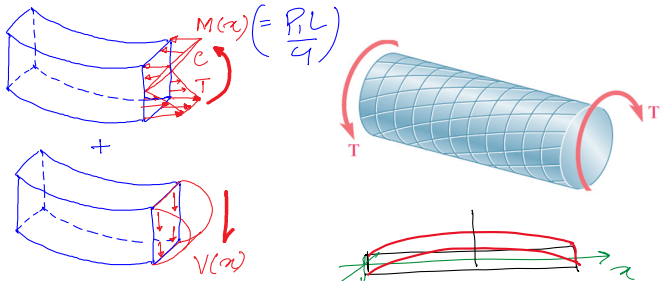
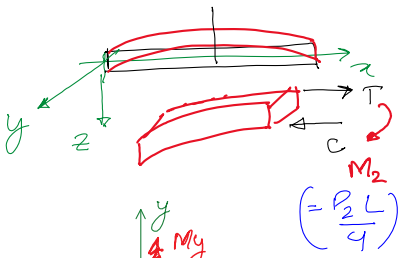
## General Combined Loading

Including
- Axial
- Bending
- Shear
- Torsion

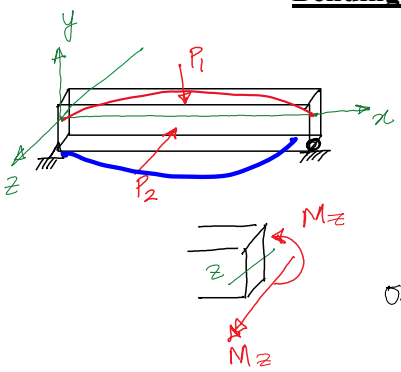


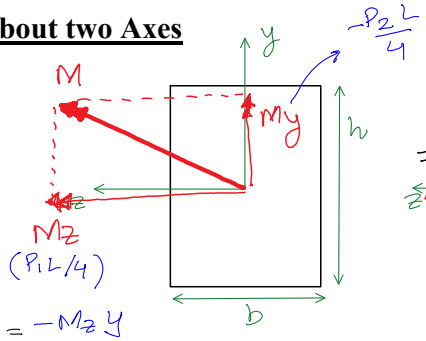



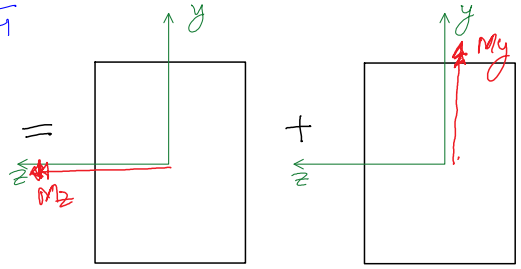

Assume:
- Proportional Limit is not exceeded (Linear behavior)
- Section is far from applied loads (St. Venant's principle)

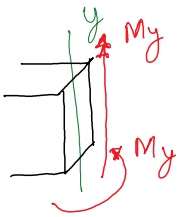
### Bending about two Axes



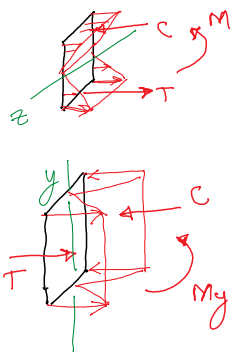




$$\sigma_x = \frac{-M_z y}{I_z}$$




$$\sigma_x = \frac{M_y z}{I_y}$$




$$\sigma_x = \frac{-M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$I_z = \frac{1}{12} b h^3 \quad ; \quad I_y = \frac{1}{12} h b^3$$

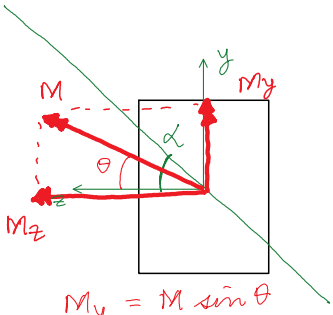
### Neutral Axis

$$\sigma_x = 0 \Rightarrow \frac{-M_z y}{I_z} + \frac{M_y z}{I_y} = 0$$

$$\Rightarrow y = \underbrace{\left(\frac{M_y}{M_z}\right)}_{\tan\theta} \cdot \underbrace{\left(\frac{I_z}{I_y}\right)}_{m} \cdot z = 0$$

$$\Rightarrow y = m z$$

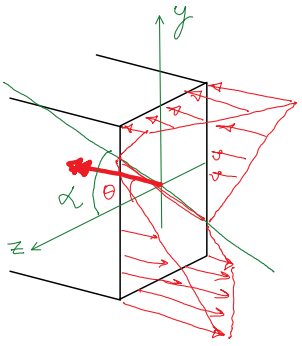
$$m = \tan\alpha$$




$$M_y = M \sin\theta$$
$$M_z = M \cos\theta$$

Note: $\alpha = \theta$ when $\frac{I_z}{I_y} = 1$ e.g. (square / circle)

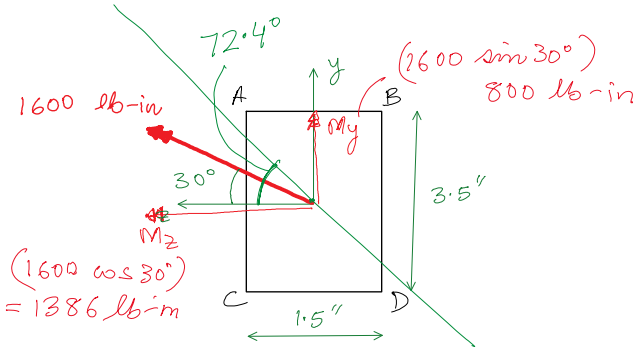
Stress distribution:



