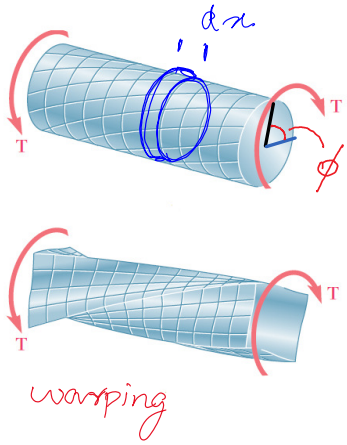
Torsional Members

Torsional moment in bars results from force-couples that lead to twisting along the longitudinal axis of the member.



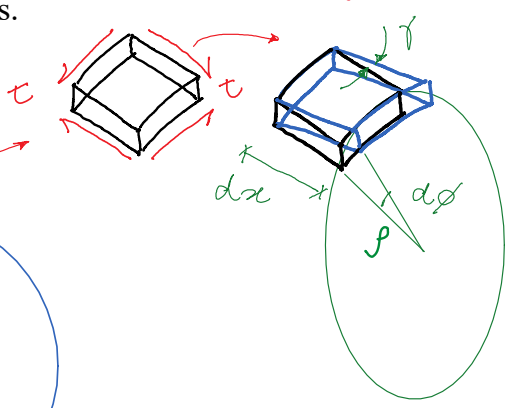


Torsional deformation:

- Circular cross-section: assume that cross-section remains plane before and after deformation
- Non-circular cross-section: warping out of plane

Note: Torsion leads to shearing deformation of the material elements.

This shearing deformation is characterized by angle of twist ϕ.
The angle of twist increases along the length of the bar.



Shear strain due to torsion:

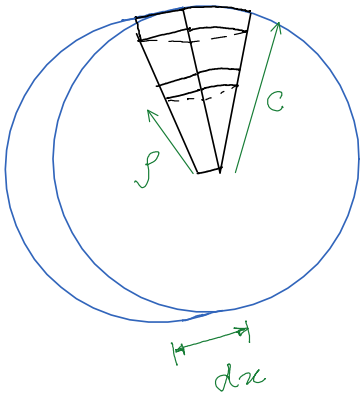
$$dx\ \gamma = \rho\, d\phi$$

$$\Rightarrow \quad \gamma = \rho \left(\frac{d\phi}{dx}\right)$$

Note:

$$\gamma_{max} = c \left(\frac{d\phi}{dx}\right)$$

$$\Rightarrow \quad \boxed{\gamma = \frac{\rho}{c}\,\gamma_{max}}$$



{ Compare with $\epsilon = \left(\frac{y}{c}\right) \epsilon_{max}$ }

Torsional Shear Stresses:

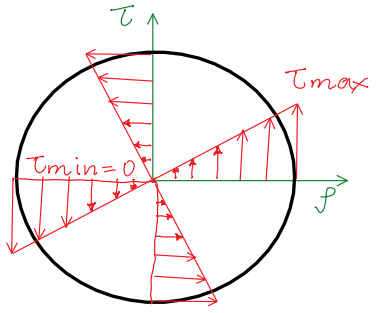
$$\tau = G\gamma = G\left(\frac{\rho}{c}\right)\gamma_{max}$$

Note:

$$\tau_{max} = G\,\gamma_{max} = G\left(c\,\frac{d\phi}{dx}\right)$$

Thus

$$\boxed{\tau = \frac{\rho}{c}\,\tau_{max}}$$

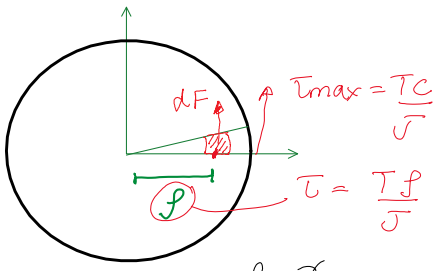


{ Compare with $\sigma = \left(\frac{y}{c}\right) \sigma_{max}$ }

Torsion Formula:

For solid shaft: $dF = \tau\, dA$

Total torsional Moment: $\int dT = \int \rho\, \underbrace{\tau\, dA}_{dF}$



$$\Rightarrow \quad T = \int dT = \int \rho \cdot \left(\frac{\rho}{c}\right) \tau_{max}\, dA$$

$$\Rightarrow \quad T = \frac{\tau_{max}}{c} \boxed{\int \rho^2 dA}$$

$\rho = r$

J: Polar Moment of Inertia $\rightarrow \int r^2 dA$

$J = I_x + I_y = \int (x^2+y^2)\, dA$ $\quad r^2 = x^2 + y^2$

$$\Rightarrow \quad T = \frac{\tau_{max}}{c} J \quad \text{or} \quad \boxed{\tau_{max} = \frac{Tc}{J}}$$

or $\quad \tau = \frac{T\rho}{J}$

Compare with:

$\sigma_{max} = -\frac{Mc}{I}$

$\sigma = -\frac{My}{I}$

Polar Moment of Inertia:

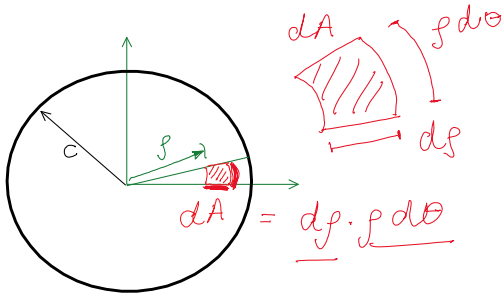
Solid Bar:

$dA = \rho\, d\theta \cdot d\rho$

$$J = \int \rho^2 dA = \int_0^c \int_0^{2\pi} \rho^2 \cdot \rho\, d\theta \cdot d\rho$$

$$= 2\pi \int_0^c \rho^3 \cdot d\rho = 2\pi \cdot \left[\frac{\rho^4}{4}\right]_0^c = \frac{2\pi \cdot c^4}{4 \cdot 2}$$

$$\Rightarrow \quad \boxed{J = \frac{\pi c^4}{2}}$$



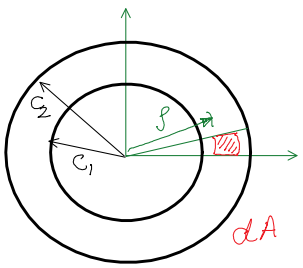
compare with $I_y = \frac{\pi c^4}{4} = I_x$ $\quad$ Note: $J = I_x + I_y$

Hollow bar:

$$J = \int \rho^2 dA = \int_{c_1}^{c_2} \int_0^{2\pi} \rho^2 \cdot \rho\, d\theta \cdot d\rho$$

$$= 2\pi \cdot \left[\frac{\rho^4}{4}\right]_{c_1}^{c_2}$$

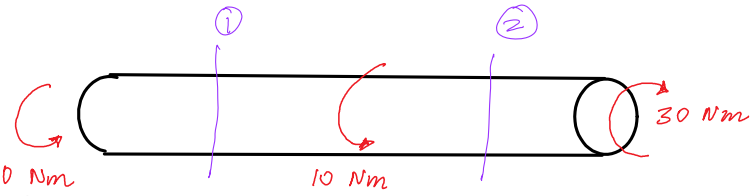
$$\Rightarrow \quad \boxed{J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right)}$$



Example

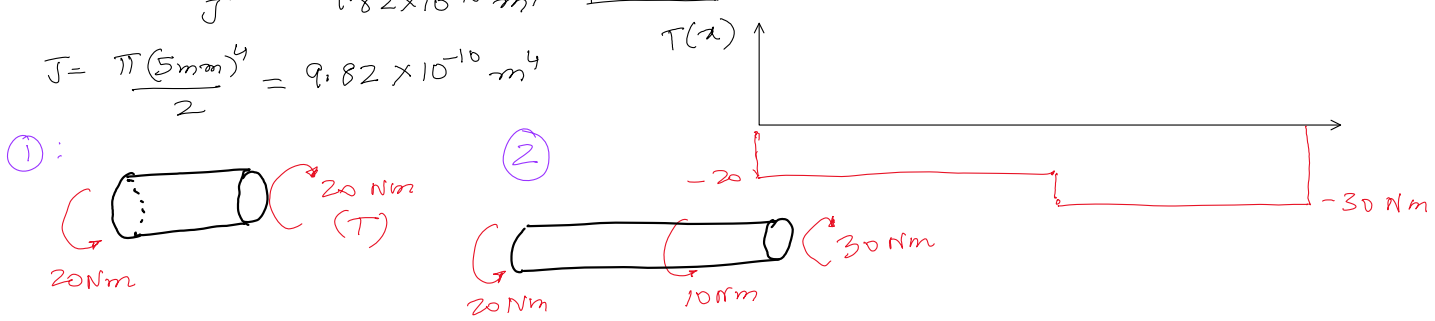
Radius: 5mm $= c$

Find τ_{max}



$$\tau_{max} = \frac{Tc}{J} = \frac{30\,Nm \cdot (5\times10^{-3}\,m)}{9.82\times10^{-10}\,m^4} = 153\,MPa$$

