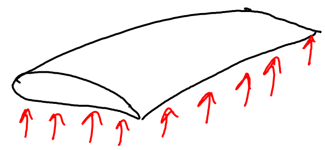
Chapter 5: Distributed Forces; Centroids and Centers of Gravity

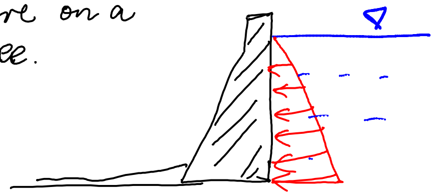
What are distributed forces?

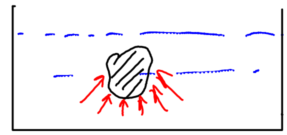
- Forces that act on a body per unit length, area or volume.
- They are not discrete forces that act at specific points. Rather they act over a continuous region.

Examples:

- Weight of any body.
- Lift force on an airplane wing.
- Buoyancy force on a submerged object.
- Hydrostatic Pressure on a dam or levee.



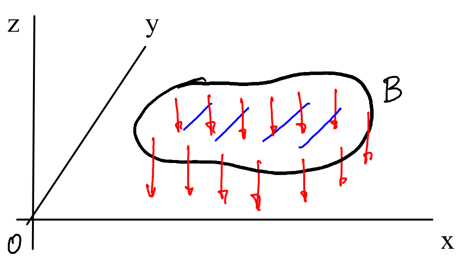


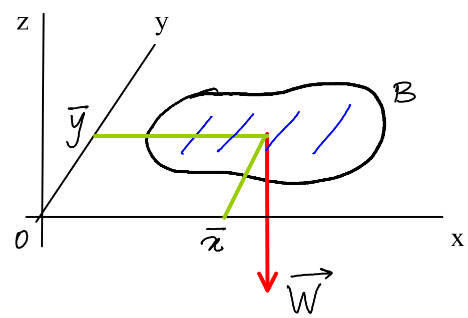


5.2 Center of Gravity

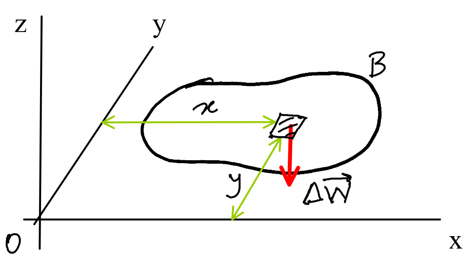
Gravity pulls each and every particle of a body vertically downwards.

What is the location of the equivalent single force that replaces all the distributed forces.





Let the location be $(\bar{x}, \bar{y})$



$$\boxed{\vec{W} = \lim_{\Delta W \to 0}\left(\sum \Delta\vec{W}\right) = \int_B d\vec{W}}$$

For the location

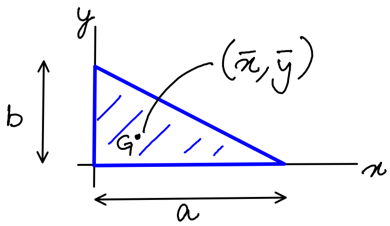
$$\sum M_{ox} = -\lim_{\Delta W \to 0}\left(\sum y\,\Delta W\right) = -\int_B y\,dW = -\bar{y}\,W$$

$$\sum M_{oy} = \lim_{\Delta W \to 0}\left(\sum x\,\Delta W\right) = \int_B x\,dW = \bar{x}\,W$$

$$\Rightarrow \boxed{\bar{y} = \frac{\int_B y\,dW}{W} \qquad \bar{x} = \frac{\int_B x\,dW}{W}}$$

Example

Find the Center of Gravity of the area shown below.



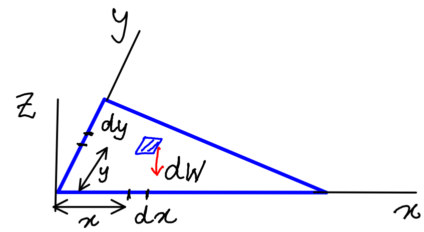
Equation of the line:

$$y = -\frac{b}{a}x + b$$

OR

$$x = -\frac{a}{b}y + a$$

Total Weight:



$$W = \int dW = \int (\underset{\text{density}}{\rho}\, g\, \underset{\text{thickness}}{t})\, dA$$

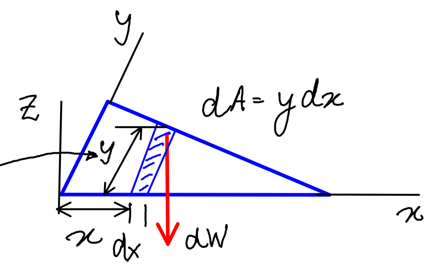
$$= \rho g t \int_0^a \left(\int_0^y dy \right) dx$$

$$= \rho g t \int_0^a y \, dx$$

$$= \rho g t \int_0^a \left(-\frac{b}{a}x + b \right) dx$$

$$= \rho g t \left[-\frac{b}{a}\left(\frac{x^2}{2}\right) + bx \right]_0^a$$

$$W = \rho g t \left[-\frac{ab}{2} + ab \right] = \boxed{\rho g t \left(\frac{1}{2}ab\right)}$$



For Center of gravity:

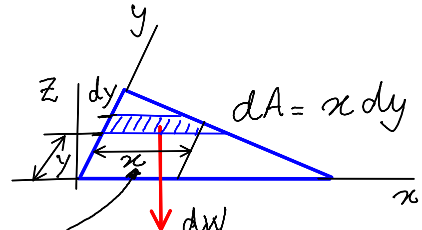
$$W\bar{x} = \int x \, dW = \rho g t \int_0^a x \underbrace{\int_0^y dy}_{y} \, dx$$

$$= \rho g t \int_0^a x\left(-\frac{b}{a}x + b\right) dx$$

$$= \rho g t \left[-\frac{b}{a}\frac{x^3}{3} + b\frac{x^2}{2} \right]_0^a$$

$$= \rho g t \left(\frac{a^2 b}{6}\right) \qquad \Rightarrow \bar{x} = \frac{\int x \, dW}{W} = \boxed{\frac{a}{3}}$$

Similarly:

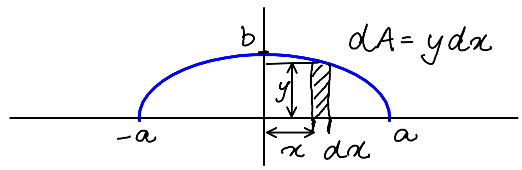


$$W\bar{y} = \int y \, dW = \rho g t \int_0^b y \underbrace{\left(\int_0^x dx\right)}_{x} dy$$

$$= \rho g t \int_0^b y\left(-\frac{a}{b}y + a\right) dy$$

$$= \rho g t \left(\frac{ab^2}{6}\right) \qquad \Rightarrow \bar{y} = \frac{\int y \, dW}{W} = \boxed{\frac{b}{3}}$$

Example:
Centroid of a Quarter or Semi Ellipse.



