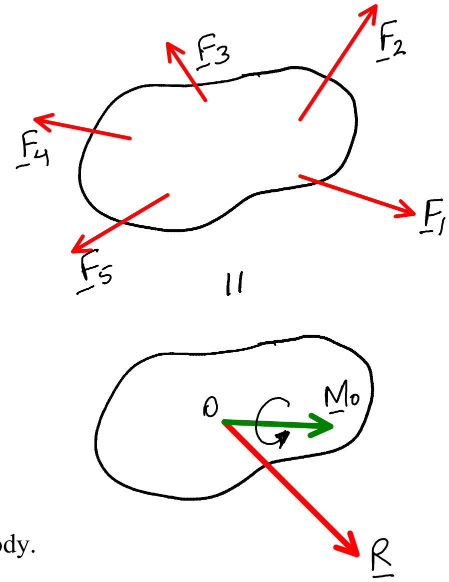
Chapter 4: Equilibrium of Rigid Bodies

A (rigid) body is said to in equilibrium if the vector sum of ALL forces and all their moments taken about any (and all) points is zero.

$$\boxed{\begin{array}{l}\sum \vec{F} = 0 \\ \sum \vec{M}_O = \sum\left(\vec{r}\times\vec{F}\right) = 0\end{array}}$$



In x, y, z components:

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$
$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

These equations give 6 independent equations in 3D space for each (rigid) body.

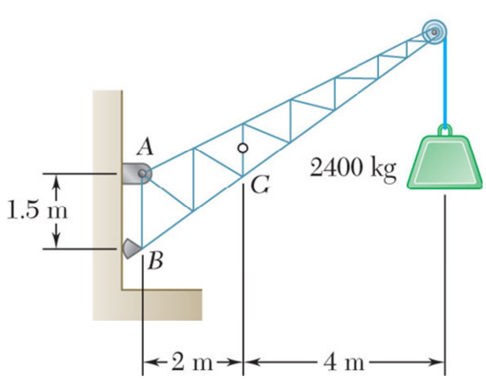
4.2 Free Body Diagrams

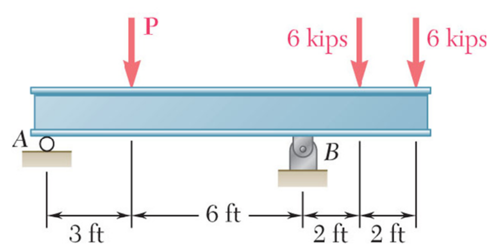
The free body diagram is a depiction of an object or a body along with **all** the external forces acting on it.

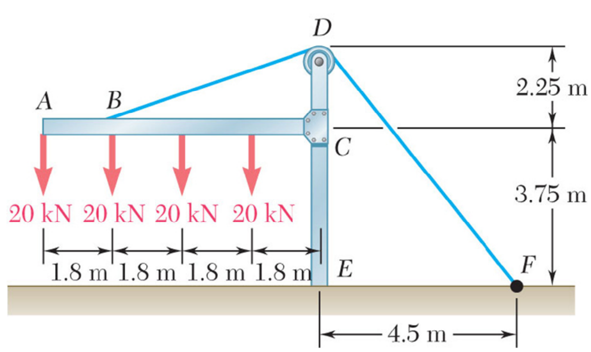
Steps in drawing a FBD of a body:

- Choose and draw the body (with dimensions). Carefully define its boundaries.
- Imagine the body in its current state and how it interacts with its surroundings.
- Draw ALL the external forces acting on the body (including self-weight).
- Any unknown forces acting on the body (required to keep it in equilibrium) must also be drawn.

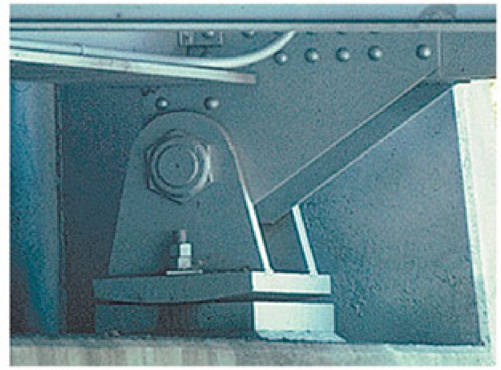
Examples







4.3 REACTIONS coming from the supports



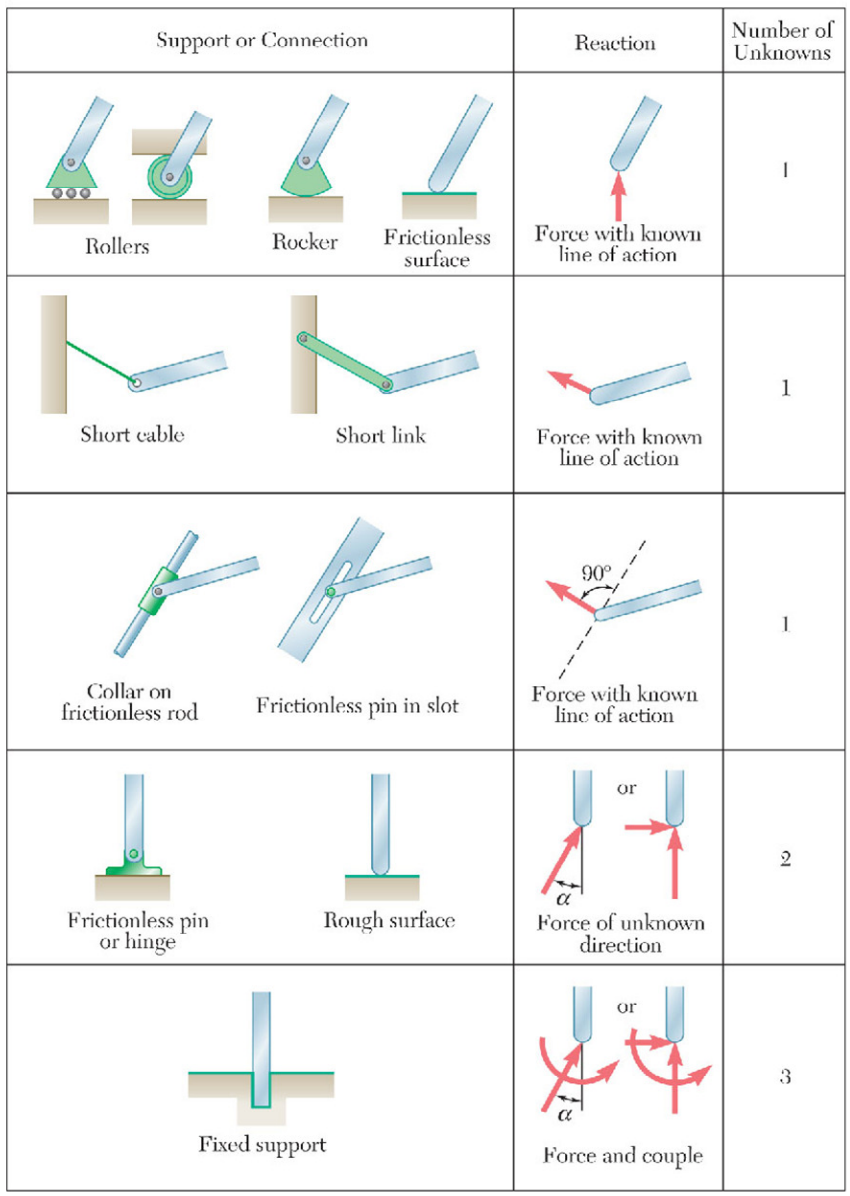
Courtesy National Information Service for Earthquake Engineering, University of California, Berkeley

Usually a body is constrained against motion using supports.