$$J = \frac{\pi (5mm)^4}{2} = 9.82\times10^{-10}\,m^4$$

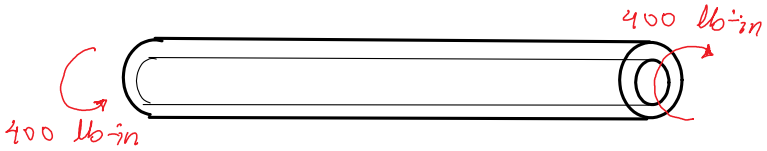


Example

Radii: 0.5in; 0.45in

a) Find τ_{Max} ; τ_{Min}

b) Radius of solid shaft for same τ_{Max} ?



a)

$$\tau_{max} = \frac{Tc_2}{J} = \frac{400\,lb\text{-}in \times 0.5in}{0.0337\,in^4} = 5930\,\frac{lb}{in^2}$$

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}(0.5^4 - 0.45^4) = 0.0337\,in^4$$

(5930 psi)

$$\tau_{min} = \frac{Tc_1}{J} = \frac{400\,lb\text{-}in \times 0.45in}{0.0337\,in^4} = 5330\,psi$$

b) $\tau_{max} = 5930\,psi$

$$\tau_{max} = \frac{Tc}{J_{solid}} = \frac{400\,lb\text{-}in \cdot c}{J_{solid}} = \frac{400\,lb\text{-}in \cdot c}{\frac{\pi c^4}{2}}$$

$$J_{solid} = \frac{\pi c^4}{2}$$

$$\Rightarrow c^3 = \frac{2\times400\,lb\text{-}in \cdot in^2}{\pi \times 5930\,lb} \Rightarrow c = 0.35in$$

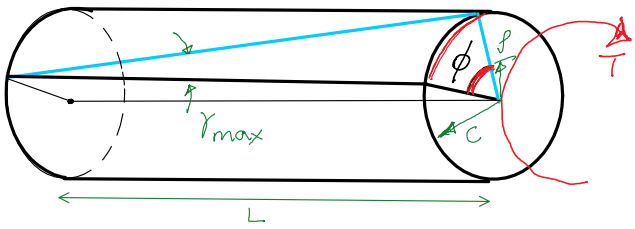
Note: Area (Hollow) $= \pi c_2^2 - \pi c_1^2 = \pi(0.5)^2 - \pi(0.45)^2 = 0.1492\,in^2$ ✓

Area (solid) $= \pi c^2 = \pi(0.35)^2 = 0.385\,in^2$

Torsional Deformation: Angle of twist:

Note: $L\gamma_{max} = c\phi$

or $(L\gamma = \rho\phi)$

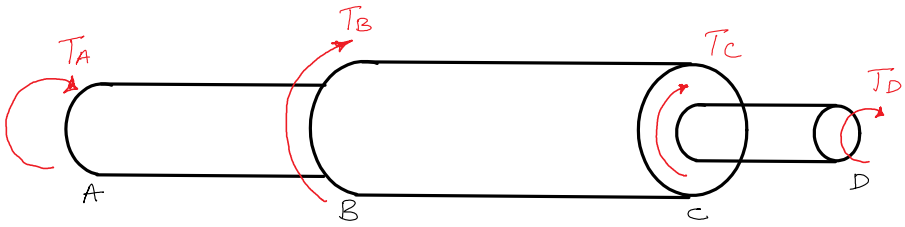


$$\Rightarrow \phi = \frac{L\gamma_{max}}{c} = \frac{L}{c}\frac{\tau_{max}}{G} = \frac{L}{c}\cdot\frac{Tc}{GJ}$$

$$\Rightarrow \boxed{\phi = \frac{TL}{GJ}}$$

{ Compare with $\delta = \frac{PL}{AE}$ }

Shafts with multiple cross-sections:



$$\phi_{TOT} = \sum_i \phi_i$$

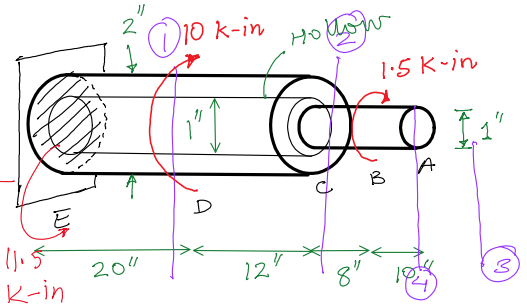
$$\phi_{TOT} = \phi_{AB} + \phi_{BC} + \phi_{CD}$$

$$= \frac{T_{AB}L_{AB}}{G_{AB}J_{AB}} + \frac{T_{BC}L_{BC}}{G_{BC}J_{BC}} + \frac{T_{CD}L_{CD}}{G_{CD}J_{CD}}$$

Example

Find ϕ_A

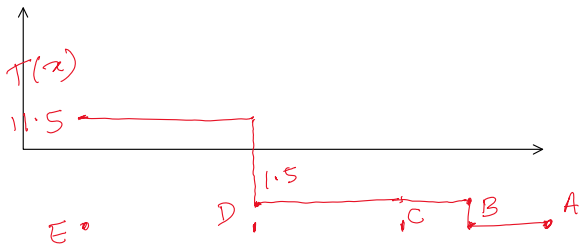
$G = 12\times10^3$ ksi (steel)



$$\phi_A = \phi_{ED} + \phi_{DC} + \phi_{CB} + \cancel{\phi_{BA}}^{\,0}$$

$$= \frac{11.5 \quad 20''}{\cancel{T_{ED}}\,\cancel{L_{ED}}} + \frac{1.5 \quad 12''}{\cancel{T_{DC}}\,\cancel{L_{DC}}} + \frac{1.5 \quad 8''}{\cancel{T_{CB}}\,\cancel{L_{CB}}}$$

$$\frac{}{G\ J_{ED}} \qquad \frac{}{G\ J_{DC}} \qquad \frac{}{G\ J_{CB}}$$

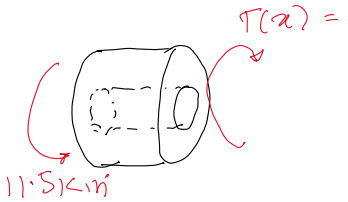


$$J_{ED} = J_{DC} = \frac{\pi}{2}\left[(1'')^4 - [0.5'']^4\right]$$

$$= 1.47\ in^4$$

$$J_{CB} = \frac{\pi}{2}(0.5)^4 = 0.0982\ in^4$$

$$\phi_A = \phi_B = 0.0242\ rad \approx 1.4^\circ$$

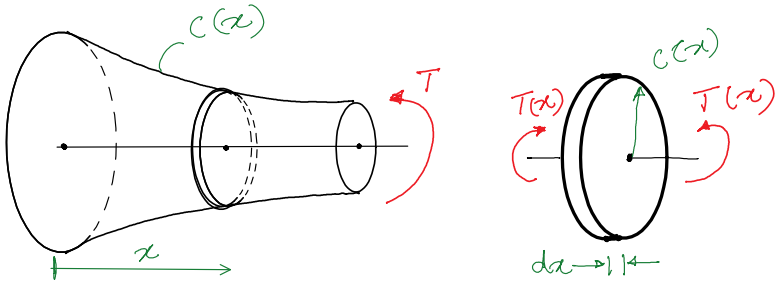




Torsion in bars with variable cross-sections:

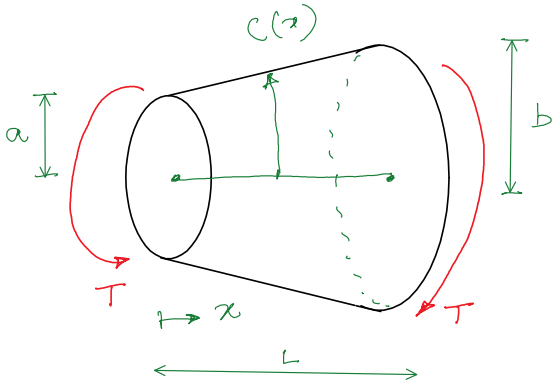
$$d\phi = \frac{T(x)\,dx}{G\,J(x)}$$

$$\Rightarrow \phi = \int d\phi = \int_0^L \frac{T(x)}{G\,J(x)}\,dx$$



Note: $J(x) = \frac{\pi}{2}\cdot[c(x)]^4$

Example: Linearly varying cross-section



Find ϕ

$$d\phi = \frac{T(x)\cdot dx}{G\,J}$$

$$\Rightarrow \phi = \int_0^L \frac{T\cdot dx}{G\,\frac{\pi}{2}\underbrace{\left[a+\left(\frac{b-a}{L}\right)x\right]}_{c(x)}^4}$$

