CE-595 : FINITE ELEMENTS IN ELASTICITY

Lec 1

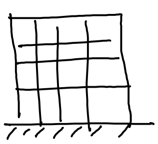
- Introduction
- FE History
- Application
- Vectors and Tensors

Ref: Z&T vol 1, Ch 1
Reddy Ch 1

- Objectives
  - Formulate the problem mathematically & derive FE equations
  - Use, Modify existing FE software
  - Read & Understand some basic literature and be able to reproduce the result.

- What is FEM
  - One of the best tools for analysis & design of common physical systems (approximate solution)

Caveat: The computed solution always has some errors.



Real-life System

→ Physical Errors : Material Behavior
Validation through Experiments

Solid / Structural Mechanics — Mathematical Model

→ Truncation Error
Verification through "Exact" solutions of the Math. Model.

FE-Mesh — Discrete Numerical Model

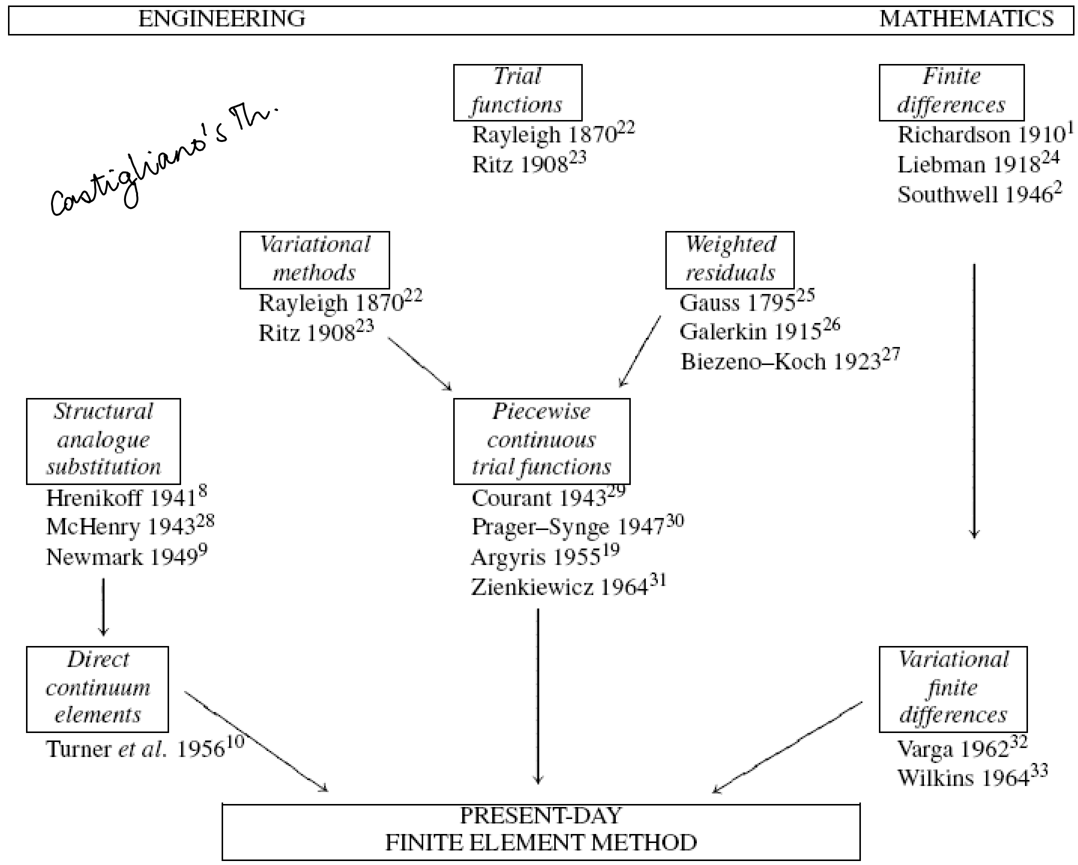
→ Roundoff Error

Approx Soln — Computer Implementation

$\boxed{\text{User Error}}$

- FE History

Table 1.1 History of approximate methods



- "Finite Element" : coined R. Clough (1960)

- Patch Test, iso-parametric FE : B. Irons

Lec 2

Applications of FEM

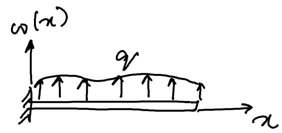
- Structural Eng. (Buildings, Bridges, Dams etc)
- Geotech. Eng. (Foundations, Tunnels etc.)
- Hydraulics & Hydrology (Ground water, sub-surface etc.)
- Mechanical Eng. (Automobiles, Aircraft, SpaceCrafts, Marine Vehicles...)
- Electrical Eng. (MEMS, Circuits etc.)

In general, FEM is a tool to solve different types of PDEs.

- Types of PDEs:
  - Hyperbolic : Wave equation (usually based on some conservation Laws)
  - Parabolic : Transient Heat conduction (usually dissipative)
  - Elliptic : Static equilibrium of solids/structures.

Examples:

- Structural Mechanics:
  - Beams:
    - Bernoulli-Euler: $(EI w'')'' = q$



    - Timoshenko: (shear)

$$\begin{aligned} M' + Q + m &= 0 \\ Q' + q &= 0 \\ N' + n &= 0 \end{aligned} \quad \Bigg| \quad \begin{aligned} M &= EI\theta' \\ Q &= GA(w' - \theta) \\ N &= EAu' \end{aligned}$$

  - Plates:
    - Kirchhoff-Love (Thin):

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = q$$

    - Reissner-Mindlin: (Thick)

$$\underbrace{\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}}_{\underset{\sim}{L}} \underbrace{\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}}_{\underline{M}} + \underbrace{\begin{Bmatrix} S_x \\ S_y \end{Bmatrix}}_{\underline{S}} = 0 \quad \Bigg| \quad \underline{M} = \underset{\sim}{D}\,\underset{\sim}{L}\,\underline{\theta}$$

$$\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} , & \frac{\partial}{\partial y} \end{bmatrix}}_{\underline{\nabla}} \underbrace{\begin{Bmatrix} S_x \\ S_y \end{Bmatrix}}_{\underline{S}} + q = 0 \quad \Bigg| \quad \underline{S} = \kappa G t \underset{\sim}{I} (\underline{\nabla} w - \underline{\theta})$$