The forces from these supports, acting on the body are external to the body and must be included in the FBD.

The magnitude of these support forces are usually unknown and are obtained by solving the equilibrium equations.

Some examples of these support reactions are:



It is very important to correctly estimate the **number** and **type** of reactions that a support can provide.

One way to determine that is:

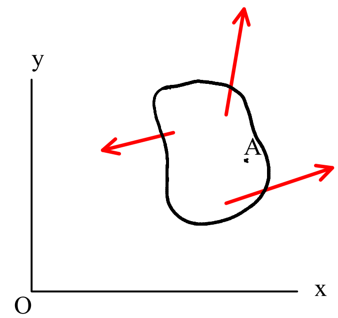
- Imagine yourself in the position of the body ().
- Now ask the question:
  "If I wanted to move in the x or y or z direction, would the support be able to stop my movement in that direction?"
- If the answer is "yes" then there will be an unknown reaction force from the support acting on the body in that direction.
  Conversely if the support cannot stop my motion in some direction, then there will not be a reaction force in that direction.

- Same thing holds true for rotations.
  "If I wanted to rotate in the x or y or z direction, would the support be able to stop my rotation in that direction?"
- If the answer is "yes" then there will be an unknown reaction moment acting on the body in that direction.
  Conversely if the support cannot stop my rotation in some direction, then there will not be a reaction moment in that direction.

4.4 Equilibrium in 2 dimensions

In the x-y plane

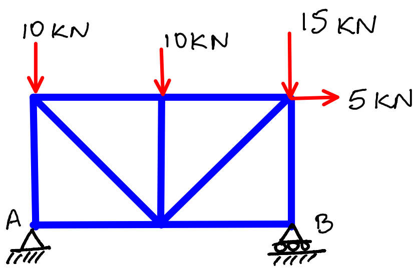
$\boxed{\sum F_x = 0}$ $\boxed{\sum F_y = 0}$ $\boxed{\sum M_A = 0}$

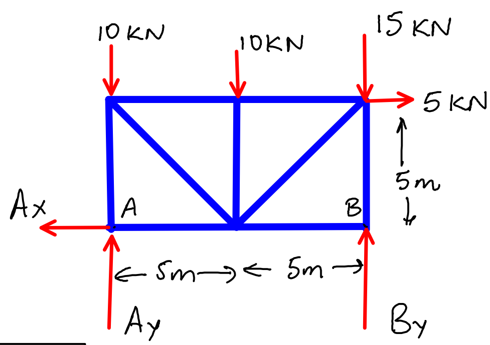


Note:

$F_z = 0 \quad M_x = M_y = 0 \quad M_z = M_O$

Example:





$\sum F_x = 0 \Rightarrow 5 - A_x = 0 \Rightarrow \boxed{A_x = 5 \text{ kN}}$

$\sum F_y = 0 \Rightarrow -35 + A_y + B_y = 0 \Rightarrow \boxed{A_y = 12.5 \text{ kN}}$

$\sum M_A = 0 \Rightarrow +B_y \times 10 - 50 - 150 - 25 = 0 \Rightarrow \boxed{B_y = 22.5 \text{ kN}}$

Interestingly:

- The 3 equilibrium equations above can also be replaced with the following equivalent equations: (only for 2 dimensions)

(A) $\sum F_x = 0 \quad \sum M_A = 0 \quad \sum M_B = 0$

(where the line AB is not perpendicular to the x-axis)

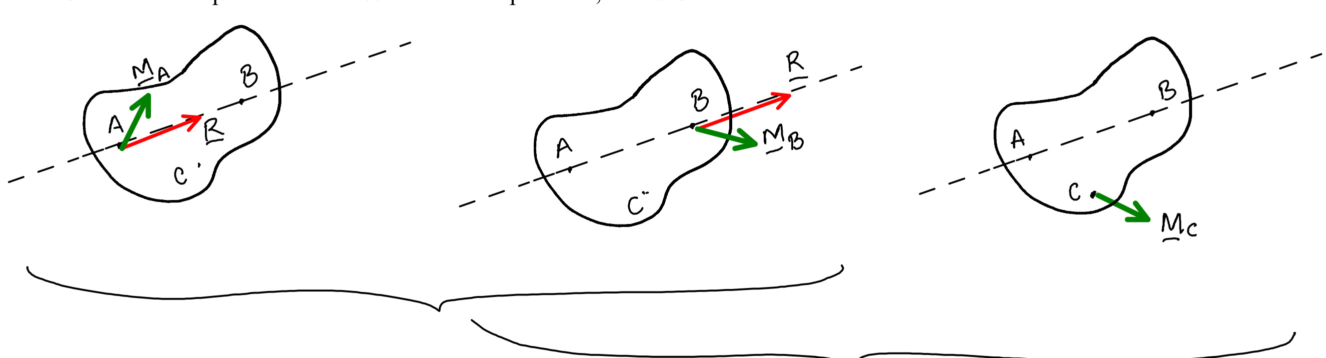
OR

(B) $\sum M_A = 0 \quad \sum M_B = 0 \quad \sum M_C = 0$

(where A, B and C are not all along the same line)

Reason is:

Consider the equivalent force & moments at points A, B and C:



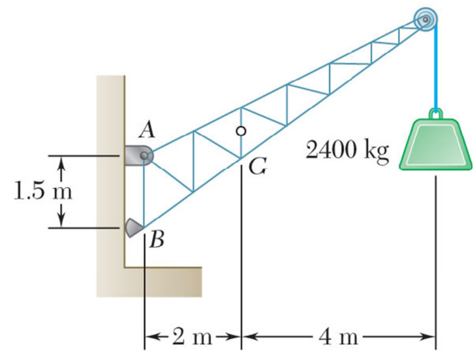
- If we show that $M_A = 0$ and $M_B = 0$, then the only thing remaining to check would be ΣF = 0 along AB.

- If in addition we say that $M_C = 0$ then that possibility is ruled out as well.