$$c(x) = a + \frac{(b-a)}{L}x$$

$$J(x) = \frac{\pi}{2}(c(x))^4 = \frac{\pi}{2}\left(a+\left(\frac{b-a}{L}\right)x\right)^4$$

Substitute $c(x)$: $\frac{dc}{dx} = \left(\frac{b-a}{L}\right)$

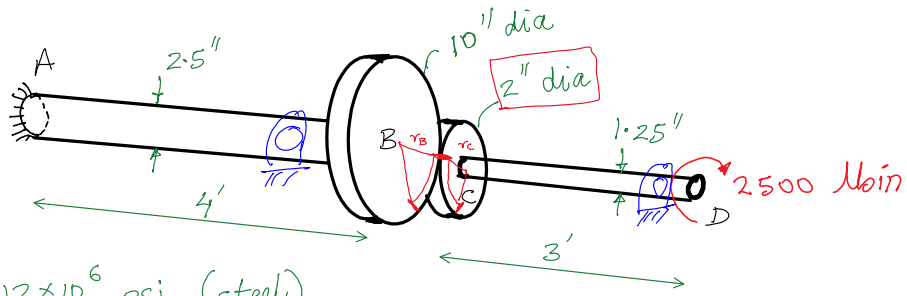
$$\Rightarrow dx = \frac{dc}{\left(\frac{b-a}{L}\right)} = \frac{L\,dc}{(b-a)}$$

$$\phi = \int_a^b \frac{2T}{G\pi c^4}\underbrace{\frac{L\,dc}{(b-a)}}_{dx} = \frac{2TL}{G\pi(b-a)}\int_a^b \frac{1}{c^4}\,dc$$

$$= \frac{2TL}{G\pi(b-a)}\left[\frac{c^{-3}}{(-3)}\right]_a^b$$

$$= \boxed{\frac{2TL}{3G\pi(b-a)}\left[\frac{1}{a^3}-\frac{1}{b^3}\right]}$$

Example



Find ϕ_D

$\phi_D = \phi_C + \phi_{CD}$ (0.065)

$\phi_{CD} = \frac{T_{CD} L_{CD}}{G_{CD} J_{CD}}$ (2500 lb in, 3')

$\gamma_C \phi_C = \gamma_B \phi_B$

$\Rightarrow \phi_C = \frac{\gamma_B}{\gamma_C} \cdot \phi_B$

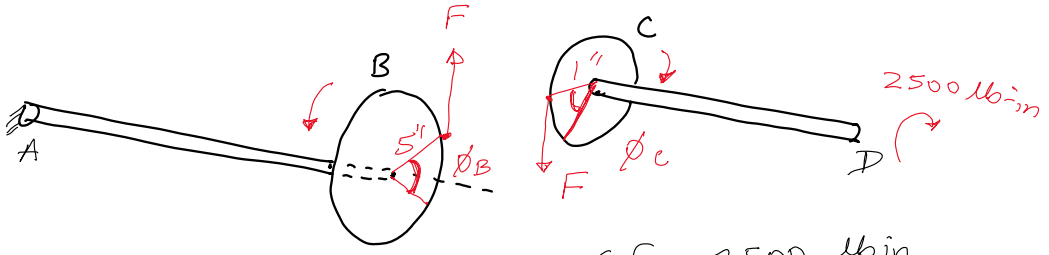
$\phi_B = \phi_A + \phi_{AB} = \frac{T_{AB} L_{AB}}{G_{AB} J_{AB}}$ ($\phi_A = 0$)

$G_{AB} = 12 \times 10^6$ psi (steel)

$J_{AB} = \frac{\pi}{2} \cdot (1.25)^4 = 3.83$ in^4

$G_{CD} = 5 \times 10^6$ psi (Brass)