- Continuum Mechanics (Solid / Fluid) :

  - Mass : $\frac{d\rho}{dt} + \rho \underline{\nabla} \cdot \underline{v} = 0$

  - Momentum : $\underline{\nabla} \cdot \underset{\sim}{\sigma} + \rho \underline{b} = \rho \frac{d\underline{v}}{dt}$

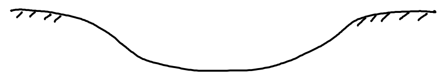
  - Energy : $\frac{\partial(\rho E)}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i H) - \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) + \frac{\partial}{\partial x_i}(\tau_{ij} u_j) - \rho f_i u_i - q_H = 0$

- Heat Conduction : $\underline{\nabla}\left(\underset{\sim}{k}(\underline{\nabla} T)\right) + \underline{f} = \rho c \frac{\partial T}{\partial t}$

- Electromagnetics : - - - -

FEM in Structural Analysis & Design

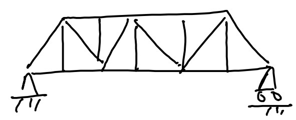
Consider a project: say Bridge



- Hydraulic Req.
- Type of Bridge. →
  - . Beam bridge

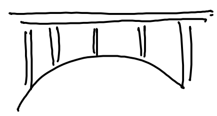
- EQ info
  - . Suspension

- Loading. ⊛
  - . Truss

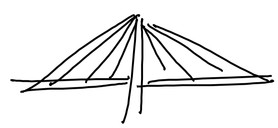


- Design Philosophy
  - assume a solution & check

. Arch



. Cable stayed



⊛ - Traffic - LL
- DL
- EQ, Wind, Snow, Temperature
- Stream / Ice
- Impact Blast *
- Fatigue.

→ ~ 50 years

Design
- ASD - Allowable Stress Design
- LRFD - Load & Resistance Factor design

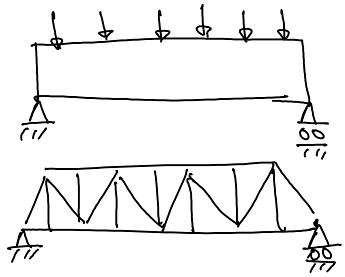
$$\gamma L < \phi R$$

γ ↑ Load Safety Factor

ϕ ↖ Strength reduction factor

Analysis Problem

- Beam
- Truss



Analyze with Force Method

→ Refine your design with a displacement-based method

— FEM

~ Design Parameters (Optimization)

- material + cut cost
- performance-based design

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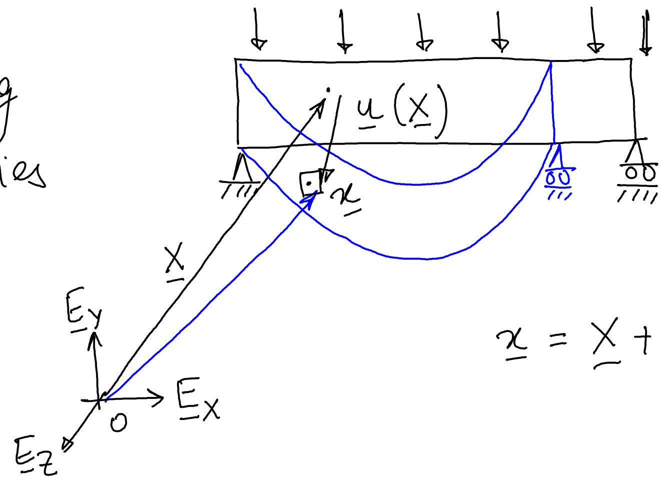
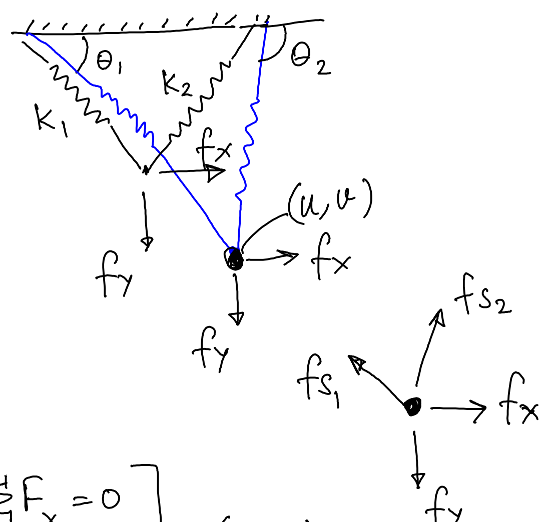
Mathematically Formulate the "Analysis" Problem

- Given
  - geometry, loading
    material properties
- Find
  $\underline{u}(\underline{X})$



$$\underline{x} = \underline{X} + \underline{u}(\underline{X})$$

Aside:



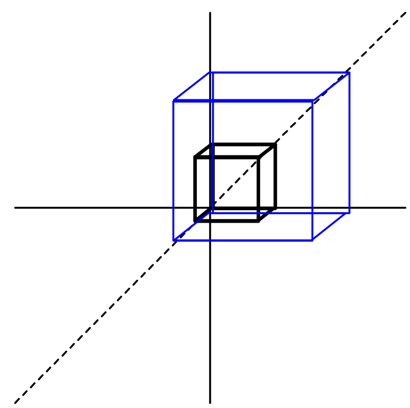
$$\left.\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \end{aligned}\right] \Rightarrow (u, v)$$

