Tuesday, November 17, 2009
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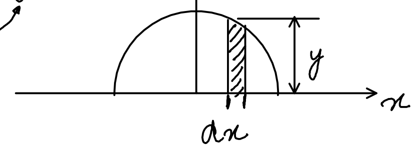
Chapter 9: Distributed Forces: Moments of Inertia

Recall from Chapter 5, we considered
- Centers of Areas, Volumes, Mass

We considered distributed forces which were proportional to the area or volume over which they act.
- The resultant Force was obtained by summing or integrating over the areas or volumes.
- The resultant Moment of the force about any axis was determined by computing the first moments of the areas or volumes about that axis.

Total Area: $A = \int dA = \int_{-r}^{r} \left(\int dy\right) dx = \int_{-r}^{r} y\, dx$

First Moment about y-axis: $Q_y = \int x\, dA = \int_{-r}^{r} x y\, dx$



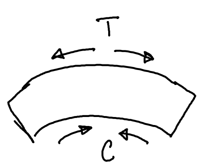
Centroid: $\bar{x} = \frac{Q_y}{A}$

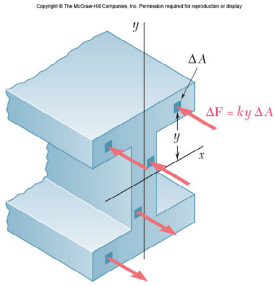
Now we will consider forces which are not only **proportional to the area or volume** over which they act but also **vary linearly with distance** from a given axis.

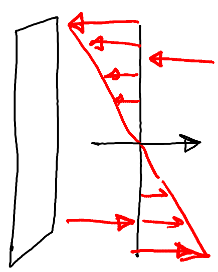
It can be shown that, when the force distribution varies linearly with distance from axis,
- The magnitude of the resultant Force is proportional to the first moment of the force distribution with respect to the axis.
- The magnitude of the resultant Moment is given by the **SECOND moments of the areas** about that axis.
- The point of application of the resultant force also depends on the second moment of the distribution with respect to the axis.

EXAMPLES:

① Beam Bending







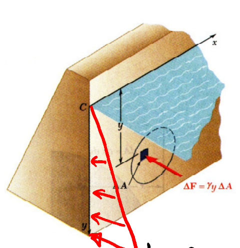
$\Delta\vec{F} = ky\Delta A$

$R = k\int y\, dA = 0 \quad \int y\, dA = Q_x$ = first moment

$M = k\int y^2 dA \quad \int y^2 dA$ = second moment

$\Delta M = \Delta F \cdot y = (k y \Delta A) y$

② Hydrostatic Pressure:



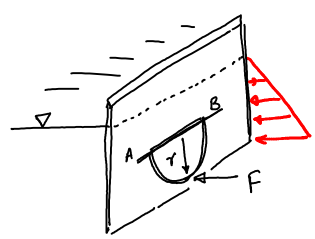
$p = \rho g y = \gamma y$

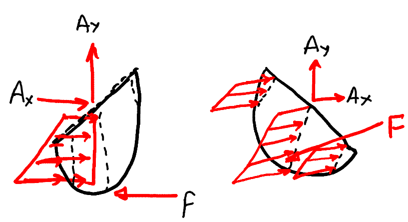
$\Delta F = p\Delta A = \gamma y \Delta A$

$R = \gamma \int y\, dA$

$M_x = \gamma \int y^2 dA = \gamma I_x$

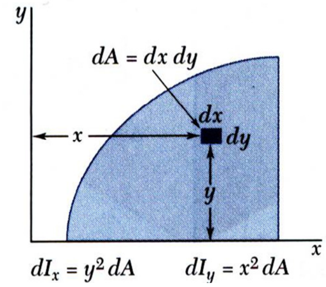
$\Delta M_x = (\gamma y \Delta A)\, y$





$\sum M_{AB} = 0 \Rightarrow F \cdot r - \gamma \cdot I_x = 0$

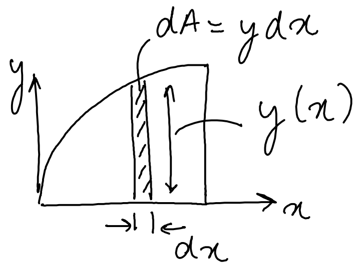
9.3 Second Moment of area by Integration

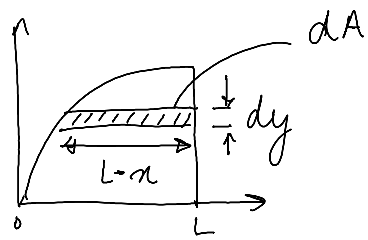


Second moments of area with respect to the *x* and *y* axes, are defined as:

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

Evaluation of the integrals is simplified by choosing dA to be a thin strip parallel to one of the coordinate axes.

