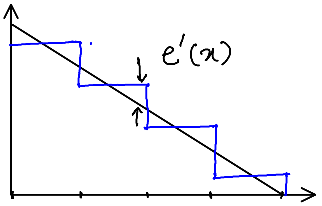
One can show (using Taylor-series):

$\boxed{e(x) \approx O(h^2)}$ for linear polynomial ($\mathbb{P}^1$) FE basis.
where $h = l_e$ (element-size).

ie. if you "refine" your 1-D "mesh" by doubling the nodes $\left(\frac{h}{2}\right)$ then the pointwise error will go down by 4 times.

- Pointwise error in the derivative (stresses / strains)

$$\boxed{e'(x) \approx O(h)}$$



However, there are points in the element where $e'(x_0) \approx O(h^2)$

The fact that (in 1-D) the displacement is exact at the nodes and stresses at certain points show higher convergence is called super-convergence.

In general, if one uses polynomial approximation of degree "p" ie. $(\mathbb{P}^p)$, then

$$\boxed{\begin{aligned} e(x) &\approx O(h^{p+1}) \\ e'(x) &\approx O(h^p) \end{aligned}}$$

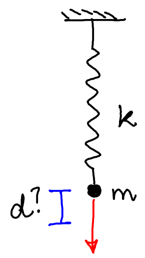
Wednesday, February 03, 2010
12:46 PM

Variational Methods

{Ref. Reddy Ch 2·3 ; 2·5
Hjelmstad Ch 9

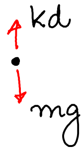
In contrast with weighted residuals, sometimes it is possible to derive the weak form from Variational (Energy) Principles.

Example



Equilibrium (MWR/PVW)

(FBD)



$\Rightarrow kd = mg$

$\Rightarrow \boxed{d = \frac{mg}{k}}$

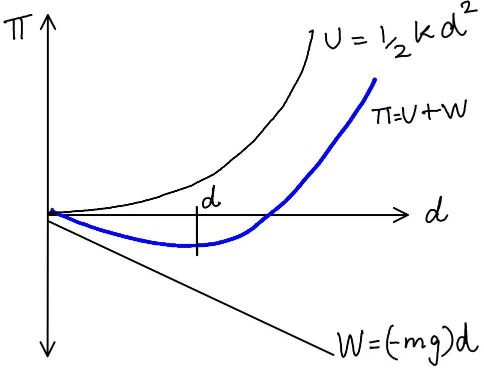
Minimization of Potential Energy

$$\Pi = U + W$$

$$\Pi = \tfrac{1}{2}kd^2 + (-mgd)$$

Minimum $\Rightarrow \frac{\partial \Pi}{\partial d} = 0$

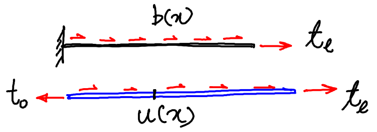
$\Rightarrow kd = mg$ (Equilibrium)



Now lets consider:

$$Cu'' + b = 0 \quad \text{for } x \in (0,\ell)$$

$$u(0) = u_0 \quad ; \quad Cu'(\ell) = t_\ell$$



Total Potential Energy

$$\Pi = U + W$$

U ↓ Strain energy

W → Potential due to conservative body forces

Recall

$$U(u) = \int_0^\ell \tfrac{1}{2}\sigma\epsilon \, dx = \int_0^\ell \tfrac{1}{2} C(u')^2 \, dx$$

$$W(u) = \int_0^\ell -ub \, dx \quad - u(\ell)\,t_\ell + u(0)\,t_0$$

Thus

$$\boxed{\Pi(u) = \int_0^\ell \left[\tfrac{1}{2}C(u')^2 - ub\right] dx}$$

$$- u(\ell)\,t_\ell + u(0)\,t_0$$

Alternative Notation
$I(u)$: Reddy
$\mathcal{E}(u)$: Hjelmstad

Wednesday, February 03, 2010
12:46 PM

To minimize the Potential Energy $\Pi(u)$ wrt $u(x)$.