Example

Find σmax

Draw stress distribution




$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

Neutral axis:

$$y = mz = \left[\tan 30^\circ \cdot \left(\frac{I_z}{I_y}\right)\right] z$$

$$\Rightarrow y = [\tan 72.4^\circ]\, z$$

$$I_z = \tfrac{1}{12}(1.5")(3.5")^3 = 5.6\ in^4$$

$$I_y = \tfrac{1}{12}(3.5")(1.5")^3 = 0.98\ in^4$$

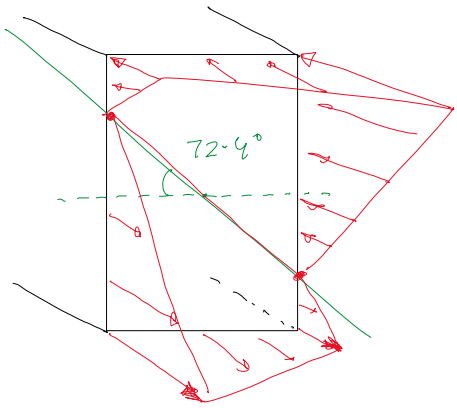
$$\tan\alpha = \cancel{\tan 30^\circ}^{0.577} \times \frac{5.6}{0.98} = 3.299$$

$$\Rightarrow \alpha = \tan^{-1}(3.299) = 72.4^\circ$$

$$\sigma_{MAX/MIN} = -\frac{(1386\ lb\text{-}in)(-1.75\ in)}{5.6\ in^4} + \frac{(800\ lb\text{-}in)(0.75)in}{0.98\ in^4}$$

(C) (B)

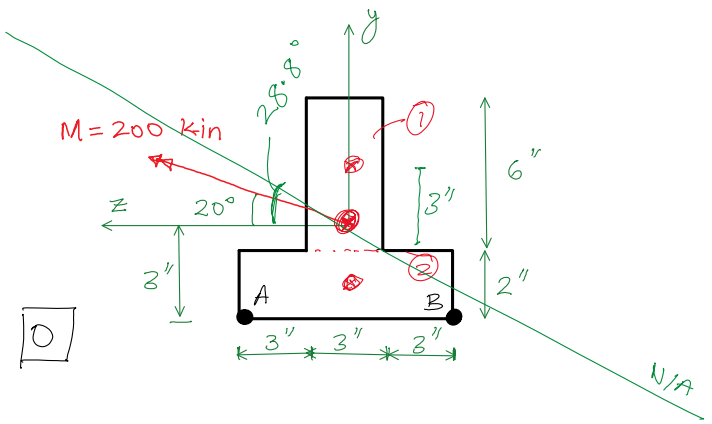
$$= \boxed{\pm 1062\ psi} \quad @\ C/B$$



Example

Find stress at A and B

Locate neutral axis




$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$\bar{y} = \frac{(3"\times 6")\,2" + (9"\times 2")(-2")}{(3"\times 6") + (9"\times 2")} = \boxed{0}$$

$$I_z = \underbrace{\frac{1}{12}(3")(6")^3 + (3"\times 6")(2")^2}_{①} + \frac{1}{12}(9")(2")^3 + (9"\times 2")(2")^2$$

$$I_y = \frac{1}{12}(6")(3")^3 + \frac{1}{12}(2")(9")^3 \quad ② \quad 135\ in^4$$

$$\Rightarrow I_z = 204\ in^4$$

$$y = mz = \underbrace{\left[\tan\theta \cdot \frac{I_z}{I_y}\right]}_{m} \cdot z$$

$M_z = M\cos 20° = 187.$

$M_y = M\sin 20° = 68.$

$$m = \tan(20°)\times\frac{204}{135} = 0.558$$

$$\Rightarrow \tan\alpha = m \Rightarrow \alpha = 28.8°$$

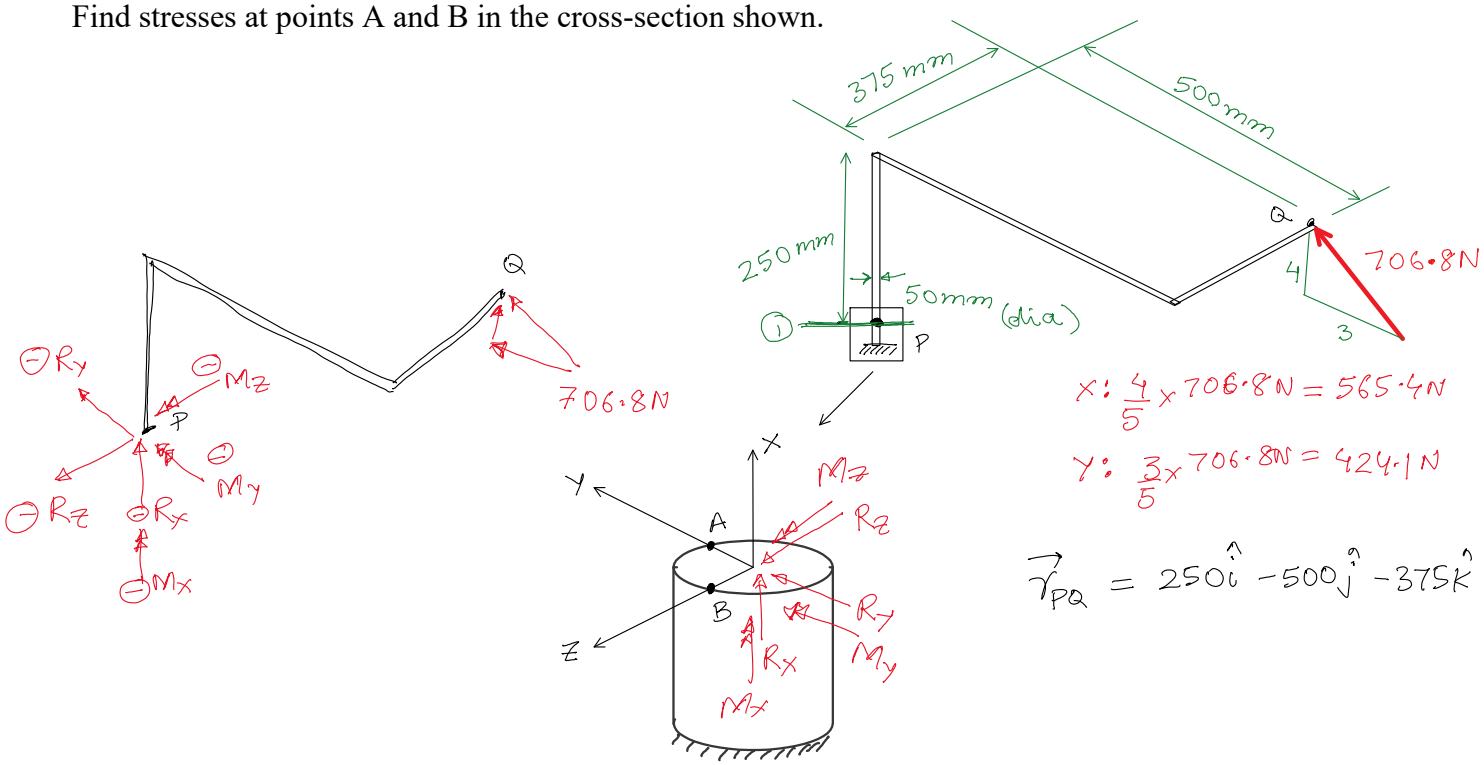
$$\sigma_A = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$= \frac{-187.9\ \text{Kin}\,(-3")}{204\ in^4} + \frac{68.4\ \text{Kin}\cdot(4.5")}{135\ in^4} = 5.04\ \text{Ksi}\ (T)$$

$$\sigma_B = \frac{-187.9\ \text{Kin}\,(3")}{204\ in^4} + \frac{68.4\ \text{Kin}\,(-4.5")}{135\ in^4} = 0.48\ \text{Ksi}\ (T)$$