Equation :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - x^2}$$

Total Area:

$$A = \int dA = \int_{-a}^{a} y\,dx = \frac{b}{a}\int_{-a}^{a}\sqrt{a^2 - x^2}\,dx$$

$$\left\{ \begin{array}{l} \text{Substitute} \\ x = a\cos\theta \\ \Rightarrow dx = -a\sin\theta\,d\theta \\ x = -a \Rightarrow \theta = \pi \\ x = a \Rightarrow \theta = 0 \end{array} \right\}$$

$$= \frac{b}{a}\int_{\pi}^{0} a\sqrt{1 - \cos^2\theta}\,(-a\sin\theta)\,d\theta$$

$$= -ab\int_{\pi}^{0}\sin^2\theta\,d\theta = -ab\int_{\pi}^{0}\left(\frac{1 - \cos 2\theta}{2}\right)d\theta$$

$$= -ab\left[\frac{1}{2}\theta\right]_{\pi}^{0} + \frac{ab}{2}\left[\frac{\sin 2\theta}{2}\right]_{\pi}^{0} \Rightarrow A = \begin{cases} \frac{\pi}{2}ab & \text{(Semi-Ellipse)} \\ \frac{\pi}{4}ab & \text{(Quarter-Ellipse)} \\ \pi ab & \text{(Ellipse)} \end{cases}$$

Moments of area:- (Quarter)

$$Q_y = \bar{x}A = \int x\,dA = \int x\,y\,dx$$

Note: about Y-axis

$$= \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\;x\,dx$$

Substitute
$u = a^2 - x^2$
$\Rightarrow \frac{du}{-2} = x\,dx$
$x = 0 \Rightarrow u = a^2$
$x = a \Rightarrow u = 0$

$$= \frac{b}{a}\int_{a^2}^{0}\sqrt{u}\;\frac{du}{-2} = \frac{-b}{2a}\left[\frac{u\sqrt{u}}{\left(\frac{3}{2}\right)}\right]_{a^2}^{0}$$

$$= \frac{-b}{3a}\left(-a^2\sqrt{a^2}\right) = \frac{a^2 b}{3}$$

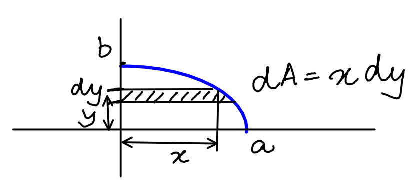
$$\Rightarrow \bar{x} = \begin{cases} \left(\frac{a^2 b}{3}\right)\Big/\left(\frac{\pi}{4}ab\right) = \boxed{\frac{4}{3}\frac{a}{\pi}} & \text{(Quarter)} \\ 0 = 0 & \text{(Semi)} \end{cases}$$

Similarly:

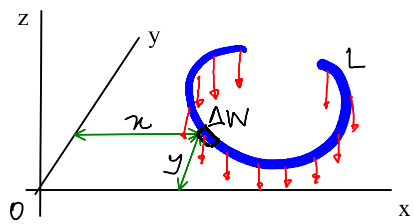
$$Q_x = \bar{y}A = \int y\,dA = \int y \cdot x \cdot dy$$

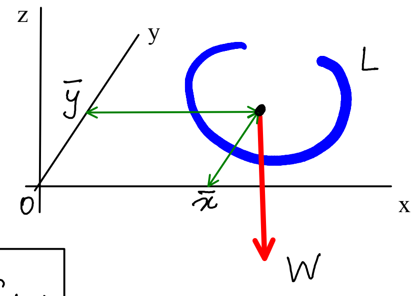
About x-axis

$$= \int_0^b \underbrace{\frac{a}{b}\sqrt{b^2 - y^2}}_{x}\;y\,dy \longrightarrow \text{Finish yourself.}$$



Centroids of Lines





Once again:

$$\boxed{W = \lim_{\Delta W \to 0} \sum \Delta W = \int_L dW \qquad \bar{x} = \frac{\int_L x\, dW}{W} \; ; \; \bar{y} = \frac{\int_L y\, dW}{W}}$$

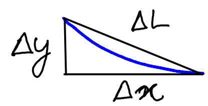
$$\int_L dW = \int_L \rho A \, dL$$

density

area of cross-section

In the limit: $dL = \sqrt{dx^2 + dy^2}$

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$\Delta L = \sqrt{\Delta x^2 + \Delta y^2}$$

Question: What if shape of the wire changes?

Exercise 5.45

Find the Centroid of the wire shown.

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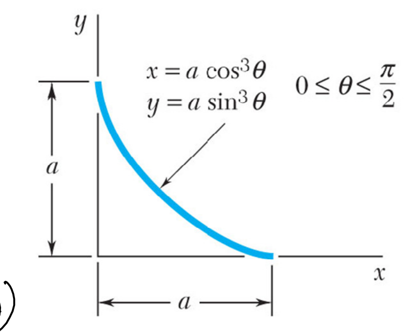


Total weight: $W = \int_L dW = \int_L \rho A \, dL$

$$= \rho A \int_L \sqrt{dx^2 + dy^2}$$

Substitute

$$x = a\cos^3\theta \Rightarrow \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

$$y = a\sin^3\theta \Rightarrow \frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\Rightarrow W = \rho A \int_0^{\pi/2} \sqrt{(3a)^2[\cos^2\theta(-\sin\theta)]^2 + (3a)^2[\sin^2\theta\cos\theta]^2}\; d\theta$$

$$= \rho A \int_0^{\pi/2} 3a\cos\theta\sin\theta \underbrace{\sqrt{\cos^2\theta + \sin^2\theta}}_{1}\, d\theta$$

$$= \rho A \int_0^{\pi/2} \frac{3a}{2}\sin 2\theta\, d\theta = \rho A\left(\frac{3a}{2}\right)\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} \Rightarrow W = \boxed{\rho A\left(\frac{3a}{2}\right)}$$

Moment about x:-

$$M_x = -\int_L y\,dW = -\bar{y}W$$

$$\Rightarrow \bar{y} = \frac{1}{W}\int_L y\,dW = \frac{1}{W}\int_0^{\pi/2} \underbrace{(a\sin^3\theta)}_{y}\ \underbrace{\rho A\, 3a \cos\theta \sin\theta\, d\theta}_{dW}$$

$$= \frac{(2)}{\cancel{\rho A}\cancel{(3a)}}\ \cancel{\rho A\, 3a}\ a\int_0^{\pi/2} \sin^4\theta \cos\theta\, d\theta$$