For this we need:

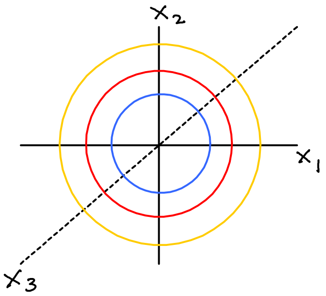
Directional Derivative (Gateaux Derivative)

For a scalar functional $J(u)$

$$D\,J(u)\cdot\bar{u}\ \underset{\substack{\uparrow\\ \text{defined as}}}{\boxed{\equiv}}\ \left[\frac{d}{d\epsilon}\left[J(u+\epsilon\bar{u})\right]\right]\Bigg|_{\epsilon=0}$$

---

eg. Consider the function



$$f(\underline{x}) = \underline{x}\cdot\underline{x} = x_i x_i$$

Gradient of f: $\nabla_{\underline{x}} f = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{bmatrix} = 2\underline{x}$

($\underline{x}$ is radial)

Directional derivative

$$D\,f(\underline{x})\cdot\underline{y} = \left[\frac{d}{d\epsilon} f(\underline{x}+\epsilon\underline{y})\right]\Bigg|_{\epsilon=0} = \left[\frac{d}{d\epsilon}(\underline{x}+\epsilon\underline{y})\cdot(\underline{x}+\epsilon\underline{y})\right]\Bigg|_{\epsilon=0}$$

$$= \left[\frac{d}{d\epsilon}\Big(\underset{\downarrow 0}{\underline{x}\cdot\underline{x}} + 2\epsilon\,\underline{x}\cdot\underline{y} + \epsilon^2\,\underline{y}\cdot\underline{y}\Big)\right]\Bigg|_{\epsilon=0} = \left[2\underline{x}\cdot\underline{y} + \underset{\downarrow 0}{2\epsilon\,\underline{y}\cdot\underline{y}}\right]\Bigg|_{\epsilon=0}$$

$$= 2\underline{x}\cdot\underline{y} = (\nabla_{\underline{x}} f)\cdot\underline{y}$$

Similarly for functions:-

$$D\,\Pi(u)\cdot\bar{u} = \left[\frac{d}{d\epsilon}\left[\Pi(u+\epsilon\bar{u})\right]\right]\Bigg|_{\epsilon=0} = \left[\frac{d}{d\epsilon}\left[\int_0^{l} \tfrac{1}{2}c\,(u'+\epsilon\bar{u}')^2 - (u+\epsilon\bar{u})b\ dx\right]\right]\Bigg|_{\epsilon=0}$$

$$= \left[\int_0^{l} \tfrac{1}{2}c\,\frac{d}{d\epsilon}\left(u'^2 + 2\epsilon u'\bar{u}' + \epsilon^2\bar{u}'^2\right) - \frac{d}{d\epsilon}(ub+\epsilon\bar{u}b)\,dx\right]\Bigg|_{\epsilon=0}$$

$$= \underbrace{\int_0^{l} c\,u'\bar{u}'\,dx}_{W_I} - \underbrace{\int_0^{l} \bar{u}b\,dx}_{W_E} \underbrace{- u(l)\,t_l + u(0)\,t_0}$$

i.e. $= W_I - W_E = G(u,\bar{u})$!

$$\boxed{D\,\Pi(u)\cdot\bar{u} = G(u,\bar{u})}$$

Wednesday, February 03, 2010
12:47 PM

Since we have obtained

$$D\Pi(u)\cdot\bar{u} = G(u,\bar{u}) = \int_0^\ell C u'\bar{u}'\,dx - \int_0^\ell \bar{u}\,b\,dx - \bar{u}(\ell)\,t_\ell + u(0)\,t_0$$

Integrate by parts (in reverse - to <u>un</u>balance the derivatives):

$$= -\int_0^\ell \bar{u}\,C u''\,dx + \left[\bar{u}\,C u'\right]_0^\ell - \int_0^\ell \bar{u}\,b\,dx - \bar{u}(\ell)\,t_\ell + u(0)\,t_0$$

$$= -\int_0^\ell \bar{u}\,(Cu''+b)\,dx + \bar{u}(\ell)\underbrace{\left[Cu'(\ell) - t_\ell\right]}_{\text{Natural BC @ } \ell} + \cancelto{0}{\bar{u}(0)}\underbrace{\left[Cu'(0)+t_0\right]}_{\text{Nat BC @ } 0}$$

Now if

$$D\Pi(u)\cdot\bar{u} = G(u,\bar{u}) = 0 \qquad \text{for } \underline{\text{all}}: \forall\ \bar{u}(x) \in H_0^1(0,\ell)$$

then

$$Ⓢ \left[\begin{array}{ll} Cu''+b = 0 & \text{at all points } x \text{ in } (0,\ell) \\ \text{and} & \\ Cu'(\ell) = \sigma(\ell) = t_\ell & \text{at } x=\ell \end{array}\right.$$

i.e. we get the governing differential equation (GDE) back from the variational principle.

