Force Method for Analysis of Indeterminate Structures

For determinate structures, the force method allows us to find internal forces (using equilibrium \textit{i.e.} based on Statics) irrespective of the material information. Material (stress-strain) relationships are needed only to calculate deflections.

However, for indeterminate structures, Statics (equilibrium) alone is not sufficient to conduct structural analysis. Compatibility and material information are essential.

Indeterminate Structures

Number of unknown Reactions or Internal forces $>$ Number of equilibrium equations
Note: Most structures in the real world are statically indeterminate.

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Smaller deflections for similar members</td>
<td>• More material $=&gt;$ More Cost</td>
</tr>
<tr>
<td>• Redundancy in load carrying capacity</td>
<td>• Complex connections</td>
</tr>
<tr>
<td>(redistribution)</td>
<td>• Initial / Residual / Settlement Stresses</td>
</tr>
<tr>
<td>• Increased stability</td>
<td></td>
</tr>
</tbody>
</table>

Methods of Analysis

Structural Analysis requires that the equations governing the following physical relationships be satisfied:

(i) Equilibrium of forces and moments
(ii) Compatibility of deformation among members and at supports
(iii) Material behavior relating stresses with strains
(iv) Strain-displacement relations
(v) Boundary Conditions

Primarily two types of methods of analysis:

\textbf{Force (Flexibility) Method}

\begin{itemize}
  \item Convert the indeterminate structure to a determinate one by removing some unknown forces / support reactions and replacing them with (assumed) known / unit forces.
  \item Using superposition, calculate the force that would be required to achieve compatibility with the original structure.
  \item Unknowns to be solved for are usually redundant forces
  \item Coefficients of the unknowns in equations to be solved are "flexibility" coefficients.
\end{itemize}

$$[A] \alpha = b$$

\textbf{Displacement (Stiffness) Method}

\begin{itemize}
  \item Express local (member) force-displacement relationships in terms of unknown member displacements.
  \item Using equilibrium of assembled members, find unknown displacements.
  \item Unknowns are usually displacements
  \item Coefficients of the unknowns are "Stiffness" coefficients.
\end{itemize}

$$[K] \delta = f$$
Example:

\[ \Delta_B + \left[ f_{BB} \right] B_y + \left[ f_{BC} \right] C_y = 0 \]
\[ \Delta_C + \left[ f_{CB} \right] B_y + \left[ f_{CC} \right] C_y = 0 \]
\[ \Rightarrow \left[ \Delta_B \right] + \left[ f_{BB} \quad f_{BC} \right] \begin{bmatrix} B_y \\ C_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Note: \[ f_{BC} = \int_0^L \frac{m_a(x) m_c(x)}{EI} \, dx \]
\[ f_{CC} = \int_0^L \frac{m_e(x) m_B(x)}{EI} \, dx \]

\[ \Rightarrow f_{BC} = f_{CC} \]

Maxwell's Theorem of Reciprocal displacements; Betti's law

For structures with multiple degree of indeterminacy

Virtual Work done by a system of forces \( P_B \) while undergoing displacements due to system of forces \( P_A \)
is equal to the

Virtual Work done by the system of forces \( P_A \) while undergoing displacements due to the system of forces \( P_B \)
Example: Determine the reactions.

Compatibility Equations. From Fig. 10-11a we require the relative rotation of one end of one beam with respect to the end of the other beam to be zero, that is,

\[ \theta_B + M_B \alpha_{BB} = 0 \]

where

\[ \theta_B = \theta_B^I + \theta_B^I \]

and

\[ \alpha_{BB} = \alpha_{BB}^I + \alpha_{BB}^I \]

\[ \Rightarrow (\theta_B^I + \theta_B^I) + \left[ \alpha_{BB}^I + \alpha_{BB}^I \right] M_B = 0 \]

The slopes and angular flexibility coefficients can be determined from the table on the inside front cover, that is,

\[ \theta_B^I = \frac{wL^3}{24EI} = \frac{120(12)^3}{24EI} = \frac{8640 \text{ lb} \cdot \text{ft}^2}{EI} \]

\[ \theta_B^I = \frac{PL^2}{16EI} = \frac{500(10)^3}{16EI} = \frac{3125 \text{ lb} \cdot \text{ft}^2}{EI} \]

\[ \alpha_{BB}^I = \frac{ML}{3EI} = \frac{1(12)}{3EI} = 4 \text{ ft} \]

\[ \alpha_{BB}^I = \frac{ML}{3EI} = \frac{1(10)}{3EI} = 3.33 \text{ ft} \]