Let $u = \sin\theta$

$\Rightarrow \frac{du}{d\theta} = \cos\theta$

limits:

$\theta = 0 \Rightarrow u = 0$

$\theta = \pi/2 \Rightarrow u = 1$

$$= 2a\int_0^1 u^4\, du = 2a\left[\frac{u^5}{5}\right]_0^1$$

$$\Rightarrow \boxed{\bar{y} = \frac{2a}{5}}$$

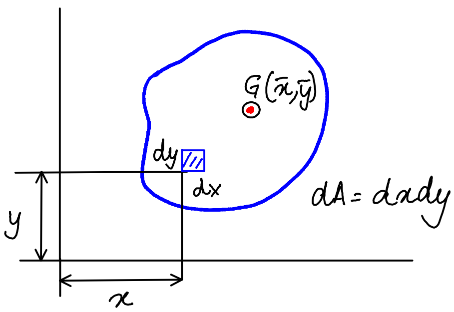
By symmetry $\boxed{\bar{x} = \frac{2a}{5}}$

5.3 - 5.4 Centroids and First Moments of Areas & Lines.

Definition:
First Moment of an Area

(about y-axis) $Q_y = \int x \, dA = \bar{x} A$

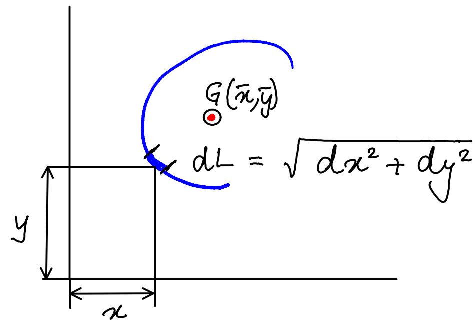
(about x-axis) $Q_x = \int y \, dA = \bar{y} A$



Definition:
First Moment of a Line

$Q_y = \int x \, dL = \bar{x} L$

$Q_x = \int y \, dL = \bar{y} L$

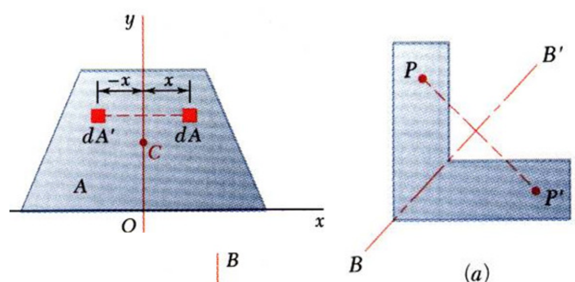


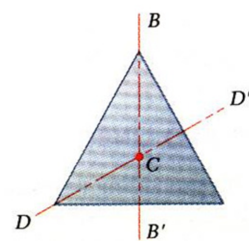
Properties of Symmetry

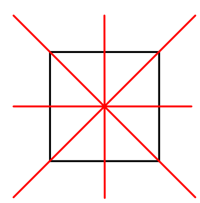
- An area is symmetric with respect to an axis *BB'* if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.
  - The first moment of an area with respect to a line of symmetry is zero.
  - If an area possesses a line of symmetry, its centroid lies on that axis
  - If an area possesses two lines of symmetry, its centroid lies at their intersection.

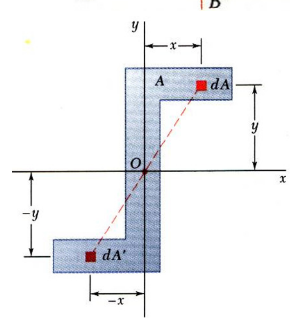
- An area is symmetric with respect to a center *O* if for every element *dA* at (*x*,*y*) there exists an area *dA'* of equal area at (-*x*,-*y*).
  - The centroid of the area coincides with the center of symmetry.

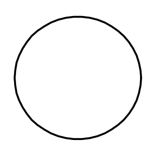
NOTE:
- Centroid of any area always exists.
- But, a center of symmetry may or may not exist.





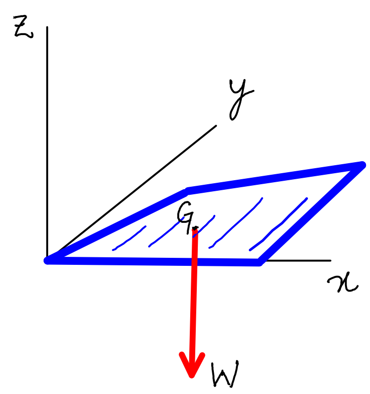


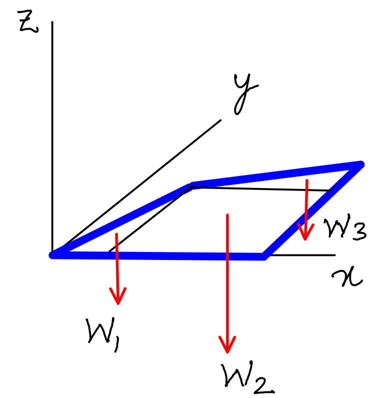




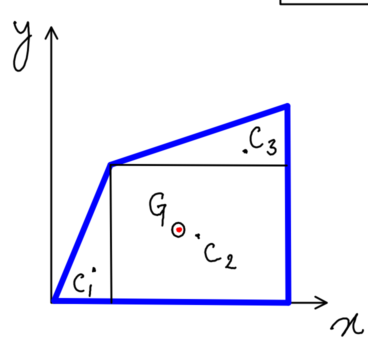
5.5 Composite Areas and Lines

The Centroid of an area (or line) that is made up of several simple shapes can be found easily using the centroids of the individual shapes.





$$\boxed{W = \sum W_i = W_1 + W_2 + W_3 \cdots}$$



$$\left.\begin{aligned} C_1 &: (\bar{x}_1, \bar{y}_1) \\ C_2 &: (\bar{x}_2, \bar{y}_2) \\ C_3 &: (\bar{x}_3, \bar{y}_3) \\ &\;\vdots \qquad \vdots \end{aligned}\right\} \quad G(\bar{X}, \bar{Y})$$

To find $G = (\bar{X}, \bar{Y})$:

$$M_y = \bar{X}\, W = \sum \bar{x}_i W_i$$

(moment about y)

$$= (\bar{x}_1 W_1 + \bar{x}_2 W_2 + \bar{x}_3 W_3 \cdots)$$

$$\Rightarrow \boxed{\bar{X} = \frac{1}{W}(\bar{x}_1 W_1 + \bar{x}_2 W_2 + \bar{x}_3 W_3 \cdots)}$$

Similarly:

$$M_x = -\bar{Y}\, W = -\sum \bar{y}_i W_i$$

(moment about x)

$$= -(\bar{y}_1 W_1 + \bar{y}_2 W_2 + \bar{y}_3 W_3 + \cdots)$$

$$\Rightarrow \boxed{\bar{Y} = \frac{1}{W}(\bar{y}_1 W_1 + \bar{y}_2 W_2 + \bar{y}_3 W_3 + \cdots)}$$

Note:
If an area is composed by adding some shapes and subtracting other shapes, then the moments of the subtracted shapes need to be subtracted as well.