In general, if you have "some" $\boxed{\Pi(u)}$
i.e. some "energy" functional

then, the governing differential equation corresponding to $\Pi(u)$ is called its <u>Euler equation</u> (or <u>Euler-Lagrange</u> equation).

So,

$$\begin{tikzcd} \Pi(u) \arrow[r] & \underset{\forall\bar{u}}{D\Pi(u)\cdot\bar{u} = G(u,\bar{u})=0} \arrow[r] & \begin{array}{l} Cu''+b=0 \\ \sigma(\ell) = t_\ell \end{array} \\ Ⓔ & Ⓦ & Ⓢ \end{tikzcd}$$

$$\boxed{\Pi(u) \xleftarrow[\boxed{?}]{} D\Pi(u)\cdot\bar{u}}\ \boxed{?}\ \underset{\forall\bar{u}}{G(u,\bar{u})=0} \xleftarrow{\text{MWR}} \underset{\text{(some GDE)}}{A(u) = f}$$

Existence of a Variational Principle is decided with the help of the <u>Vainberg's Theorem</u>:

Vainberg's Theorem

(Existence of $\Pi(u)$ or Ⓔ)

Given a functional $G(u,\bar{u})$:

If

- $G(\cdot,\cdot)$ is <u>linear</u> in the <u>second</u> argument:

ie $$G(u,(\alpha\bar{u}_1+\beta\bar{u}_2)) = \alpha\, G(u,\bar{u}_1)+\beta\, G(u,\bar{u}_2)$$

- Directional derivative is <u>symmetric</u> in the <u>second</u> argument:

ie $$D\,G(u,\bar{u}_1)\cdot\bar{u}_2 = D\,G(u,\bar{u}_2)\cdot\bar{u}_1$$

$$\left\{ \text{where } \ DG(u,\bar{u}_1)\cdot\bar{u}_2 \equiv \left[\frac{d}{d\epsilon}G\left((u+\epsilon\bar{u}_2),\bar{u}_1\right)\right]\Bigg|_{\epsilon=0} \right\}$$

Then

$$\boxed{\Pi(u) = \int_0^1 G(tu,u)\,dt \ + c}$$

such that $D\Pi(u)\cdot\bar{u} = G(u,\bar{u})$.

<u>Example</u>: Consider the Weak form for the 1-D problem:-

$$G(u,\bar{u}) = \int_0^l \bar{u}'\,C\,u'\,dx - \int_0^l \bar{u}\,b\,dx - \bar{u}(l)\,t_l$$

Check:

- $G(u,(\alpha\bar{u}_1+\beta\bar{u}_2)) = \alpha\, G(u,\bar{u}_1)+\beta\, G(u,\bar{u}_2)$ ✓
- $$D\,G(u,\bar{u}_1)\cdot\bar{u}_2 = \frac{d}{d\epsilon}\left[\int_0^l \bar{u}_1'\left(C(u'+\epsilon\bar{u}_2')\right)dx - \int_0^l \bar{u}_1 b - \bar{u}_1(l)\,t_l\right]\Bigg|_{\epsilon=0}$$
$$= \int_0^l \bar{u}_1'\,C\,\bar{u}_2'\,dx \quad ⊛$$

$$\& \quad D\,G(u,\bar{u}_2)\cdot\bar{u}_1 = \frac{d}{d\epsilon}\left[\int_0^l \bar{u}_2'\left(C(u'+\epsilon\bar{u}_1')\right)dx - \int_0^l \bar{u}_2 b - \bar{u}_2(l)\,t_l\right]\Bigg|_{\epsilon=0}$$
$$= \int_0^l \bar{u}_2'\,C\,\bar{u}_1'\,dx = ⊛ \qquad ✓$$

