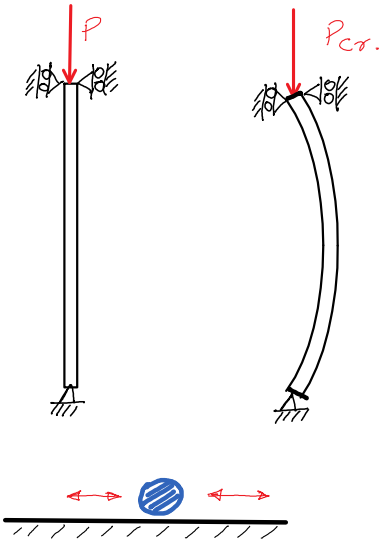
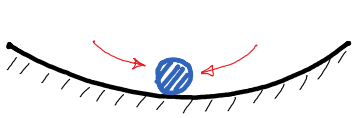
Column Buckling

Columns behave primarily as axial members in compression, but they undergo bending when an eccentric load is applied. There is always some eccentricity due to imperfections.

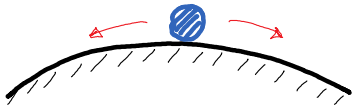
For a sufficiently high compressive axial load, called critical load, columns can become unstable and bend / buckle out of vertical and collapse in an uncontrolled manner.



Equilibrium and Stability:



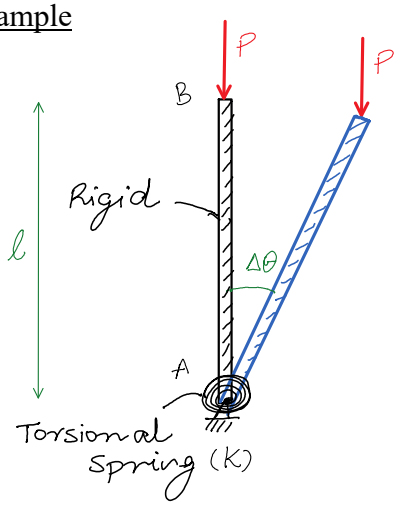
Stable equilibrium

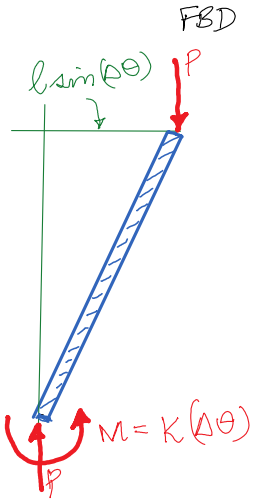


Unstable equilibrium

Neutral Equilibrium

Example





$\sum M_A = 0$

$\Rightarrow K(\Delta\theta) - P\,l\sin(\Delta\theta) = 0$

$K(\Delta\theta)$: Restoring moment. $P\,l\sin(\Delta\theta)$: Destabilizing moment

(I) If $K(\Delta\theta) > P\,l\sin(\Delta\theta)$ Stable equilibrium.

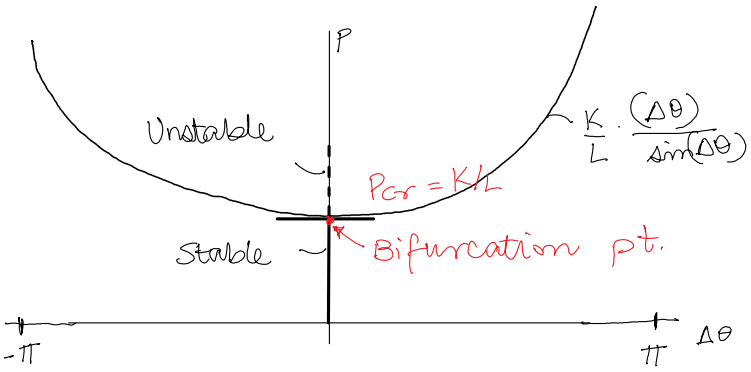
(II) $K(\Delta\theta) < P\,l\sin(\Delta\theta)$ Unstable equilibrium.

