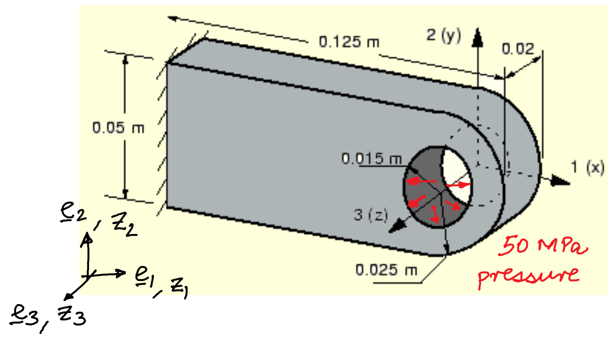
Chapter 3: Stress and Equilibrium of Deformable Bodies

When structures / deformable bodies are acted upon by loads, they build up internal forces (stresses) within them to be able to carry those loads (equilibrium) without breaking apart / failing. These internal forces are usually electromagnetic forces between atoms and molecules of the constituent material. However, we will assume that the body is a continuum and these forces are *distributed* uniformly over surfaces / volumes.



Example:
Given: body / geometry, boundary conditions, material properties, loads
Find: Solution (displacements, strains, stresses etc.) everywhere in the body. $\underset{\sim}{u}(\underset{\sim}{z})$ $\underset{\sim}{\epsilon}(\underset{\sim}{z})$ $\underset{\sim}{S}(\underset{\sim}{z})$

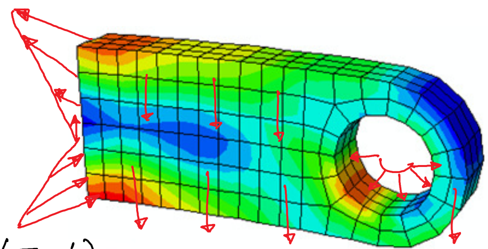
Using: Governing partial differential equations (PDEs)

$$\mathrm{div}(\underset{\sim}{S}) + \underset{\sim}{b} = \rho \ddot{\underset{\sim}{u}} \qquad (+\text{BCs})$$

$$\underset{\sim}{\epsilon} = \tfrac{1}{2}\left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T\right)$$

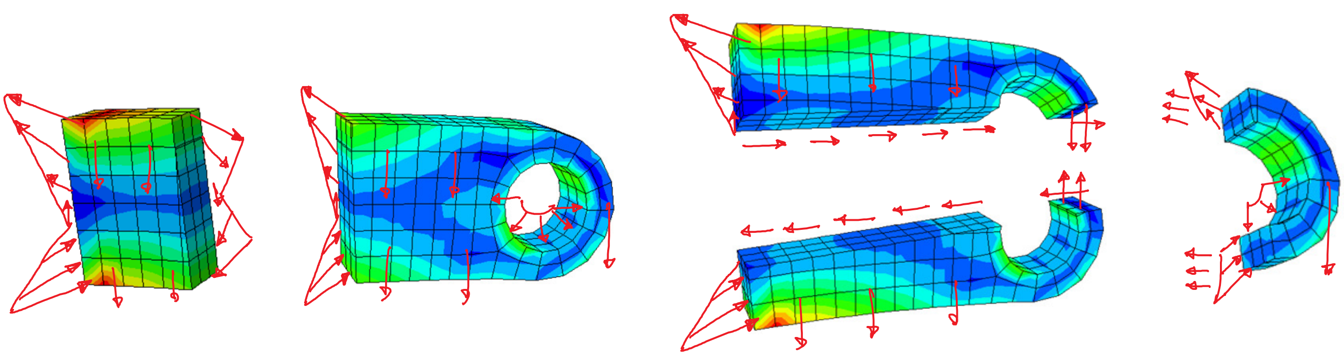
$$\underset{\sim}{S} = \lambda\, tr(\underset{\sim}{\epsilon})\, \underset{\sim}{I} + 2\mu\, \underset{\sim}{\epsilon}$$

(for small strain linear elasticity, for example)



In addition, if anything is changing with time, then find everything at all times of interest! $\underset{\sim}{u}(\underset{\sim}{z},t)$; $\underset{\sim}{\epsilon}(\underset{\sim}{z},t)$; $\underset{\sim}{S}(\underset{\sim}{z},t)$

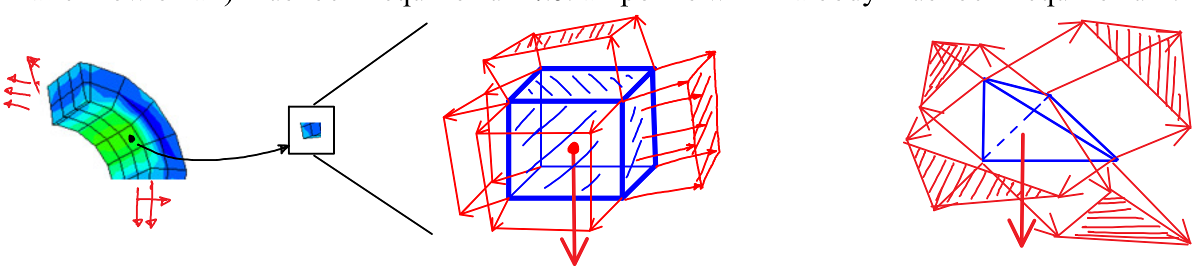
Free body diagrams:
FBDs are one of the most important tools to determine if a structure / body is in equilibrium or not.
In FBDs we draw a specific body (or a specific part of a body) and mark all the *external* forces that are acting on it.



Newton's Laws:

1. If sum of all the external forces acting on a body (or a specific part of a body) is **0**, then that body (or that specific part of the body) is at rest or constant velocity (in an inertial frame of reference).
2. If sum of all the external forces acting on a body (or a specific part of a body) is NOT **0**, then the instantaneous acceleration of the body (or a specific part of a body) is given by: $\boldsymbol{F} = m\, \boldsymbol{a}$
3. All forces in the universe occur as pairs of equal and opposite forces between two interacting bodies.

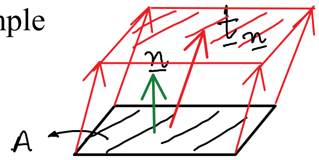
=> In order for a structure / deformable body to be in equilibrium, each and every part of the body (no matter how small) must be in equilibrium *i.e.* all points within a body must be in equilibrium:



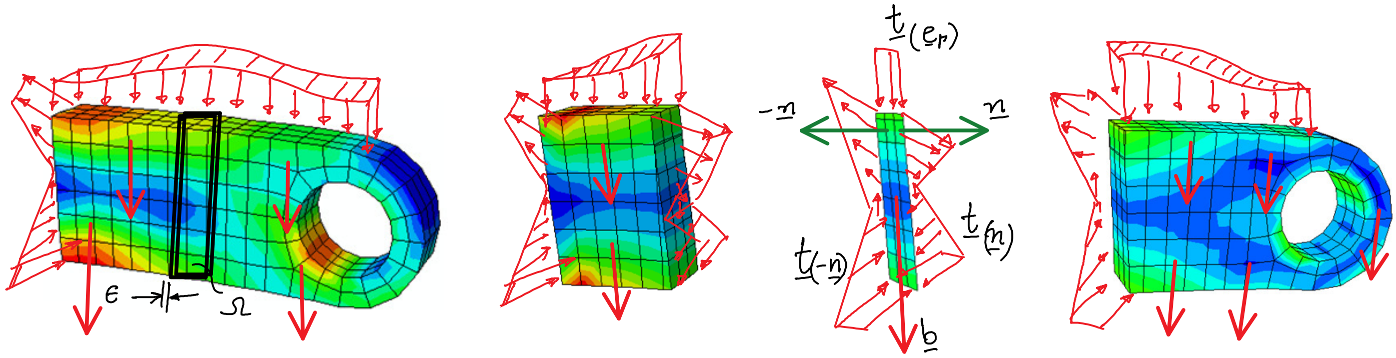
Traction vector (at a point)

Traction is the distributed force per unit area acting at a point on a surface passing through that point either on an outside (boundary) surface or any surface within the body.