Example

Find stresses at points A and B in the cross-section shown.




$X: \frac{4}{5} \times 706.8N = 565.4N$

$Y: \frac{3}{5} \times 706.8N = 424.1N$

$\vec{r}_{PQ} = 250\hat{i} - 500\hat{j} - 375\hat{k}$

(3) $\begin{cases} \sum F_x = 0 \Rightarrow R_x = -565.4N \\ \sum F_y = 0 \Rightarrow R_y = -424.1N \\ \sum F_z = 0 \Rightarrow R_z = 0 \end{cases}$

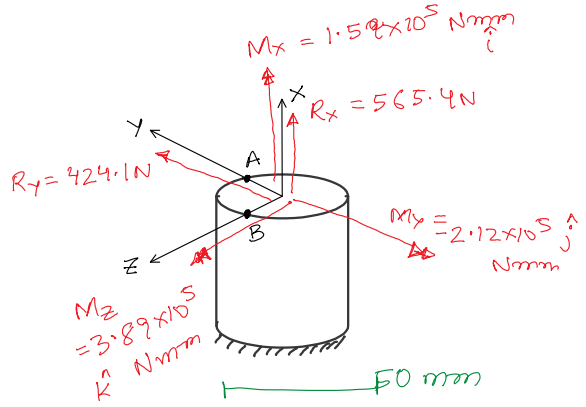
(3) $\sum \vec{M}_P = \vec{0} \Rightarrow \left[ \vec{r}_{PQ} \times (565.4N\,\hat{i} + 424.1N\,\hat{j}) \right] = \left[ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 250 & -500 & -375 \\ 565.4 & 424.1 & 0 \end{vmatrix} \right] = \vec{0}$

$\left[ \mp (M_x\hat{i} + M_y\hat{j} + M_z\hat{k}) \right]$

$\Rightarrow \hat{i}\left(-500\times 0 - (-375\times 424.1) \mp M_x\right) = 0 \Rightarrow M_x = 1.59\times10^5\ Nmm$

$-\hat{j}\left(250\times 0 - (-375\times 565.4) \pm M_y\right) = 0 \Rightarrow M_y = -2.12\times10^5\ Nmm$

$+\hat{k}\left(250\times 424.1 - (-500\times 565.4) \mp M_z\right) = 0 \Rightarrow M_z = 3.887\times10^5\ Nmm$



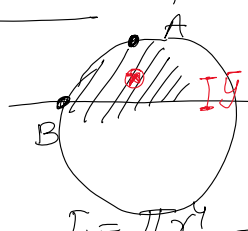

(A): $R_x$: $\sigma_{xx} = \frac{R_x}{A} = \frac{565.4\,N}{1963\,mm^2} =$

(B): $R_x$: $\sigma_{xx} = \frac{R_x}{A} = \frac{565.4\,N}{1963\,mm^2} =$

(A): $R_y$: $\frac{VQ_A}{It} \Rightarrow \tau_{xy} = 0$

(B): $R_y$: $\frac{VQ_B}{It} \Rightarrow \tau_{xy} = \frac{424.1\,N \times 1.042 \times 10^4\,mm^3}{3.068 \times 10^5\,mm^4 \times 50\,mm} =$

$A = \frac{\pi}{4}(50)^2 = 1963\,mm^2$

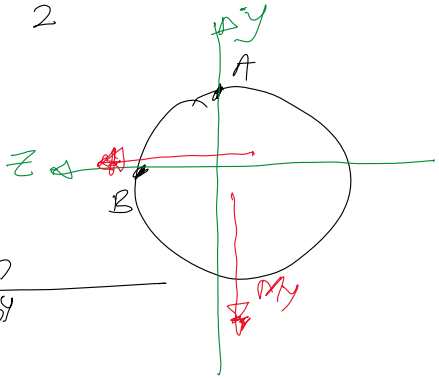



$Q_A = 0$

$Q_B = \left(\frac{1}{2}\pi r^2\right)\bar{y} = 1.042 \times 10^4\,mm^3$

(A): $M_x$: $\tau_{xz} = \frac{Tc}{J} = \frac{1.59 \times 10^5\,Nmm \times 25\,mm}{6.13 \times 10^5\,mm^4}$ $I = \frac{\pi r^4}{4} = 3.068 \times 10^5\,mm^4$

(B): $M_y$: $\tau_{xy} = \frac{-Tc}{J} = \frac{-1.59 \times 10^5 \times 25}{6.13 \times 10^5}$ $J = \frac{\pi r^4}{2} = 6.13 \times 10^5\,mm^4$



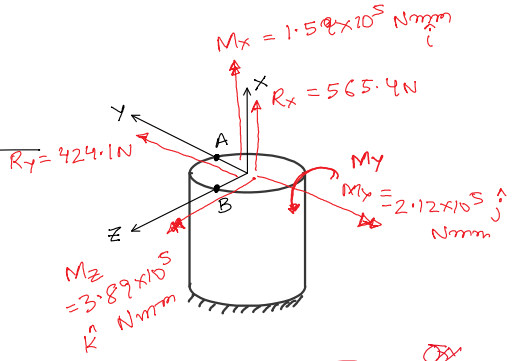

(A): $M_y$: $\sigma_{xx} = \frac{M_y z}{I_y} = 0$

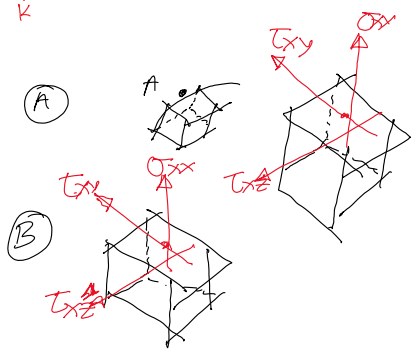
(B): $M_y$: $\sigma_{xx} = \frac{M_y \cdot z}{I_y} = \frac{2.12 \times 10^5\,Nmm \times 25\,mm}{3.068 \times 10^5\,mm^4}$