(III) $K(\Delta\theta) = P_{cr}\,l\sin(\Delta\theta)$ (neutral)

$\Rightarrow P_{cr} = \frac{K}{l}\cdot\left[\frac{(\Delta\theta)}{\sin(\Delta\theta)}\right] = \boxed{\frac{K}{L}}$

If $\Delta\theta \to 0$ $\Rightarrow 1$

Bifurcation Diagram

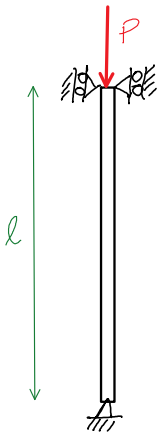


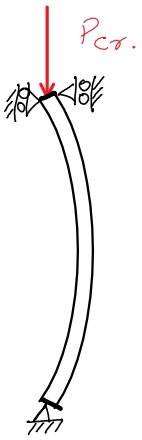
Deformation:

If $P > P_{cr} \Rightarrow P = \frac{K}{L}\frac{(\Delta\theta)}{\sin(\Delta\theta)}$ (solve for $\Delta\theta$)

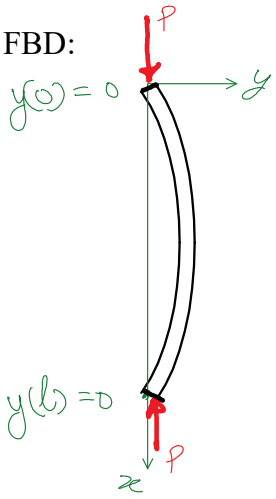
If $P < P_{cr} \Rightarrow \Delta\theta = 0$ (only solution)

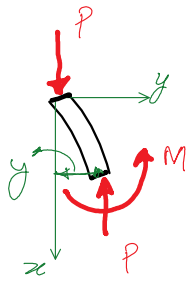
Euler's Column Buckling for Pin-Pin supports





FBD:





Destabilizing moment : $+P.y$

Stabilizing moment : $-M$

If $\Sigma M > P_y$: Stable

If $\Sigma M < P_y$: Unstable.

Solving for column deformations using beam theory:

$$\frac{M}{EI} = \frac{d^2y}{dx^2} \quad \text{(Curvature)}$$

For P_{cr} : $-M = P_{cr}\,y$

$$\Rightarrow M = EI\frac{d^2y}{dx^2} \Rightarrow +EI\frac{d^2y}{dx^2} + P_{cr}\,y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \left(\frac{P_{cr}}{EI}\right)\cdot y = 0 \Rightarrow y(x) = A\sin\left[\underbrace{\left(\frac{P_{cr}}{EI}\right)}_{p=\sqrt{\frac{P_{cr}}{EI}}}x\right] + B\cos\left[\underbrace{\left(\frac{P_{cr}}{EI}\right)}_{p}x\right]$$

Verify:

$$\frac{d^2y}{dx^2} = -p^2\left[A\sin px + B\cos(px)\right]$$

To solve for A & B:

$$y(0) = 0 \Rightarrow B = 0 \Rightarrow y(x) = A\sin(px)$$
$$y(\ell) = 0 \Rightarrow 0 = A\sin(p\ell) = 0$$

Either $A = 0$ or $\sin(p\ell) = 0 \Rightarrow p\ell = \pi, 2\pi, 3\pi \ldots$

$$p\ell = n\pi$$

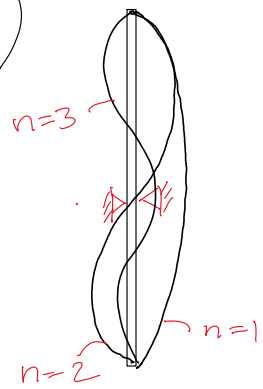
$$\Rightarrow \left(\sqrt{\frac{P_{cr}}{EI}}\right)\cdot\ell = n\pi \Rightarrow \underbrace{P_{cr} = \frac{n^2\pi^2}{\ell^2}\cdot EI}_{\text{Euler's buckling load}}$$

Lowest P_{cr} $(n=1)$

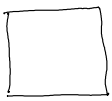
$$\Rightarrow \boxed{P_{cr} = \frac{\pi^2 EI}{\ell^2}}$$

Deformation of the column

$$y(x) = A \sin\left(\sqrt{\frac{P_{cr}}{EI}} \cdot x\right) = A \sin\left(\frac{\pi}{l} x\right)$$



Gross-section:

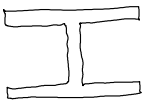




$I_{max} = I_{min}$

Any direction of buckling is possible





buckles in the weak axis $\Rightarrow$ Use I_{min}.