Thus

\[ \frac{8640 \text{ lb} \cdot \text{ft}^2}{EI} + \frac{3125 \text{ lb} \cdot \text{ft}^2}{EI} + M_B \left( \frac{4 \text{ ft}}{EI} + \frac{3.33 \text{ ft}}{EI} \right) = 0 \]

\[ M_B = -1604 \text{ lb} \cdot \text{ft} \]
The negative sign indicates \( M_B \) acts in the opposite direction to that shown in Fig. 10-11c. Using this result, the reactions at the supports are calculated as shown in Fig. 10-11d. Furthermore, the shear and moment diagrams are shown in Fig. 10-11e.
Examples

Support B settles by 1.5 in.
Find the reactions and draw the Shear Force and Bending Moment Diagrams of the beam.

\[ (+) \ \nu (x) = \frac{p b x}{6L \ E I} \left( L^2 - b^2 - x^2 \right) \]

\[ \Delta_B = \nu \left( \frac{L}{2} \right) = \frac{-20 \times 12 \times 24 \left( 48^2 - 12^2 - 24^2 \right)}{6 \times 48 \times E I} \]

\[ = \frac{-31680}{EI} \ K \cdot \text{ft}^3 \]

\[ \Delta_B' = f_{BB} B_y = \left( \frac{L^3}{48EI} \right) B_y = \frac{2304}{EI} \ K \cdot \text{ft}^3 \]

Compatibility equation:

\[ \Delta_B + \Delta_B' = -1.5 \]

\[ \Rightarrow B_y = \frac{-1.5 - \Delta_B}{f_{BB}} \Rightarrow B_y = +5.56 \ K \]
Example: Frames

**EXAMPLE 10.5**

The frame, or bent, shown in the photo is used to support the bridge deck. Assuming $EI$ is constant, a drawing of it along with the dimensions and loading is shown in Fig. 10–12a. Determine the support reactions.

![Frame diagram](Image)

**Indeterminate Degree = 1.**

**PART-1:** Actual Loads

**PART-2:** Redundant Force $A_x$

**Compatibility Equation.** Reference to point $A$ in Fig. 10–12b requires

$$0 = \Delta A + A_x f_{AA}$$

**PART-1 (ΔA Deflection under Actual Loads)**

1(a) Real:

![Frame diagram](Image)

1(b) Virtual:

![Frame diagram](Image)
Principle of virtual work ⇒

\[ \Delta_A = \int_0^5 \left( \frac{m_3}{E} \right) dx = 2 \int_0^5 (1x_1) dx_1 + 2 \int_0^5 (200x_2)(-5) dx_2 \\
+ 2 \int_0^5 (1000 + 200x_3 - 20x_3^2)(-5) dx_3 \]

\[ = 0 - \frac{25000}{EI} - \frac{66666.7}{EI} = -\frac{91666.7}{EI} \]

**PART-2:** (f_{AA} Deflection under Redundant force):

2(a) Real:

![Real Diagram](image1)

2(b) Virtual:

![Virtual Diagram](image2)

---

**EXAMPLE 10.5 CONTINUED**

For \( f_{AA} \) we require application of a real unit load and a virtual unit load acting at \( A \), Fig. 10-12f. Thus,

\[ f_{AA} = \int_0^5 \left( \frac{m_3}{E} \right) dx = 2 \int_0^5 (1x_1)^2 dx_1 + 2 \int_0^5 (5)^2 dx_2 + 2 \int_0^5 (5)^2 dx_3 \]

\[ = \frac{583.33}{EI} \]

Substituting the results into Eq. (1) and solving yields

\[ \Delta_A + f_{AA} A_x = 0 \]

\[ 0 = -\frac{91666.7}{EI} + A_x \left( \frac{583.33}{EI} \right) \]

\[ A_x = 157 \text{ kN} \]

**Ans.**
**EXAMPLE 10.7**

Determine the force in member AC of the truss shown in Fig. 10-14a. AE is the same for all the members.

**SOLUTION**

**Principle of Superposition.** By inspection the truss is indeterminate to the first degree. Since the force in member AC is to be determined, member AC will be chosen as the redundant. This requires “cutting” this member so that it cannot sustain a force, thereby making the truss statically determinate and stable. The principle of superposition applied to the truss is shown in Fig. 10-14b.