Thus $\Pi(u)$ exists:

$$\Pi(u) = \int_0^1 \underbrace{G(tu,u)}\,dt \ + \cancel{c} \nearrow 0 \ \text{(say)}$$

$$= \int_0^1 \left[\underbrace{\int_0^l u'\,C(tu)'\,dx - \int_0^l u\,b\,dx - u(l)\,t_l}\right]dt$$

$$= \underbrace{\left(\int_0^1 t\,dt\right)}\left[\int_0^l u'\,C\,u'\,dx\right] - \int_0^l u\,b\,dx - u(l)\,t_l$$

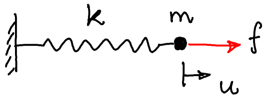
$$\Bigg| \ \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \tfrac{1}{2}$$

$$\boxed{\Pi(u) = \left[\int_0^l \tfrac{1}{2}\epsilon\sigma\,dx\right] - \int_0^l u\,b\,dx - u(l)\,t_l}$$

Friday, February 05, 2010
10:42 AM

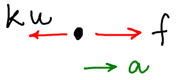
Hamilton's Principle for dynamics

Consider



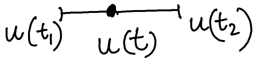
Equation of motion: (Dynamic Equilibrium F = ma)

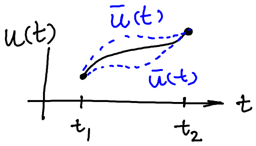
FBD:



$$m\ddot{u} + ku - f = 0$$

Using Energy principles (Variational methods):





$K(\dot{u}) = \frac{1}{2} m \dot{u}^2$; $\Pi(u) = \frac{1}{2} k u^2 - f.u$

Define: Lagrangian $\boxed{\mathcal{L}(u,\dot{u}) = K(\dot{u}) - \Pi(u)}$

Hamiton's Principle:

$$D\left[\int_{t_1}^{t_2} \mathcal{L}(u,\dot{u})\right]\cdot\bar{u} = 0 \qquad \forall\, \bar{u}(t)$$

$$\Rightarrow \frac{d}{d\epsilon}\left[\int_{t_1}^{t_2}\left[\frac{1}{2} m(\dot{u}+\epsilon\dot{\bar{u}})^2 - \frac{1}{2} k (u+\epsilon\bar{u})^2 - f.(u+\epsilon\bar{u})\right] dt\right]\Bigg|_{\epsilon=0} = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\left(m\dot{u}\dot{\bar{u}} - ku\bar{u} - f\bar{u}\right) dt = 0 \qquad \forall\, \bar{u}(t)$$

(Integrate by parts in t)

$$\Rightarrow \int_{t_1}^{t_2}\left(-\bar{u}\, m\ddot{u} - ku\bar{u} - f\bar{u}\right) dt + \underset{0}{\left[\cancel{m\dot{u}}\bar{u}\right]_{t_1}^{t_2}} = 0 \qquad \forall\, \bar{u}(t)$$

$(\bar{u}(t_1) = \bar{u}(t_2) = 0)$

$$\Rightarrow \boxed{m\ddot{u} + ku - f = 0}$$

This is the Euler-Lagrange equation corresponding to the Lagrangian above.

Using the same Lagrangian $\mathcal{L}(u,\dot{u}) = K(\dot{u}) - \Pi(u)$

For 1-D problem:

$$\sigma' + b = \rho\ddot{u}$$

For 2-D & 3D problems:

$$\mathrm{div}\, \underset{\sim}{\sigma} + \underline{b} = \rho\, \underline{\ddot{u}}$$

Friday, February 05, 2010
10:42 AM

In class assignment:

Consider $-\dfrac{d}{dx}\left(u\dfrac{du}{dx}\right) + f = 0$ for $x \in (0, l)$

BC $\left(u\dfrac{du}{dx}\right)\Big|_{x=0} = 0$ and $u(l) = 1$ ← EBC on u

$\bar{u}(l) = 0$ ← HEBC on $\bar{u}$

$$G(u,\bar{u}) = \int_0^l \bar{u}\,(uu')'\,dx - \int_0^l \bar{u} f\,dx$$

$$= -\int_0^l \bar{u}'(uu')\,dx + \left[\underset{0}{\cancel{\bar{u}}}(uu')\right]_0^l - \int_0^l \bar{u} f\,dx$$

Does $\Pi(u)$ exist?