Buckling stress

$$P_{cr} = \frac{\pi^2 EI}{l^2} \Rightarrow \sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{l^2} \cdot \left(\frac{I}{A}\right)$$

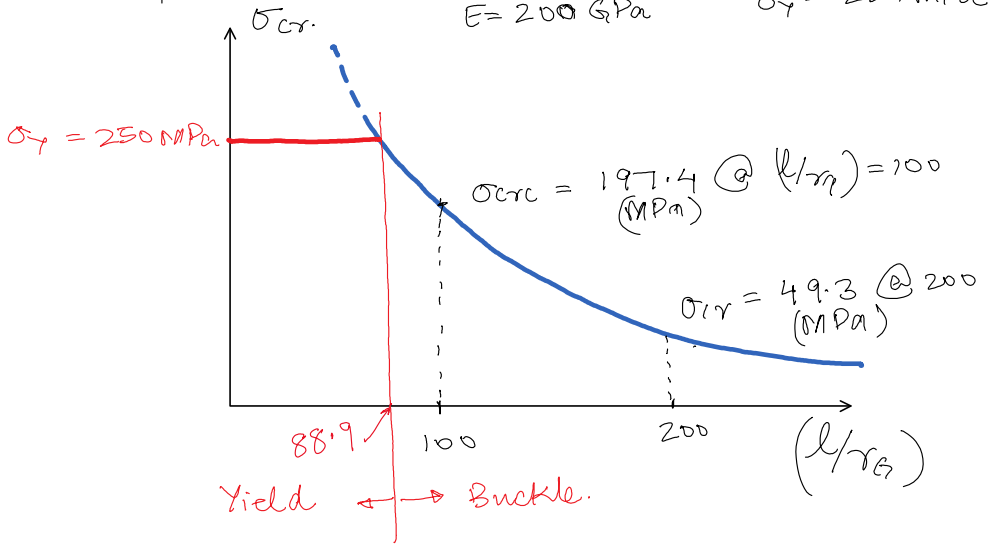
r_G^2

Radius of Gyration

$$r_G = \sqrt{\frac{I}{A}}$$

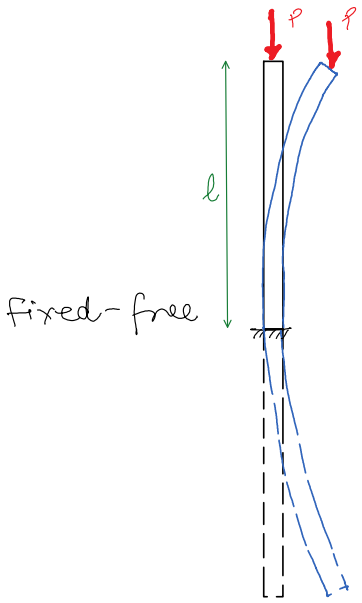
$$\Rightarrow \boxed{\sigma_{cr} = \frac{\pi^2 E}{(l/r_G)^2}}$$

$\frac{l}{r_G}$: slenderness ratio.



Extension of Euler's formula for other end-conditions

$$P_{cr} = \frac{\pi^2 EI}{(l_e)^2}$$ l_e : Effective length



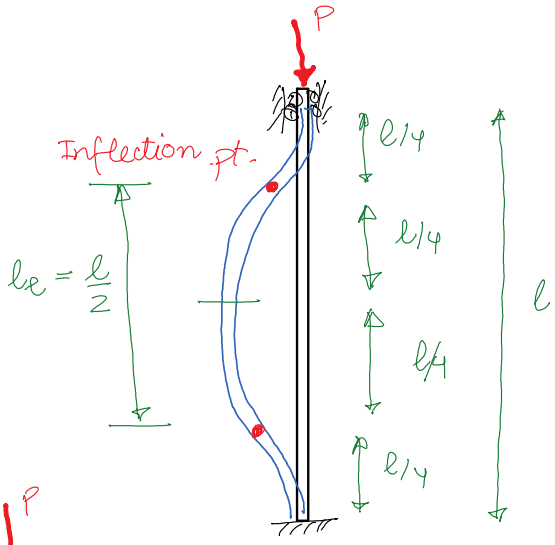
Fixed-free column: $l_e = 2l$.

$$P_{cr} = \frac{\pi^2 EI}{(2l)^2} \qquad \sigma_{cr} = \frac{\pi^2 E}{(l_e/r_g)^2}$$