(A): $M_z$: $\sigma_{xx} = \frac{-M_z y}{I_z} = \frac{3.89 \times 10^5\,Nmm \times 25\,mm}{3.068 \times 10^5\,mm^4}$

(B): $M_z$: $\sigma_{xx} = \frac{-M_z \bar{y}}{I_z} = 0$

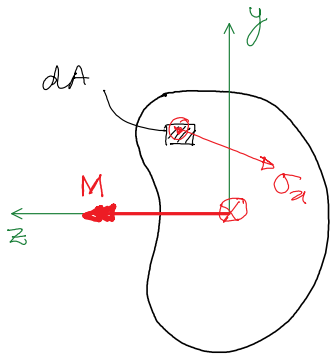
| | (A) $\sigma_{xx}$ | (A) $\tau_{xz}$ | (A) $\tau_{xy}$ | (B) $\sigma_{xx}$ | (B) $\tau_{xz}$ | (B) $\tau_{xy}$ |
|---|---|---|---|---|---|---|
| $R_x$ | $R_x/A$ | — | — | $R_x/A$ | — | — |
| $R_y$ | — | — | $\frac{VQ_A}{It}$ | — | — | $\frac{VQ_B}{It}$ |
| $R_z$ | — | — | — | — | — | — |
| $M_x$ | — | $\frac{Tc}{J}$ | — | — | — | $\frac{-Tc}{J}$ |
| $M_y$ | $\frac{M_z}{I_y}$ | — | — | $\frac{M_z}{I_y}$ | — | — |
| $M_z$ | $\frac{-M_y}{I_z}$ | — | — | $\frac{-M_y}{I_z}$ | — | — |
| | $-31.39$ MPa | $6.48$ MPa | $0$ | $-17$ MPa | $0$ | $-6.19$ MPa |



## Beams with non-symmetric cross-sections

Is $\sigma_x = -\frac{M_z y}{I_z}$ still valid ?




Total cross-sectional force :

$$P = \int_A \sigma_x \, dA = 0 \quad \text{—— (1)}$$

$$M_z = -\int_A \sigma_x \cdot y \, dA = M \quad \text{—— (2)}$$

$$M_y = +\int \sigma_x \cdot z \, dA = 0 \quad \text{—— (3)}$$

Condition (1) : $P = \int_A -\frac{M_z y}{I_z} \cdot dA = -\frac{M_z}{I_z} \underbrace{\int y \cdot dA}_{=0} = 0$

As long as z-axis passes through the centroid, ok. ✓

Condition (2) :

$$M = -\int \left(-\frac{M_z}{I_z}\right) \cdot y^2 \, dA = \frac{M_z}{I_z} \cdot \left(\int y^2 \, dA\right) = M \quad \checkmark$$

($I_z$)

Condition (3) :

$$M_y = \int \left(-\frac{M_z}{I_z}\right) \cdot y \cdot z \cdot dA = 0$$

$$= \left(-\frac{M_z}{I_z}\right) \underbrace{\left(\int yz \cdot dA\right)}_{I_{yz}} = 0$$

$I_{yz}$ : Product Moment of Inertia

Note :

$I_z = \int y^2 \, dA$

$I_y = \int z^2 \, dA$

$I_{yz} = \int yz \, dA$

for (3) to hold $I_{yz} = 0$

Note:

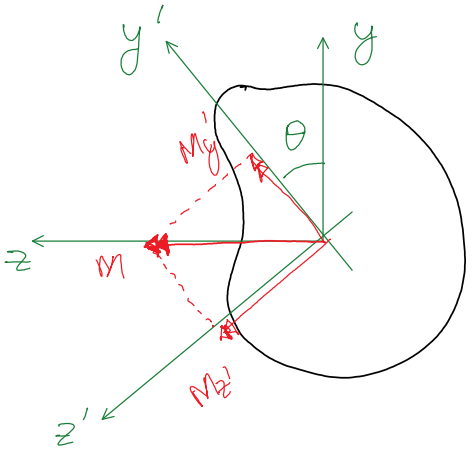
for $\sigma_x = -\frac{M_z}{I_z} y$ to hold

- Origin must be at the centroid.

- y, z axis must be principal axes

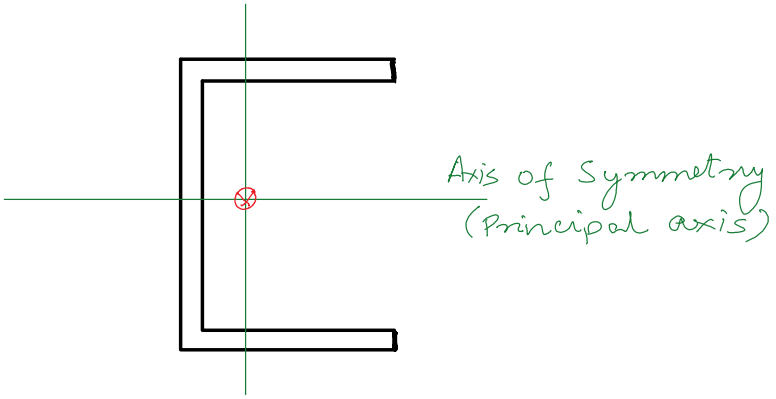
- If y & z are not principal axes, then we must transform y-z to y'z' which are principal axes:

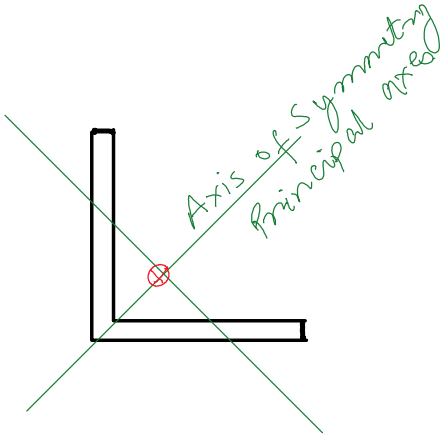
- Moments of Inertia transform like stresses




- Principal axes are perpendicular.