Exercise 5.7

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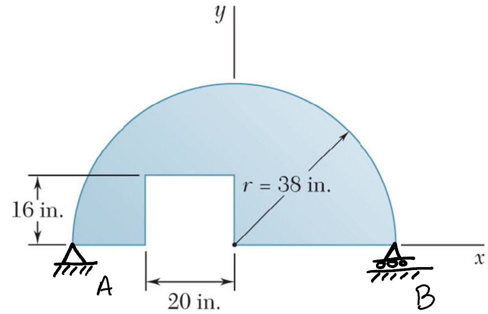
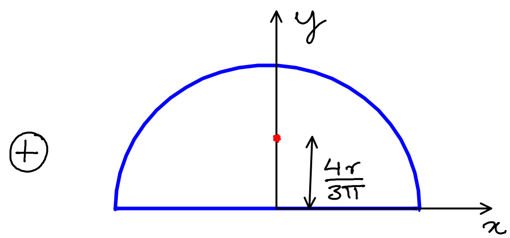
Find the centroid of the figure shown.

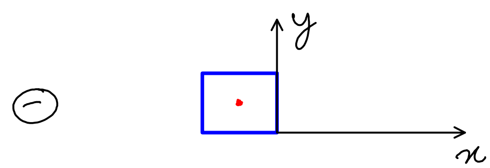
Find the reactions at A & B.
(specific weight γ = 0.28 lb/in ^{3}; thickness t = 1 in)



	Area	$\bar{x}$	$\bar{y}$	Q_y	Q_x
⊕ A1:	$\frac{\pi r^2}{2}$	0	$\frac{4r}{3\pi}$	0	$\left(\frac{4r}{3\pi}\right)\left(\frac{\pi r^2}{2}\right)$
	2268.23	0	16.13	0	36581.33
⊖ A2:	320	−10	8	−3200	2560.0
	1948.23	$\bar{X}$	$\bar{Y}$	+3200	34021.33



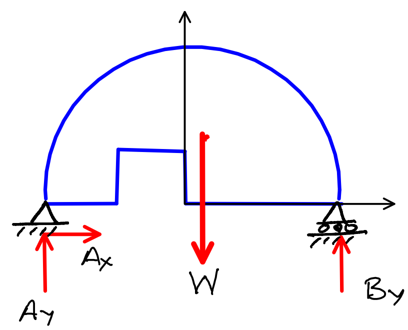
$$\bar{X} = \frac{Q_y}{A} = 1.643 \text{ in}$$



$$\bar{Y} = \frac{Q_x}{A} = 17.463 \text{ in}$$

To find the reactions:

FBD:



$\sum F_x = 0 \Rightarrow \quad A_x = 0$

$\sum F_y = 0 \Rightarrow \quad -W + A_y + B_y = 0$

$\Rightarrow -\gamma A t + A_y + B_y = 0$

$\Rightarrow -545.5 + A_y + B_y = 0$

$\sum M_A = 0 \Rightarrow \quad -W(38 + \bar{X}) + B_y(76) = 0$

$\Rightarrow \boxed{B_y = 284.5 \text{ lb}} \Rightarrow \boxed{A_y = 260.96 \text{ lb}}$

Exercise 5.28

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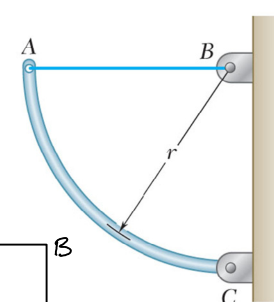
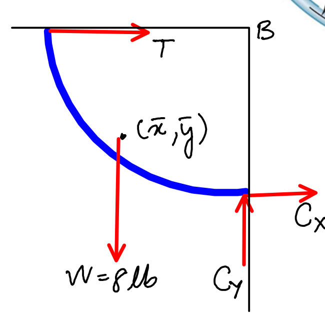
A uniform circular rod of weight 8 lb and radius r=10 in is shown.
Determine the tension in the cable AB & the reaction at C.



$$\bar{x} = \bar{y} = -\frac{2r}{\pi} = -6.3662 \text{ in}$$

FBD:



$\sum F_x = 0 \Rightarrow \quad T + C_x = 0$

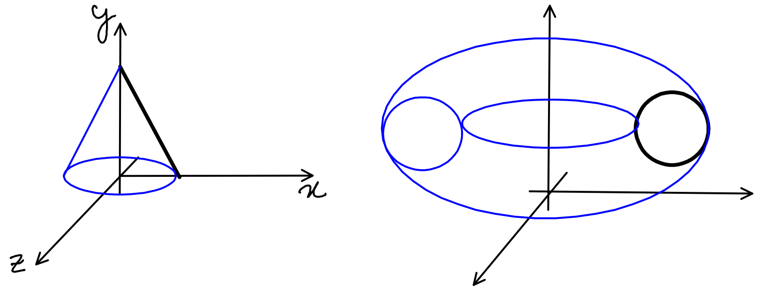
$\sum F_y = 0 \Rightarrow \quad -W + C_y = 0 \quad \Rightarrow \boxed{C_y = 8 \text{ lb}}$

$\sum M_c = 0 \Rightarrow \quad -T \times 10 + W\left(\frac{2r}{\pi}\right) = 0$

$\Rightarrow \boxed{T = 5.093 \text{ lb}} \quad \boxed{C_x = -5.093 \text{ lb}}$

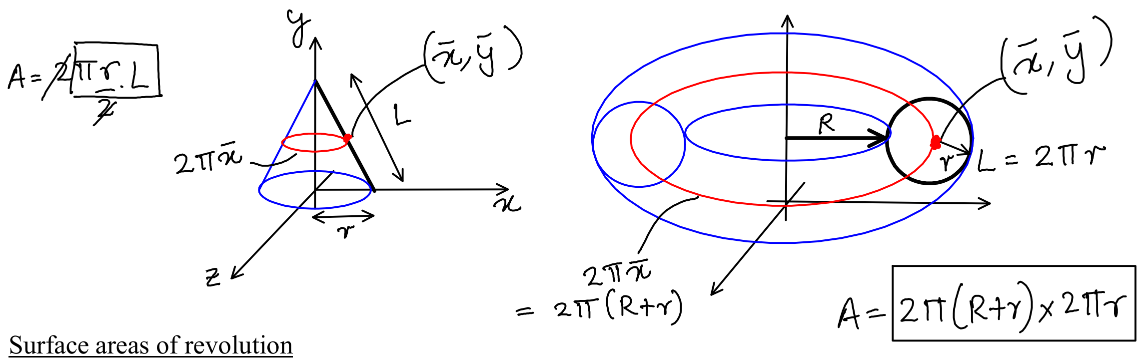
5.7 Surfaces & Volumes of Revolution: Theorems of Pappus-Guldinus

Surfaces of revolution are obtained when one "sweeps" a 2-D curve about a fixed axis.