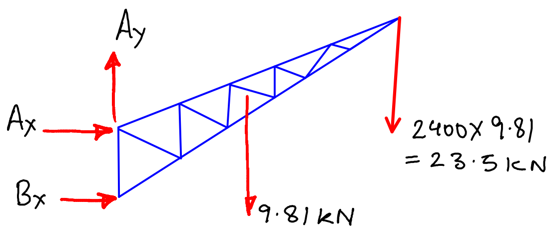
Example 4.1

A fixed crane has a mass of 1000 kg and is used to lift a 2400 kg crate. It is held in place by a pin at *A* and a rocker at *B*. The center of gravity of the crane is located at *G*.

Determine the components of the reactions at *A* and *B*.



FBD



$\Sigma F_x = 0 \Rightarrow A_x + B_x = 0$ —(1)

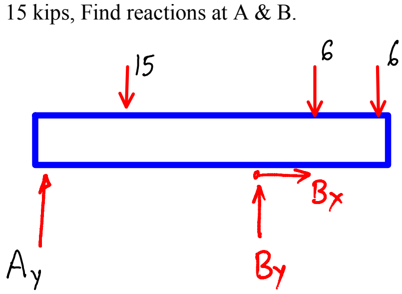
$\Sigma F_y = 0 \Rightarrow A_y - 9.81 - 23.5 = 0$ —(2)

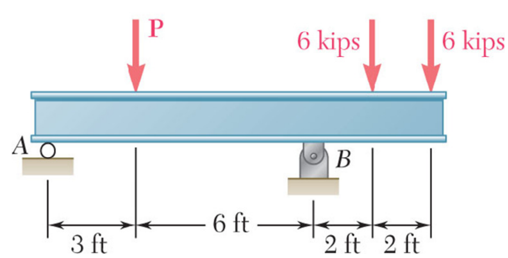
$\Sigma M_A = 0 \Rightarrow B_x \times 1.5 - 9.81 \times 2 - 23.5 \times 6 = 0$ —(3)

Example 4.2

P = 15 kips, Find reactions at A & B.

FBD





$\Sigma F_x = 0 \Rightarrow B_x = 0$

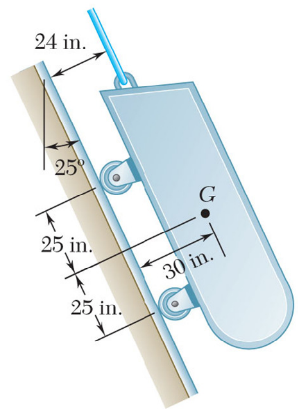
$\Sigma F_y = 0 \Rightarrow -15 - 6 - 6 + A_y + B_y = 0$

$\Sigma M_B = 0 \Rightarrow -A_y \times 9 + 15 \times 6 - 6 \times 2 - 6 \times 4 = 0$

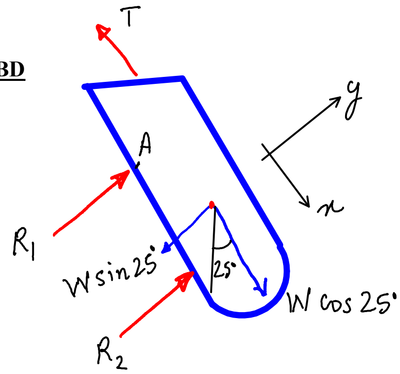
Example 4.3

A loading car is at rest on an inclined track. The gross weight of the car and its load is 5500 lb, and it is applied at at G. The cart is held in position by the cable.

Determine the tension in the cable and the reaction at each pair of wheels.



FBD



$\sum F_x = 0 \Rightarrow -T + W\cos 25° = 0$

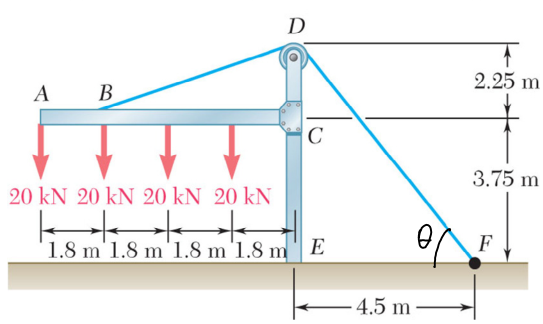
$\sum F_y = 0 \Rightarrow R_1 + R_2 - W\sin 25° = 0$

$\sum M_A = 0 \Rightarrow T \times 24 + 50 \times R_2 - W\sin 25° \times 25 - W\cos 25° \times 30 = 0$

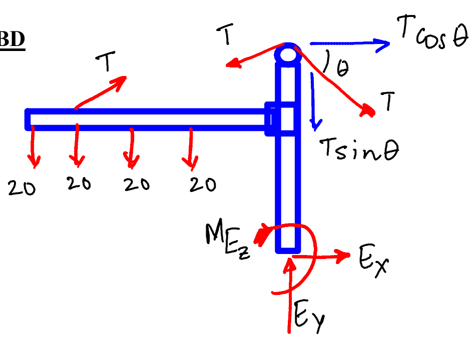
Example 4.4

The frame supports part of the roof of a small building. The tension in the cable is 150 kN.

Determine the reaction at the fixed end E.



FBD



$\sqrt{4.5^2 + 6^2} = 7.5$

$\cos\theta = \left(\frac{4.5}{7.5}\right)$

$\sin\theta = \left(\frac{6}{7.5}\right)$

$\sum F_x = 0 \Rightarrow T\left(\frac{4.5}{7.5}\right) + E_x = 0$

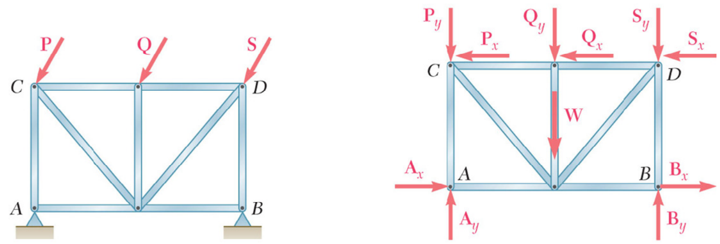
$\sum F_y = 0 \Rightarrow -80 - T\left(\frac{6}{7.5}\right) + E_y = 0$

$\sum M_E = 0 \Rightarrow 20\,(1.8 + 3.6 + 5.4 + 7.2) - T\cos\theta \times 6 + M_{Ez} = 0$