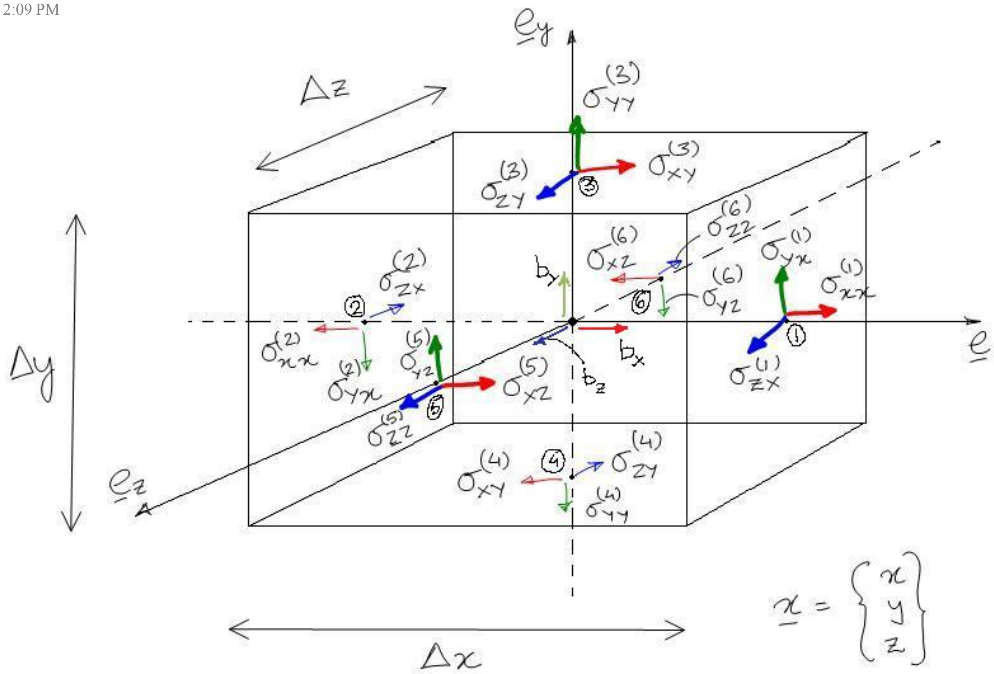
$$f_{s_1} = k_1 \Delta_1(u, v)$$

$$f_{s_2} = k_2 \Delta_2(u, v)$$

Example:

$$\underline{u}(\underline{X}) = \begin{Bmatrix} u_x(X, Y, Z) \\ u_y(X, Y, Z) \\ u_z(X, Y, Z) \end{Bmatrix} = \underline{X}$$





$$\underset{\sim}{x} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

① : ⊕ x-face
② : ⊖ x-face

③ : ⊕
④ : ⊖ y faces

⑤ : ⊕
⑥ : ⊖ z-faces

Note from the Cauchy relationship :

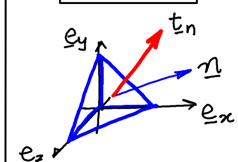
$$\underbrace{\begin{Bmatrix} t_{n_{1x}} \\ t_{n_{1y}} \\ t_{n_{1z}} \end{Bmatrix}}_{\underset{\sim}{t}_{n_1}} = \underbrace{\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}}_{\boxed{\underset{\sim}{\sigma}\ \text{Cauchy Stress}}} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\underset{\sim}{n}_1} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yx} \\ \sigma_{zx} \end{Bmatrix}$$

(traction at point ①)

$\underset{\sim}{n}_1$ ↖ unit normal to face ① at point ①

(Ref: Hjelmstad Ch 3)

$$\boxed{\underset{\sim}{t}_n = \underset{\sim}{\sigma}\,\underset{\sim}{n}}$$



Also Note :

$$\sigma_{xx}^{(1)} = \sigma_{xx}^{(2)} + \frac{\partial \sigma_{xx}}{\partial x} \cdot \Delta x + \cdots$$ (going from ② to ① only change in Δx)

$$\sigma_{xy}^{(3)} = \sigma_{xy}^{(4)} + \frac{\partial \sigma_{xy}}{\partial y} \Delta y + \cdots$$ (going from ③ to ④ only change in Δy)

$$\sigma_{xz}^{(5)} = \sigma_{xz}^{(6)} + \frac{\partial \sigma_{xz}}{\partial z} \Delta z + \cdots$$ (going from ⑤ to ⑥ only change in Δz)

Finally

$$\sum F_x = 0$$

$$\Rightarrow \sigma_{xx}^{(1)} \Delta y \Delta z - \sigma_{xx}^{(2)} \Delta y \Delta z$$

$$+ \sigma_{xy}^{(3)} \Delta x \Delta z - \sigma_{xy}^{(4)} \Delta x \Delta z$$

$$+ \sigma_{xz}^{(5)} \Delta x \Delta y - \sigma_{xz}^{(6)} \Delta x \Delta y + b_x (\Delta x \Delta y \Delta z) = 0$$

$$(\text{or } \rho \ddot{u}_x \Delta x \Delta y \Delta z)$$

Thus

$\sum F_x = 0$

$$\Rightarrow \quad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0 \quad \text{or } \rho \ddot{u}_x$$

$\sum F_y = 0$

$$\Rightarrow \quad \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial y} + b_y = 0 \quad \text{or } \rho \ddot{u}_y$$

$\sum F_z = 0$

$$\Rightarrow \quad \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \quad \text{or } \rho \ddot{u}_z$$

In matrix form, this may be expressed as:

$$\left\{ \left\{ \partial/\partial x \quad \partial/\partial y \quad \partial/\partial z \right\} \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \right\}^T + \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} = \rho \begin{Bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_z \end{Bmatrix}$$

In Indicial notation:

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \ddot{u}_i$$

or simply as $\boxed{\sigma_{ij,j} + b_i = \rho \ddot{u}_i}$ *

* with summation implied by repeated indices "j"
* derivative $\partial/\partial x_j$ expressed as $,j$

In vector / tensor co-ordinate independent form:

$$\boxed{\operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \rho \underline{\ddot{u}}}$$

$\sum \underline{M} = 0$

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$$

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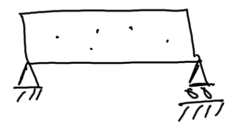
Lec 3.