Theorem 1

Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.



Surface areas of revolution

Rotating about y-axis: $\boxed{A = 2\pi \bar{x} L}$

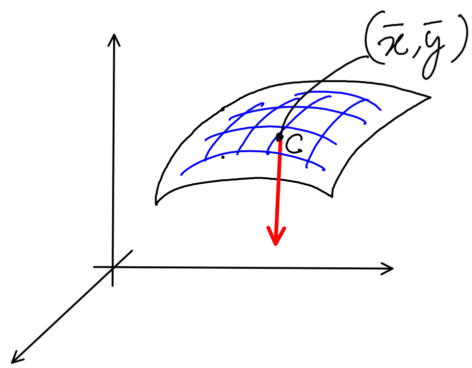
Rotating about x-axis: $\boxed{A = 2\pi \bar{y} L}$

General 3D surfaces (aside)

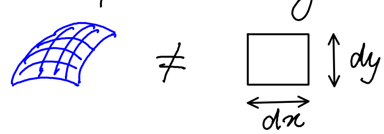
The concepts of area, centers of areas, and Moments of areas can also be extended to general 3D surfaces.

The same integral formulas still hold:

1. Area : $A = \int dA$
2. Moment about x: $Q_x = \int y\,dA = \bar{y} A$
3. Moment about y: $Q_y = \int x\,dA = \bar{x} A$
4. Centroid: $\bar{x} = \dfrac{\int x\,dA}{A}$, $\bar{y} = \dfrac{\int y\,dA}{A}$



Note: However "dA" is not as simple as "dxdy"



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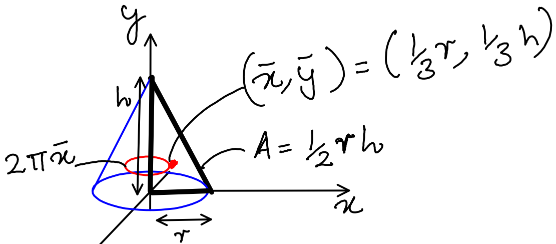
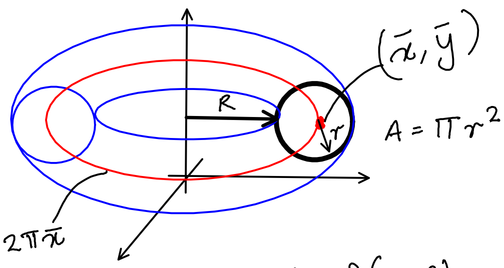
Theorem 2

Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.



$V = \cancel{2}\pi \frac{r}{3} \cdot \frac{1}{\cancel{2}} r h = \boxed{\frac{1}{3}\pi r^2 h}$



$V = 2\pi(R+r)(\pi r^2)$

Volumes of Revolution

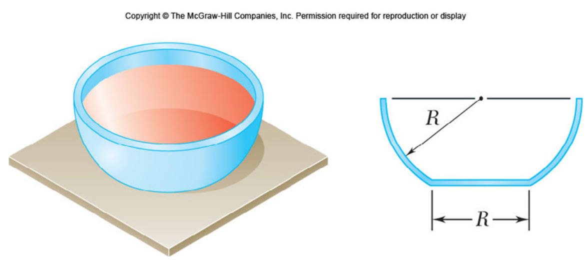
Rotating about y-axis: $\boxed{V = 2\pi\bar{x}A}$

Rotating about x-axis: $\boxed{V = 2\pi\bar{y}A}$

Exercise 5.59

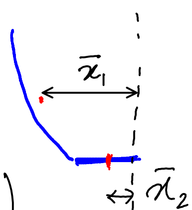
Find the internal surface area and the volume of the punch bowl.

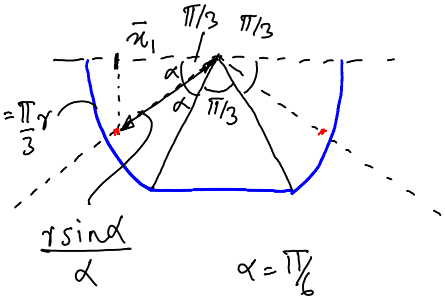
Given R= 250 mm.



$\bar{x}_1 = \left(\frac{r\sin\alpha}{\alpha}\right)(\cos\alpha)$

$\bar{x}_2 = \frac{r}{4}$





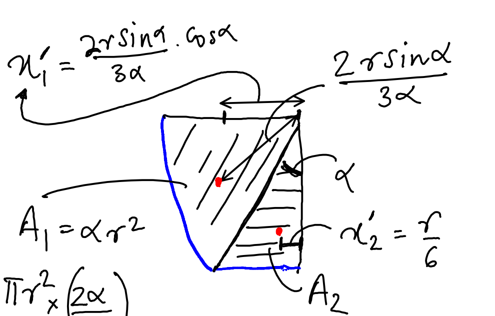
Surface Area

$A = 2\pi\bar{x}_1 L_1 + 2\pi\bar{x}_2 L_2$

$= 2\pi\left(\frac{r\sin(\pi/6)}{(\pi/6)}\right)\cdot\cos(\pi/6)\cdot\left(\frac{\pi}{3}r\right)$