$J_{CD} = \frac{\pi}{2} (0.625")^4 = 0.24$ in^4



$T_{AB} = 2500$ lb $\times$ 5 in $= 12500$ lb-in

$\gamma_C F = 2500$ lb in

$F = \frac{2500 \text{ lb in}}{1"} = 2500$ lb

$\phi_B = \frac{T_{AB} L_{AB}}{G_{AB} J_{AB}} = \frac{12500 \text{ lb in} \times 48"}{12 \times 10^6 \text{ psi} \times 3.83 \text{ in}^4}$

$= 0.013$ rad

$\Rightarrow \phi_C = \frac{\gamma_B \phi_B}{\gamma_C} = \frac{5" \times 0.013}{1"} = 0.065$ rad

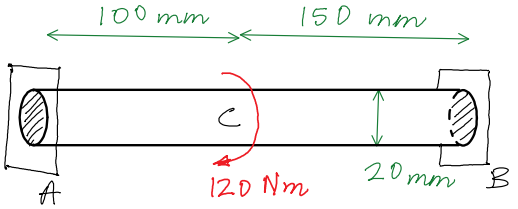
$\phi_D = \phi_C + \frac{T_{CD} L_{CD}}{G_{CD} J_{CD}} = 0.065 + \underbrace{\frac{2500 \text{ lb in} \times 36"}{5 \times 10^6 \text{ psi} \times 0.24 \text{ in}^4}}_{0.075 \text{ rad}} = 0.14$ rad

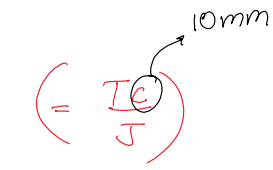
Statically Indeterminate Torsion Problems:

Can't be solved using statics alone. Response depends upon material properties.
Solved by imposing compatibility of deformation.

Example

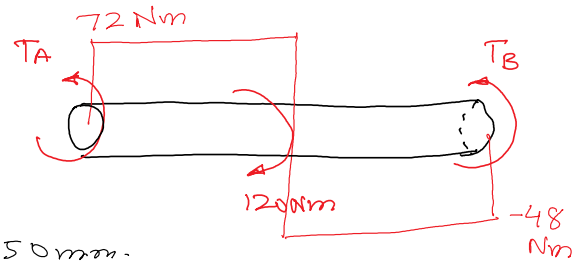
- Find reactions
- Maximum shear stress τ_{max} $\left(= \frac{Tc}{J}\right)$





Equm: $T_A + T_B = 120\ Nm.$ — (1)

$\phi_{C/A} = \phi_{C/B}$ (Compatibility)



$$\frac{T_A L_{AC}}{GJ} = \frac{T_B L_{BC}}{GJ} \Rightarrow T_A \cdot 100 = T_B\ 150mm.$$

$$\Rightarrow T_A = 1.5\ T_B \quad — (2)$$

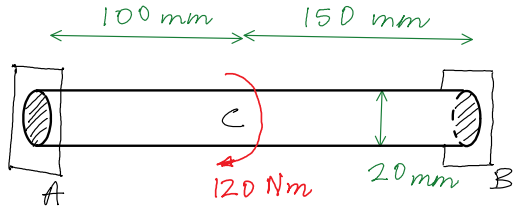
Eq (1): $2.5\ T_B = 120\ Nm \Rightarrow \underline{T_B = 48Nm} \Rightarrow \underline{T_A = 72\ Nm.}$

$$\tau_{max\ (AC)} = \frac{72\ Nm \times 0.01\,m}{1.57\times10^{-8}\ m^4 \times 2} = \underline{45.8\ MPa}$$

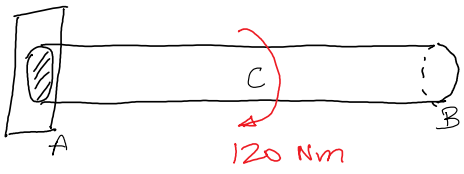
$$J = \frac{\pi}{2} \cdot c^4 = \frac{\pi}{2}(10)^4$$

$$= 1.57\times10^{-8}\ m^4$$

Alternatively: (Superposition)

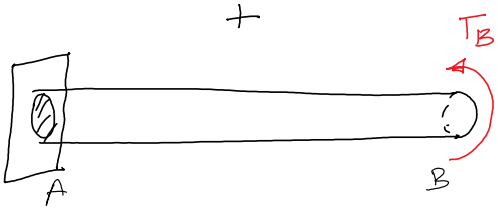


=



(1)