Check $G(u,\bar{u}) = -\displaystyle\int_0^l \bar{u}'\,uu'\,dx$

- Linear in $\bar{u}$: YES ✓
- $DG(u,\bar{u}_1)\cdot\bar{u}_2 = DG(u,\bar{u}_2)\cdot\bar{u}_1$?

LHS:
$$= \frac{d}{d\epsilon}\left[-\int_0^l \bar{u}_2'\,(u+\epsilon\bar{u}_1)\,(u'+\epsilon\bar{u}_1')\,dx\right]\Bigg|_{\epsilon=0}$$

$$= -\int_0^l \bar{u}_2'\left(u\bar{u}_1' + \bar{u}_1 u'\right)dx = -\int_0^l \Big(\underbrace{\bar{u}_1'\bar{u}_2'u}_{✓} + \underbrace{\bar{u}_1\,\bar{u}_2'u'}_{\times}\Big)dx$$

$\circledast$

RHS:
$$= \frac{d}{d\epsilon}\left[-\int_0^l \bar{u}_1'\,(u+\epsilon\bar{u}_2)(u'+\epsilon\bar{u}_2')\,dx\right]\Bigg|_{\epsilon=0}$$

$$= -\int_0^l \bar{u}_1'\left(u\bar{u}_2' + \bar{u}_2 u'\right)dx = -\int_0^l\Big(\underbrace{\bar{u}_1'\bar{u}_2'u}_{✓} + \underbrace{\bar{u}_1'\,\bar{u}_2u'}_{\times}\Big)dx$$

$\neq \circledast$

For them to be equal: $\displaystyle\int_0^l \bar{u}_1\bar{u}_2'u'\,dx = \int_0^l \bar{u}_1'\bar{u}_2\,u'\,dx$

ie $\displaystyle\int_0^l \bar{u}_2^2\left[\frac{\bar{u}_1'\bar{u}_2 - \bar{u}_1\bar{u}_2'}{\bar{u}_2^2}\right]u'\,dx = 0 \quad \forall\ \bar{u}_1\ \bar{u}_2$

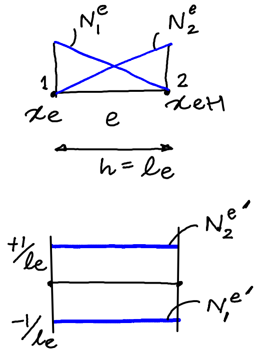
$\Rightarrow \displaystyle\int_0^l \bar{u}_2^2\underbrace{\left(\frac{\bar{u}_1}{\bar{u}_2}\right)'}_{\bar{w}}u'\,dx = 0 \quad \forall\ \bar{u}_1, \bar{u}_2$ (or $\bar{w}$)

ie u' would have to be zero. Thus $\Pi(u)$ does <u>not</u> exist.

3-Node Quadratic 1-D Finite Element

We have developed 1-D finite elements with linear polynomials.

The solution was approximated as:-



$$u(x) \approx u_e^h(x) = \sum_{a=1}^{2} N_a^e(x)\, d_a^e$$

$$= \left[N_1^e(x) \;\vdots\; N_2^e(x) \right] \begin{bmatrix} d_1^e \\ \hdashline d_2^e \end{bmatrix} = \underset{\sim}{N}\, \underline{d}$$

$$\bar{u}(x) \approx \bar{u}_e^h(x) = \underset{\sim}{N}\, \underline{\bar{d}}$$

Similarly

$$\epsilon(x) \equiv u'(x) \approx u_e'^h(x) = \left[N_1^{e'}(x) \;\vdots\; N_2^{e'}(x) \right] \begin{bmatrix} d_1^e \\ \hdashline d_2^e \end{bmatrix} = \underset{\sim}{B}\, \underline{d}$$

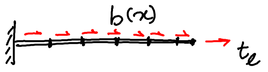
This led to:

$$\underset{\sim}{K}^e = \int_{x_e}^{x_{e+1}} \underset{\sim}{B}^T C \underset{\sim}{B}\, dx \qquad \underline{f}^e = \int_{x_e}^{x_{e+1}} \underset{\sim}{N}^T b\, dx$$

If C is constant : If b is constant :

$$\underset{\sim}{K}^e = \frac{CA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \underline{f}^e = b l_e \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Galerkin Weak Form:



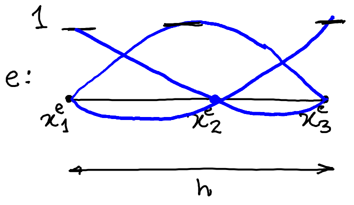
$$\tilde{G}^h(\underline{d}, \underline{\bar{d}}) = \underline{\bar{d}}^T \left(\underset{\sim}{K}\, \underline{d} - \underline{f} \right) = 0 \quad \text{for all } \underline{\bar{d}}$$

where

$$\underset{\sim}{K} = \overset{M}{\underset{e=1}{\mathsf{A}}} \underset{\sim}{K}^e \quad ; \quad \underline{f} = \overset{M}{\underset{e=1}{\mathsf{A}}} \underline{f}^e + \underline{f}^N$$

Nodal loads

One can also develop higher order approximation "shape" functions. Consider a 3node element. We can generate shape functions using Lagrange polynomials.



$$N_1^e(x) = \frac{(x - x_2^e)(x - x_3^e)}{(x_1^e - x_2^e)(x_1^e - x_{\textcircled{3}}^e)}$$

$$N_2^e(x) = \frac{(x - x_1^e)(x - x_3^e)}{(x_2^e - x_1^e)(x_2^e - x_3^e)}$$

$$N_3^e(x) = \frac{(x - x_1^e)(x - x_2^e)}{(x_3^e - x_1^e)(x_3^e - x_2^e)}$$

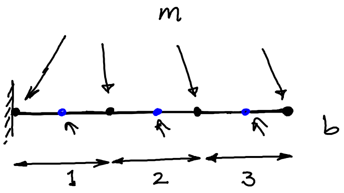
Monday, February 08, 2010
12:24 PM

Thus the approximation is

$$u(x) \approx u_e^h(x) = \underset{\sim}{N}\, \underline{d}^e = \left[N_1^e \,\vdots\, N_2^e \,\vdots\, N_3^e \right] \begin{bmatrix} d_1^e \\ d_2^e \\ d_3^e \end{bmatrix}$$

$$\bar{u}(x) \approx \bar{u}_e^h(x) = \underset{\sim}{N}\, \underline{\bar{d}}^e$$

$$\epsilon(x) \approx u_e^{h\prime}(x) = \underset{\sim}{B}\, \underline{d}^e$$



Weak form

$$G(\underline{d}, \underline{\bar{d}}) = \sum_{e=1}^{M} \left\{ \underline{\bar{d}}^{e^T} \left(\underset{\sim}{K}^e \underline{d}^e - \underline{f}^e \right) \right\} + \underline{\bar{d}}^{G^T} \underline{f}^N = 0 \quad \forall \underline{\bar{d}}$$

where

$$\underset{\substack{\sim \\ 3\times 3}}{K^e} = \int_{x_1^e}^{x_3^e} \underset{\sim}{B}^T C \underset{\sim}{B}\, dx \quad ; \quad \underline{f}^e = \int_{x_1^e}^{x_3^e} \underset{\sim}{N}^T b\, dx$$

Thus

$$G(\underline{d}, \underline{\bar{d}}) = [\bar{d}_1^1 \;\; \bar{d}_2^1 \;\; \bar{d}_3^1] \left(\begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 \end{bmatrix} \begin{bmatrix} d_1^1 \\ d_2^1 \\ d_3^1 \end{bmatrix} - \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \end{bmatrix} \right) +$$

$$[\bar{d}_1^2 \;\; \bar{d}_2^2 \;\; \bar{d}_3^2] \left(\begin{bmatrix} K_{11}^2 & K_{12}^2 & K_{13}^2 \\ K_{21}^2 & K_{22}^2 & K_{23}^2 \\ K_{31}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix} \begin{bmatrix} d_1^2 \\ d_2^2 \\ d_3^2 \end{bmatrix} - \begin{bmatrix} f_1 \\ f_2^2 \\ f_3 \end{bmatrix} \right) +$$

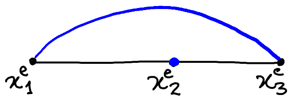
$$\vdots \qquad\qquad\qquad \vdots$$

$$[\bar{d}_1^M \;\; \bar{d}_2^M \;\; \bar{d}_3^M] \left(\begin{bmatrix} K_{11}^M & K_{12}^M & K_{13}^M \\ K_{21}^M & K_{22}^M & K_{23}^M \\ K_{31}^M & K_{32}^M & K_{33}^M \end{bmatrix} \begin{bmatrix} d_1^M \\ d_2^M \\ d_3^M \end{bmatrix} - \begin{bmatrix} f_1^M \\ f_2^M \\ f_3^M \end{bmatrix} \right) + \underline{\bar{d}}^T \underline{f}^N = 0$$