Example



Total force (on A) $\quad \underline{f} = \int_A \underline{t}_{(\underline{n})}\, dA$



Sum of forces for the slice of material ("pillbox"):

$$\int_\Gamma \underline{t}_{(\underline{e}_\Gamma)} \underbrace{(\cancel{\epsilon}\, ds)}_{dA} + \int_\Omega \underline{t}_{(\underline{n})}(\underline{x})\, dA + \int_\Omega \underline{t}_{(-\underline{n})}(\underline{x})\, dA + \int_\Omega \underline{b}\, \underbrace{(\cancel{\epsilon}\, dA)}_{dV} = \underline{0}$$

As $\epsilon \to 0 \quad \Rightarrow \quad \int_\Omega \left(\underline{t}_{(\underline{n})} + \underline{t}_{(-\underline{n})}\right) dA = 0$ for any arbitrary Ω

$$\Rightarrow \quad \boxed{\underline{t}_{(\underline{n})} = -\underline{t}_{(-\underline{n})}}$$

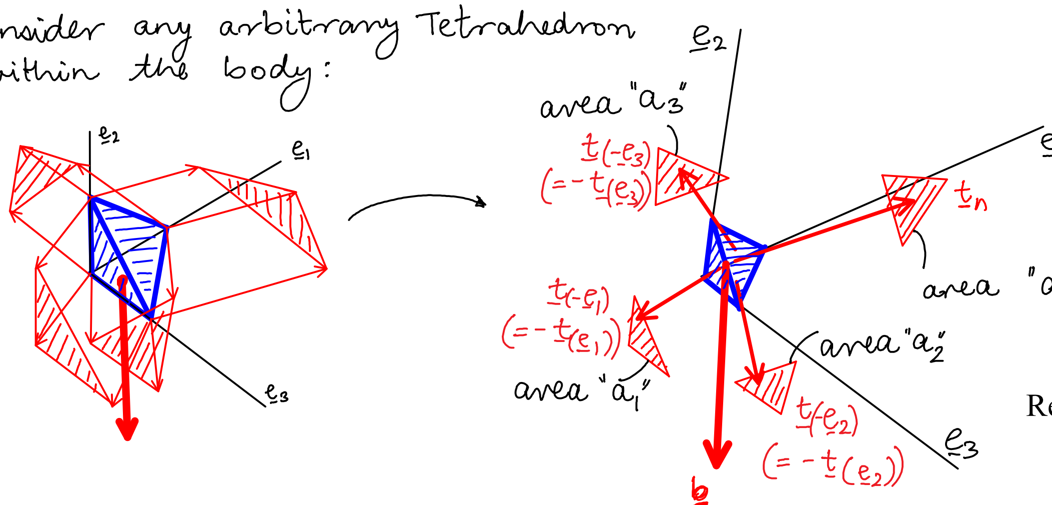
(at all points)

Cauchy reciprocal theorem. (Follows from Newton's 3rd Law)

Stress tensor

Tractions at a point is related to internal forces that develop within a body to maintain equilibrium. These internal forces within a body are best represented with a stress tensor field within the body.

Consider any arbitrary Tetrahedron within the body:



Resultant tractions

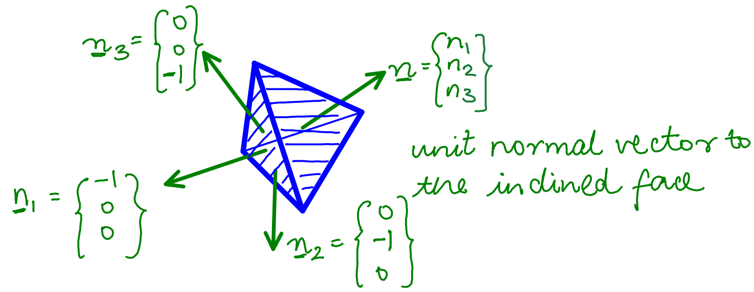
As we consider smaller & smaller tetrahedrons, the traction on each face becomes more & more uniform / constant:

$$\int_{a_n} \underline{t}_{(\underline{n})}\, dA \approx \underline{t}_{(\underline{n})}\, a_n$$

The Cauchy Stress-Traction relation

Note: From geometry, it can be shown that:

$a_1 = n_1 \, a_n$
$a_2 = n_2 \, a_n$
$a_3 = n_3 \, a_n$ ← components of $\underset{\sim}{n}$



Now equilibrium of this tetrahedron

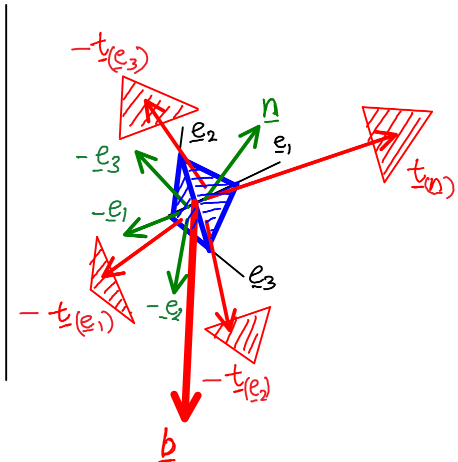
$\Rightarrow$ Sum of all 4 (tractions × area) + (Body force × volume) = 0

$\Rightarrow \underset{\sim}{t}_{(n)} a_n - \underset{\sim}{t}_{(e_1)} a_1 - \underset{\sim}{t}_{(e_2)} a_2 - \underset{\sim}{t}_{(e_3)} a_3 + \underset{\sim}{b}(v_n) = \underset{\sim}{0}$

$\Rightarrow \frac{\cancel{a_n}}{\cancel{a_n}}\left(\underset{\sim}{t}_{(n)} - \underset{\sim}{t}_{(e_1)} n_1 - \underset{\sim}{t}_{(e_2)} n_2 - \underset{\sim}{t}_{(e_3)} n_3\right) + \underset{\sim}{b}\left(\frac{v_n}{a_n}\right) = \underset{\sim}{0}$, $\frac{v_n}{a_n} \to 0$

$\Rightarrow \underset{\sim}{t}_{(n)} = \underset{\sim}{t}_{(e_1)}(\underset{\sim}{n}\cdot\underset{\sim}{e}_1) + \underset{\sim}{t}_{(e_2)}(\underset{\sim}{n}\cdot\underset{\sim}{e}_2) + \underset{\sim}{t}_{(e_3)}(\underset{\sim}{n}\cdot\underset{\sim}{e}_3)$

$= (\underset{\sim}{t}_{(e_1)} \otimes \underset{\sim}{e}_1)\underset{\sim}{n} + (\underset{\sim}{t}_{(e_2)} \otimes \underset{\sim}{e}_2)\underset{\sim}{n} + (\underset{\sim}{t}_{(e_3)} \otimes \underset{\sim}{e}_3)\underset{\sim}{n}$



$$\Rightarrow \boxed{\underset{\sim}{t}_{(n)} = \underbrace{\left[\sum_{i=1}^{3}\left(\underset{\sim}{t}_{(e_i)} \otimes \underset{\sim}{e}_i\right)\right]}_{\underset{\sim}{S}\ :\ \text{Cauchy Stress}} \underset{\sim}{n} = \underset{\sim}{S}\,\underset{\sim}{n}}$$