$+ 2\pi\left(\frac{r}{4}\right)\cdot\left(\frac{r}{2}\right)$

$= 364631.1 \text{ mm}^2$



$A_2 = \frac{1}{2}\left(\frac{r}{2}\right) r\cos\alpha$

Volume

$V = 2\pi x_1' A_1 + 2\pi x_2' A_2$

$= 2\pi\left(\frac{2r\sin\alpha}{3\alpha}\right)\cos\alpha\cdot(\alpha r^2)$

$+ 2\pi\left(\frac{r}{6}\right)\left(\frac{1}{2}\frac{r}{2}r\cos\alpha\right)$

$= 31.88$ litres

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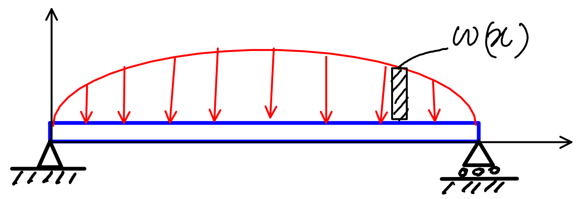
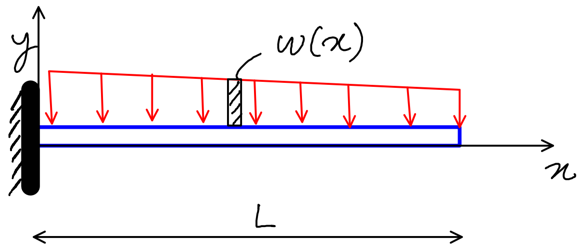
5.8 Distributed Loads on Beams

In several applications, engineers have to design beams that carry distributed loads along their length.
$w(x)$ is weight per unit length.



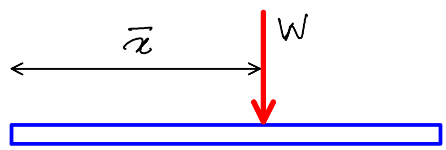
Simply supported beam.



Cantilever Beam

Total weight:

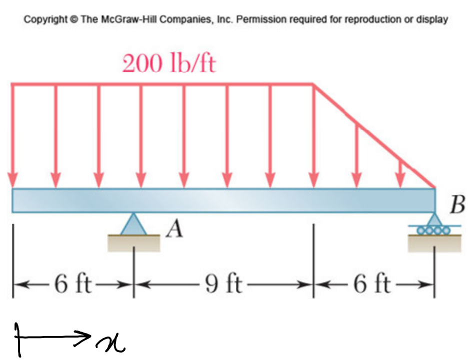
$$W = \int_0^L w\,dx$$



Point of action:

$$\bar{x} = \frac{Q_y}{W} = \frac{\int_0^L x\,w\,dx}{W}$$

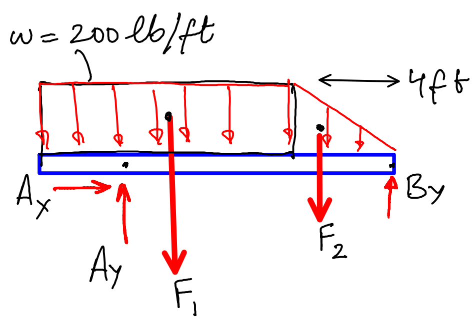
Exercise 5.70



Find the reactions at the supports.

$$F_1 = \int_0^{15} w\,dx = \int_0^{15} 200\,dx = 200 \times 15 = \underline{3000\ lb}$$

$$F_2 = \int_{15}^{21} w\,dx = \underline{600 lb}$$



$\Sigma F_x = 0$

$\Rightarrow A_x = 0$

$\Sigma F_y = 0 \Rightarrow$

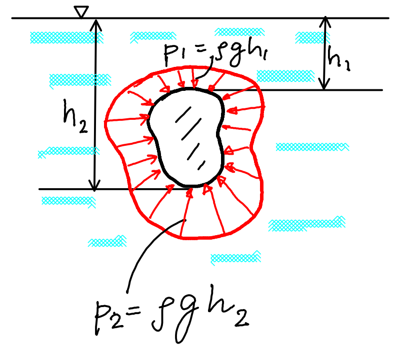
$A_y + B_y - F_1 - F_2 = 0$

$\Sigma M_A = 0$

$\Rightarrow B_y \times 15 - F_1 \times 1.5 - F_2 \times 11 = 0$

3 Unknowns (A_x, A_y, B_y); 3 equations

5.9 Distributed forces on submerged surfaces.



Objects that are submerged in water (or in any liquid) are subjected to distributed force per unit area which is called pressure.

In water, this pressure always acts perpendicular (normal) to the submerged surface and its magnitude is given by:

$$p = \rho g h$$

p: hydrostatic pressure; ρ: density (of liquid); g: gravity; h: depth below waterline

For water:

$\rho = 1000\ kg/m^3$

$\gamma = \rho g = 62.4\ lb/ft^3$

Note:
The buoyancy force is the resultant of all these distributed forces acting on the body. Recall the buoyancy force is equal to the weight of the water displaced.

Aside: If the liquid is viscous, then in addition the normal pressure the viscous fluid may also apply a tangential traction to the body. This traction is also a force per unit area and is a more general form of pressure.

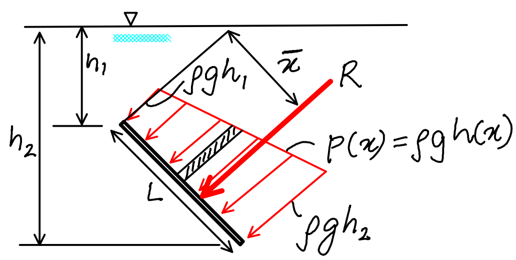
Resultant force

To obtain the resultant force acting on a submerged surface:

$$\vec{F} = \int_A \vec{p}\, dA$$

For inclined surfaces:

$(dA = b\,dx)$ — b: width of the plate (into the plane)



$$R = \int_L p\, b\, dx$$

$$\bar{x} = \frac{\int_L x\, p\, b\, dx}{R}$$

Exercise 5.82

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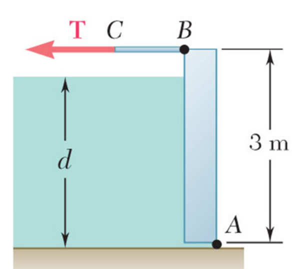
3m x 4m wall of the tank is hinged at A and held by rod BC.

Find the tension in the rod as a function of the water depth d.