Thus for each element :

$$[K_{21}^e \;\; K_{22}^e \;\; K_{23}^e] \begin{bmatrix} d_1^e \\ d_2^e \\ d_3^e \end{bmatrix} \boxed{=} \{ f_2^e \}$$

only for bubbles



"Bubble" functions

$$\Rightarrow \quad d_2^e = K_{22}^{e^{-1}} \left\{ f_2^e - [K_{21}^e \;\; K_{23}^e] \begin{bmatrix} d_1^e \\ d_3^e \end{bmatrix} \right\}$$

$$\begin{bmatrix} K_{mm}^e & K_{mb}^e \\ K_{bm}^e & K_{bb}^e \end{bmatrix} \begin{bmatrix} d_m^e \\ d_b^e \end{bmatrix} - \begin{bmatrix} f_m^e \\ f_b^e \end{bmatrix}$$

$$d_b^e = K_{bb}^{-1} \left(f_b^e - K_{bm}^e d_m^e \right)$$

Substitute in the remaining equations :
(for each element)

$$\begin{bmatrix} K_{11}^e & K_{13}^e \\ K_{31}^e & K_{33}^e \end{bmatrix} \begin{Bmatrix} d_1^e \\ d_3^e \end{Bmatrix} \ominus \begin{bmatrix} K_{12}^e \\ K_{32}^e \end{bmatrix} [K_{22}^e]^{-1} [K_{21} \;\; K_{23}] \begin{bmatrix} d_1^e \\ d_{\textcircled{3}}^e \end{bmatrix}$$

In general

$$\bar{d}_m^e \left\{ \underbrace{\left(K_{mm}^e - \overbrace{K_{mb}^e K_{bb}^{e^{-1}} K_{bm}^e}^{\to \text{ Schur Complement of bubble}} \right)}_{[\tilde{K}_{mm}^e]} d_m^e - \underbrace{\left(f_m^b - K_{mb}^e K_{bb}^{e^{-1}} f_b^e \right)}_{\{\tilde{f}_m^e\}} \right\}$$

Now the "bubble" degrees of freedom have been eliminated by <u>static</u> <u>condensation</u>, we can assemble the Global equations as before.

Thus $$\tilde{G}^h(\underline{d}, \underline{\bar{d}}) = \underline{\bar{d}}_m^T \left(\underset{\sim}{\tilde{K}}_{mm}^G \underline{d}_m - \underline{\tilde{f}}_m \right)$$

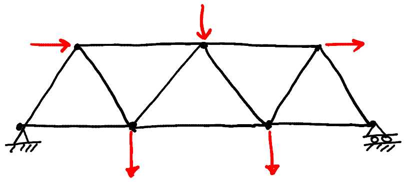
Note the Total dofs of global problem has <u>not</u> changed!

1-D Application - Trusses

{Ref : Reddy § 4.6 }

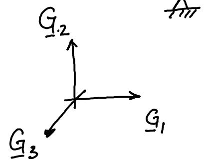
These 1-D finite elements can also be applied to 2D & 3D structures whose individual structural components (elements) behave as 1-D finite elements.

Example : Trusses :



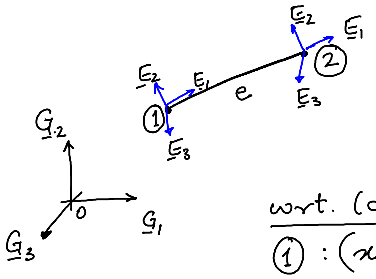
Note :

- All connections are "pins" (2D)
  "ball-socket" (3D)
- body force "b" ≈ Nodal loads $\underset{\sim}{f}^N$ only.



