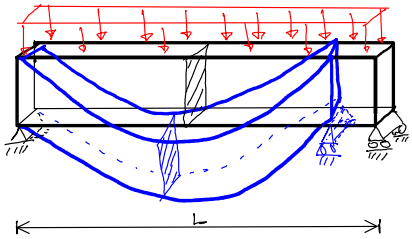
Bending Elements : Beams Theory

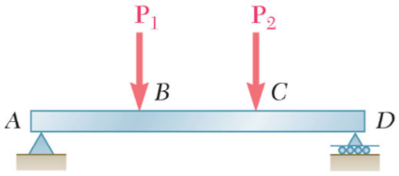
Beams are long straight members with constant cross-section that carry loads transverse to their longitudinal axis by bending.

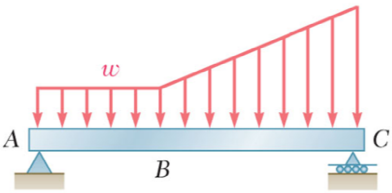


Beam theory (that describes the stresses and strains in a beam) is the backbone of modern structural analysis.

Beam Supports and External Reactions:

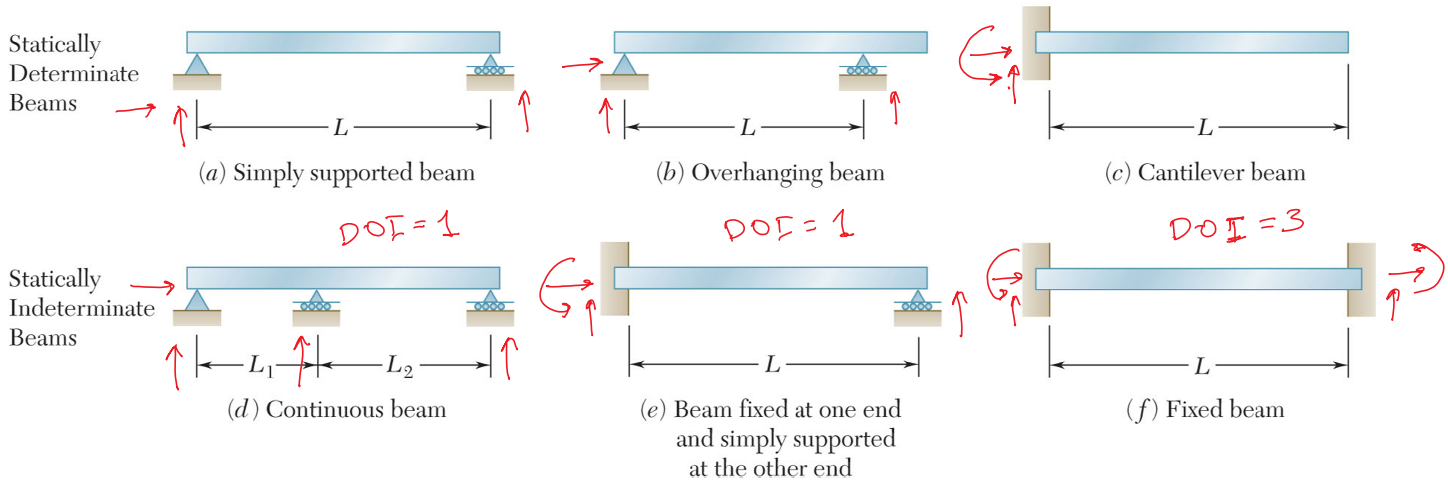
Beams can have point loads or distributed loads acting on them.





Beams may also be externally determinate or indeterminate depending upon the type of support.

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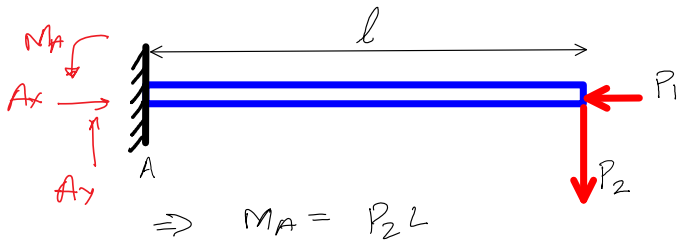
For statically determinate beams, we can find the external forces using the FBD of the entire beam. For example, consider the cantilever beam shown with an end load.