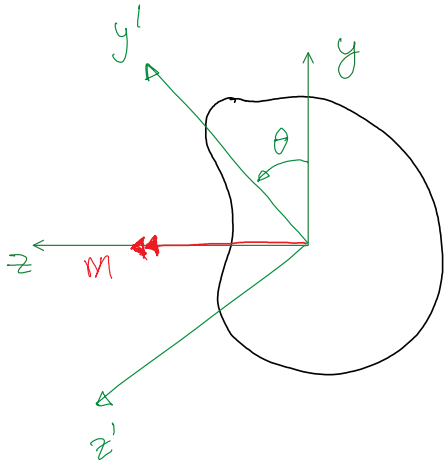
- Axes of symmetry is principal axis. => $I_{yz} = 0$

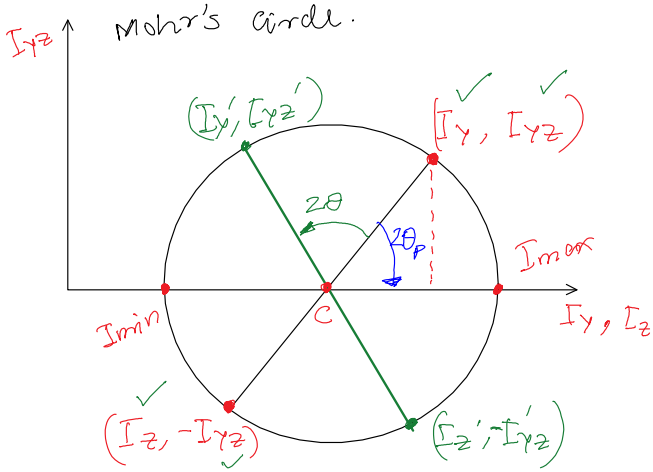


## Transformation of second moments of area:

Given $I_y$ ; $I_z$ ; $I_{yz}$




C : Center : $\left(\frac{I_y + I_z}{2}, 0\right)$

$$R = \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + (I_{yz})^2}$$

$$I_{y'} = \left(\frac{I_y + I_z}{2}\right) + \left(\frac{I_y - I_z}{2}\right)\cos 2\theta - I_{yz}\sin 2\theta$$

$$I_{z'} = \left(\frac{I_y + I_z}{2}\right) - \left(\frac{I_y - I_z}{2}\right)\cos 2\theta + I_{yz}\sin 2\theta$$

$$I_{y'z'} = \frac{I_y - I_z}{2}\sin 2\theta + I_{yz}\cos 2\theta$$

- Finding Principal Axes:

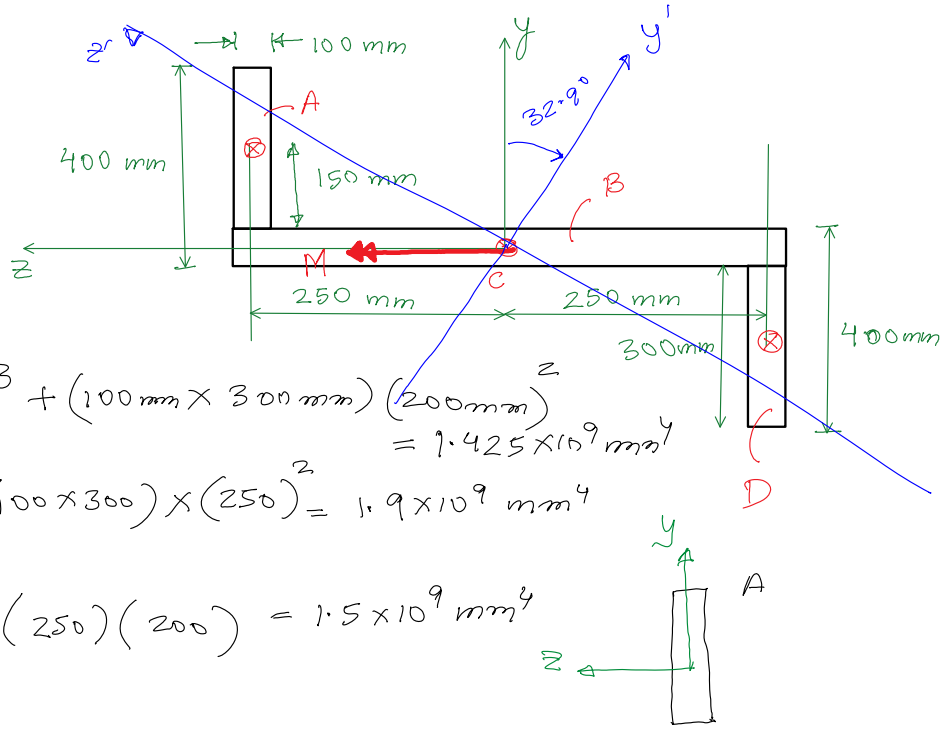
$$I_{max}/I_{min} = \left(\frac{I_y + I_z}{2}\right) \pm R$$

$$\tan(2\theta_p) = \frac{-I_{yz}}{(I_y - I_z)/2}$$

## Example

Find Principal Axes:

$I_{max}$ ; $I_{min.}$ ; $\theta_p$




$I_y$ ; $I_z$ ; $I_{yz}$

A:

$$I_z = \frac{1}{12}(100mm)(300mm)^3 + (100mm \times 300mm)(200mm)^2 = 1.425 \times 10^9\ mm^4$$

$$I_y = \frac{1}{12}(300)(100)^3 + (100 \times 300) \times (250)^2 = 1.9 \times 10^9\ mm^4$$

$$I_{yz} = 0 + (100 \times 300)(250)(200) = 1.5 \times 10^9\ mm^4$$




Note:

Parallel Axis Theorem for $I_{yz} = \bar{I}_{yz} + A(d_y)(d_z)$

B:

$$I_z = \frac{1}{12}(600)(100)^3 + 0$$

$$I_y = \frac{1}{12}(100)(600)^3 + 0$$

$$I_{yz} = 0$$

D:

$$I_z = \frac{1}{12}(100mm)(300mm)^3 + (100mm \times 300mm)(\ominus 200mm)^2 = 1.425 \times 10^9\ mm^4$$

$$I_y = \frac{1}{12}(300)(100)^3 + (100 \times 300) \times (\ominus 250)^2 = 1.9 \times 10^9\ mm^4$$

$$I_{yz} = 0 + (100 \times 300)(\ominus 250)(\ominus 200) = 1.5 \times 10^9\ mm^4$$