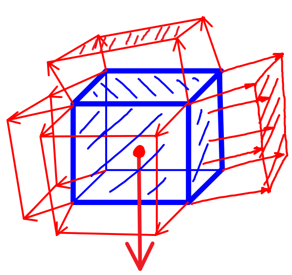
Cauchy Stress-Traction relationship

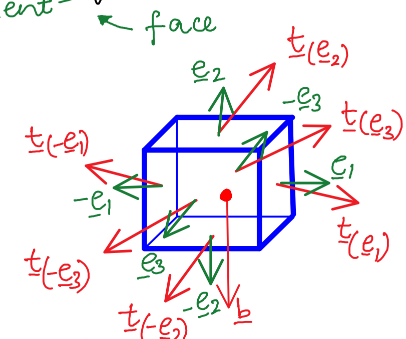
$$\Rightarrow \underbrace{\begin{Bmatrix} t_{1(n)} \\ t_{2(n)} \\ t_{3(n)} \end{Bmatrix}}_{\underset{\sim}{t}_{(n)}} = \underbrace{\left[\sum_{i=1}^{3}\begin{Bmatrix} t_1(e_i) \\ t_2(e_i) \\ t_3(e_i) \end{Bmatrix}\underbrace{\{\delta_{i1} \quad \delta_{i2} \quad \delta_{i3}\}}_{\underset{\sim}{e}_i}\right]}_{\underset{\sim}{S}} \underbrace{\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}}_{\underset{\sim}{n}} = \underbrace{\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}}_{\underset{\sim}{S}} \underbrace{\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}}_{\underset{\sim}{n}}$$

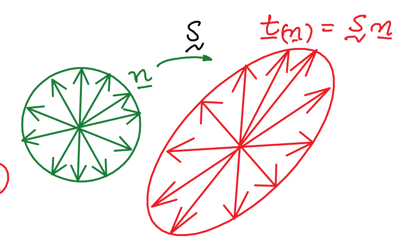
Note: $\underset{\sim}{t}_{(e_1)} = \underset{\sim}{S}\,\underset{\sim}{e}_1 \sim \begin{Bmatrix} S_{11} \\ S_{21} \\ S_{31} \end{Bmatrix}$; $\underset{\sim}{t}_{(e_2)} = \underset{\sim}{S}\,\underset{\sim}{e}_2 \sim \begin{Bmatrix} S_{12} \\ S_{22} \\ S_{32} \end{Bmatrix}$; $\underset{\sim}{t}_{(e_3)} = \underset{\sim}{S}\,\underset{\sim}{e}_3 \sim \begin{Bmatrix} S_{13} \\ S_{23} \\ S_{33} \end{Bmatrix}$

Physical interpretation of the components of *S* : $S_{ij} = \underset{\sim}{e}_i \cdot \underset{\sim}{S}\,\underset{\sim}{e}_j$

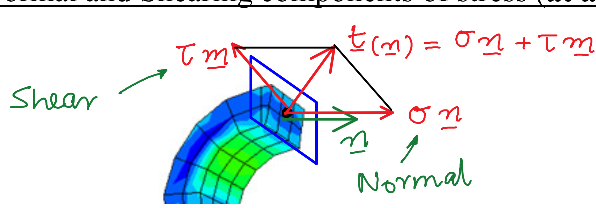
component → i ; face → j







Normal and Shearing components of stress (**at a point**)

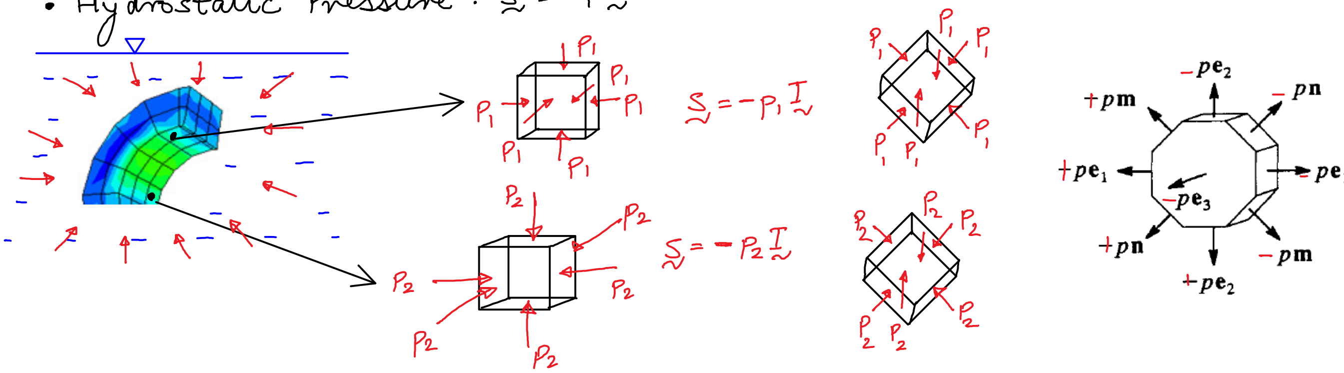


$\sigma = \underset{\sim}{n}\cdot\underset{\sim}{t}_{(n)} = (\underset{\sim}{n}\cdot\underset{\sim}{S}\,\underset{\sim}{n})$

$\Rightarrow \tau \underset{\sim}{m} = \underset{\sim}{S}\,\underset{\sim}{n} - \sigma\underset{\sim}{n} = (\underset{\sim}{I} - \underset{\sim}{n}\otimes\underset{\sim}{n})\,\underset{\sim}{t}_{(n)}$

Examples of a State of Stress (**at a point**):

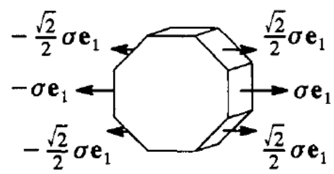
- Hydrostatic Pressure : $\underset{\sim}{S} = -p\underset{\sim}{I}$

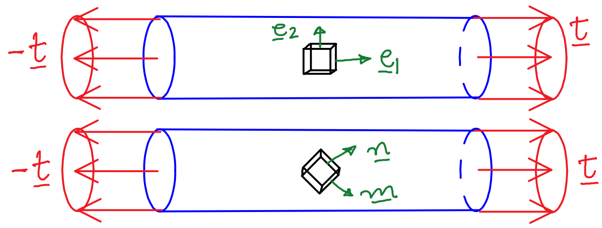


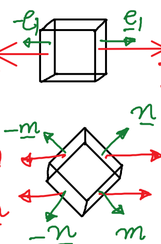
$$\underset{\sim}{S} = -p_1\underset{\sim}{I} \qquad \underset{\sim}{S} = -p_2\underset{\sim}{I}$$

Note: Convention for (outward) normal (tension positive)

- Uniaxial Tension : $\underset{\sim}{S} = \sigma\, \underline{e}_1 \otimes \underline{e}_1 \sim \begin{bmatrix}\sigma & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$







$$-\underline{t}_{(-\underline{e}_1)} = -(\underset{\sim}{S}\underline{e}_1) \qquad \underline{t}_{(\underline{e}_1)} = \underset{\sim}{S}\underline{e}_1$$

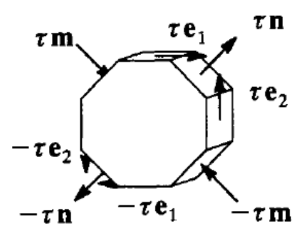
$$\underline{t}_{(-\underline{m})} = -\underset{\sim}{S}\underline{m} \qquad \underline{t}_{(\underline{n})} = \underset{\sim}{S}\underline{n} = \frac{\sqrt{2}}{2}\sigma\,\underline{e}_1$$

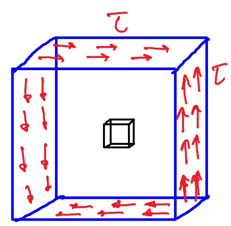
$$\underline{t}_{(-\underline{n})} = -\underset{\sim}{S}\underline{n} \qquad \underline{t}_{(\underline{m})} = \underset{\sim}{S}\underline{m} = \frac{\sqrt{2}}{2}\sigma\,\underline{e}_1$$