External :

$$\Sigma F_x = 0 \Rightarrow A_x - P_1 = 0$$

$$\Sigma F_y = 0 \Rightarrow A_y - P_2 = 0$$

$$\Sigma M_{A_z} = 0 \Rightarrow -P_2 L + M_A = 0$$

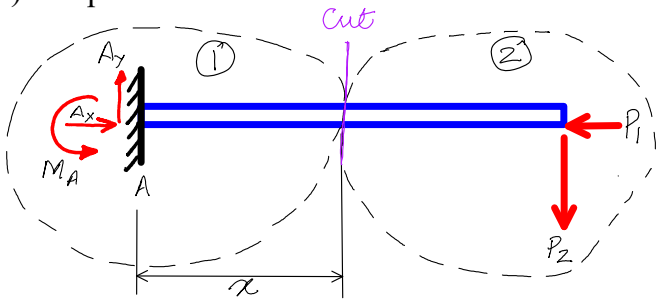


$$\Rightarrow M_A = P_2 L$$

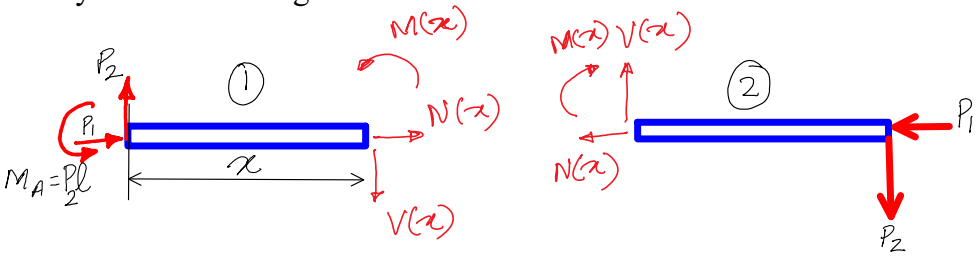
However we may also want to find out the **internal forces** (and **moments**) at different points of the beam. This will help us decide if the beam can support the applied load or not.

Internal Forces and Moments in Beams

Imagine two (or more) sub-parts of the beam as shown.



For any location *x* along the beam:



N(x): Axial force
V(x): SHEAR Force
M(x): BENDING Moment

FBD ①

$\Sigma F_x = 0 \Rightarrow N(x) + P_1 = 0 \Rightarrow N(x) = -P_1$ (C)

$\Sigma F_y = 0 \Rightarrow -V(x) + P_2 = 0 \Rightarrow V(x) = P_2$

$\Sigma M_A = 0 \Rightarrow -V(x)\,x + (P_2 L) + M(x) = 0$

$\Rightarrow M(x) = -P_2 L + P_2 x$

$= -P_2 (L - x)$

FBD ②

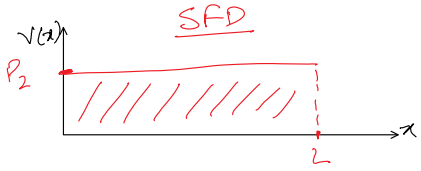
$\Sigma F_x = 0 \Rightarrow N(x) = -P_1$ (C)

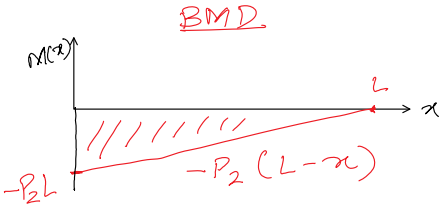
$\Sigma F_y = 0 \Rightarrow V(x) = P_2$

$\Sigma M_x = 0 \Rightarrow M(x) = -P_2 (L - x)$

Plots:

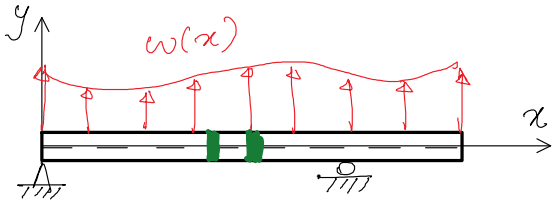




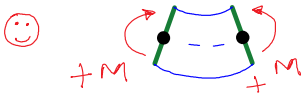


Sign convention for beams:

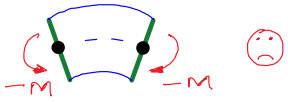
Positive Load: Up
Positive Axial: Tension
Positive Shear: Left Face Up
(Right Face Down)
Positive Moment: Left Face Clockwise
(Right Face Anti-Clockwise)