4.5 Statically INDETERMINATE Reactions

Case 1: Indeterminate

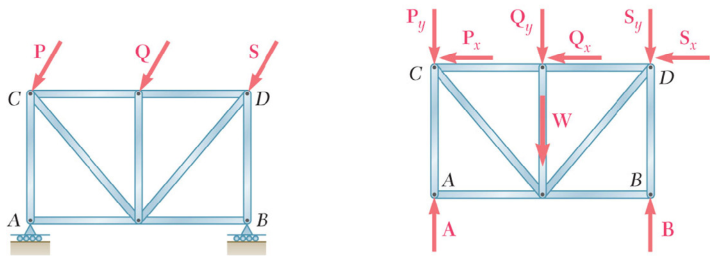
- More unknowns; Not enough equations



(a) (b)

Case 2: Partially Restrained

- Fewer unknowns than equations



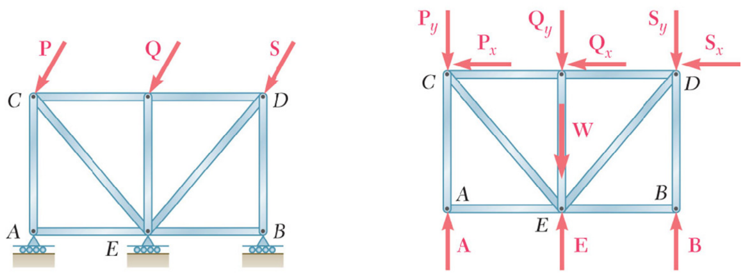
(a) (b)

Case 3: Improperly Restrained

Both:

- Indeterminate AND
- Partially restrained

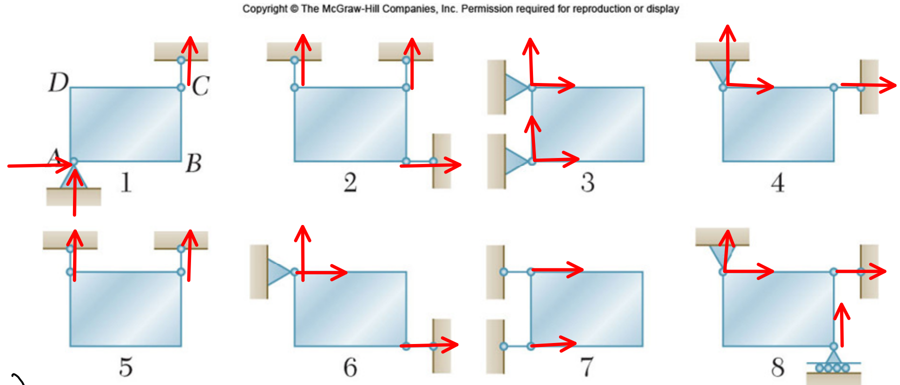
IMPORTANT:
In this case, ALL the reactions are either concurrent or parallel.



(a) (b)

Exercise 4.59

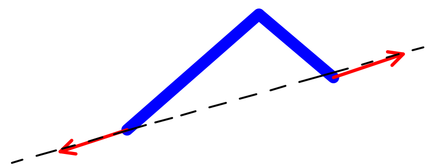
(1) Determinate – OK
(2) ″
(3) INdeterminate (Case I)
(4) IMPROPER (Case III)
(5) Partial Restrained (Case II)
(6) Determinate – OK
(7) Partial Restraint (Case II)
(8) Indeterminate (Case I)

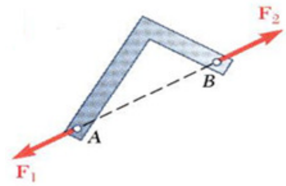


4.6 - 4.7 Two Force members & Three Force members

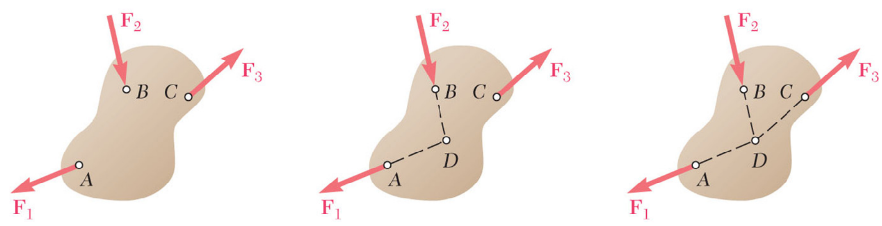
Sometimes, there are bodies (or parts of bodies) that have EXACTLY **TWO** or **THREE** external forces acting on them. For such cases, the equilibrium equations are much more simplified.

Case 1: **2** force member





Case 2: **3** Force member

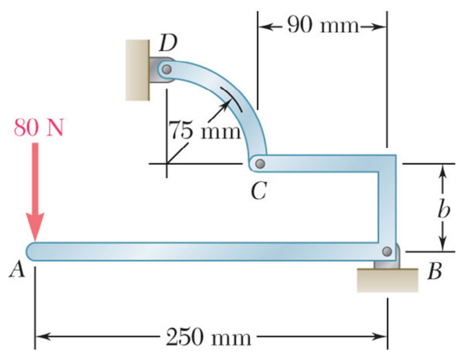


(a) (b) (c)

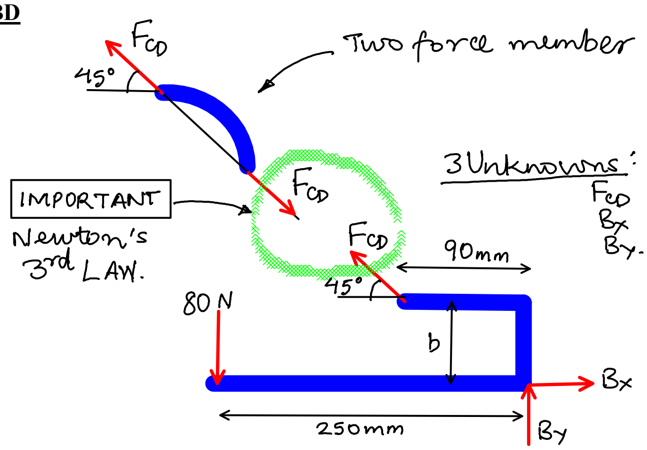
Exercise 4.66 & 4.67

Determine the reactions at B and D

- $b = 60$ mm
- $b = 120$ mm



FBD



From the equilibrium of Member ABC:

$$\Sigma F_x = 0 \Rightarrow -F_{CD}\cos 45° + B_x = 0$$

$$\Sigma F_y = 0 \Rightarrow -80 + F_{CD}\sin 45° + B_y = 0 \Rightarrow B_y =$$

$$\Sigma M_B = 0 \Rightarrow +(80 \times 250) + (F_{CD}\cos 45° \times b) - (F_{CD}\sin 45° \times 90) = 0$$

$$\Rightarrow F_{CD} = \frac{80 \times 250}{(90\sin 45° - b\cos 45°)} = \boxed{\begin{array}{ll} +942.8\text{ N} & \text{for } b = 60\text{ mm} \\ -942.8\text{ N} & \text{for } b = 120\text{mm} \end{array}}$$