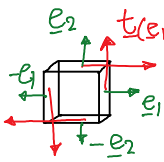
$$\underline{n} = \frac{\sqrt{2}}{2}(\underline{e}_1 + \underline{e}_2)$$

$$\underline{m} = \frac{\sqrt{2}}{2}(\underline{e}_1 - \underline{e}_2)$$

- Pure Shear : $\underset{\sim}{S} = \tau(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1) \sim \begin{bmatrix}0 & \tau & 0\\ \tau & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

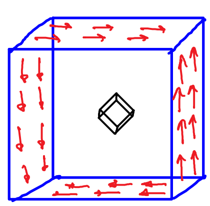


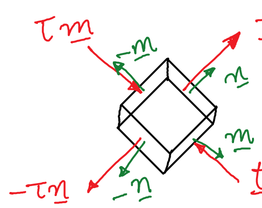




$$\underline{t}_{(\underline{e}_1)} = \tau\,\underline{e}_2 \qquad \underline{t}_{(\underline{e}_2)} = \tau\,\underline{e}_1$$

$$\underline{t}_{(-\underline{e}_2)} = -\tau\,\underline{e}_1 \qquad \underline{t}_{(-\underline{e}_1)} = -\tau\,\underline{e}_2$$



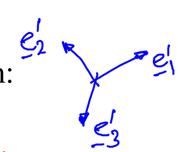


$$\underline{t}_{(\underline{n})} = \tau\left(\frac{\sqrt{2}}{2}\underline{e}_1 + \frac{\sqrt{2}}{2}\underline{e}_2\right) = \tau\,\underline{n}$$

$$\underline{t}_{(\underline{m})} = \tau\left(-\frac{\sqrt{2}}{2}\underline{e}_1 + \frac{\sqrt{2}}{2}\underline{e}_2\right) = -\tau\,\underline{m}$$

Transformation of Stress (Coordinate change)

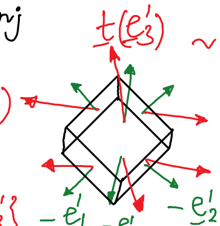
In order to obtain the components of stress in a different coordinate system:



One can either use the usual coordinate transformation formula (for any tensor) $\quad Q_{ij} = \underline{e}_i' \cdot \underline{e}_j$

$$S_{ij}' = Q_{im}\, S_{mn}\, (Q^T)_{nj}$$

Or calculate the tractions on the faces of the infinitesimal cube aligned along the new coordinate axis, and interpret these tractions as the components of the stress tensor as shown previously.



$$\underline{t}(\underline{e}_3') \sim \begin{bmatrix}S_{31}'\\ S_{32}'\\ S_{33}'\end{bmatrix}\{\underline{e}_1'\ \underline{e}_2'\ \underline{e}_3'\}$$

$$-\underline{t}(\underline{e}_2') \sim \begin{bmatrix}S_{21}'\\ S_{22}'\\ S_{23}'\end{bmatrix}\{\underline{e}_1'\underline{e}_2'\underline{e}_3'\}$$

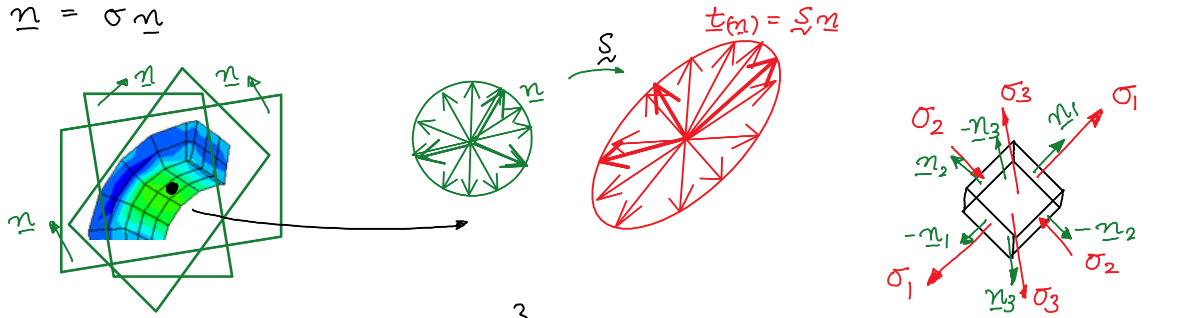
$$\underline{t}(\underline{e}_1') \sim \begin{bmatrix}S_{11}'\\ S_{21}'\\ S_{31}'\end{bmatrix}\{\underline{e}_1'\ \underline{e}_2'\ \underline{e}_3'\}$$

Principal (Eigen) values of Stress

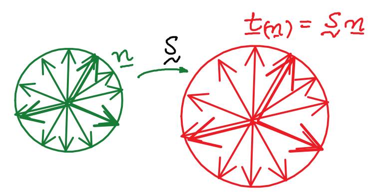
Like any tensor, the stress tensor $\boldsymbol{S}$ also has the same interpretations of the Eigenvalues & Eigenvectors:

- Values and directions associated with maximum tractions
- Values and directions associated with only *normal* tractions / *no shear* tractions

$$\underset{\sim}{S}\,\underline{n} = \sigma\,\underline{n}$$



- Spectral decomposition : $\underset{\sim}{S} = \sum_{i=1}^{3} \sigma_i \left(\underline{n}_i \otimes \underline{n}_i\right)$



Examples:

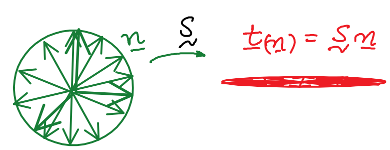
- Hydrostatic Pressure : $\underset{\sim}{S} = -p\underset{\sim}{I}$

  $\Rightarrow \quad \sigma_1 = \sigma_2 = \sigma_3 = -p$ Any direction is an Eigenvector

- Uniaxial tension : $\underset{\sim}{S} = \sigma\left(\underline{e}_1 \otimes \underline{e}_1\right)$

  $\Rightarrow \quad \sigma_1 = \sigma \quad ; \quad \sigma_2 = \sigma_3 = 0$

  $\underline{n}_1 = \underline{e}_1 \quad ; \quad \underline{n}_2, \underline{n}_3 \perp \underline{e}_1$

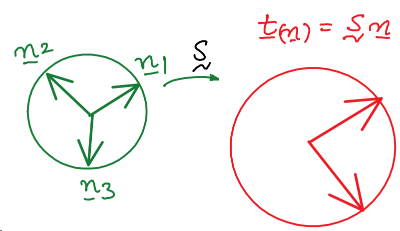


- Pure shear : $\underset{\sim}{S} = \tau\left(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_2\right) \sim \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

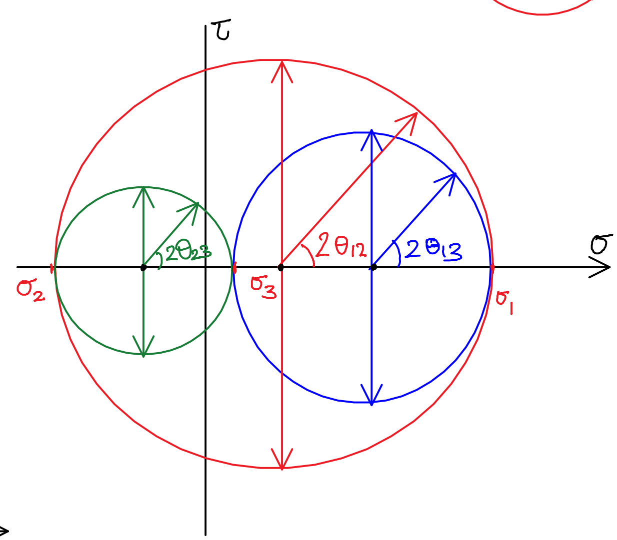
$$\det\left(\underset{\sim}{S} - \sigma\underset{\sim}{I}\right) = -\sigma\left(\sigma^2 - \tau^2\right) = 0$$

$$\Rightarrow \quad \sigma_1 = \tau \quad ; \quad \sigma_2 = -\tau \quad ; \quad \sigma_3 = 0$$