Smiley positive

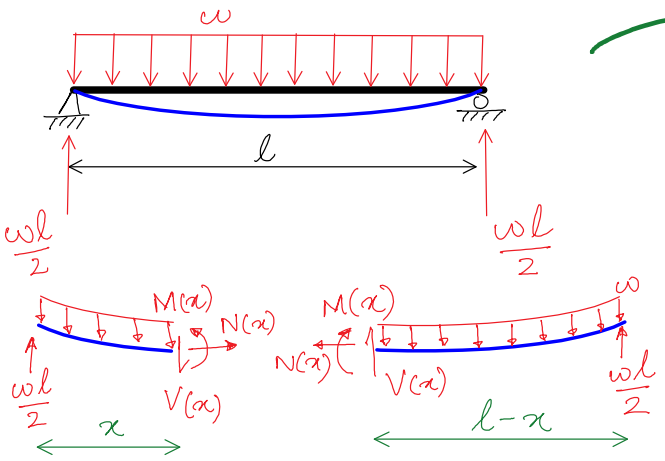


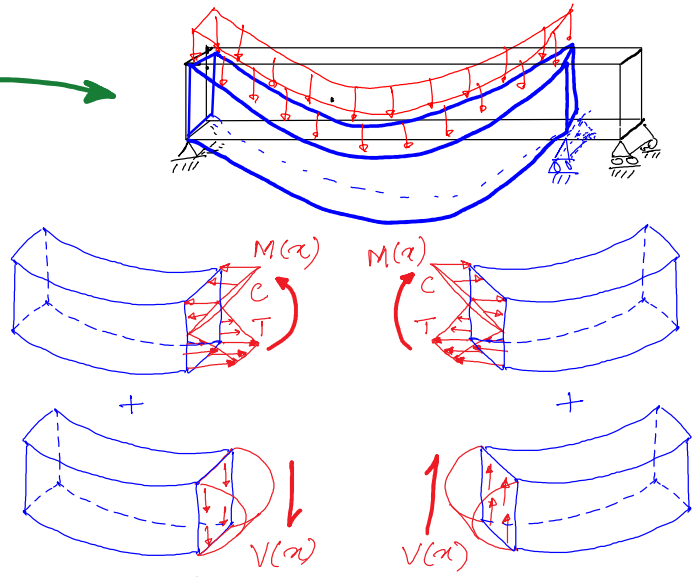
Frowny negative



Shear Force and Bending Moment from Stress and Traction in Beams

Simply supported beam:





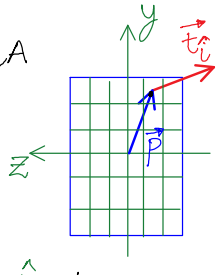
Stress Resultants

- Axial force

$$N(x) \equiv \int_{\text{Area}} (\vec{t}_{\hat{i}} \cdot \hat{i})\, dA$$

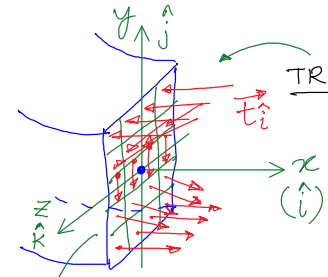
- Shear force:

$$V(x) \equiv -\int_{\text{Area}} (\vec{t}_{\hat{i}} \cdot \hat{j})\, dA$$



- Bending moment

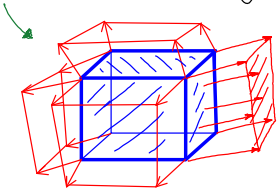
$$M(x) \equiv \int_{\text{Area}} (\vec{p} \times \vec{t}_{\hat{i}}) \cdot \hat{k}\, dA$$



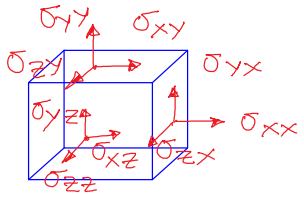
TRACTION on surface (î)

Distributed forces (per unit area) (need not be normal to the surface)

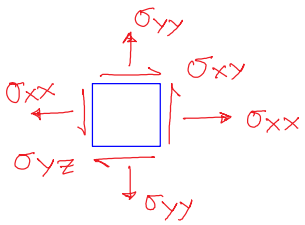
Equilibrium must hold for any elemental volume



(on all 6 faces)



Recall in 2D:



Stress

$$\underset{\sim}{\sigma} \rightarrow \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

components of tractions on all 6 faces

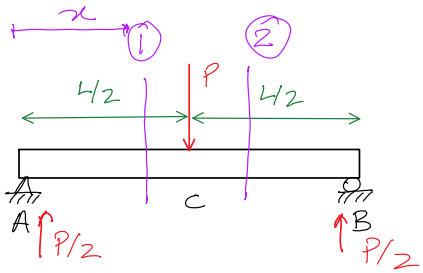
Shear Force (SF) and Bending Moment (BM) **Diagrams**

In general we want to find internal Shear force and Bending moments everywhere along the length of a beam (to find maximums and to make sure that the beam can carry these loads).