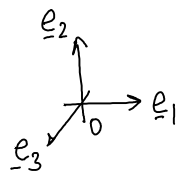
Vectors & Tensors

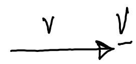
(Ref. Hjelmstad Ch1)

- Scalar : one number (magnitude) eg. Temp (T)
  - Scalar field: $T(\underset{\sim}{X})$



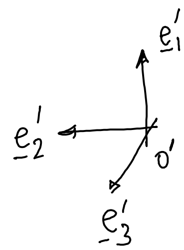
- Vector : magnitude + direction eg. velocity
  - Vector field $\underset{\sim}{v}(\underset{\sim}{X})$





$$\underset{\sim}{V} = V\underset{\sim}{e}_1 = \begin{Bmatrix} V \\ 0 \\ 0 \end{Bmatrix}_{(0,\, \underset{\sim}{e}_1\, \underset{\sim}{e}_2\, \underset{\sim}{e}_3)}$$

3×1

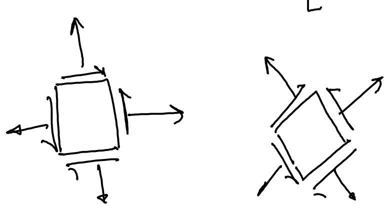


$$\underset{\sim}{V} = -V\underset{\sim}{e}_2' = \begin{Bmatrix} 0 \\ -V \\ 0 \end{Bmatrix}_{(0',\, \underset{\sim}{e}_1',\, \underset{\sim}{e}_2',\, \underset{\sim}{e}_3')}$$

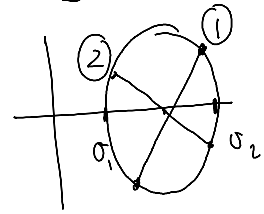
- Vector space : a set of all possible vectors
  eg $\mathbb{R}^3$

- Tensor : Operates on vector $\longrightarrow$ vector
  Linear map from one vector space to another
  Example: $\underset{\sim}{\sigma}$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

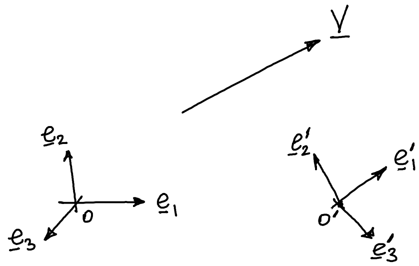


① ②



- Co-ordinate Transformation

$$\underline{V} = v_i \underline{e}_i = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3$$



ie $\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{(o, \underline{e}_1, \underline{e}_2, \underline{e}_3)}$

Also

$$\underline{V} = v'_i \underline{e}'_i = v'_1 \underline{e}'_1 + v'_2 \underline{e}'_2 + v'_3 \underline{e}'_3$$

i.e. $\begin{Bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{Bmatrix}_{(o', \underline{e}'_1, \underline{e}'_2, \underline{e}'_3)}$

To find v'_i :

$$v'_i \underbrace{(\underline{e}'_i \cdot \underline{e}'_j)}_{\delta_{ij}} = v_i \underbrace{(\underline{e}_i \cdot \underline{e}'_j)}_{Q_{ji}}$$

$$\delta_{ij} = \begin{cases} 0 & : i \neq j \\ 1 & : i = j \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $\boxed{v'_j = Q_{ji} \; v_i}$

In matrix form

$$\begin{Bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

Vector products:

$$\underline{u} = u_i \underline{e}_i \quad ; \quad \underline{v} = v_j \underline{e}_j$$

- Dot product :

$$\underline{u} \cdot \underline{v} = u_i v_i = \underline{u}^T \underline{v}$$
$$= (u_1 v_1 + u_2 v_2 + u_3 v_3) = \{u_1 \; u_2 \; u_3\} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

- Cross Product :

$$\underline{u} \times \underline{v} = \underbrace{(\epsilon_{ijk} \; u_i \; v_j)}_{\omega_k} \underline{e}_k = \omega_k \underline{e}_k$$

$$\underline{\omega} = \underline{u} \times \underline{v} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- <u>Tensor Product</u> $\otimes$:

$$\underset{\sim}{T} = \underline{u} \otimes \underline{v}$$

$$= u_i v_j \left(\underline{e}_i \otimes \underline{e}_j\right) \rightsquigarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}}_{T_{ij}}$$

$$\underset{\sim}{T} = T_{ij} \left(\underline{e}_i \otimes \underline{e}_j\right)$$

Property :

$$\underset{\sim}{T}\, \underline{x} = \left(\underline{u} \otimes \underline{v}\right) \underline{x} \rightsquigarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \left(\underline{v} \cdot \underline{x}\right) \underline{u}$$

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Kinematics of Deformation

(Ref: Hjelmstad Ch 2)

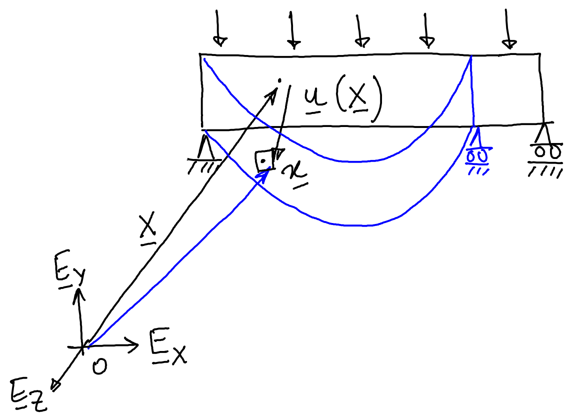
Given: geometry, load, material properties

Find: $\underset{\sim}{u}(\underset{\sim}{X})$ that satisfies

$$\operatorname{div} \underset{\sim}{\sigma} + \underset{\sim}{b} = \underset{\sim}{0}$$

$\Rightarrow$ We need

- $\underset{\sim}{\sigma}(\underset{\sim}{\epsilon}) \leftarrow$ Material Model
- $\underset{\sim}{\epsilon}(\underset{\sim}{u}) \leftarrow$ Strain-displacement (Deformation)



Recall:

$$\underset{\sim}{x} = \underset{\sim}{X} + \underset{\sim}{u}(\underset{\sim}{X})$$

ie $\boxed{\underset{\sim}{x} = \underset{\sim}{\phi}(\underset{\sim}{X})}$ Deformation Map

Strains are related to changes in the deformation.