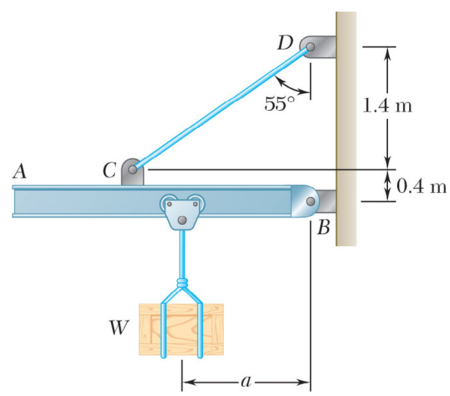
$$\Rightarrow B_x = F_{CD}\cos 45° = \boxed{\begin{array}{ll} +666.7\text{ N} & \text{for } b = 60\text{ mm} \\ -666.7\text{ N} & \text{for } b = 120\text{ mm} \end{array}}$$

$$\Rightarrow B_y = 80 - F_{CD}\sin 45° = \boxed{\begin{array}{ll} -586.7\text{ N} & \text{for } b = 60\text{mm} \\ -746.7\text{ N} & \text{for } b = 120\text{mm} \end{array}}$$

Exercise 4.69

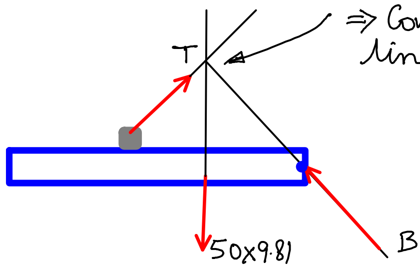
50 kg crate is attached to a trolley beam as shown.
Given a = 1.5 m.

Determine
(a) Tension in the cable CD
(b) Reaction at B



3 force member
⇒ Concurrent lines of action

FBD



2 Unknowns: T, B

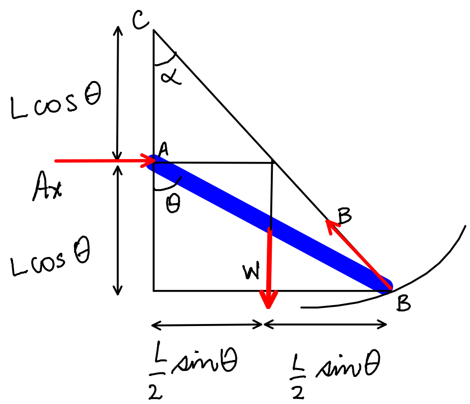
2 Equations

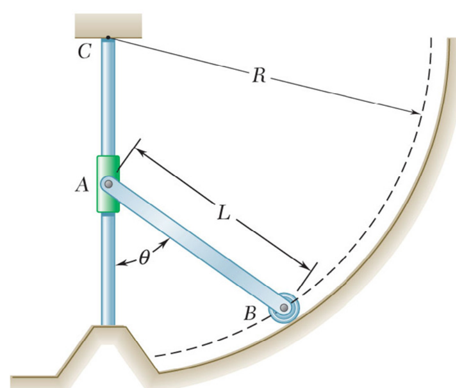
$\Sigma F_x = 0$; $\Sigma F_y = 0$

($\Sigma M = 0$ is already in the "3 force" member concept)

Exercise 4.90 (HW-13):

Find an equation in R, L and θ that governs Equilibrium.





At this point θ can be found simply from geometry.

- Vertical distance: $2L\cos\theta = R\cos\alpha$
- Horizontal distance: $L\sin\theta = R\sin\alpha$

Eliminate "α" to get an equation in θ, R, L.

Alternatively:

Without using the fact that this is a 3-force member:

FBD

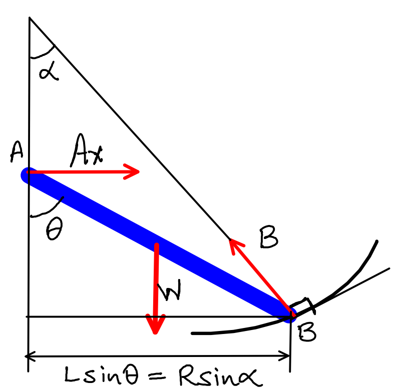
Geometry:

$[L\sin\theta = R\sin\alpha]$ (gives α in terms of θ)

+

$$\begin{bmatrix} \Sigma F_x = 0 \\ \Sigma F_y = 0 \\ \Sigma M_B = 0 \end{bmatrix}$$

3 equations
3 unknowns (A_x, B, θ)



Friday, October 02, 2009
5:45 AM

4.8 - 4.9 Equilibrium of Bodies in 3D space

- Draw the **FBD**
- Equations of equilibrium are given by:

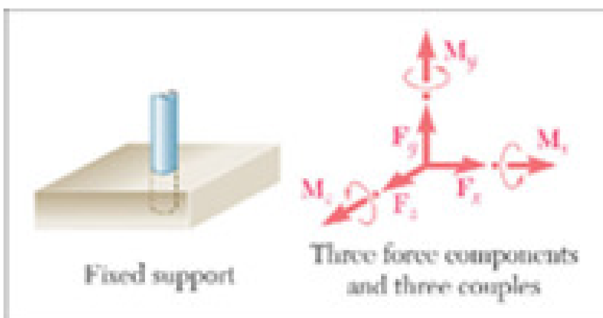
$$\boxed{\begin{array}{l} \sum \vec{F} = 0 \\ \sum \vec{M}_O = \sum \left(\vec{r} \times \vec{F} \right) = 0 \end{array}}$$

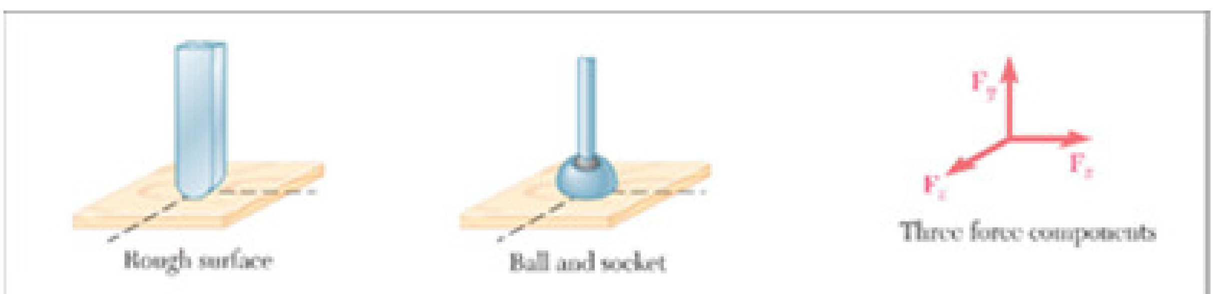
- 6 scalar equations are required to express the conditions for the equilibrium of a rigid body in the general three dimensional case.