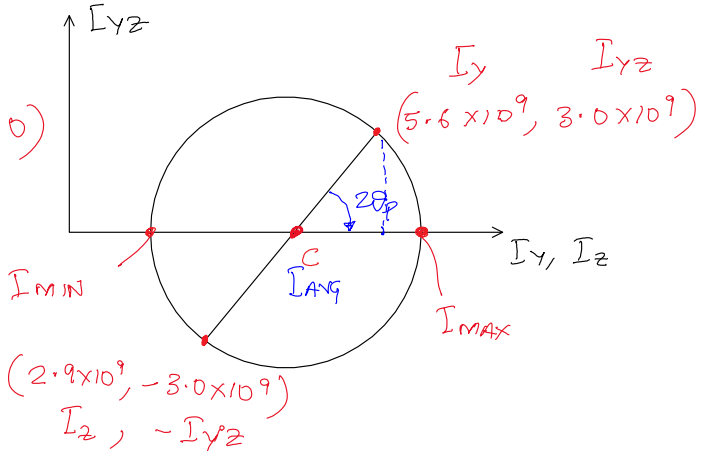
for entire cross section:

$$I_{y\,tot} = 5.6 \times 10^9\ mm^4$$

$$I_{z\,tot} = 2.9 \times 10^9\ mm^4$$

$$I_{yz\,tot} = 3.0 \times 10^9\ mm^4$$

$$C = \frac{1}{2}(5.6+2.9)\times 10^9\ mm^4 = (4.25\times 10^9\ mm^4,\ 0)$$




$$R = \sqrt{(5.6-4.25)^2 + (3.0)^2}\ \times 10^9\ mm^4$$

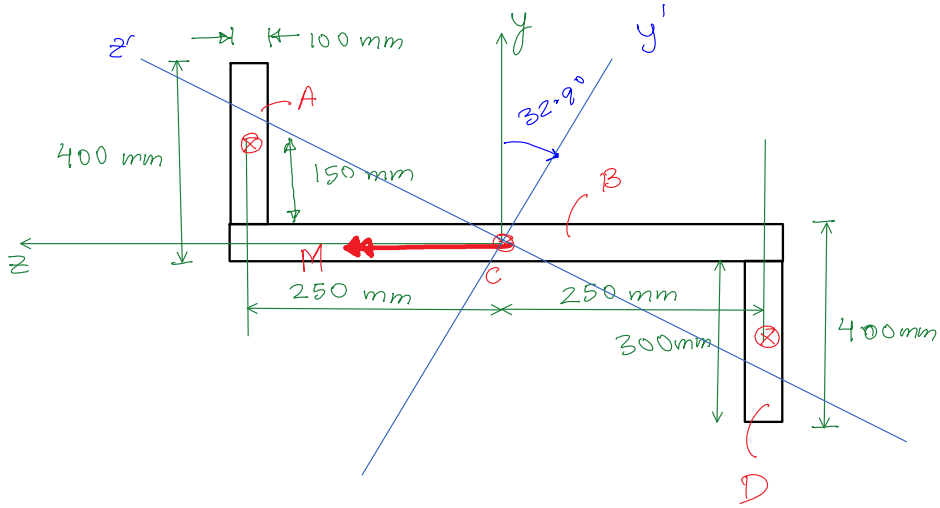
$$I_{max},\ I_{min} = C \pm R$$

$$= 7.54\times10^9\ mm^4 \ /\ 0.96\times10^9\ mm^4$$

$$\tan 2\theta_p = \left(\frac{I_{YZ}}{I_y - I_{AVG}}\right)$$

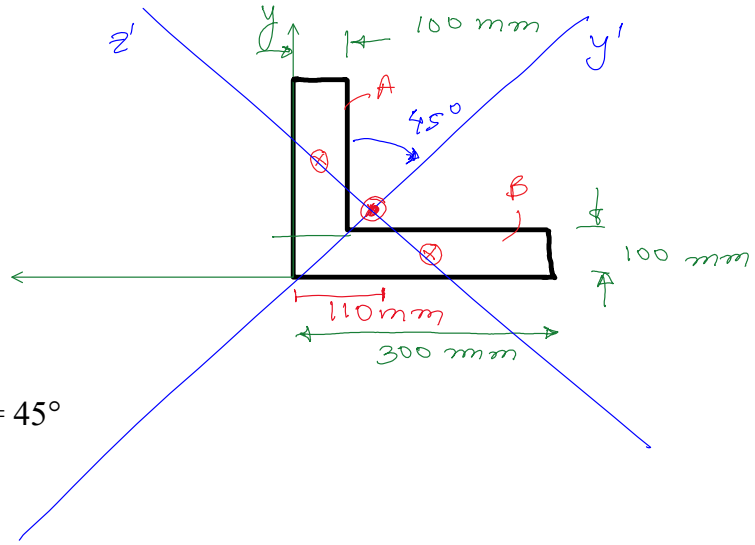
$$\Rightarrow \theta_p = -32.9^\circ$$

$$\sigma_x = -\frac{M_{z'}\,y}{I_{z'}} + \frac{M_{y'}\,z}{I_{y'}}$$



Example:

Find $\theta_p$




$$\bar{y} = \bar{z} = \frac{(100\times200)200 + (100\times300)50}{(100\times200) + (100\times300)}$$

$$= 110 \text{ mm}$$

Complete the following yourself and Verify that $\theta_P = 45°$

A: $I_z =$

$I_y =$

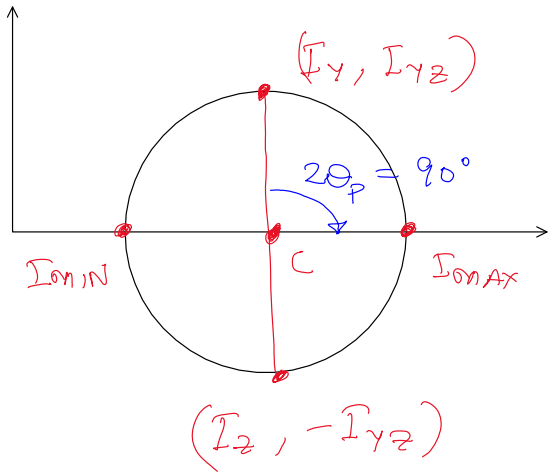
$I_{yz} =$

B: $I_z =$

$I_y =$

$I_{yz} =$

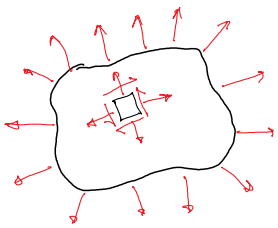
$\theta_{p_1} =$



## Thin-walled Pressure Vessels

(Ref: Ch-8, Hibbeler)