Define: Deformation Gradient: $\underset{\sim}{F}$ (tensor)

$$\underset{\sim}{F} = \frac{d\underset{\sim}{x}}{d\underset{\sim}{X}} = \nabla_{\underset{\sim}{X}} \underset{\sim}{x} = \frac{\partial x_i}{\partial X_I} \, \underset{\sim}{e}_i \otimes \underset{\sim}{E}_I$$

(Also $\underset{\sim}{F} = \underset{\sim}{I} + \nabla \underset{\sim}{u}$)

ie. $\underset{\sim}{F} = F_{iI} \, \underset{\sim}{e}_i \otimes \underset{\sim}{E}_I \rightsquigarrow \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix}$

Interpret as: $\boxed{\Delta \underset{\sim}{x} = \underset{\sim}{F}(\Delta \underset{\sim}{X})}$

ie $\underset{\sim}{F}$ acts on $\Delta \underset{\sim}{X}$ (undeformed) $\longrightarrow$ $\Delta \underset{\sim}{x}$ (deformed)

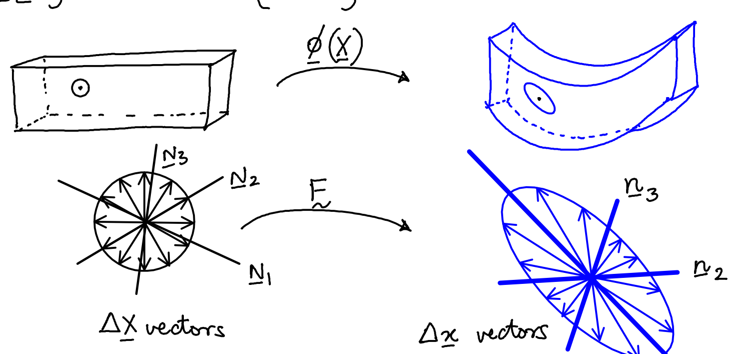
$$\begin{bmatrix} F_{xX} & F_{xY} & F_{xZ} \\ F_{yX} & F_{yY} & F_{yZ} \\ F_{zX} & F_{zY} & F_{zZ} \end{bmatrix} \begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix} \longrightarrow \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix}$$

$$\underset{\sim}{F} \underset{\sim}{N}_i = \lambda \underset{\sim}{n}_i$$

$$\underset{\sim}{F} = \sum_{i=1}^{3} \lambda_i \left(\underset{\sim}{n}_i \otimes \underset{\sim}{N}_i \right)$$

(Eigenvalues of $\cancel{[\underset{\sim}{F}]}$) $\underset{\sim}{U}$

(Spectral decomposition)



$$\boxed{\Delta \underset{\sim}{x} = \underset{\sim}{F} \, \Delta \underset{\sim}{X}}$$

Strain

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$\underset{\sim}{F}$ has rotation + stretching (Polar decomposition)

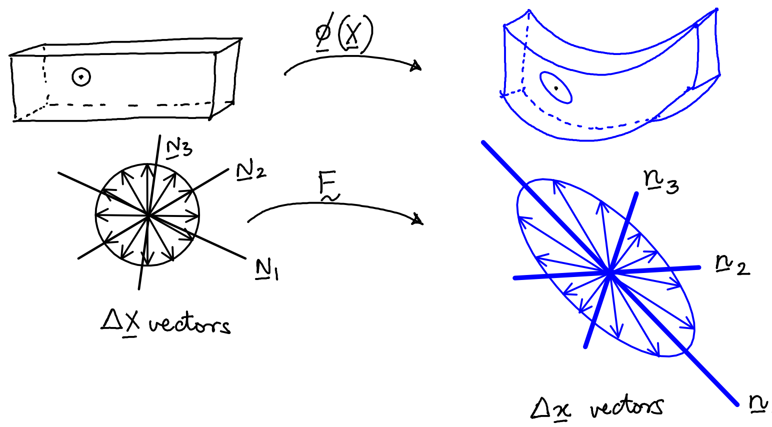
$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} = \underset{\sim}{V}\,\underset{\sim}{R}$

pure rotation — symmetric

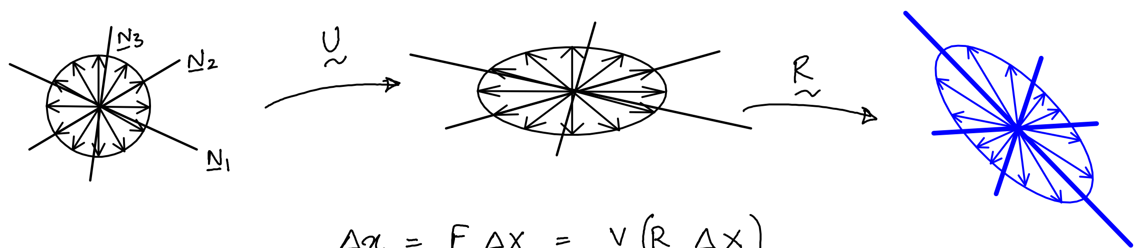
$\underset{\sim}{F}\,\underline{N}_i = \lambda\,\underline{n}_i$

$\underset{\sim}{F} = \sum_{i=1}^{3} \lambda_i \left(\underline{n}_i \otimes \underline{N}_i\right)$

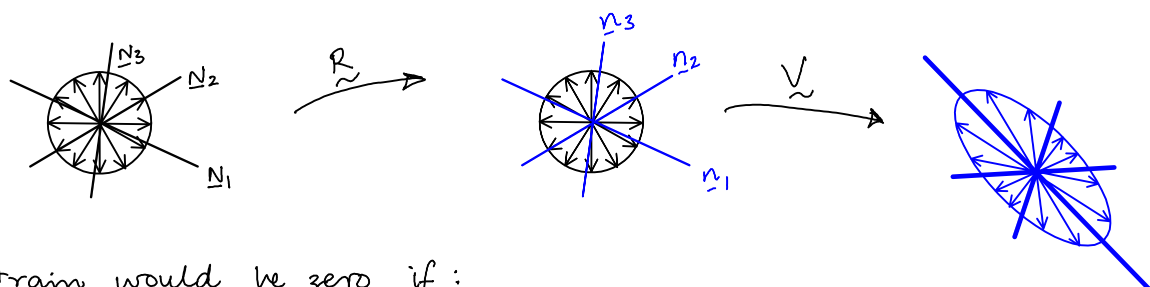
(Spectral decomposition)