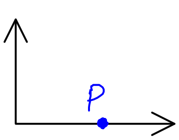
$$\underline{n}_1 = \frac{1}{\sqrt{2}}\left(\underline{e}_1 + \underline{e}_2\right) \; ; \quad \underline{n}_2 = \frac{1}{\sqrt{2}}\left(-\underline{e}_1 + \underline{e}_2\right) \; ; \quad \underline{n}_3 = \underline{e}_3$$



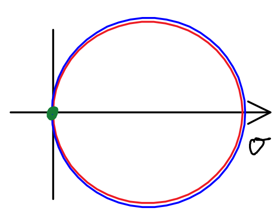
Relation to Mohr circle representation



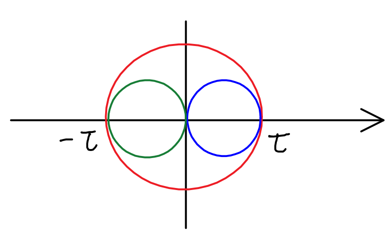
- Hydrostatic Pressure :



- Uniaxial tension :

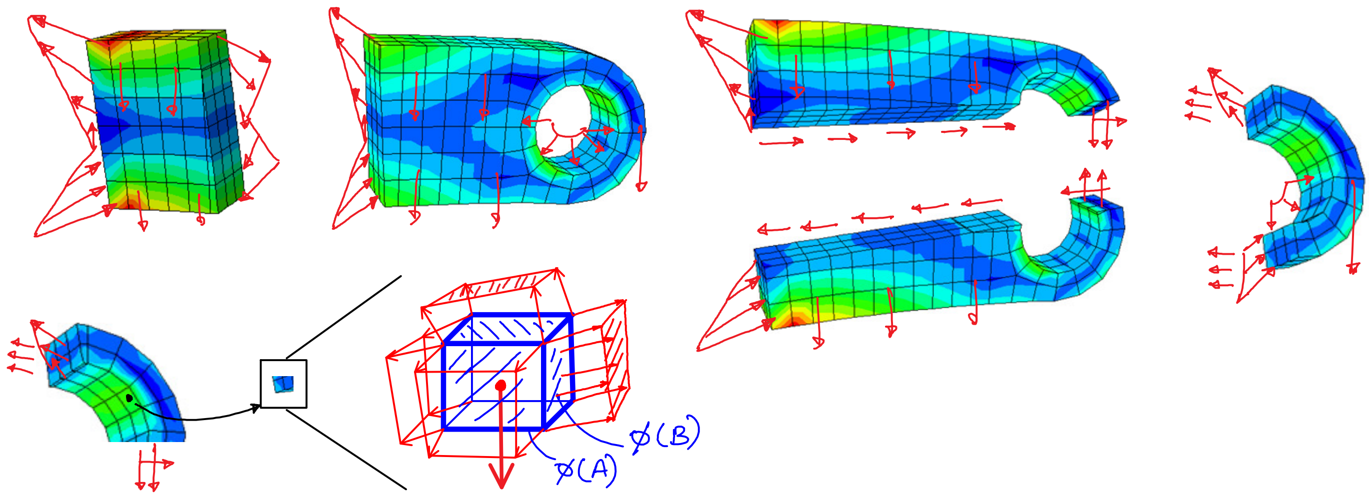


- Pure shear (in $\underline{e}_1, \underline{e}_2$)



Partial Differential Equations for Equilibrium

In order for a body / structure to be in equilibrium, *every* sub-part of that body must also be equilibrium.



Equilibrium of body ϕ(B) with surface ϕ(A): $\sum \underline{F} = \underline{0} \Rightarrow \int_{\phi(A)} \underline{t}_{(\underline{n})}\, da + \int_{\phi(B)} \underline{b}\, dv = \underline{0}$

Using $\underline{t}_{(\underline{n})} = \underset{\sim}{S}\, \underline{n}$ and divergence theorem $\int_{\phi(B)} \operatorname{div} \underset{\sim}{T}\, dV = \int_{\phi(A)} \underset{\sim}{T}\underline{n}\, da$

$$\Rightarrow \int_{\phi(B)} \left(\operatorname{div} \underset{\sim}{S} + \underline{b}\right) dv = \underline{0}$$ (True for any volume ϕ(B))

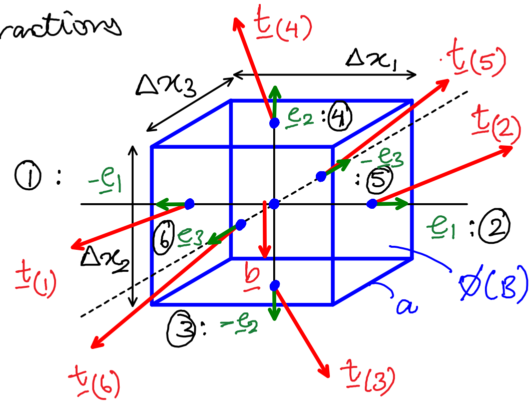
PDE for equilibrium: $\boxed{\operatorname{div} \underset{\sim}{S} + \underline{b} = \underline{0}}$ at all points "$\underline{x}$" $\in \phi(B)$

In indicial notation: $\frac{\partial S_{ij}}{\partial x_j} + b_i = 0$ Note: $\frac{\partial}{\partial x_j}$ (deformed coords)

Alternative (conventional/long) way: Resultant Tractions

Consider 6 faces: ①, ②, ③, ④, ⑤, ⑥

normals: $-\underline{e}_1, \underline{e}_1, -\underline{e}_2, \underline{e}_2, -\underline{e}_3, \underline{e}_3$



Resultant Traction on a face "i": $\underline{t}_{(i)}$

$$\underline{t}_{(1)} = \underset{\sim}{S}(\underline{x}_1)\,(-\underline{e}_1)$$

$$\underline{t}_{(2)} = \underset{\sim}{S}(\underline{x}_2)\,(\underline{e}_1) \cong \left[\underset{\sim}{S}(\underline{x}_1) + \frac{\partial \underset{\sim}{S}(\underline{x}_1)}{\partial x_1} \cdot \Delta x_1\right] \underline{e}_1$$

$$\underline{t}_{(3)} = \underset{\sim}{S}(\underline{x}_3)\,(-\underline{e}_2)$$

$$\underline{t}_{(4)} = \underset{\sim}{S}(\underline{x}_4)\,(\underline{e}_2) \cong \left[\underset{\sim}{S}(\underline{x}_3) + \frac{\partial \underset{\sim}{S}(\underline{x}_3)}{\partial x_2} \cdot \Delta x_2\right] \underline{e}_2$$

$$\underline{t}_{(5)} = \underset{\sim}{S}(\underline{x}_5)\,(-\underline{e}_3)$$

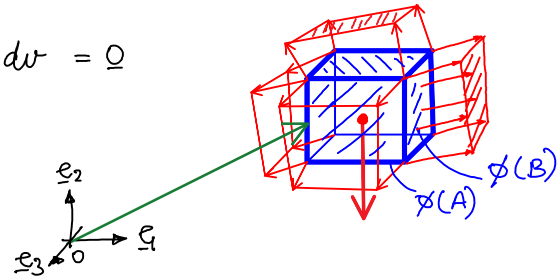
$$\underline{t}_{6} = \underset{\sim}{S}(\underline{x}_6)\,(\underline{e}_3) \cong \left[\underset{\sim}{S}(\underline{x}_5) + \frac{\partial \underset{\sim}{S}(\underline{x}_5)}{\partial x_3} \cdot \Delta x_3\right] \underline{e}_3$$