**Compatibility Equation.** With reference to member AC in Fig. 10-14b, we require the relative displacement $\Delta_{AC}$, which occurs at the ends of the cut member AC due to the 400-lb load, plus the relative displacement $F_{AC}f_{ACAC}$ caused by the redundant force acting alone, to be equal to zero, that is,

$$0 = \Delta_{AC} + F_{AC}f_{ACAC}$$

*(Applying Eq. 3-1, $b + r > 2f$ or $6 + 3 > 2(4), 9 > 8, 9 - 8 = 1$st degree.)*

---

**PART 1 (\(\Delta_{AC}\) Relative Deflection of A & C due to Actual Loads)**

1(a) Real:

![Diagram of actual truss with loads and deflections]

**Principle of virtual work:**

$$W' = \Sigma N' L'$$

$$\Delta_{AC} = \Sigma \frac{nNL}{AE}$$

1(b) Virtual:

![Diagram of virtual truss with forces and virtual work]

$$= 2\left[\frac{(-0.8)(400)(8)}{AE}\right] + \frac{(-0.6)(0)(6)}{AE} + \frac{(-0.6)(300)(6)}{AE}$$

$$+ (1)(-500)(10) + \frac{(1)(0)(10)}{AE}$$

$$= \frac{11200}{AE}$$

---

*Force Method Page 8*
PART - 2 (from Relative Delegation of A & C due to Redundant force)

2a) Real:

2b) Virtual:

Principle of Virtual Work

We' = U's'

EXAMPLE 10.7 CONTINUED

\[ F_{AC} = \sum \frac{n_i L_i}{AE} = 2 \left[ \frac{(-0.8)^2(8)}{AE} \right] + 2 \left[ \frac{(-0.6)^2(6)}{AE} \right] + 2 \left[ \frac{(1)^2(10)}{AE} \right] = \frac{34.56}{AE} \]

Substituting the data into Eq. (1) and solving yields

\[ 0 = \frac{11200}{AE} + \frac{34.56}{AE} F_{AC} \]

\[ F_{AC} = 324 \text{ lb (T)} \]

Compatibility:

\[ \Delta_{AC} + [f_{AC}] F_{AC} = 0 \]

\[ \text{Ans.} \]

Since the numerical result is positive, AC is subjected to tension as assumed, Fig. 10-14b. Using this result, the forces in the other members can be found by equilibrium, using the method of joints.

Aside: If we also want to find the actual horizontal displacement of C, then we can use the method of virtual work.

In addition to the above "real" problem, we solve the following "virtual" problem:

(by scaling the response by \( \frac{P'}{400} \))

Using the principle of virtual work:

\[ \Delta_c = \sum_{m=1}^{N'} \frac{N_i}{EA} N_i \frac{L_i}{AE} = \frac{1}{AE} \left[ 0.352 \times 40.8 \times 8 + 0.264 \times 105.6 \times 6 + 0.81 \times 324 \times 10 + (0.44) \times (-176) \times 10 + (-0.486) \times (-194.4) \times 6 + 0.352 \times 140.8 \times 8 \right] \]

\[ = \frac{4925.93}{EA} \]
EXAMPLE 10.9

The simply supported queen-post trussed beam shown in the photo is to be designed to support a uniform load of 2 kN/m. The dimensions of the structure are shown in Fig. 10–16a. Determine the force developed in member CE. Neglect the thickness of the beam and assume the truss members are pin connected to the beam. Also, neglect the effect of axial compression and shear in the beam. The cross-sectional area of each strut is 400 mm², and for the beam I = 20(10⁶) mm⁴. Take E = 200 GPa.

Fig. 10–16

SOLUTION

Principle of Superposition. If the force in one of the truss members is known, then the force in all the other members, as well as in the beam, can be determined by statics. Hence, the structure is indeterminate to the first degree. For solution the force in member CE is chosen as the redundant. This member is therefore sectioned to eliminate its capacity to sustain a force. The principle of superposition applied to the structure is shown in Fig. 10–16b.