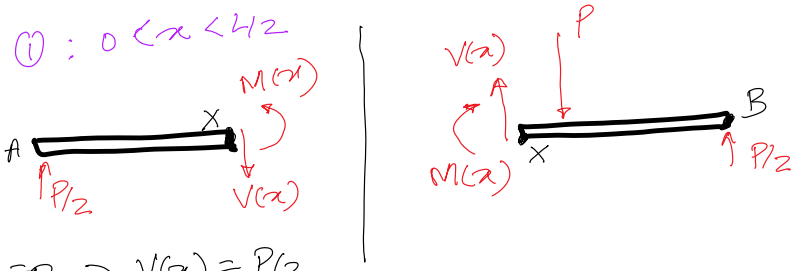
This is done by taking sections at different distances "x" along the length of the beam and expressing Shear Force $V(x)$ and Bending Moment $M(x)$ as functions of x.

Example

Draw SFD and BMD for the beam:

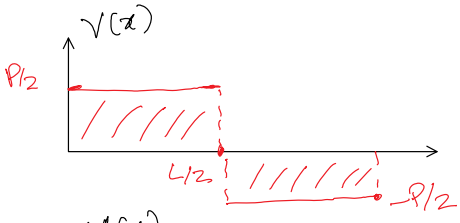


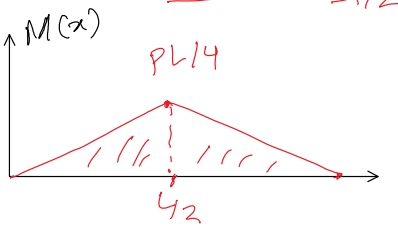
Cut (1) : $0 < x < L/2$



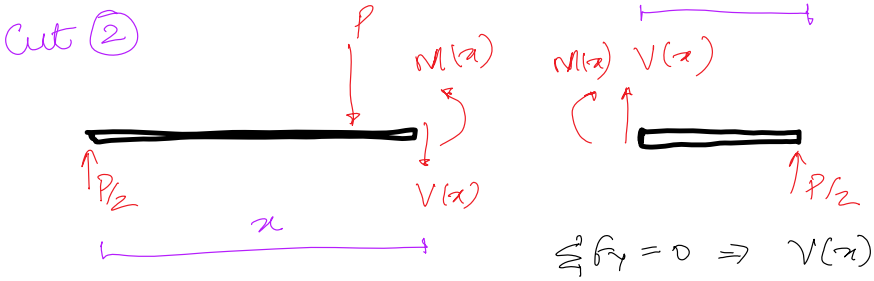
$\sum F_y = 0 \Rightarrow V(x) = P/2$

$\sum M_x = 0 \Rightarrow M(x) - \frac{P}{2}x = 0 \Rightarrow M(x) = \frac{P}{2}x$





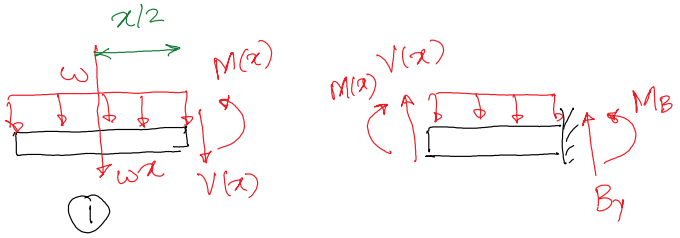
Cut (2)



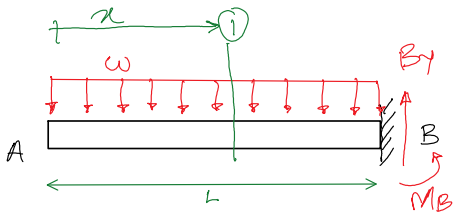
$\sum F_y = 0 \Rightarrow V(x) = -P/2$

$\sum M_x = 0 \Rightarrow M(x) = \frac{P}{2}(L - x)$

Example: Find reactions and draw SFD and BMD



$0 < x < L$



$\sum F_y = 0 \Rightarrow -\omega x - V = 0 \Rightarrow V(x) = -\omega x$

$\sum M_x = 0 \Rightarrow (\omega x)\left(\frac{x}{2}\right) + M(x) = 0 \Rightarrow M(x) = -\frac{\omega x^2}{2}$