Now equilibrium:

$$\Rightarrow \sum \underline{F} = \underline{0} \Rightarrow \left[\overbrace{\left(\frac{\partial \underset{\sim}{S}}{\partial x_1} \underline{e}_1 + \frac{\partial \underset{\sim}{S}}{\partial x_2} \underline{e}_2 + \frac{\partial \underset{\sim}{S}}{\partial x_3} \underline{e}_3 \right)}^{\operatorname{div} \underset{\sim}{S}} + \underline{b} \right] \Delta x_1 \Delta x_2 \Delta x_3 = \underline{0}$$

Equilibrium of Moments:

$$\sum \underline{M}_0 = \underline{0} \quad \Rightarrow \quad \int_{\partial(A)} \underline{x} \times \underline{t}_{(\underline{n})}\, da + \int_{\partial(B)} \underline{x} \times \underline{b}\, dv = \underline{0}$$



Note: $\underline{x} \times \underline{t}_{(\underline{n})} = \underline{x} \times \underset{\sim}{S}\, \underline{n}$

$$= \epsilon_{ijk}\, x_i\, S_{jl}\, n_l\, \underline{e}_k$$

Note, Divergence Theorem:

$$\Rightarrow \int_{\partial(A)} \epsilon_{ijk}\, x_i\, S_{jl}\, n_l\, da = \int_{\partial(B)} \frac{\partial}{\partial x_l}\left(\epsilon_{ijk}\, x_i\, S_{jl}\right) dv$$

$$= \int_{\partial(B)} \epsilon_{ijk}\left(\delta_{il}\, S_{jl} + x_i\, S_{jl,l}\right) dv$$

$$\Rightarrow \sum M_{0_k} = \int_{\partial(B)} \epsilon_{ijk}\left(S_{ji} + x_i\, S_{jl,l}\right) dv + \int_{\partial(B)} \epsilon_{ijk}\, x_i\, b_j\, dv$$

$$= \int_{\partial(B)} \epsilon_{ijk}\, S_{ji}\, dv + \int_{\partial(B)} \epsilon_{ijk}\, x_i \overbrace{\cancel{\left(S_{jl,l} + b_j\right)}}^{0} dv \quad \Rightarrow \quad \boxed{\epsilon_{ijk}\, S_{ji} = 0} \quad \text{—(1)}$$

$$\Rightarrow -\epsilon_{jik}\, S_{ji} = -\epsilon_{ijk}\, S_{ij} = 0 \quad \text{—(2)}$$

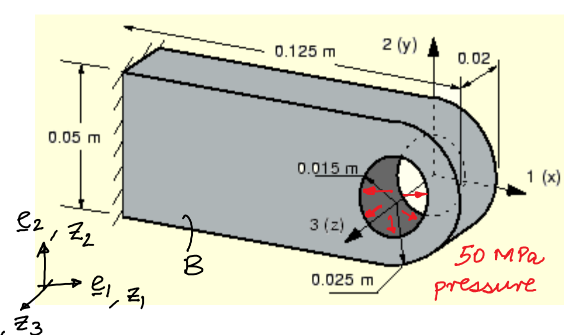
Adding (1) & (2): $\epsilon_{ijk}\left(S_{ji} - S_{ij}\right) = 0 \Rightarrow \boxed{S_{ij} = S_{ji}} \Rightarrow \boxed{\underset{\sim}{S} = \underset{\sim}{S}^T}$

Thus governing equations of equilibrium of a structure / body:

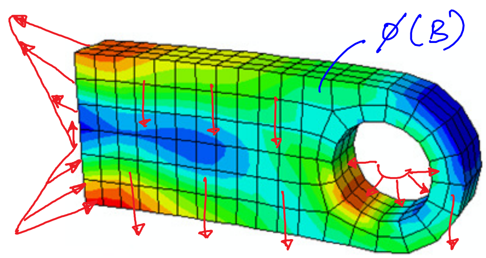
$$\mathrm{div}\, \underset{\sim}{S} + \underline{b} = \underline{0} \qquad \forall\, \underline{x} \in \partial(B)$$

$$\underset{\sim}{S} = \underset{\sim}{S}^T \qquad \forall\, \underline{x} \in \partial(B)$$

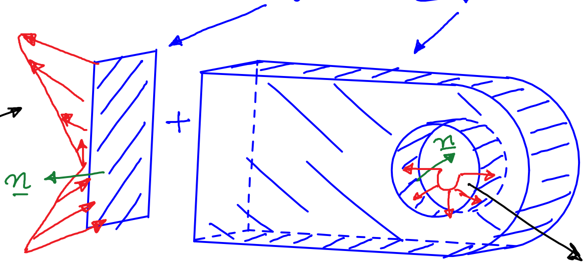
$$\underset{\sim}{S}\, \underline{n} = \underline{t}_{(\underline{n})} \qquad \forall\, \underline{x} \in \underbrace{\partial(A)}$$



$$\partial(A) = \underbrace{\partial(A_D)} \cup \underbrace{\partial(A_N)}$$



Unknown Reactions



$\underline{t}_{(\underline{n})} = \underset{\sim}{S}\, \underline{n}$

Given External Tractions

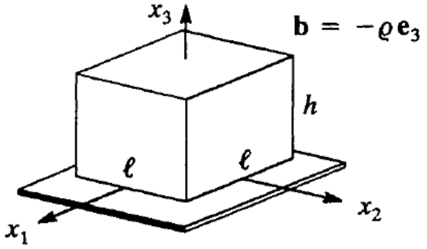
$\underline{t}_{(\underline{n})} = \underset{\sim}{S}\, \underline{n}$

$(= -p\, \underline{n}$ for this problem$)$

Examples of stress fields in equilibrium

Look at Examples 19 and 20 from the textbook (Hjelmstad 2005)

Example 19: Rigid Block under self-weight



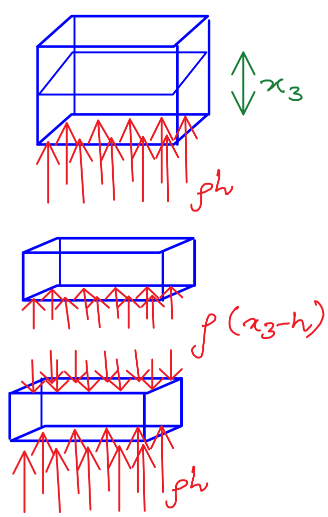
Stress field : $\mathbf{S}(\mathbf{x}) = \varrho(x_3 - h)[\mathbf{e}_3 \otimes \mathbf{e}_3]$

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho(x_3-h) \end{bmatrix}$$

Verify :

GDE: $\operatorname{div} \underset{\sim}{S} = \dfrac{\partial S_{ij}}{\partial x_j} \underline{e}_i \sim \begin{bmatrix} 0 \\ 0 \\ \rho \end{bmatrix} \Rightarrow \operatorname{div} \underset{\sim}{S} + \underline{b} \sim \begin{bmatrix} 0 \\ 0 \\ \rho \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\rho \end{bmatrix} \sim \underline{0}$

BC : $\underset{\sim}{S}\,\underline{n}$ on 6 faces:



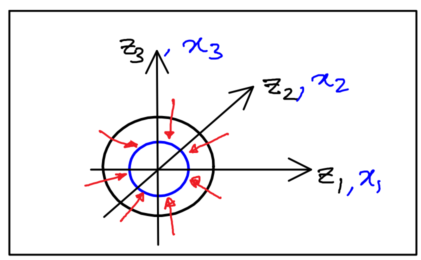
- For $\underline{n} = \pm \underline{e}_1$ and $\pm \underline{e}_2 \Rightarrow \underset{\sim}{S}\,\underline{n} = \underline{0}$
- For $\underline{n} = \underline{e}_3$ (at $x_3 = h$) $\Rightarrow \overset{\underline{0}}{\cancel{\underset{\sim}{S}}}\,\underline{n} = \underline{0}$

  $\underline{n} = -\underline{e}_3$ (at $x_3 = 0$) $\Rightarrow \underset{\sim}{S}\,\underline{n} \sim \begin{bmatrix} 0 \\ 0 \\ \rho h \end{bmatrix}$

Note: • $\rho h(\ell^2)$ = weight of block.