Compatibility Equation. With reference to the relative displacement of the cut ends of member CE, Fig. 10–16b, we require

$$0 = \Delta_{CE} + F_{CE}f_{CE,CE}$$

(1)
Part 1

**Displacement** $\Delta CE$

**Real Loads**

\[
\Delta_{CE} = \int_0^L \frac{Mm}{EI} \, dx + \sum \frac{nNL}{AE} = 2 \int_0^L (6x_1 - x_1)(-0.5x_1) \, dx_1 \\
+ 2 \int_2^3 (6x_2 - x_2^2)(-1) \, dx_2 \\
+ 2 \left( \frac{-0.5(0)(1)}{AE} \right) + \left( \frac{1(0)(2)}{AE} \right) \\
= -\frac{12}{EI} - \frac{17.33}{EI} + 0 + \frac{0.5}{AE} + \frac{2}{AE} \\
= \frac{-29.33(10^9)}{200(10^9)(20)(10^{-6})} = -7.333(10^{-3}) \text{ m}
\]

\[
\frac{d^2 \Delta}{dx^2} = 0
\]

\[
\Delta_{CE} = \int_0^L \frac{n^2L}{EI} \, dx + \sum \frac{n^2L}{AE} = 2 \int_0^L (-0.5x_1)^2 \, dx_1 \\
+ 2 \left( \frac{1.118^2(\sqrt{5})}{AE} \right) + 2 \left( \frac{-0.5^2(1)}{AE} \right) + \left( \frac{1.118^2(1)}{AE} \right) \\
= \frac{1.3333}{EI} + \frac{2}{EI} + \frac{5.590}{AE} + \frac{0.5}{AE} + \frac{2}{AE} \\
= \frac{3.333(10^9)}{200(10^9)(20)(10^{-6})} + \frac{8.090(10^9)}{400(10^{-6})(200)(10^9)} \\
= 0.9345(10^{-3}) \text{ m/kN}
\]

Substituting the data into Eq. (1) yields

\[
0 = -7.333(10^{-3}) \text{ m} + F_{CE}(0.9345(10^{-3}) \text{ m/kN}) \\
F_{CE} = 7.85 \text{ kN}
\]

Part 2

**Flexibility coeff** $f_{CECE}$

\[
f_{CECE} = \frac{AC}{BC} = \frac{1}{2}
\]

\[
AC = \frac{1}{2}
\]

\[
BC = \frac{1}{2}
\]
Systematic Analysis using the Force (Flexibility) Method

For structures with large number of redundant unknowns

Note: Maxwell's Theorem (Betti's Law) => Flexibility matrix is symmetric!

Example:

Member Properties: For any member "m" connected nodes I and J

\[
\begin{align*}
\vec{m}_I &= \frac{(x_J - x_I)}{\ell} \hat{i} + \frac{(y_J - y_I)}{\ell} \hat{j} = m_{Ix} \hat{i} + m_{Iy} \hat{j} \\
\vec{m}_J &= \frac{(x_I - x_J)}{\ell} \hat{i} + \frac{(y_I - y_J)}{\ell} \hat{j} = m_{Jx} \hat{i} + m_{Jy} \hat{j} \\
\ell &= \sqrt{(x_J - x_I)^2 + (y_J - y_I)^2}
\end{align*}
\]

<table>
<thead>
<tr>
<th>m</th>
<th>I</th>
<th>J</th>
<th>m_{Ix} (-m_{Iy})</th>
<th>m_{Iy} (-m_{Jy})</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>2</td>
<td>3/5</td>
<td>4/5</td>
</tr>
<tr>
<td>D</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>E</td>
<td>2</td>
<td>5</td>
<td>3/5</td>
<td>-4/5</td>
</tr>
<tr>
<td>F</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>G</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
**Equilibrium of nodes**

Given the forces and moments at node A:

\[ \begin{align*}
F_A & = 10 \text{ kN} \\
F_B & = 10 \text{ kN}
\end{align*} \]

And the forces at node B:

\[ \begin{align*}
F_C & = 3/5 \text{ kN} \\
F_D & = -4/5 \text{ kN}
\end{align*} \]

Similarly, for all other nodes:

\[ \begin{bmatrix}
1 & 0 & 1 \\
0 & -1 & 1 \\
-3/5 & 0 & 3/5 \\
0 & -4/5 & -4/5 \\
0 & 3/5 & 1 \\
1 & 4/5 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
-3/5 & -1 & 1/4/5 \\
4/5 & 0 \\
\end{bmatrix} \begin{bmatrix}
F_A \\
F_B \\
F_C \\
F_D \\
F_E \\
F_F \\
F_G \\
1_x \\
1_y \\
3_x \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
10 \\
0 \\
0 \\
0 \\
0 \\
0 \\
-10 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[ [A] \{N\} + \{f\} = \{0\} \]

\[ i.e. \]
\[ [A] \{N\} = -\{f\} \]
Compatibility equations for original indeterminate truss:

\[
\begin{align*}
\Delta_{5x} &= 0 \quad \Rightarrow \quad \Delta'_{5x} \text{ ACTUAL} + \Delta''_{5x} \text{ REDUNDANT} = 0 \\
\Delta_{5y} &= 0 \quad \Rightarrow \quad \Delta'_{5y} \text{ ACTUAL} + \Delta''_{5y} \text{ REDUNDANT} = 0
\end{align*}
\]

\[
[A] \begin{bmatrix} \{N_A\} \end{bmatrix} = -\{f_A\}
\]

\[
[N_R] = \begin{bmatrix} N_{R1} & N_{R2} & \cdots & N_{RN} \end{bmatrix}
\]

\[
[R_d] = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 1 \end{bmatrix}
\]

\[
[A] = \begin{bmatrix} 0 & -1.5000 & 0 & -1.0000 & 0 & 1.2500 & 0 & 0 & -1.2500 & 1.0000 & 0.7500 & 0 & 1.5000 & 0 & -1.0000 & -1.0000 & -1.5000 \end{bmatrix}
\]

\[
5x = 0 \quad 5y = 0
\]
Using the Principle of virtual work:

\[
\Delta_A = \sum_{m=1}^{M} N_{R1}^m \begin{bmatrix} N_{A_m} & L_m \end{bmatrix} \frac{A_m E_m}{m} = \{N_A\}^T \begin{bmatrix} F_m \end{bmatrix} \{N_{R1}\}^T \quad \text{where} \quad [F_m] = \frac{1}{AE} A \begin{bmatrix}
A & B & C & D & E & F & G \\
4 & A & B & C & D & E & F \\
5 & 4 & A & B & C & D & E \\
3 & 5 & 3 & A & B & C & D
\end{bmatrix}
\]

\(\text{global (un assembled) flexibility matrix.}\)

\[
\Delta_R = \sum_{m=1}^{M} N_{R1}^m \begin{bmatrix} N_{R1} & L_m \end{bmatrix} \frac{A_m E_m}{m} = \{N_{R1}\}^T \begin{bmatrix} F_m \end{bmatrix} \{N_{R1}\}
\]

Thus from the compatibility equation:

\[
\{\Delta_A\}^T_{Nd \times 1} + \left[ \Delta_R \right]^T_{Nd \times Nd} \{\text{diag}(fR)\}_{Nd \times 1} = 0
\]

\[
\frac{1}{AE} \begin{bmatrix} 0 \\
-196.8750
\end{bmatrix}^{2 \times 1} + \frac{1}{AE} \begin{bmatrix} 6.0000 & 4.5000 \\
4.5000 & 29.7500
\end{bmatrix}^{2 \times 2} \begin{bmatrix} S_x \\
S_y
\end{bmatrix}^{2 \times 2} = \begin{bmatrix} 0 \\
0
\end{bmatrix}^{2 \times 1}
\]

\[\Rightarrow \begin{bmatrix} S_x \\
S_y
\end{bmatrix} = \begin{bmatrix} -5.5983 \\
7.4645
\end{bmatrix} \text{ kN}\]
Analysis of Symmetric structures

Symmetry: Structure, Boundary Conditions, and Loads are symmetric.

Anti-symmetric: Structure, Boundary Conditions are symmetric, Loads are anti-symmetric.

Symmetry helps in reducing the number of unknowns to solve for.

Examples:

![Diagram of symmetric loading](attachment:image1)

![Diagram of antisymmetric loading](attachment:image2)

Figure 10_18

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Influence lines for *Determinate* structures

Influence line is a diagram that shows the variation for a particular force/moment at specific location in a structure as a unit load moves across the entire structure.

\[
V_c = \begin{cases} 
-\frac{x}{8} & 0 < x < 4 \\
1 - \frac{x}{8} & 4 < x < 12 
\end{cases}
\]

\[
M_c = \begin{cases} 
\left(1 - \frac{x}{8}\right)4 - 1(8-x) & 0 < x < 4 \\
\left(1 - \frac{x}{8}\right)4 = \left(1 - \frac{x}{2}\right) & 4 < x < 12 
\end{cases}
\]

Müller-Breslau Principle

The influence of a certain force (or moment) in a structure is given by (i.e. it is equal to) the deflected shape of the structure in the absence of that force (or moment) and when given a corresponding unit displacement (or rotation).