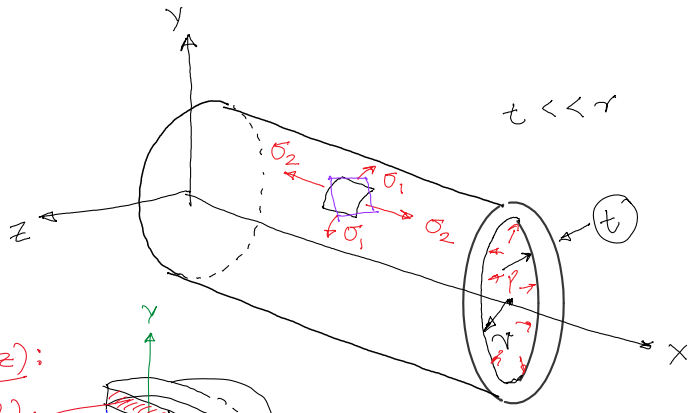
- No bending
- Stresses are in-plane





Cylindrical Pressure Vessel

- Axisymmetric behavior
- Principal stresses along axis (no shear stress)
  - σ1 - Hoop stress (circumference)
  - σ2 - Longitudinal stress (axial)

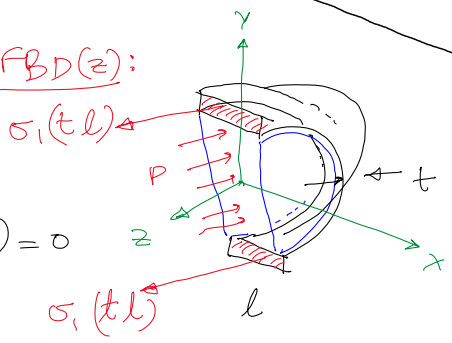



σ1 - Hoop stress

$$\sum F_x = 0 \Rightarrow +\sigma_2(\pi r t) - \sigma_2(\pi r t) = 0$$

$$\sum F_z = 0 \Rightarrow 2\sigma_1(tl) - p(2rl) = 0$$

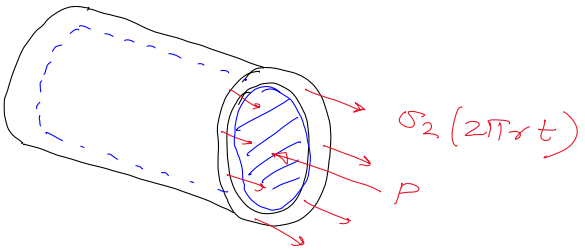
FBD(z):




$$\Rightarrow \boxed{\sigma_1 = \frac{pr}{t}}$$

$r >> t$

σ2 - Longitudinal stress



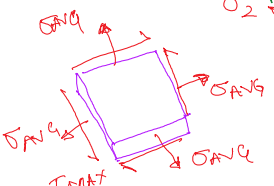

$$\sum F_x = 0$$

$$\Rightarrow \sigma_2(2\pi r t) - p(\pi r^2) = 0$$

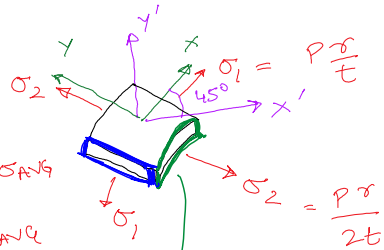
$$\Rightarrow \boxed{\sigma_2 = \frac{pr}{2t}}$$

Shear stresses:

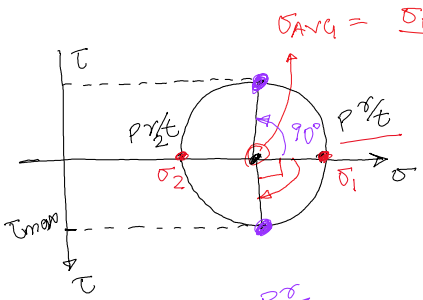
In-plane:




$\tau_{MAX} = \frac{pr}{4t}$




(@ 45° to x-y plane $\sigma_1, \sigma_2$ plane)



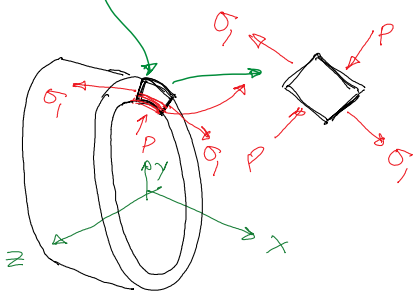

$\sigma_{AVG} = \frac{\sigma_1 + \sigma_2}{2} = \frac{3}{4}\frac{pr}{t}$

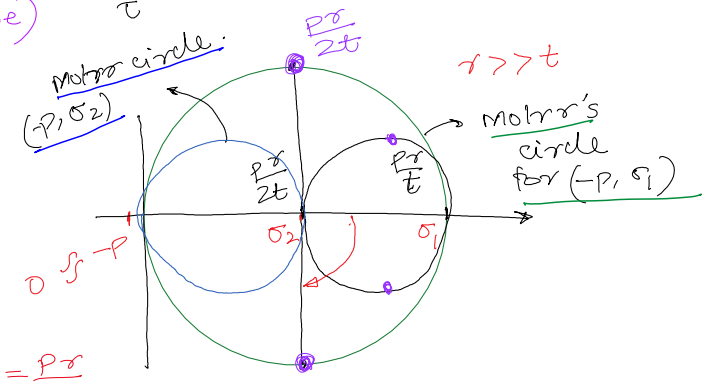
$R = \frac{pr}{4t}$

Out-of-plane:

$\tau_{MAX} = \frac{pr}{2t}$ (out-of-plane)

(@ 45° in y-z plane)