Fixed-fixed column $l_e = \frac{l}{2}$

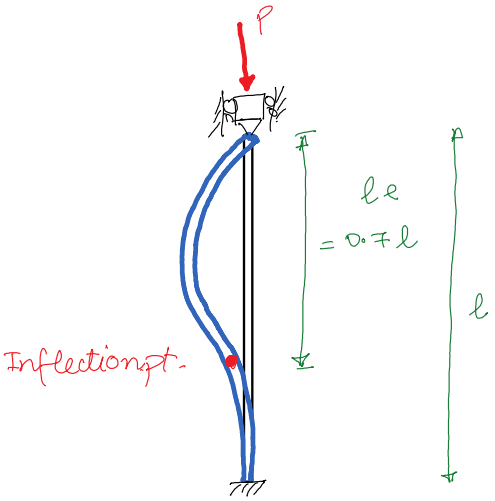
$$P_{cr} = \frac{\pi^2 EI}{(l_e)^2} = \frac{4\pi^2 EI}{l^2}$$

$$\sigma_{cr} = \frac{\pi^2 E}{(l_e/r_g)^2} \qquad r_g = \sqrt{\frac{I}{A}}$$



Fixed-pinned column:

$$P_{cr} = \frac{\pi^2 EI}{(l_e)^2} = \frac{\pi^2 EI}{(0.7l)^2}$$



Summary: $P_{cr} = \frac{\pi^2 EI}{(l_e)^2}$

Condition	K		l_e
Pin-Pin	1		1.0 l
Fixed-free	2		2.0 l
Fixed-fixed	0.5		0.5 l
Fixed-pin	0.7		0.7 l

$l_e = (K)l$

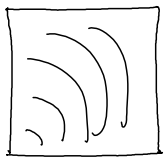
Example

Wood Column with Square cross-section

$E = 13$ GPa (Parallel to the grain)

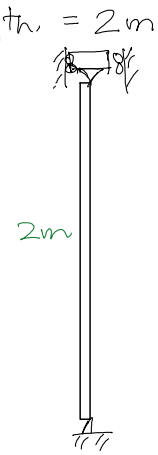
$\sigma_{All} = 12$ MPa (include F.S)

Buckling factor of safety $= 2.5$



$I = \frac{a^4}{12}$

find a for $P = 100$ kN and $P = 200$ kN.



a) $P = 100$ kN

$$P_{cr} = \frac{\pi^2 EI}{l^2} > (100\,kN \times 2.5)$$

$$\Rightarrow \frac{\pi^2 E a^4}{12(2m)^2} > 250\,kN$$

$$\Rightarrow a > \left[\frac{250\,kN \times 12 \times (2m)^2}{\pi^2 \times 13\,GPa}\right]^{1/4} = 98.3\,mm$$

Use $a = 100$ mm ✓

$$\sigma_{comp} = \frac{P}{A} = \frac{100\,kN}{(100\,mm)^2} = \frac{10^5\,N}{10^4\,mm^2} = 10\,MPa \; (< 12\,MPa\ \sigma_{ALLOW})$$

b) $P = 200$ kN

$$P_{cr} = \frac{\pi^2 EI}{l^2} > 200 \times 2.5 = 500\,kN$$

$$\Rightarrow \frac{\pi^2 E a^4}{12 l^2} > 500\,kN \Rightarrow a > \left[\frac{500\,kN \times 12 \times (2m)^2}{\pi^2 \times 13\,GPa}\right]^{1/4}$$

$$\Rightarrow a = 116.95 \rightarrow \text{say } a = 117\,mm$$

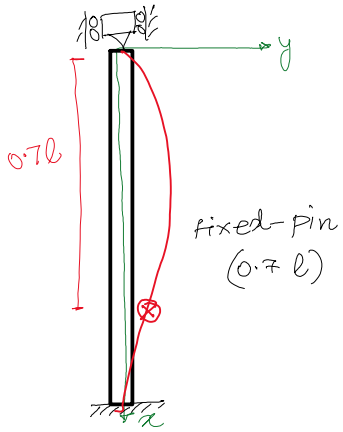
$$\sigma_{comp} = \frac{200\,kN}{(117\,mm)^2} = 14.6\,MPa \quad > 12\,MPa\ (\sigma_{ALLOW}) \times$$

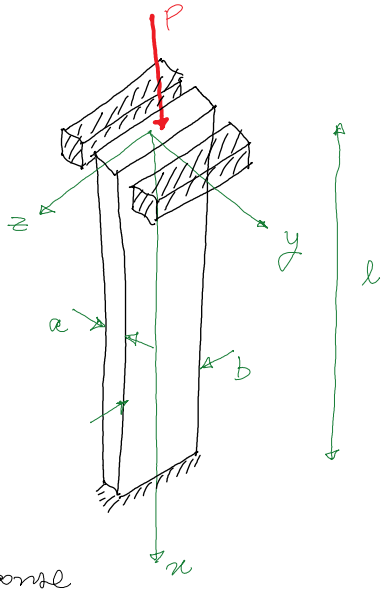
$$\sigma_{ALLOW} = \frac{200\,kN}{a^2\,mm^2} < 12\,MPa \Rightarrow a^2 > \frac{200\,kN}{12\,MPa} \Rightarrow a > 129.1\,mm$$

say 130 mm

Example







$I_y = \frac{1}{12} a b^3$

$I_z = \frac{1}{12} a^3 b$

Design (find a, b) for most efficient response against buckling. FoS = 2.5

$\ell = 20$ in. $\quad E = 10 \times 10^6$ psi $\quad P = 5$ Kips.