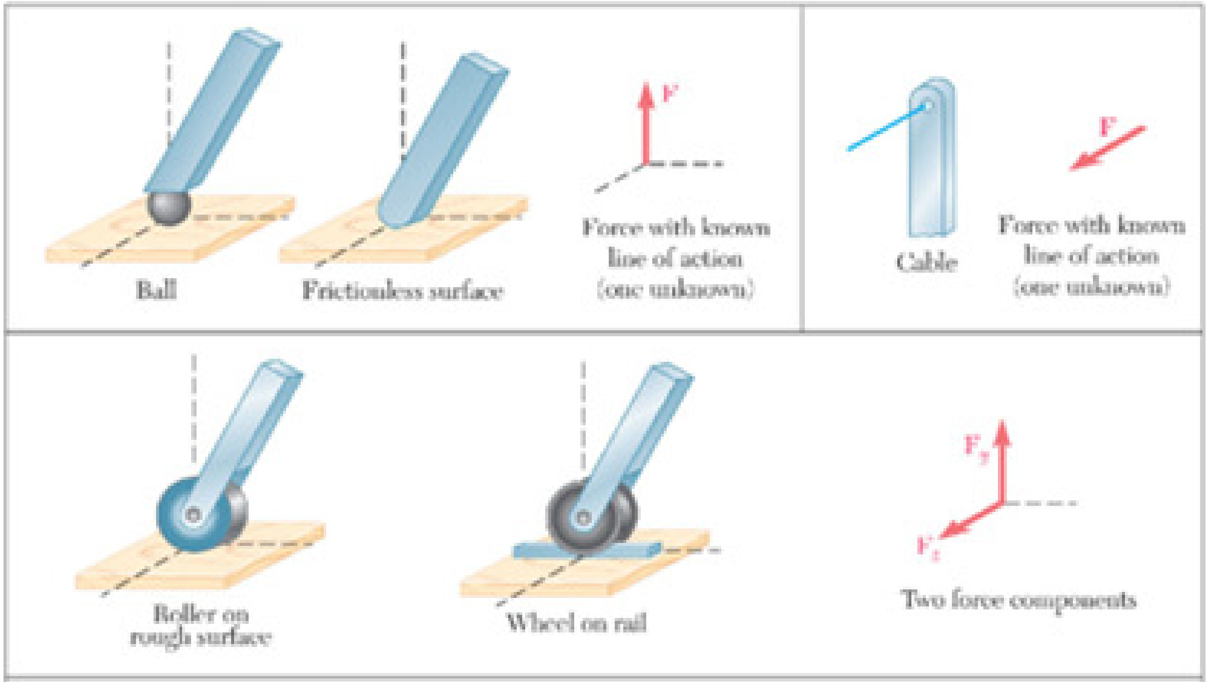
$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$
$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

- 6 unknown reactions can be solved for.

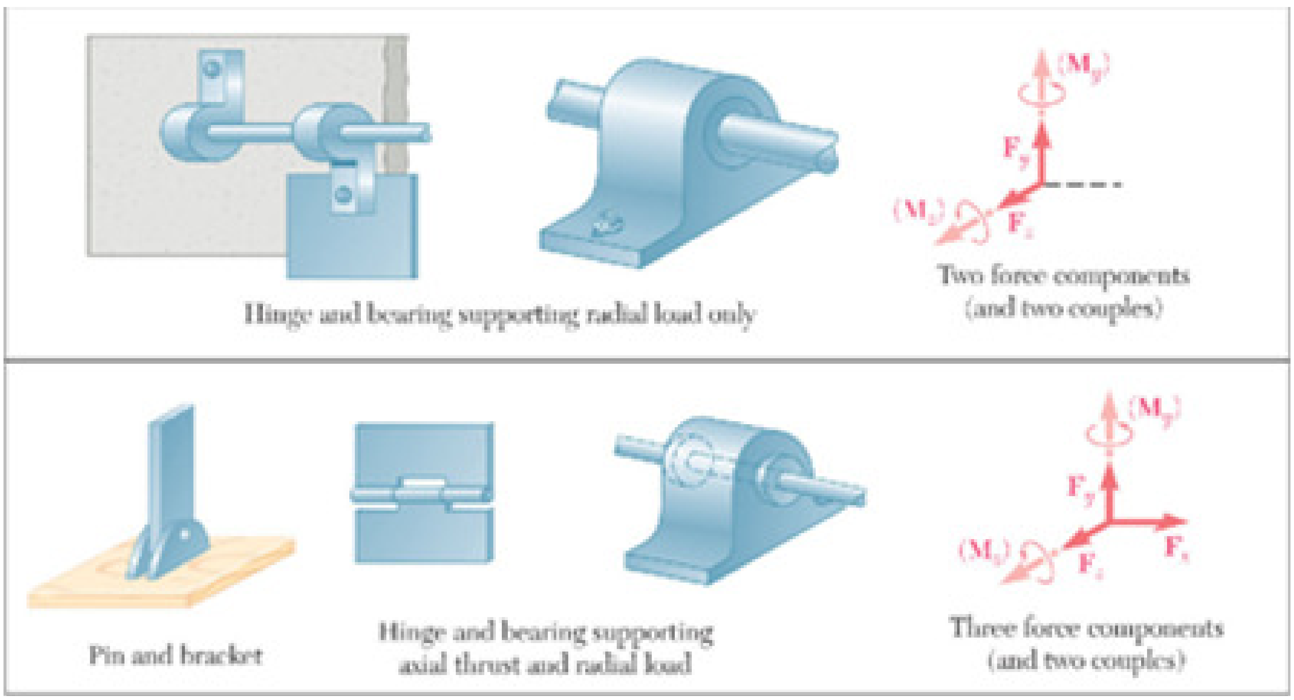
Some unknown reactions in 3D:



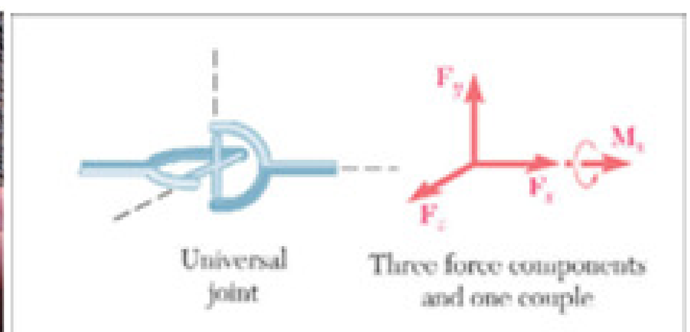




Ball
Frictionless surface
F
Force with known line of action (one unknown)
Cable
F
Force with known line of action (one unknown)
Roller on rough surface
Wheel on rail
F_y
F_z
Two force components