+



(2)

$\phi_B = 0$ (Compatibility)

$$\phi_{B_1} = \frac{TL}{GJ} = \frac{120\ Nm \times 0.1\ m}{GJ}$$

$$\phi_{B_2} = \left(-\frac{TL}{GJ}\right) = \left(-\frac{T_B \cdot 0.25\ m}{GJ}\right)$$

$$\phi_B = \phi_{B_1} + \phi_{B_2} = 0 \Rightarrow T_B = \frac{120 \times 0.1}{0.25}$$

$$\Rightarrow \underline{T_B = 48Nm}$$

$$\underline{T_A = 72\ Nm}$$

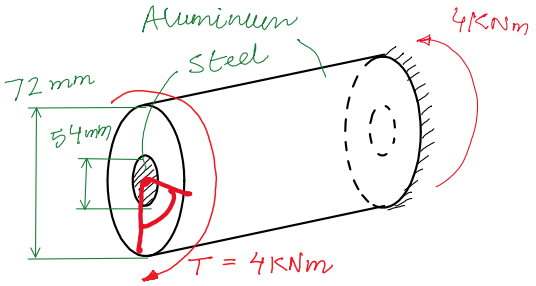
Example

$G_{St} = 77\ GPa$

$G_{Al} = 27\ GPa$

Find $\tau_{max(St)}$; $\tau_{max(Al)}$; ϕ

($\tau_{max} = Tc/J$; $\phi = TL/GJ$)



$c_{ST} = 27\ mm$

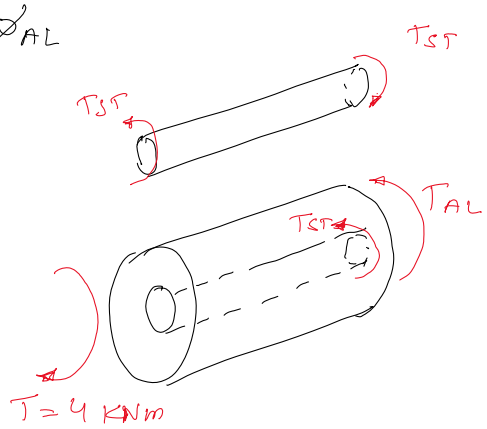
$c_{AL} = 36\ mm$

Equilibrium $\Rightarrow T = T_{ST} + T_{AL} = 4\ KNm$ — (1)

Compatibility $\Rightarrow \phi_{ST} = \phi_{AL}$

$$\Rightarrow \frac{T_{ST} L_{ST}}{(GJ)_{ST}} = \frac{T_{AL} L_{AL}}{(GJ)_{AL}}$$

$$T_{ST} = T_{AL} \frac{(GJ)_{ST}}{(GJ)_{AL}}$$



$$J_{AL} = \frac{\pi}{2}(r_2^4 - r_1^4)$$

$$= \frac{\pi}{2}(36^4 - 27^4)$$

$$= 1.804 \times 10^6\ mm^4$$

$$J_{ST} = \frac{\pi}{2} r_1^4$$

$$= \frac{\pi}{2} \times 27^4$$

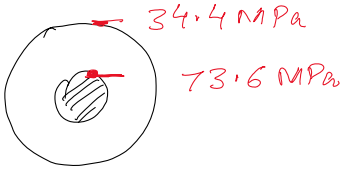
$$= 8.348 \times 10^5\ mm^4$$

$$T_{ST} = T_{AL} \cdot \frac{77\ GPa \times 8.348 \times 10^5\ mm^4}{27\ GPa \times 1.804 \times 10^6\ mm^4} \Rightarrow \text{(2)}$$

$$T_{ST} = 2.276\ KNm$$

$$T_{AL} = 1.724\ KNm$$

$$(\tau_{MAX})_{ST} = \frac{2.276\ KNm \times 27\ mm}{8.348 \times 10^5\ mm^4} = 73.6\ MPa$$



$$(\tau_{MAX})_{AL} = \frac{1.724\ KNm \times 36\ mm}{1.804 \times 10^6\ mm^4} = 34.4\ MPa$$

$$\phi = \phi_{ST} = \phi_{AL} = \frac{2.276\ KNm \times 2.5\ m}{77\ GPa \times 8.348 \times 10^5\ mm^4} = 0.085\ rad$$

ϕ_{ST} $= \phi_{AL}$