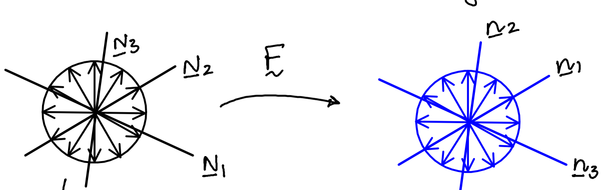
$$\Delta\underline{x} = \underset{\sim}{F}\,\Delta\underline{X} = \underset{\sim}{R}\left(\underset{\sim}{U}\,\Delta\underline{X}\right)$$



$$\Delta\underline{x} = \underset{\sim}{F}\,\Delta\underline{X} = \underset{\sim}{V}\left(\underset{\sim}{R}\,\Delta\underline{X}\right)$$



- Strain would be zero if :



$\Delta\underline{X}\cdot\Delta\underline{X} = \epsilon^2$ $\qquad$ $\Delta\underline{x}\cdot\Delta\underline{x} = \epsilon^2$

$$\Delta\underline{X}\cdot\Delta\underline{X} = \left(\underset{\sim}{F}\,\Delta\underline{X}\right)\cdot\left(\underset{\sim}{F}\,\Delta\underline{X}\right) = \Delta\underline{X}\cdot\left(\underset{\sim}{F}^T\underset{\sim}{F}\right)\Delta\underline{X}$$

ie $\boxed{\Delta\underline{X}\cdot\left(\underset{\sim}{F}^T\underset{\sim}{F} - \underset{\sim}{I}\right)\Delta\underline{X} = 0}$

- <u>Cauchy-Green</u> <u>Deformation</u> <u>Tensor</u> $\boxed{\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}}$

$$= (\boldsymbol{R}\boldsymbol{U})^T (\boldsymbol{R}\boldsymbol{U})$$

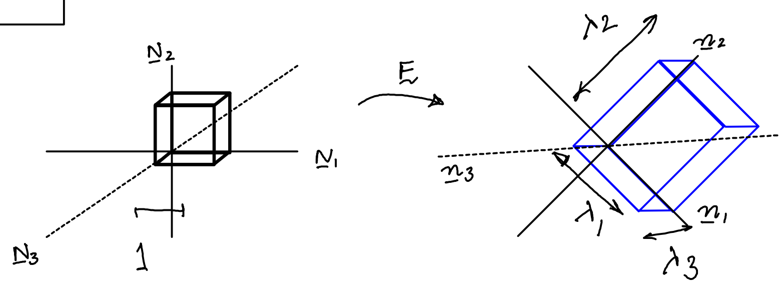
$$\boxed{\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \left(\boldsymbol{N}_i \otimes \boldsymbol{N}_i\right)}$$

$$\boldsymbol{C} = \boldsymbol{U}^T \underbrace{(\boldsymbol{R}^T \boldsymbol{R})}_{\boldsymbol{I}} \boldsymbol{U} = \boldsymbol{U}^2$$

$$\boxed{\boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \left(\boldsymbol{N}_i \otimes \boldsymbol{N}_i\right)}$$

Properties of $\boldsymbol{C}$:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{SYM} & & C_{33} \end{bmatrix}$$



In terms of <u>invariants</u>

$$I_C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = tr(\boldsymbol{C}) = C_{ii}$$

$$II_C = (\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2 + (\lambda_3\lambda_1)^2 = \tfrac{1}{2}\left((tr\,\boldsymbol{C})^2 - tr(\boldsymbol{C}^2)\right) = \tfrac{1}{2}\left(C_{ii}C_{jj} - C_{ij}C_{ji}\right)$$

$$III_C = (\lambda_1\lambda_2\lambda_3)^2 = det(\boldsymbol{C}) = \tfrac{1}{6}\,\epsilon_{ijk}\,\epsilon_{lmn}\,C_{il}\,C_{jm}\,C_{kn}$$

$$\epsilon_{ijk} = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$$

$i \rightarrow j \rightarrow k \rightarrow i$

- Lagrangian - <u>Strain</u> Tensor

$$\boxed{\boldsymbol{E} = \tfrac{1}{2}\left(\boldsymbol{C} - \boldsymbol{I}\right)}$$

$$= \tfrac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) = \tfrac{1}{2}\left((\boldsymbol{I} + \nabla\boldsymbol{u})^T(\boldsymbol{I} + \nabla\boldsymbol{u}) - \boldsymbol{I}\right)$$

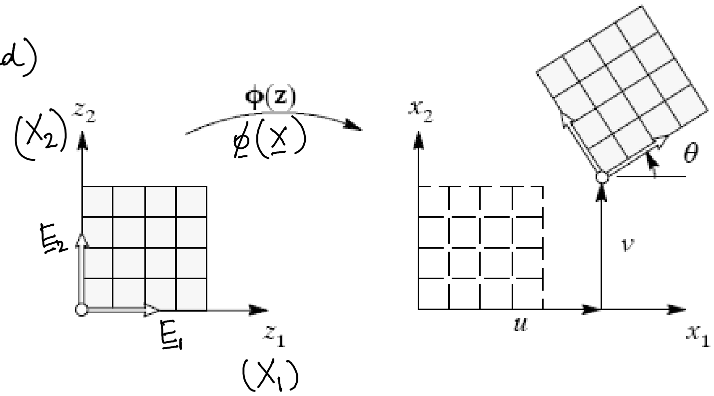
$$\Rightarrow \boxed{\boldsymbol{E} = \tfrac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T + \underbrace{\nabla\boldsymbol{u}^T\,\nabla\boldsymbol{u}}_{\text{Non-linear}}\right)}$$

- Linearized Strain Tensor:

$$\boxed{\boldsymbol{\epsilon} = \tfrac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)}$$

Example

(Ref : Pg 76, Hjelmstad)



$$\underline{x} = \underline{\phi}(\underline{X}) = \underbrace{\left(u + X_1 \cos\theta - X_2 \sin\theta\right)}_{x_1} \underline{E}_1 + \underbrace{\left(v + X_1 \sin\theta + X_2 \cos\theta\right)}_{x_2} \underline{E}_2 + \underbrace{X_3}_{x_3} \underline{E}_3$$

$$\underset{\sim}{F} \rightsquigarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{E} = \underset{\sim}{0}$$

Linearized:

$$\underset{\sim}{F} \rightsquigarrow \begin{bmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} \rightsquigarrow \begin{bmatrix} 1+\theta^2 & 0 & 0 \\ 0 & 1+\theta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underset{\sim}{E} \neq \underset{\sim}{0}$! (but $\underset{\sim}{\epsilon} = \underset{\sim}{0}$)

Compatibility of Strains

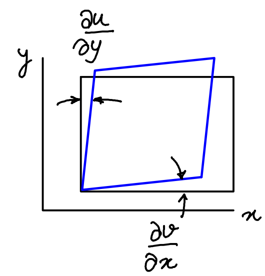
(Linearized / Small strain)

Meaning of compatibility $\underset{\sim}{\epsilon}$

Given $\underset{-}{\phi}(\underset{-}{X}) \rightarrow \underset{\sim}{\epsilon}(\underset{-}{u})$ Automatically satisfied.