Hinge and bearing supporting radial load only
(M_y)
F_y
(M_z)
F_z
Two force components (and two couples)
Pin and bracket
Hinge and bearing supporting axial thrust and radial load
(M_y)
F_y
(M_z)
F_z
F_x
Three force components (and two couples)



Universal joint
F_y
M_x
F_x
F_z
Three force components and one couple

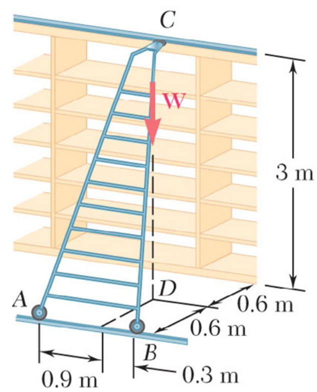
Friday, October 02, 2009
10:45 AM

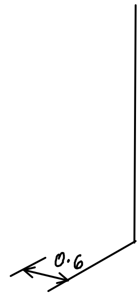
Examples 4.7

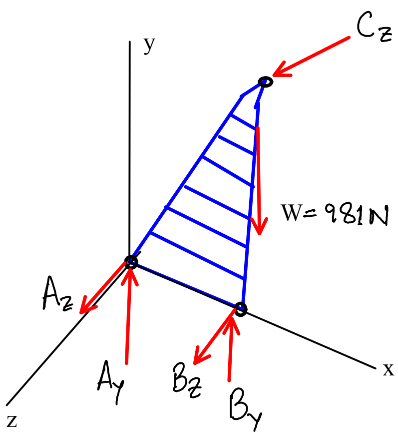
Given W = Ladder + Person
= 100 x 9.81 = 981 N

The wheels at A & B are flanged while the wheel at C is unflanged.

Determine reactions at A, B and C







$\vec{r}_{AW} = 0.9\underline{i} - 0.6\underline{k}$

Note: Unrestrained in "x".

5 unknowns
5 equations

$\boxed{\Sigma \vec{F} = 0}$

$\Rightarrow \quad 0\underline{i} + (A_y + B_y - W)\underline{j} + (A_z + B_z + C_z)\underline{k} = \underline{0}$] 2 equations

$\boxed{\Sigma \vec{M} = \Sigma \vec{r} \times \vec{F} = \underline{0}}$

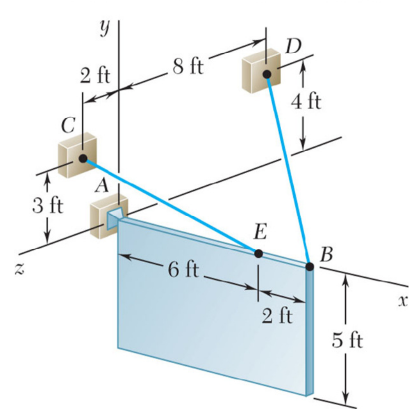
$\Rightarrow \Sigma \vec{M}_A = (\vec{r}_{AB} \times \vec{F}_B) + (\vec{r}_{AC} \times \vec{F}_C) + (\vec{r}_{AW} \times \vec{W})$

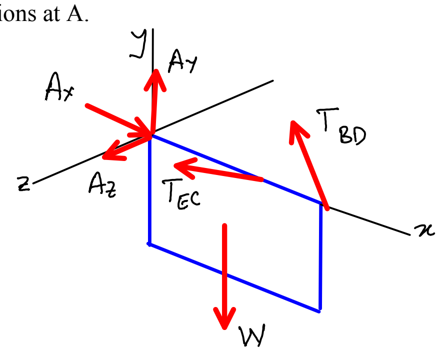
(about A) $= [1.2\underline{i} \times (B_y\underline{j} + B_z\underline{k})] + [(0.6\underline{i} + 3\underline{j} - 1.2\underline{k}) \times C_z\underline{k}]$
$+ [(0.9\underline{i} - 0.6\underline{k}) \times (-W\underline{j})]$] 3-equations

Example 4.8

Given W = 270 lbs

Determine
- Tensions in AE and BD.
- Reactions at A.





$\vec{T}_{EC} = T_{EC}\ \vec{EC}/|EC|$

$\vec{T}_{BD} = T_{BD}\ \vec{BD}/|BD|$

Note: Unrestrained in rotation about x.
Unknowns: $A_x, A_y, A_z, T_{EC}, T_{BD}$

$$\Sigma \vec{F} = 0$$

$$\Rightarrow A_x \underline{i} + A_y \underline{j} + A_z \underline{k} + \vec{T}_{EC} + \vec{T}_{BD} - 270 \underline{j} = 0 \quad \Big] \text{ 3 equations}$$

$$\Sigma \vec{M}_A = 0$$

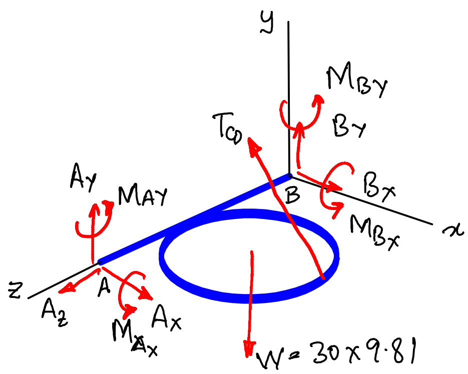
$$\Rightarrow \left(\vec{r}_{AE} \times \vec{T}_{EC}\right) + \left(\vec{r}_{AB} \times \vec{T}_{BD}\right) + \left(4\underline{i} \times (-270\underline{j})\right) = 0 \quad \Big] \text{ 2 equations}$$

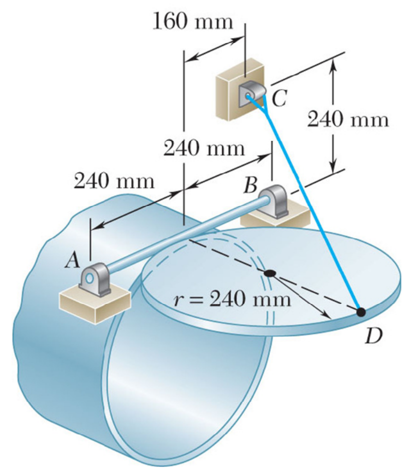
$$\Rightarrow \left(6\underline{i} \times \vec{T}_{EC}\right) + \left(8\underline{i} \times \vec{T}_{BD}\right) + \left(4\underline{i} \times -270\underline{j}\right) = 0$$

<u>Example 4.9</u>

Given mass of the cover: 30 kg
Assume no axial reaction at B.