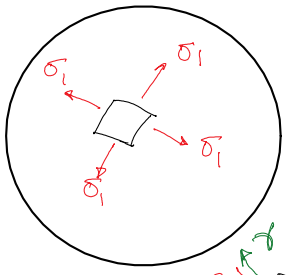

r >> t

$\sigma_{MAX} = \sigma_1 = \frac{pr}{t}$

$\tau_{MAX} = \frac{pr}{2t}$

Spherical Pressure Vessels:

- All points have same state of stress
- Iso-tropic: same in all directions
- No shear stress (in-plane)

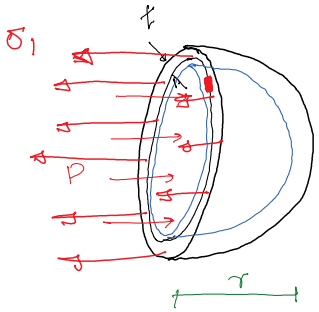


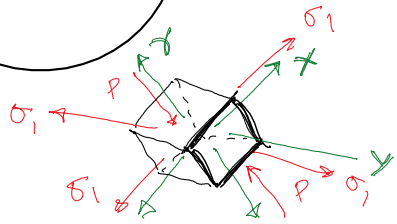

In-plane stresses:

$\sum F_x = 0$

$P(\pi r^2) - \sigma_1(2\pi r t) = 0$

$\Rightarrow \quad \sigma_1 = \frac{pr}{2t}$



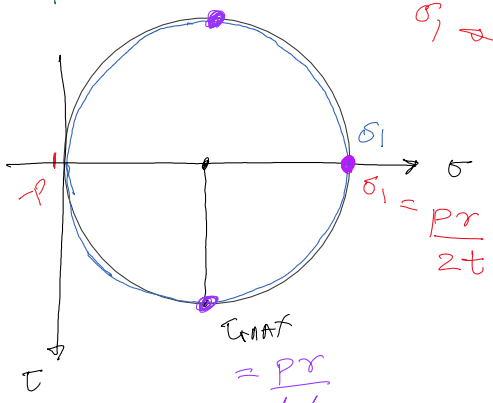



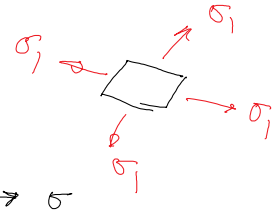

r >> t

No-inplane shear stresses!

Shear stresses (out-of-plane)

$\tau_{MAX} = \frac{\sigma_1}{2} = \frac{pr}{4t}$




$\sigma_1 = \frac{pr}{2t}$

$= \frac{pr}{4t}$

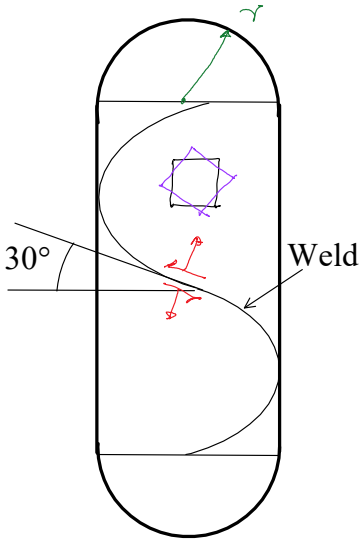
$\sigma_{MAX} = \sigma_1 = \frac{pr}{2t}$

$\tau_{MAX} = \frac{pr}{4t}$

Example:

Cylindrical tank with spherical cap and spiral weld:

$t$ = 12 mm
$r$ = 1250 mm
$p$ = 1.5 Mpa




Find:

a) - Normal and shear stresses in Spherical Cap
b) - Normal and shear stresses in Cylindrical body
c) - Normal and shear stresses across the weld

a) Spherical:

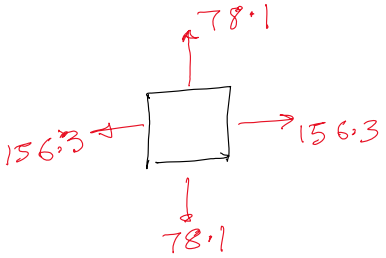
$\sigma_{MAX} = pr/2t = \frac{1.5(N/mm^2)\cdot 1250mm}{2\times 12mm} = 78.1 MPa$

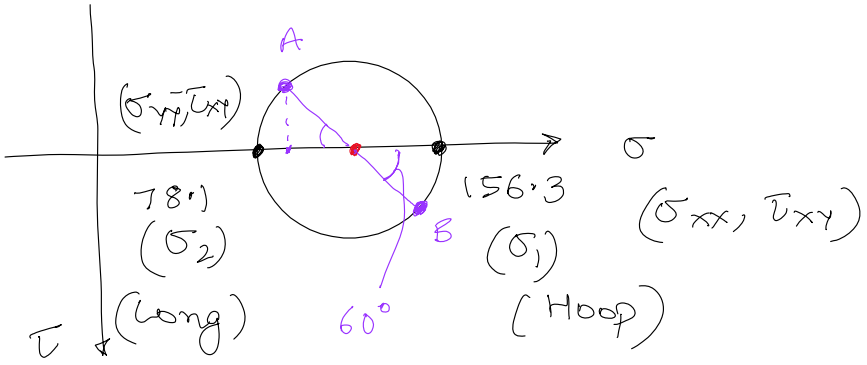
$\tau_{MAX} = pr/4t = \frac{78.1}{2} = 39.1 MPa$

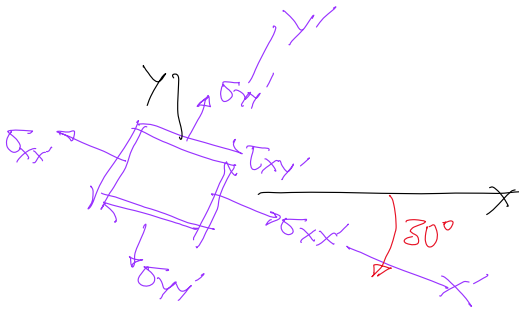
b) Cylinder:

$\sigma_{MAX} = \frac{pr}{t} = 78.1 \times 2 = 156.3 MPa$

$\tau_{MAX} = \frac{pr}{2t} = 78.1 MPa$




center = (156.3 + 78.1)/2 = 117.2 MPa
R = 78.1/2 = 39.1 MPa

Pt (A): $\sigma_{yy}' = \sigma_{AVG} - R\cos 60° = 97.65$ MPa

$\tau_{xy}' = R\sin 60° = 39.1 \times \frac{\sqrt{3}}{2} = 33.86 MPa$