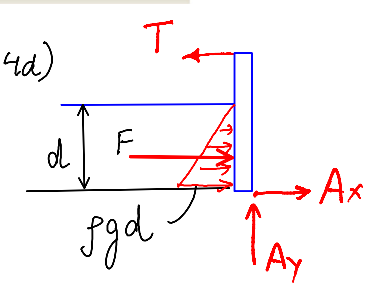
$\circlearrowleft \sum M_A = 0$

$\Rightarrow +3T - F\,d/3 = 0$

$\Rightarrow T = \frac{1}{9} F d$

$= \frac{1}{9} \underbrace{\left(\tfrac{1}{2}\rho g d\right) \cdot d \cdot 4}_{F}\ d.$

$F = \frac{1}{2}(\rho g d)(4d)$



$$\boxed{T = \frac{2}{9} \rho g d^3}$$

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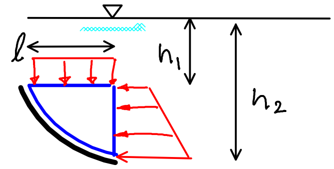
For Curved Surfaces:

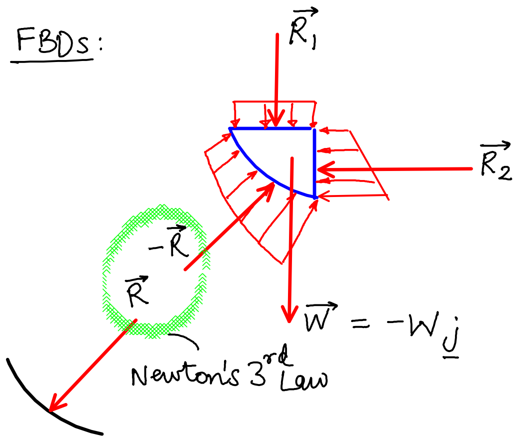
Forces on curved submerged surfaces can he obtained by using equilibrium of a surrounding portion of water.



$$\vec{R}_1 = (\rho g h_1)\, \ell\, b\, (-\underline{j})$$

$$\vec{R}_2 = (\rho g h_1)(h_2 - h_1)\, b\, (-\underline{i}) + \frac{1}{2}(\rho g (h_2 - h_1))(h_2 - h_1)\, b\, (-\underline{i})$$

FBDs:



$\vec{W}$ = weight of small body of water

Equilibrium of the small body of water:

$$\Sigma \vec{F} = 0$$

$$\Rightarrow \vec{R}_1 + \vec{R}_2 + \vec{W} + (-\vec{R}) = 0$$

$$\Rightarrow \vec{R} = \vec{R}_1 + \vec{R}_2 + \vec{W}$$

Example 5.10

FBD 1:

1 ft. deep

$\gamma_{conc} = 150\ lb/ft^3$

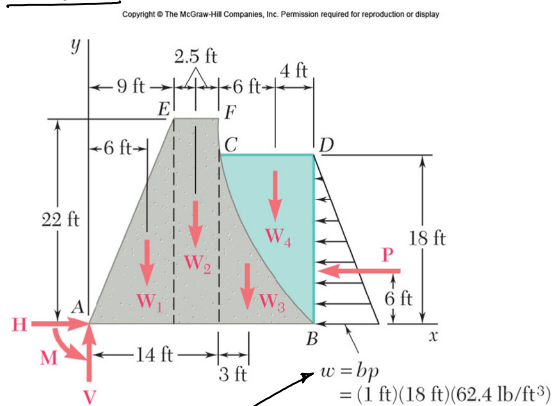
$\gamma_{water} = 62.4\ lb/ft^3$

$W_1 = \frac{1}{2} \times 9 \times 22 \times 1 \times \gamma_c = 14{,}850\ lb$

$W_2 = 5 \times 22 \times 1 \times \gamma_c = 16{,}500\ lb$

$W_3 = \frac{ah}{3} \times 1 \times \gamma_c = 9000\ lb$

$W_4 = \frac{2}{3} ah \times 1 \times \gamma_w = 7488\ lb.$



$$P = \frac{1}{2}\overbrace{(\gamma_w h \times 1)}^{w} \times h = 10{,}109\ lb$$

Note: The reactions H, V, M are not actual reactions.
The reactions at the bottom of the dam are also distributed.
H, V, M are the resultants at A of these distributed reactions.

$\Sigma F_x = 0 \Rightarrow H = P = \underline{10109\ lb}$

$\Sigma F_y = 0 \Rightarrow V = W_1 + W_2 + W_3 + W_4 = \underline{47{,}840\ lb}$

$\Sigma M_A = 0 \Rightarrow M - W_1 \times 6 - W_2 \times 11.5 - W_3 \times 17 - W_4 \times 20 + P \times 6$

$\Rightarrow M = \underline{520{,}960}\ \underline{lb\text{-}ft}$

Single Equivalent force The resultant reaction acts at a point "d" from A :

$V d = M$

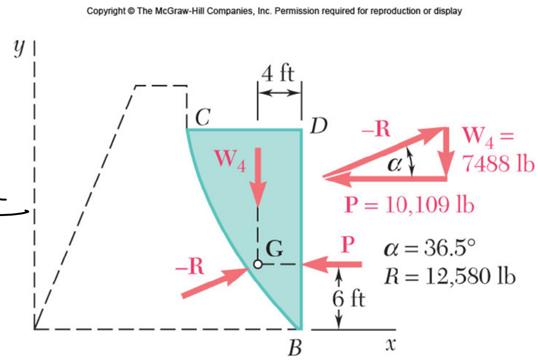
$\Rightarrow d = \frac{M}{V} = 10.89$ ft.

Resultant of water pressure:

Equilibrium of Body of water

(Note: 3 force member.)

FBD 2

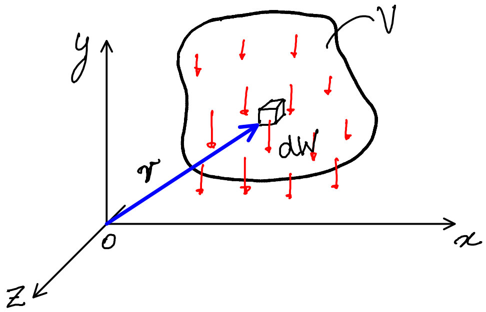


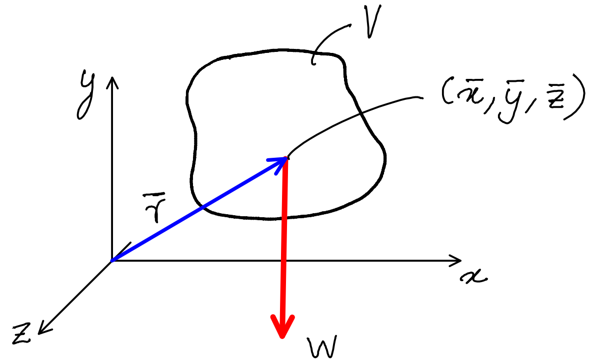
$\Rightarrow \quad (-\vec{R}) + \vec{W_4} + \vec{P} = 0$

$\Rightarrow$ Resultant water pressure $\vec{R} = 10109\,\underline{i} + 7488\,\underline{j}$ lb

5.10 Center of Gravity in 3D space; Center of volume

The formulas for center of gravity in 2 D can be easily generalized to 3D as follows:



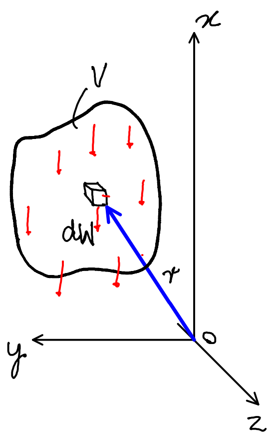


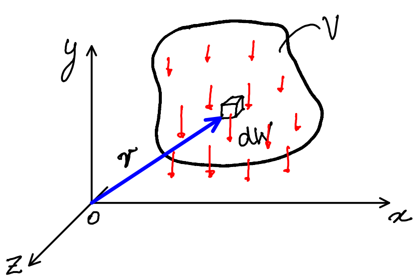
$$\sum \vec{F} = \int_V d\vec{W} = \int_V dW\,(-\underline{j})$$

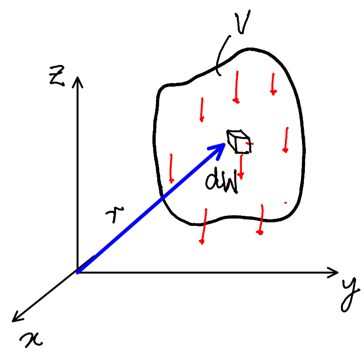
$$\sum \vec{M}_0 = \int_V \vec{r} \times d\vec{W} = \vec{\bar{r}} \times \vec{W}$$

$$W = \int_V dW$$

$$\bar{r} = \bar{x}\,\underline{i} + \bar{y}\,\underline{j} + \bar{z}\,\underline{k}$$







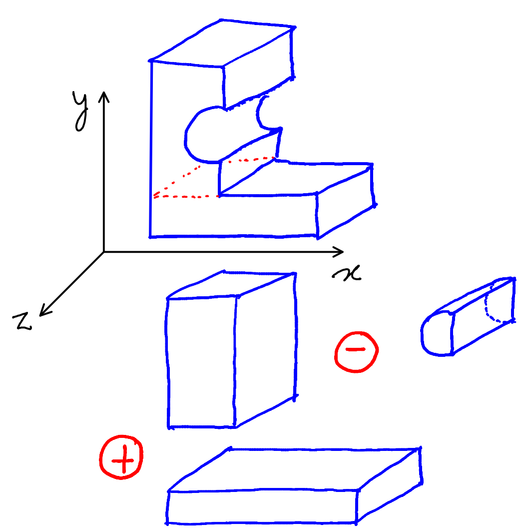
$$\boxed{\bar{x} = \frac{\int x\,dW}{W}} \qquad \boxed{\bar{y} = \frac{\int y\,dW}{W}} \qquad \boxed{\bar{z} = \frac{\int z\,dW}{W}}$$

5.11 Composition of Volumes

$$\bar{X} = \frac{\sum \bar{x}_i W_i}{W}$$

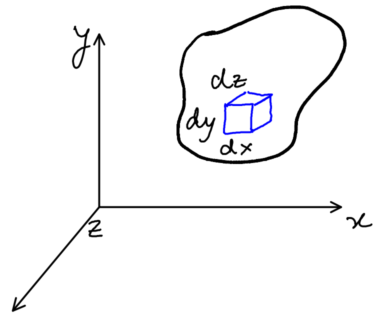
$$\bar{Y} = \frac{\sum \bar{y}_i W_i}{W}$$

$$\bar{Z} = \frac{\sum \bar{z}_i W_i}{W}$$



Examples 5.11 and 5.12 in the book.

5.12 Center of Volume by integration.



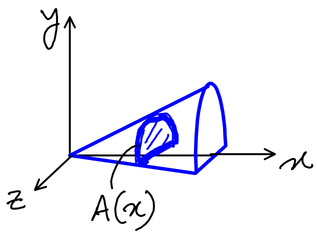
Volume $V = \iiint dx\,dy\,dz$

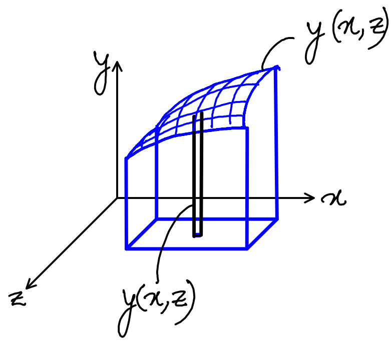
Center of Volume:

$$\bar{x} = \frac{\int x\,dV}{V} \qquad \bar{y} = \frac{\int y\,dV}{V} \qquad \bar{z} = \frac{\int z\,dV}{V}$$

For complex 3D shapes, triple integrals can be difficult to evaluate exactly.

For some special cases one can find the centroid as follows:
(i) Bodies of revolution
(ii) Volume under a surface

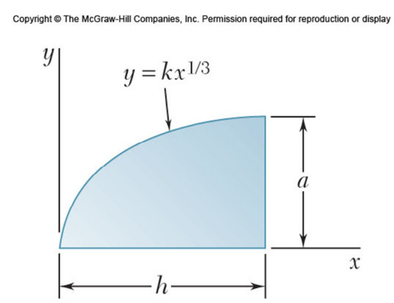




(i) $V = \int A(x)\,dx$ (ii) $V = \iint y(x,z)\,dx\,dz$

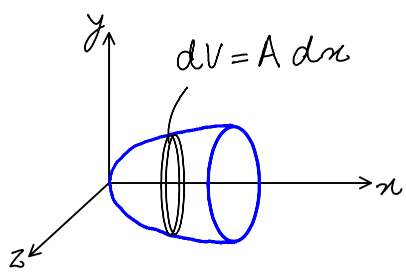
Read Example 5.13

Exercise 5.126
Find the centroid of the volume obtained by rotating the shaded area about the x-axis.



Note $\bar{y} = \bar{z} = 0$

To find $\bar{x}$:



$A(x) = \pi y^2 = \pi\left(k x^{1/3}\right)^2$

$$\bar{x} = \frac{\int x\,dV}{V}$$

$$\bar{x} = \frac{\int x\,A(x)\,dx}{\int A(x)\,dx} = \frac{\int x\,\pi\left(k x^{1/3}\right)^2 dx}{\int \pi\left(k x^{1/3}\right)^2 dx} = \frac{\pi k^2 \int x^{5/3}\,dx}{\pi k^2 \int x^{2/3}\,dx}$$

$$\Rightarrow \bar{x} = \left.\frac{\dfrac{x^{8/3}}{8/3}}{\dfrac{x^{5/3}}{5/3}}\right|_0^h = \boxed{\frac{5}{8}h}$$