For x-z plane:

$$P_{cr}\,(x\text{-}z) = \frac{\pi^2 E I_y}{(\ell e)^2}$$

x-y plane

$$P_{cr}\,(x\text{-}y) = \frac{\pi^2 E I_z}{(\ell e)^2}$$

for most efficient bucking $P_{cr}(x\text{-}z) = P_{cr}(x\text{-}y)$

$$\frac{\pi^2 E \left(\frac{1}{12} a b^3\right) b^2}{(2\ell)^2} = \frac{\pi^2 E \left(\frac{1}{12} a^3 b\right) a^2}{(0.7\ell)^2}$$

$$\Rightarrow \frac{b^2}{4} = \frac{a^2}{(0.7)^2} \Rightarrow \boxed{a = 0.35\, b}$$

(FoS) (P) $\quad P_{cr}$ $\quad (0.35\,b)$

$$(2.5)\ 5\text{kip} = \frac{\pi^2 E I_y}{(2\ell)^2} = \frac{\pi^2 \cdot 10^7 \text{ psi} \cdot \frac{1}{12} a b^3}{(2 \times 20)^2 \text{ in}^2} = 5000 \text{ lbs}(2.5$$

6.9477

$$\Rightarrow b^4 = \Rightarrow \underline{b = 1.62 \text{ in}}$$

$$\Rightarrow a = 0.35\, b = \underline{0.567 \text{ in}}$$

Check $\quad \frac{5 \text{ kip}}{ab} = \frac{5}{(1.62 \times 0.567)} = 5.44 \text{ ksi} < \sigma_{ALLOW}$ (for Al ~ 20 ksi)

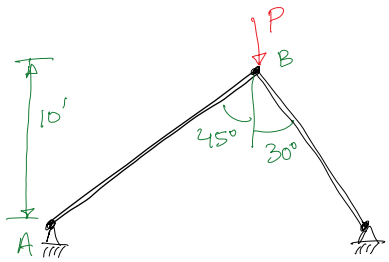
Example

$E = 29000$ ksi

$\sigma_y = 50$ ksi

diameter (AB) (BC) = 5/8"

Find P_{cr}?

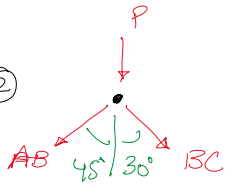


$I_{AB} = I_{BC} = \frac{\pi r^4}{4} = \frac{\pi (5/8)^4}{4 \times 16}$

$l_{BC} = \frac{10'}{\cos 30°}$

$\sum F_x = 0 \Rightarrow -AB \sin 45° + BC \cos 30° = 0$ — (1)

$\sum F_y = 0 \Rightarrow -P - AB \cos 45° - BC \cos 30° = 0$ — (2)



(1) $\Rightarrow BC = \sqrt{2}\, AB$

(2) $\Rightarrow AB \cdot \frac{1}{\sqrt{2}} + BC \cdot \frac{\sqrt{3}}{2\sqrt{2}} = -P$

$\Rightarrow AB \left(\frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}} \right) = -P \Rightarrow AB = -0.52\,P$

$BC = -0.73\,P$

$P_{cr}(AB) = \frac{\pi^2 E I}{(l_e)^2_{AB}} = \frac{\pi^2 \; 29000 \text{ ksi} \times \pi/4 (5/16)^4}{(10\sqrt{2} \times 12 \text{ in})^2} = ___ \Rightarrow P_{cr}^{TOT} = \frac{P_{cr}(AB)}{0.52}$

$P_{cr}(BC) = \frac{\pi^2 E I}{(l_{e_{BC}})^2} = \frac{\pi^2 \; 29000 \text{ ksi} \times \pi/4 (5/16)^4}{(10/\cos 30°)^2} = ___ \Rightarrow P_{cr}^{TOT} = \frac{P_{cr}(BC)}{0.73}$