2D : planar trusses
3D : space trusses

Consider 1 truss member :



w.r.t. (o, G_1, G_2, G_3)

① : (x_1, y_1, z_1)
② : (x_2, y_2, z_2)

Note :

- The global co-ordinates of the nodes ① & ② fix the direction E_1
- In 2D E_2 is $\perp$ to E_1
- In 3D E_2, E_3 can be any 2 mutually perpendicular directions
  So, one must define E_2 as the "orientation" of the element in 3D
  Then $E_3 = E_1 \times E_2$

Now w.r.t $\{E_1\}$ we found that $\underset{\sim}{K}^e_{\{E_1\}} = \frac{CA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

So w.r.t $\{E_1, E_2\}$ in 2D :

$$\underset{\sim}{K}^e_{\{E_1 E_2\}} = \frac{CA}{l_e}\begin{array}{c} \begin{array}{cccc} ①_x & ①_y & ②_x & ②_y \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

w.r.t $\{E_1\, E_2\, E_3\}$ in 3D :

$$\underset{\sim}{K}^e_{\{E_1 E_2 E_3\}} = \frac{CA}{l_e}\begin{array}{c} \begin{array}{cccccc} ①_x & ①_y & ①_z & ②_x & ②_y & ②_z \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

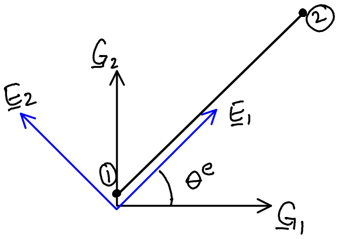
Monday, February 08, 2010
9:30 AM

Note that the coordinate axes are related as:

Let $\underline{v}$ be a vector:

$$\underline{v} = v_i^G \underline{G}_i = v_i^e \underline{E}_i \quad \text{and} \quad \left\{ v_i^e \right\} = \left[Q_{ij} \right] \left\{ v_j^G \right\} \quad \text{where} \quad Q_{ij} = \left(\underline{G}_j \cdot \underline{E}_i \right)$$

Thus the displacement at a node:



$$\begin{Bmatrix} d_x^e \\ d_y^e \\ d_z^e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 \\ -\sin\theta_e & \cos\theta_e & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_x^G \\ d_y^G \\ d_z^G \end{Bmatrix}$$

In general

$$\begin{Bmatrix} \underline{d}_1^e \\ \underline{d}_2^e \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{Q}^e & 0 \\ 0 & \underset{\sim}{Q}^e \end{bmatrix} \begin{Bmatrix} \underline{d}_1^G \\ \underline{d}_2^G \end{Bmatrix} \qquad \text{ie} \qquad \boxed{\underline{d}^e = \underset{\sim}{T}^e \underline{d}^G}$$

$\underset{\sim}{Q}$: Orthogonal rotation matrix. $\left(\underset{\sim}{Q} \underset{\sim}{Q}^T = \underset{\sim}{Q}^T \underset{\sim}{Q} = \underset{\sim}{I} \right)$

Substitute in the weak form (for trusses):

$$\tilde{G}^h(\underline{d}, \bar{\underline{d}}) = \sum_{e=1}^{M} \bar{\underline{d}}^{e^T} \left(\underset{\sim}{K}^e \underline{d}^e - \underline{f}^e \right) + \bar{\underline{d}}^{G^T} \underline{f}^N$$

Before we can "Assemble" the equations, we need to convert the element dofs to global dofs.

ie

$$\tilde{G}^h(\underline{d}, \bar{\underline{d}}) = \sum_{e=1}^{M} \underbrace{\bar{\underline{d}}^{G^T} \begin{bmatrix} \underset{\sim}{Q}^{e^T} & 0 \\ 0 & \underset{\sim}{Q}^{e^T} \end{bmatrix}}_{\bar{\underline{d}}^{e^T}} \underset{\sim}{K}^e \underbrace{\begin{bmatrix} \underset{\sim}{Q}^e & 0 \\ 0 & \underset{\sim}{Q}^e \end{bmatrix} \underline{d}^G}_{\underline{d}^e} + \bar{\underline{d}}^{G^T} \underline{f}^N$$

$$\tilde{G}^h(\underline{d}, \bar{\underline{d}}) = \bar{\underline{d}}^{G^T} \left(\underset{\sim}{K}^G \underline{d}^G - \underline{f}^N \right) = 0 \qquad \text{for all } \bar{\underline{d}}$$

where

$$\boxed{\underset{\sim}{K}^G = \mathop{\mathsf{A}}_{e=1}^{M} \left\{ \underset{\sim}{T}^{e^T} \underset{\sim}{K}^e \underset{\sim}{T}^e \right\}}$$