- FBD of a part of a block at height x_3:

Example: Soft deformable solid sphere of negligible weight floating in a pressurized chamber:



Stress field : $\underset{\sim}{S}(\underline{x}) = -p\underset{\sim}{I} \quad \forall\, \underline{x} \in B$

Verify:

GDE : $\operatorname{div}(\underset{\sim}{S}) = \underline{0}$

$\Rightarrow \operatorname{div} \underset{\sim}{S} + \overset{\underline{0}}{\cancel{\underline{b}}} = \underline{0} \qquad \forall\, \underline{x} \in B$

BCs : $\underline{n} = \dfrac{\underline{x}}{\|\underline{x}\|} \qquad \forall\, \underline{x} \in A_N \qquad (A_D = \phi)$

null

$$\underset{\sim}{S}\,\underline{n} = -p\underset{\sim}{I}\,\frac{\underline{x}}{\|\underline{x}\|} = -p\,\underline{n} \qquad \forall\, \underline{x} \in A_N$$

Aside: To find the Deformation map:

Assume $\underline{\phi}(\underline{z}) = \alpha\,\underline{z} \Rightarrow \underset{\sim}{F} = \alpha\underset{\sim}{I}$; $\underset{\sim}{C} = \alpha^2 \underset{\sim}{I}$; $\underset{\sim}{E} = \left(\tfrac{1}{2}\alpha^2 - 1\right)\underset{\sim}{I}$

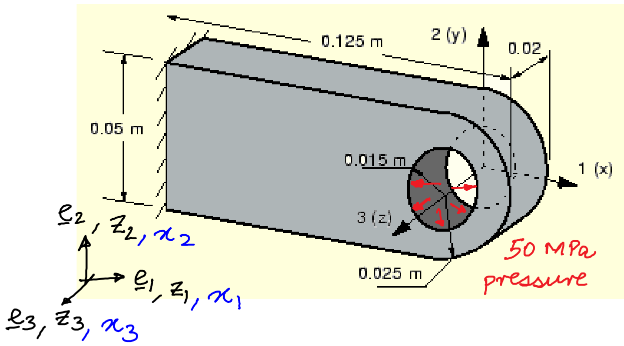
Noting that $\underset{\sim}{S} = -p\underset{\sim}{I}$ and $\underset{\sim}{E} = \left(\tfrac{1}{2}\alpha^2 - 1\right)\underset{\sim}{I}$

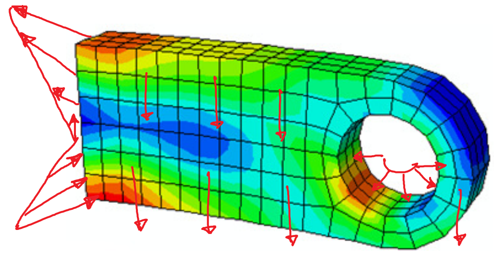
we can find $\alpha(p)$ if we know material properties $\underset{\sim}{S}(\underset{\sim}{E})$.

First and Second Piola-Kirchhoff stresses

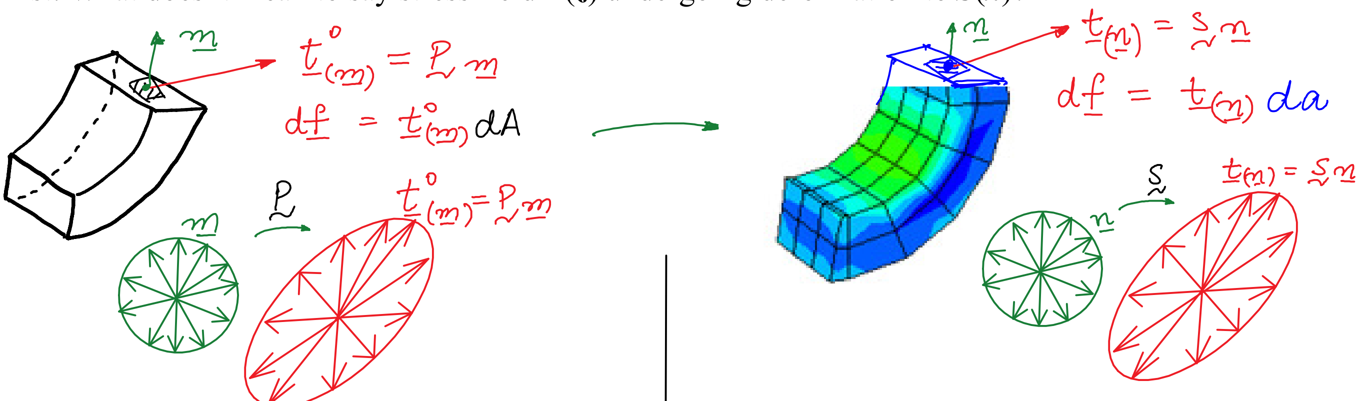
Cauchy stress tensor (field) is defined over the *deformed* configuration of a structure / body and is directly related to the governing equations of equilibrium and boundary conditions.

However, the deformed configuration of a body is usually *unknown* (and it is usually what we aim to calculate). Thus, sometimes it is beneficial to try to express the equations of equilibrium on the *undeformed* configuration of a body. This raises the question:
"Is there a stress tensor field $\boldsymbol{P}(\boldsymbol{z})$ (defined on the *undeformed* configuration) that after undergoing *deformation*, produces Cauchy stress field $\boldsymbol{S}(\boldsymbol{x})$ satisfying the governing PDEs of equilibrium and BCs?"





First: What does it mean to say stress field $\boldsymbol{P}(\boldsymbol{z})$ undergoing deformation to $\boldsymbol{S}(\boldsymbol{x})$?



$$\sum \underset{\sim}{F} = \underset{\sim}{0} \Rightarrow \int_A \underset{\sim}{t}^{\circ}_{(m)}\, dA + \int_B \underset{\sim}{b}^{\circ}\, dV = \underset{\sim}{0}$$

i.e. $$\int_A \underset{\sim}{P}\, \underset{\sim}{m}\, dA + \int_B \underset{\sim}{b}^{\circ}\, dV = \underset{\sim}{0}$$

$$\int_{\phi(A)} \underset{\sim}{t}_{(n)}\, da + \int_{\phi(B)} \underset{\sim}{b}\, dv = \underset{\sim}{0}$$

$$\int_{\phi(A)} \underset{\sim}{S}\, \underset{\sim}{n}\, da + \int_{\phi(B)} \underset{\sim}{b}\, dv = \underset{\sim}{0}$$

Recall: $\underset{\sim}{n}\, da = \det(\underset{\sim}{F})\, \underset{\sim}{F}^{-T} \underset{\sim}{m}\, dA$ and $dv = \det(\underset{\sim}{F})\, dV$

$$\Rightarrow \int_A \underbrace{\underset{\sim}{S}\,(\det \underset{\sim}{F})\,\underset{\sim}{F}^{-T}}_{\underset{\sim}{P}} \underset{\sim}{m}\, dA + \int_B \underbrace{\underset{\sim}{b}\,\det(\underset{\sim}{F})}_{\underset{\sim}{b}^{\circ}}\, dV$$