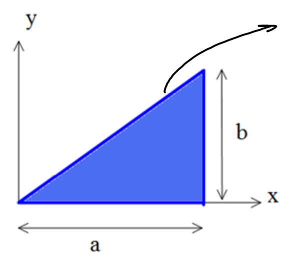




$$I_y = \int x^2 dA = \int x^2 y(x)\,dx \qquad I_x = \int y^2 dA = \int y^2 (L - x(y))\,dy$$

Example

Find the second moment of Area and the radii of gyration about the x-axis and the y-axis.



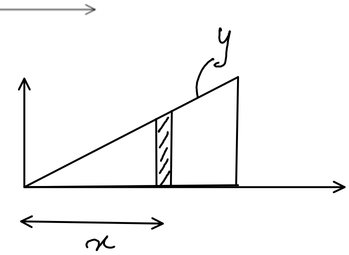
$y = \frac{b}{a}x$

$x = \frac{a}{b}y$

Moment of inertia about y-axis

$$I_y = \int_0^a x^2 y\,dx = \int_0^a \frac{b}{a}x^3 dx$$

$$= \frac{b}{\cancel{a}}\frac{a^{\cancel{4}3}}{4} \Rightarrow \boxed{I_y = \frac{a^3 b}{4}}$$



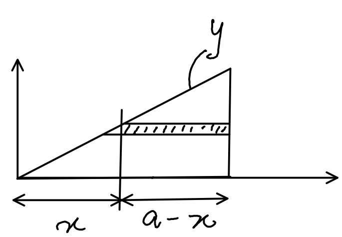
Radius of gyration :

$$I_y = (k_y)^2 A \Rightarrow k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{a^3 b}{4}\,\frac{2}{ab}} = \boxed{\frac{a}{\sqrt{2}}}$$

Moment of inertia about x-axis:

$$I_x = \int y^2 dA = \int_0^b y^2 (a - x)\,dy$$

$$= \int_0^b y^2\left(a - \frac{a}{b}y\right)dy = a\frac{b^3}{3} - \frac{a}{\cancel{b}}\frac{b^{\cancel{4}3}}{4} = \boxed{\frac{ab^3}{12}}$$



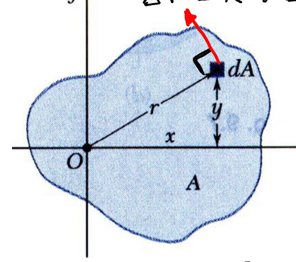
Radius of gyration: $I_x = (k_x)^2 A \Rightarrow k_x = \sqrt{\frac{I_x}{A}} = \boxed{\frac{b}{\sqrt{6}}}$

9.4 Polar moment of Inertia

For problems involving **torsion** (or twisting), the resisting forces are proportional to the polar moment of inertia, defined as:

$$J_0 = \int r^2 dA \qquad (r^2 = x^2 + y^2)$$

$\Delta F = k\, r\, \Delta A$



$\Delta M = \Delta F\, r$

$$M = k \underbrace{\int r^2 dA}_{J_0}$$

The polar moment of inertia is related to the rectangular moments of inertia:

$$J_0 = \int (x^2+y^2)\, dA = \underbrace{\int x^2 dA}_{I_y} + \underbrace{\int y^2 dA}_{I_x}$$

$$J_0 = I_y + I_x$$

9.5 Radius of Gyration

Radius of gyration (***k***) is the distance from the axis where an area can be thought of as being concentrated for purposes of evaluating the second moment of area / moment of inertia.

(Similar to the concept of center of area: where one can think of the area as being concentrated for evaluating the first moment of area)

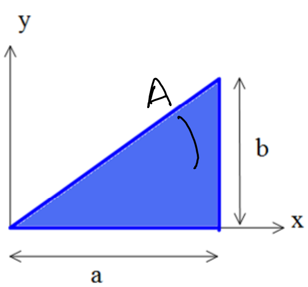
Analogous to:

$I_x = k_x^2 A$	$\boxed{k_x = \sqrt{\dfrac{I_x}{A}}}$	Radius of gyration about x-axis	$Q_x = \bar{y} A$
$I_y = k_y^2 A$	$\boxed{k_y = \sqrt{\dfrac{I_y}{A}}}$	Radius of gyration about y-axis	$Q_y = \bar{x} A$
$J_O = k_O^2 A$	$\boxed{k_O = \sqrt{\dfrac{J_O}{A}}}$	Polar Radius of gyration	

Note:

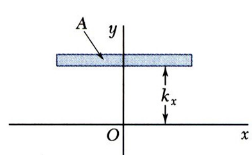
$$k_O^2 = k_x^2 + k_y^2$$

For example, consider the triangle:

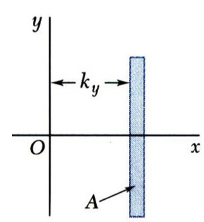


Equivalent concentrated areas for

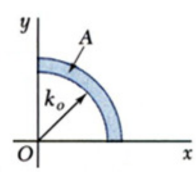
I_x



I_y



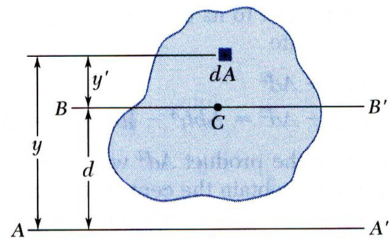
J_0



9.6 Parallel Axis Theorem

Consider moment of inertia I of an area A with respect to the axis AA'

$$I = \int y^2 dA$$

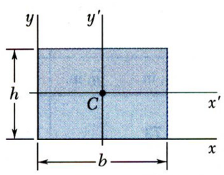
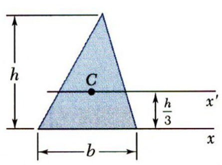
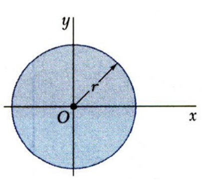
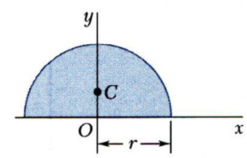
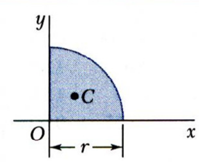
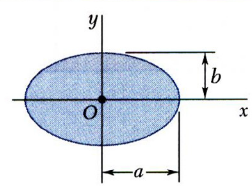


This moment of inertia can be calculated using the moment of inertia about the **centroidal axis**:

$$I_{AA'} = \int (y'+d)^2 dA = \underbrace{\int y'^2 dA}_{I_{BB'}} + 2d \underbrace{\cancel{\int y' dA}}_{Q_x \to 0} + d^2 \underbrace{\int dA}_{A}$$

$$\boxed{I = \bar{I} + Ad^2}$$

$\boxed{\text{IMP}}$ only if BB' is a centroidal axis

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $+ \frac{bh^3}{4}$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2+h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3 b$ $J_O = \frac{1}{4}\pi ab(a^2+b^2)$

9.7 Moment of Inertia of a composite Areas

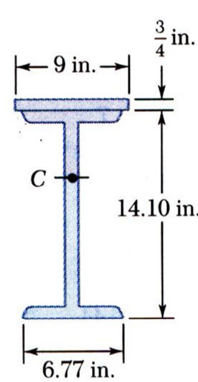
The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, ... , with respect to the same axis.

You may have to refer to Tables 9.12 and 9.13 A & B in your book to find the individual areas and moment of inertias.

Example 9.4

The strength of a W14x38 rolled steel beam is increased by attaching a plate to its upper flange.

Determine the moment of inertia and radius of gyration with respect to an axis which is parallel to the plate and passes through the centroid of the section.

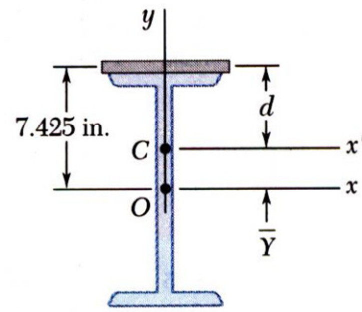


SOLUTION:

1) Determine location of the centroid of composite section with respect to a coordinate system with origin at the centroid of the beam section.

2) Apply the parallel axis theorem to determine moments of inertia of beam section and plate with respect to composite section centroidal axis.

3) Calculate the radius of gyration from the moment of inertia of the composite section.



Step (1)

Section	A, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
Plate	6.75	7.425	50.12
Beam Section	11.20	0	0
	$\sum A = 17.95$		$\sum \bar{y}A = 50.12$

$$\bar{Y}\sum A = \sum \bar{y}A \qquad \bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{50.12\text{ in}^3}{17.95\text{ in}^2} = 2.792\text{ in.}$$

Step (2)

$$I_{x',\text{beam section}} = \bar{I}_x + A\bar{Y}^2 = 385 + (11.20)(2.792)^2$$
$$= 472.3\text{ in}^4$$

$$I_{x',\text{plate}} = \bar{I}_x + Ad^2 = \tfrac{1}{12}(9)\left(\tfrac{3}{4}\right)^3 + (6.75)(7.425 - 2.792)^2$$
$$= 145.2\text{ in}^4$$

$$I_{x'} = I_{x',\text{beam section}} + I_{x',\text{plate}} = 472.3 + 145.2$$

$$\boxed{I_{x'} = 618\text{ in}^4}$$

Step (3)

$$k_{x'} = \sqrt{\frac{I_{x'}}{A}} = \frac{617.5\text{ in}^4}{17.95\text{ in}^2}$$

$$\boxed{k_{x'} = 5.87\text{ in.}}$$