$\boxed{?} \leftarrow \underset{\sim}{\epsilon}$

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$



$$\epsilon_{xx} = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial^2 \epsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y}$$

$$\gamma_{xy} = 2\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}}$$

Similarly 2 more equations

In addition:

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \Rightarrow \frac{\partial \epsilon_{xy}}{\partial z} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial x \partial z}\right) \quad \oplus$$

Similarly:

$$\frac{\partial \epsilon_{yz}}{\partial x} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial z \partial x}\right) \quad \ominus$$

$$\frac{\partial \epsilon_{xz}}{\partial y} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 w}{\partial x \partial y}\right) \quad \oplus$$

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial \epsilon_{xy}}{\partial z} - \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{xz}}{\partial y}\right) = \frac{\partial^2 \epsilon_{xx}}{\partial y \partial z}}$$

Similarly 2 more equations

$\Rightarrow$ Total 6 equations of compatibility.

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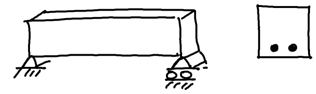
Material Behavior : Constitutive Models

(Ref. Hjelmstad Ch 4)

Need for constitutive equations : (At every point)



- Indeterminacy

\# of Scalar Equations

$$\operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \underline{0} \qquad 3$$

$$\underset{\sim}{\epsilon} = \tfrac{1}{2}\left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T\right) \qquad \underline{6}$$

(or $\underset{\sim}{E}$) $\qquad \boxed{9} \quad + 6\,?$

Unknowns

$$\underbrace{\underline{\phi}(\underline{X}) \text{ or } \underline{u}(\underline{X})}_{3} \;;\; \underset{\sim}{\epsilon} \;;\; \underset{\sim}{\sigma} \qquad 3 + 6 + 6 = \boxed{15}$$

$\underset{\sim}{\sigma}(\underset{\sim}{\epsilon})$

$\underset{\sim}{\sigma} = C\,\underset{\sim}{\epsilon}\,?$

General Principles for Material models

- Deterministic
- Local action
- Observer Objectivity (Material Frame Indifference)
- Laws of Thermodynamics

Classes of Material Models

- Elasticity
  - Hyper-elasticity (Rate independent) $\qquad \underset{\sim}{\sigma} = \underset{\approx}{C}\,\underset{\sim}{\epsilon}$ (4th order Tensor)
  - Hypo-elasticity $\qquad \dot{\underset{\sim}{\sigma}} = \underset{\approx}{C}\,\dot{\underset{\sim}{\epsilon}}$
  - Visco-elasticity
- Inelasticity
  - Rate independent Plasticity
  - Damage Plasticity
  - Viscoplasticity

Hyper-Elasticity:

If the potential energy stored at every material point can be expressed as a scalar function :

Strain energy density : $\psi(\underline{X}, \underset{\sim}{\epsilon}(\underline{X}))$

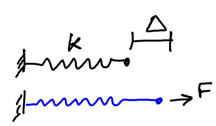
eg.

$$\psi(\underline{X}, \underset{\sim}{\epsilon}(\underline{X})) \equiv \tfrac{1}{2}\,\underset{\sim}{\epsilon} : (\underset{\approx}{C}\,\underset{\sim}{\epsilon})$$

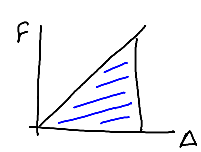
$$= \tfrac{1}{2}\,\epsilon_{ij}\, C_{ijkl}\, \epsilon_{kl}$$

$$\boxed{\underset{\sim}{\sigma} = \frac{\partial \psi}{\partial \underset{\sim}{\epsilon}}} = \frac{\partial \psi}{\partial \epsilon_{ij}} (\underline{e}_i \otimes \underline{e}_j)$$

Recall: Linear spring



$\psi = \tfrac{1}{2} k \Delta^2$



For Finite deformations:

$$\bar{\psi}(\underset{\sim}{F}) \quad \Rightarrow \quad \underset{\sim}{P} = \frac{\partial \bar{\psi}}{\partial \underset{\sim}{F}}$$

$$\hat{\psi}(\underset{\sim}{C}) = \tilde{\psi}(\underset{\sim}{E}) \quad \Rightarrow \quad \underset{\sim}{S} = 2\frac{\partial \hat{\psi}}{\partial \underset{\sim}{C}} = \frac{\partial \tilde{\psi}}{\partial \underset{\sim}{E}}$$

$$\begin{cases} \underset{\sim}{P} = J\,\underset{\sim}{\sigma}\,\underset{\sim}{F}^{-T} \\ \underset{\sim}{S} = J\,\underset{\sim}{F}^{-1}\,\underset{\sim}{\sigma}\,F^{-T} \end{cases}$$

where $J = \det(\underset{\sim}{F})$

For isotropic

Usually the strain energy function is written in terms of invariants.

i.e. $\hat{\psi}(I_c, II_c, III_c)$

eg. · Mooney - Rivlin

$$\hat{\psi}(I_c, II_c) = a(I_c - 3) + b(II_c - 3)$$

- Isotropy

If $\hat{\psi}(I_c, II_c, III_c) = \hat{\hat{\psi}}(\lambda_1, \lambda_2, \lambda_3)$

then isotropy

$$\Rightarrow \hat{\hat{\psi}}(\lambda_1, \lambda_2, \lambda_3) = \hat{\hat{\psi}}(\lambda_2, \lambda_1, \lambda_3) = \hat{\hat{\psi}}(\lambda_3, \lambda_1, \lambda_2) \ \ldots$$