Find Tension in CD and reactions at A & B.





$$\vec{T}_{DC} = T_{DC} \frac{\vec{DC}}{|DC|}$$

$$\Sigma \vec{F} = 0$$

$$\Rightarrow \left(A_x \underline{i} + A_y \underline{j} + A_z \underline{k}\right) + \left(B_x \underline{i} + B_y \underline{j}\right) + \vec{T}_{DC} - 30 \times 9.81 \underline{j} = 0 \quad \Big] \text{ 3 equations}$$

$$\Sigma \vec{M}_A = 0$$

$$\Rightarrow \left[r_{AB} \times \left(B_x \underline{i} + B_y \underline{j}\right)\right] + \left[\vec{r}_W \times \left(-30 \times 9.81 \underline{j}\right)\right] + \left(\vec{r}_{AD} \times \vec{T}_{DC}\right)$$

$$\left(M_{Ax} \underline{i} + M_{Ay} \underline{j}\right) + \left(M_{Bx} \underline{i} + M_{By} \underline{j}\right) = 0 \quad \Big] \text{ 3 equations}$$

Unknowns: $\underbrace{A_x, A_y, A_z, B_x, B_y, T_{DC}}_{\text{Forces}}$ $\underbrace{M_{Ax}, M_{Ay}, M_{Bx}, M_{By}}_{\text{Moments.}}$

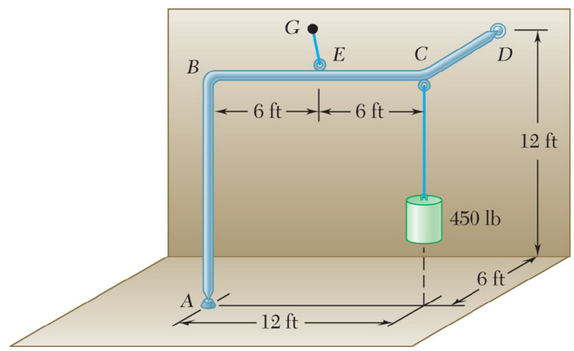
<u>Note</u>: The book assumes that $M_{Ax} = M_{Ay} = M_{Bx} = M_{By} = 0$.
This is <u>not</u> a good assumption.

Example 4.10

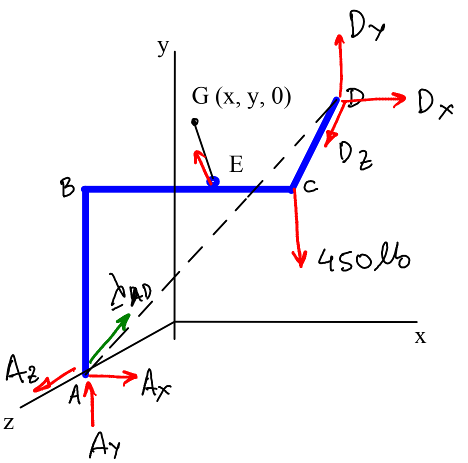
Given W = 450 lb.

Find

- Location of G so that the tension EG is minimum
- This minimum value of tension



Note: Determinate/Indeterminate



$$\vec{AD} = 12\underline{i} + 12\underline{j} - 6\underline{k}$$

$$\vec{\lambda}_{AD} = \tfrac{2}{3}\underline{i} + \tfrac{2}{3}\underline{j} - \tfrac{1}{3}\underline{k}$$

$$\vec{EG} = (x-6)\,\underline{i} + (y-12)\,\underline{j} - 6\,\underline{k}$$

$$\vec{T}_{EG} = T\,\vec{\lambda}_{EG} = T\,\frac{\vec{EG}}{|EG|}$$

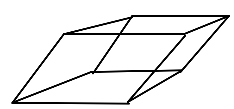
Moment about AD:

$$\Sigma M_{AD} = \vec{\lambda}_{AD}\cdot\left[\left(\vec{r}_{AE}\times\vec{T}_{EG}\right) + \left(\vec{r}_{AC}\times\vec{W}\right)\right] = 0$$

$$= \vec{\lambda}_{AD}\cdot\left(\vec{r}_{AE}\times\vec{T}_{EG}\right) + \frac{\vec{AD}}{|AD|}\cdot\left[\left(12\underline{i}+12\underline{j}\right)\times\left(-450\underline{j}\right)\right] = 0$$

$$= \vec{\lambda}_{AD}\cdot\left(\vec{r}_{AE}\times\vec{T}_{EG}\right) + \left(\tfrac{2}{3}\underline{i} + \tfrac{2}{3}\underline{j} - \tfrac{1}{3}\underline{k}\right)\cdot\left(-5400\,\underline{k}\right) = 0$$

$$= \lambda_{AD}\cdot\left(\vec{r}_{AE}\times\vec{T}_{EG}\right) + 1800 = 0$$



ie $\vec{T}_{EG}\cdot\left(\vec{\lambda}_{AD}\times\vec{r}_{AE}\right) = -1800$ (Cyclic Permutation)

$$T\,\vec{\lambda}_{EG}\cdot\left(\tfrac{2}{3}\underline{i} + \tfrac{2}{3}\underline{j} - \tfrac{1}{3}\underline{k}\right)\times\left(6\underline{i} + 12\underline{j}\right) = -1800$$

$$T\,\vec{\lambda}_{EG}\cdot\begin{vmatrix} i & j & k \\ 2/3 & 2/3 & -1/3 \\ 6 & 12 & 0 \end{vmatrix} = -1800$$

$$\Rightarrow \underbrace{T}_{\text{Minimize}}\;\underbrace{\vec{\lambda}_{EG}\cdot\left(4\underline{i} - 2\underline{j} + 4\underline{k}\right)}_{\text{Maximize}} = -1800 \quad \text{—— (1)}$$

Minimize ⇒ Maximize ⇒ $\vec{\lambda}_{EG}$ and $(4\underline{i} - 2\underline{j} + 4\underline{k})$ should point in the same (or opposite) direction.

i.e. we can express

$$\vec{EG} = c\left(4\underline{i} - 2\underline{j} + 4\underline{k}\right) \quad \begin{Bmatrix}\text{for some} \\ \text{unknown } c\end{Bmatrix}$$

$$\Rightarrow (x-6)\,\underline{i} + (y-12)\,\underline{j} - 6\,\underline{k} = c\left(4\,\underline{i} - 2\,\underline{j} + 4\,\underline{k}\right)$$

Equating "$\underline{k}$" components: $-6 = c4 \Rightarrow c = -3/2$

$$\Rightarrow \quad x - 6 = \left(-3/2\right)4 \quad \Rightarrow \boxed{x = 0}$$

$$\Rightarrow \quad y - 12 = \left(-3/2\right)(-2) \quad \Rightarrow \boxed{y = 15\ ft}$$

$$\Rightarrow \text{From } (1) \quad T\left(\frac{-6\underline{i} + 3\underline{j} - 6\underline{k}}{\sqrt{81}}\right) \cdot \left(4\underline{i} - 2\underline{j} + 4\underline{k}\right) = -1800$$

$$\Rightarrow \quad T\left(\frac{-54}{9}\right) = -1800 \Rightarrow \boxed{T = 300\ lb}$$