Example:

Principle of virtual work \( A_y \delta y - 1 \delta y' = 0 \)

\( \Rightarrow \) For influence line of \( A_y \) choose \( \delta y = 1 \) \( \Rightarrow A_y = \delta y' \)

Examples:

\( V_c \)

\( V_c \delta y - 1 \delta y' = 0 \)

\( \delta y = 1 \Rightarrow V_c = \delta y' \)

\( M_c \)

\( M_c \delta \theta - 1(\delta y) = 0 \)

\( \delta \theta = 1 \Rightarrow M_c = \delta y' \)
Example

Draw the influence lines for the reaction and bending-moment at point C for the following beam.

Using the Müller-Breslau principle:

**Reaction at C:**

\[ M_A = 0 \]

\[ (V_B \times 5) - 1 \times x = 0 \Rightarrow V_B = \frac{x}{5} \]

\[ M_D = 0 \]

\[ (V_B \times 10) - (C_y \times 5) = 0 \Rightarrow C_y = \frac{2x}{5} \]

\[ M_D = 0 \]

\[ 1 \times (15-x) - C_y \times 5 = 0 \Rightarrow C_y = \frac{3-x}{5} \]

**Bending Moment at C:**

\[ \theta = 1 = \frac{h}{5} \Rightarrow h = 5 \]

\[ M_c = -V_B \times 5 = -\frac{x}{5} \]

\[ M_c = -1 (10-x) = x - 10 \]

\[ M_c = 0 \]

**Influence line:**
Example

- Draw the influence lines for the shear-force and bending-moment at point C for the following beam.
- Find the maximum bending moment at C due to a 400 lb load moving across the beam.

Using Müller–Breslau principle:

\[
V_C = 0
\]

\[
\sum M_B = 0 \\
\Rightarrow -1 \times (x - 5) + D_y \times 10 \Rightarrow D_y = \frac{x}{10} - \frac{1}{2}
\]

\[
V_C = -D_y = \frac{V_2}{2} - \frac{x}{10}
\]

\[
\sum M_B = 0 \\
\Rightarrow 1 \times (15 - x) - V_B \times 10 = 0 \Rightarrow V_B = \frac{3}{2} - \frac{x}{10}
\]

\[
V_C = V_B = \frac{3}{2} - \frac{x}{10}
\]

⇒ Influence line:

\[
M_C = 0 \\
\Rightarrow M_C = V_B \times 5 = \frac{15}{2} - \frac{x}{2}
\]

\[
M_C = \frac{h_2}{2} \\
\Rightarrow h = \frac{15}{8}
\]
Influence lines for indeterminate structures

The Müller-Breslau principle also holds for indeterminate structures.

For statically determinate structures, influence lines are straight.

For statically indeterminate structures, influence lines are usually curved.

Examples:

Note: compatibility

\[ -1 \cdot f_{AD} + A_y \cdot f_{AA} = 0 \]

\[ A_y = \frac{f_{AD}}{f_{AA}} = \frac{f_{DA}}{f_{AA}} \]

Alternatively, using Müller-Breslau principle:

\[ P_{VW} \Rightarrow A_y \delta y - 1 \delta y' = 0 \]

choose \( \delta y = 1 \) ⇒ \( A_y = \delta y' \)

To find \( \delta y' \) when \( \delta y = 1 \):

\[ \delta y' = \frac{f_{DA}}{f_{AA}} \]
Example

Draw the influence line for

- Vertical reaction at A
- Moment at A

Using Müller-Breslau principle:

(i) Remove vertical reaction at A and give unit displacement

(ii) To find the equation of the deflected curve: Consider AB as a cantilever

Using table in the book:

\[ u(x) = -\frac{p}{6EI} \left( x^3 - 3Lx^2 \right) = -\frac{1}{2} \left( \frac{x^3}{L^3} - \frac{3x^2}{L^2} \right) \]

For \( 0 < x < 3 \):

\[ \Rightarrow \text{Influence line} = \Delta(x) = 1 - u(x) = 1 - \frac{1}{2} \left( \frac{x^3}{L^3} - \frac{3x^2}{L^2} \right) \]

For \( 3 < x < 6 \):

\[ \text{Influence line} = \Delta(x) = -\Theta_B (x - L) = -\frac{pL^2}{2EI} (x - L) \]

\[ = -\frac{1}{2} \left( \frac{x^3}{L^3} - \frac{3x^2}{L^2} \right) \]

Similarly for Moment at A:

For \( 0 < x < 3 \)

\[ \Delta(x) = \frac{(L-x)(x^2)}{2L} \]

For \( 3 < x < 6 \)

\[ \Delta(x) = -\Theta_B (x - L) = -\frac{1}{2} (x - L) \]