For small strain

- Hooke's Model : $\boxed{\underset{\sim}{\sigma} = \lambda \, tr(\underset{\sim}{\epsilon})\underset{\sim}{I} + 2\mu \underset{\sim}{\epsilon}}$

(2 constants)

In terms of other constants:

$$\lambda = \frac{2\mu\nu}{1-2\nu} = \frac{\mu(C-2\mu)}{3\mu - C} = \frac{C\nu}{(1+\nu)(1-2\nu)} = \frac{3K\nu}{1+\nu}$$

$$K = \lambda + \tfrac{2}{3}\mu = \frac{\mu C}{3(3\mu - C)} = \frac{\lambda(1+\nu)}{3\nu} = \frac{C}{3(1-2\nu)}$$

$$C = 2\mu(1+\nu) = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9K\mu}{3K+\mu}$$

$$\mu = \frac{C}{2(1+\nu)} = \tfrac{3}{2}(K-\lambda) = \frac{3K(1-2\nu)}{2(1+\nu)} = \frac{\lambda(1-2\nu)}{2\nu}$$

$$\nu = \frac{\lambda}{2(\lambda+\nu)} = \frac{C}{2\mu} - 1 = \frac{3K-2\mu}{2(3K-\mu)} = \frac{3K-C}{6K}$$

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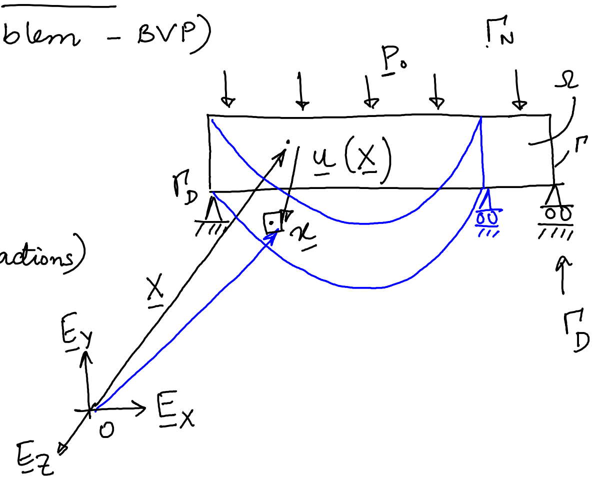
Final Problem Formulation

(Boundary Value Problem - BVP)



Given

- geometry : Ω (& boundary Γ)
- loads : $\underline{b}$ (self-weight)
  - : $\underline{p}_o$ (surface loads-tractions)
- Material(s)
  - $\underset{\sim}{\sigma} : \underset{\sim}{\sigma}(\underset{\sim}{\epsilon})$

Find $\boxed{\underline{\phi}(\underline{x})}$ or $\underline{u}(\underline{X})$

that satisfies :

$$\left.\begin{aligned} \operatorname{div} \underset{\sim}{\sigma} + \underline{b} &= \underline{0} \\ \underset{\sim}{\sigma} &= \underset{\sim}{\sigma}^T \\ \underset{\sim}{\epsilon} &= \tfrac{1}{2}\left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T\right) \end{aligned}\right] \quad \text{at } \underline{\text{all}} \ \underline{X} \text{ in } \Omega$$

+ Boundary conditions $\Gamma = \underline{\Gamma_D \cup \Gamma_N}$ $\quad (\Gamma_D \cap \Gamma_N = \emptyset)$

$\underline{\phi}(\underline{X}) = \underline{\phi}_D(\underline{X})$ on Γ_D (Dirichlet)

$\underline{t} = \underset{\sim}{\sigma}\,\underline{n} = \underline{p}_o$ on Γ_N (Neumann)

1-D Problem

Given

geometry : ℓ

density : ρ

cross-section : $(A = 1)$ say.

Material : $\sigma = C\,\epsilon$

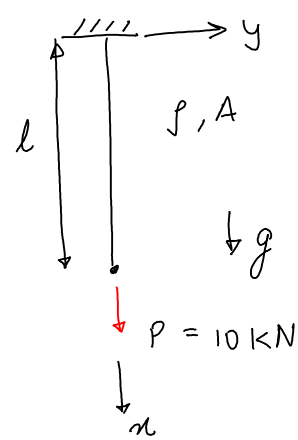
Find

$u(x)$

such that

$$\left.\begin{aligned} \sigma' + b &= 0 \\ \sigma &= C\,\epsilon \\ \epsilon &= u' \end{aligned}\right] \quad \forall x \in (0,\ell)$$

$b = \rho g$



BC

$u(0) = 0$ on $\Gamma_D = (x=0)$

$\sigma(\ell) = P$ on $\Gamma_N = (x=\ell)$

(i.e. $C\,u'(\ell) = P$)

Solution

$$\frac{d(Cu')}{dx} = -\rho g$$

$$\Rightarrow Cu' = -\rho g x + c_1$$

BC

$Cu'(l) = P$

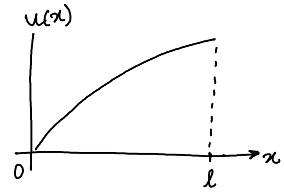
$\Rightarrow c_1 = P + \rho g l$

$$u(x) = -\frac{\rho g}{C}\frac{x^2}{2} + \frac{c_1}{C}x + c_2$$

$u(0) = 0$

$\Rightarrow c_2 = 0$

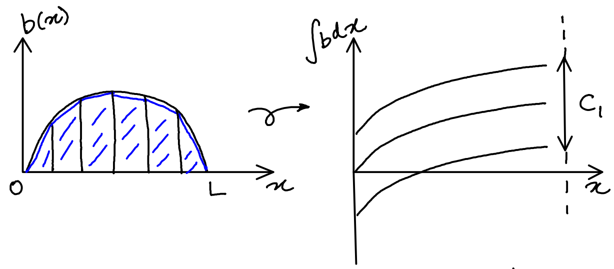
i.e. $$\boxed{u(x) = -\frac{1}{2}\frac{\rho g}{C}x^2 + \frac{(P+\rho g l)}{C}x}$$



HW2 Hint

$$\int d(Cu') = \int -b(x)\,dx$$

$$Cu' = -\underbrace{\int b(x)\,dx} + c_1$$



All these curves have same derivative: b(x).
The particular curve depends upon c_1.