Thus: $$\boxed{\underset{\sim}{P} = \overbrace{\det(\underset{\sim}{F})}^{J}\, \underset{\sim}{S}\, \underset{\sim}{F}^{-T}}$$ and $$\boxed{\underset{\sim}{b}^{\circ} = \det(\underset{\sim}{F})\, \underset{\sim}{b} = J \underset{\sim}{b}}$$

1st Piola Stress   Equivalent Body Force

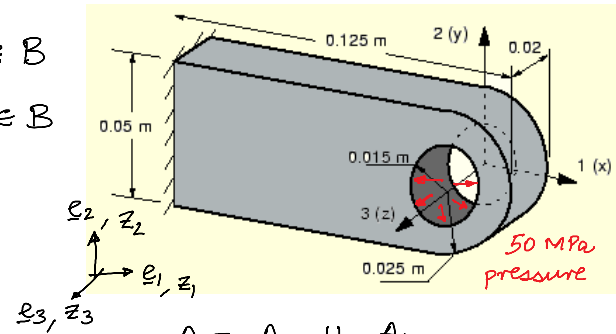
Note: Another way to view the 1st Piola stress tensor is to interpret it as the stress field resulting from a simple change of variables from $\boldsymbol{x}$ to $\boldsymbol{z}$.

Governing equations in the *undeformed* configuration (for equilibrium in the *deformed* configuration)

$$\sum \underset{\sim}{F} = \underset{\sim}{0} \quad \Rightarrow \quad \mathrm{DIV}\ \underset{\sim}{P} + \underset{\sim}{b}^{o} = \underset{\sim}{0} \quad \forall\ \underset{\sim}{z} \in B$$

$$\sum \underset{\sim}{M} = \underset{\sim}{0} \quad \Rightarrow \quad \underset{\sim}{P}\,\underset{\sim}{F}^{T} = \underset{\sim}{F}\,\underset{\sim}{P}^{T} \quad \forall\ \underset{\sim}{z} \in B$$

Note : $(J \underset{\sim}{S} \cancel{\underset{\sim}{F}^{-T}})\ \cancel{\underset{\sim}{F}^{T}} = \cancel{\underset{\sim}{F}}\,(J\, \cancel{\underset{\sim}{F}^{-1}}\, \underset{\sim}{S})$

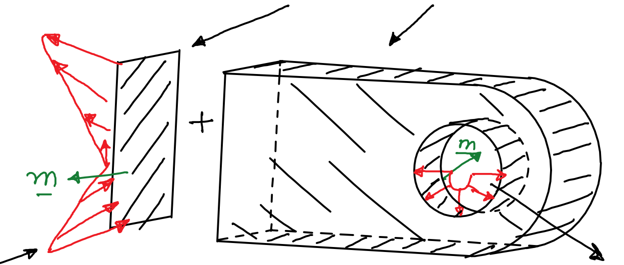


BCs : $\quad \underset{\sim}{P}\,\underset{\sim}{m} = \underset{\sim}{t}^{o}_{(\underset{\sim}{m})} \quad \forall\ \underset{\sim}{z} \in A$

$A = A_D \cup A_N$

Note: $\quad \underset{\sim}{P} = P_{ij}\ \underset{\sim}{e}_i \otimes \underset{\sim}{g}_j$

$$\mathrm{DIV}\ \underset{\sim}{P} \equiv \frac{\partial P_{ij}}{\partial z_j}\ \underset{\sim}{e}_i$$

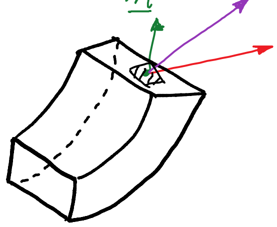


Unknown Reactions $\quad \underset{\sim}{t}^{o}_{(\underset{\sim}{m})} = \underset{\sim}{P}\,\underset{\sim}{m}$

Given External Tractions $\quad \underset{\sim}{t}^{o}_{(\underset{\sim}{m})} = \underset{\sim}{P}\,\underset{\sim}{m} \quad (\neq -p\,\underset{\sim}{m})$

Second Piola Kirchhoff Stress tensor

Since the 1st Piola Kirchhoff stress tensor is *not* symmetric, one can create a symmetric tensor as:

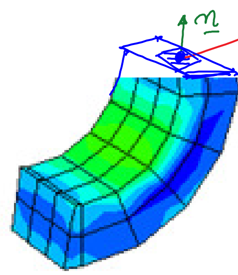


$d\underset{\sim}{f}^{o} = \underset{\sim}{F}^{-1}\, d\underset{\sim}{f}$

$\underset{\sim}{t}^{o}_{(\underset{\sim}{m})} = \underset{\sim}{P}\,\underset{\sim}{m}$

$d\underset{\sim}{f} = \underset{\sim}{t}^{o}_{(\underset{\sim}{m})}\, dA$

$$\xrightarrow[\underset{\sim}{F}]{\phi}$$



$\underset{\sim}{t}_{(\underset{\sim}{n})} = \underset{\sim}{S}\,\underset{\sim}{n}$

$d\underset{\sim}{f} = \underset{\sim}{t}_{(\underset{\sim}{n})}\, da$

$$d\underset{\sim}{f}^{o} = \underset{\sim}{F}^{-1}\, d\underset{\sim}{f} = \underbrace{\underset{\sim}{F}^{-1}\,\underset{\sim}{P}}_{\underset{\sim}{\Sigma}}\,\underset{\sim}{m}\, dA \qquad (= \underset{\sim}{F}^{-1}\,\underset{\sim}{S}\,\underset{\sim}{n}\, da)$$

$$\Rightarrow \quad \boxed{\underset{\sim}{\Sigma} = \underset{\sim}{F}^{-1}\,\underset{\sim}{P} = J\,\underset{\sim}{F}^{-1}\,\underset{\sim}{S}\,\underset{\sim}{F}^{-T}}$$

Note : $\quad \underset{\sim}{\Sigma}^{T} = \underset{\sim}{P}^{T}\,\underset{\sim}{F}^{-T} = J\,\underset{\sim}{F}^{-1}\,\underset{\sim}{S}\,\underset{\sim}{F}^{-T} = \underset{\sim}{\Sigma} \qquad$ (symmetric)

The second Piola stress tensor was "concocted" to be a *symmetric* tensor. This is sometimes useful in doing computations (for instance using the finite element method for large deformation problems).

Objective Stress Rates:

Cauchy stress $\underset{\sim}{S}$ is objective: $\qquad S'_{ij} = Q_{im}\, S_{mn}\, Q^{T}_{nj}$

But $\dot{\underset{\sim}{S}}$ is not objective.

To express rate-dependent behavior one must use an objective stress rate such as:

- Co-rotational rate : $\quad \overset{o}{\underset{\sim}{u}} \equiv \dot{\underset{\sim}{u}} - \underset{\sim}{W}\,\underset{\sim}{u} \quad ; \quad \overset{o}{\underset{\sim}{T}} \equiv \dot{\underset{\sim}{T}} - \underset{\sim}{W}\,\underset{\sim}{T} + \underset{\sim}{T}\,\underset{\sim}{W} \quad$ (Jaumann)
- Convected rate : $\quad \hat{\underset{\sim}{u}} \equiv \dot{\underset{\sim}{u}} + \underset{\sim}{L}^{T}\underset{\sim}{u} \quad ; \quad \hat{\underset{\sim}{T}} \equiv \dot{\underset{\sim}{T}} + \underset{\sim}{L}^{T}\underset{\sim}{T} + \underset{\sim}{T}\,\underset{\sim}{L} \quad$ (Cotter-Rivlin)
- Oldroyd rate ; Truesdell rate ; Green-Naghdi rate . . .