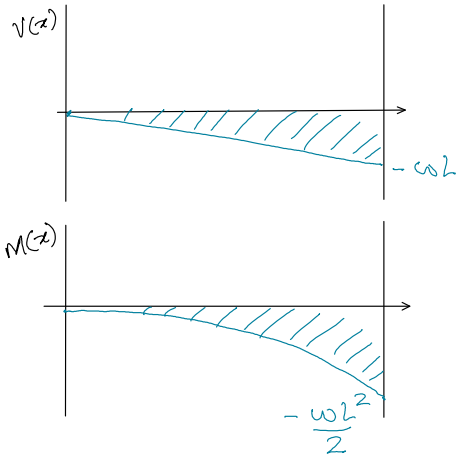
$B_y = -V(L) = +\omega L$

$M_B = M(L) = -\frac{\omega L^2}{2}$

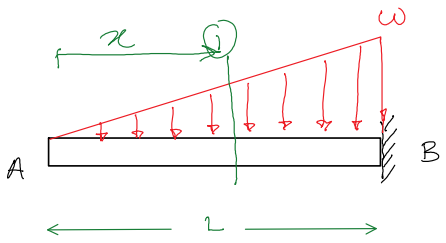


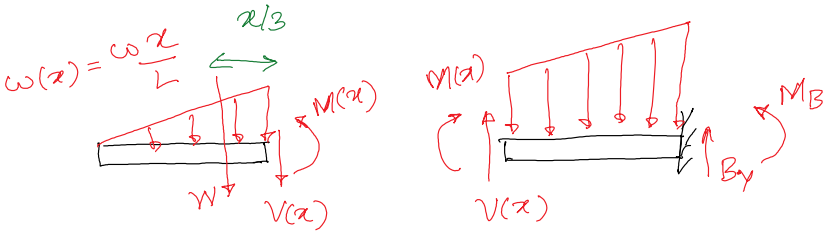
$M(L) = M_B$

$V(L) = -B_y$



Example: Find reactions and draw SFD and BMD

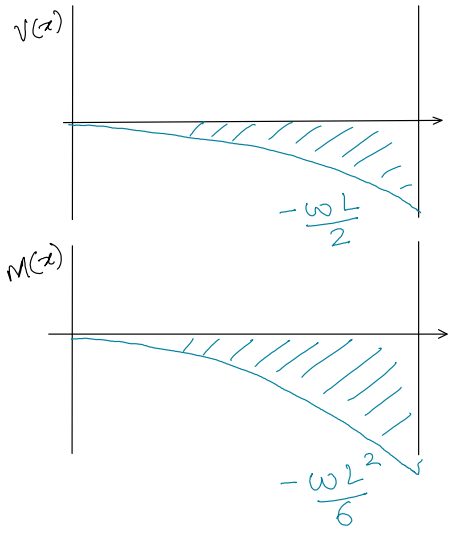




$\sum F_y = 0 \Rightarrow -\frac{1}{2} \cdot x \cdot \left(\frac{\omega x}{L}\right) - V(x) = 0 \Rightarrow V(x) = -\frac{\omega x^2}{2L}$

$\sum M_x = 0 \Rightarrow \underbrace{\left(\frac{1}{2} x \cdot \frac{\omega x}{L}\right)}_{W} \cdot \left(\frac{x}{3}\right) + M(x) = 0$

$\Rightarrow M(x) = -\frac{\omega x^3}{6L}$

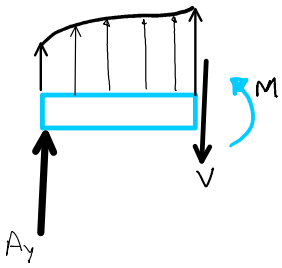


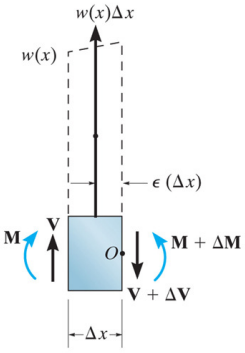
$B_y = -V(L) = \frac{\omega L}{2}$

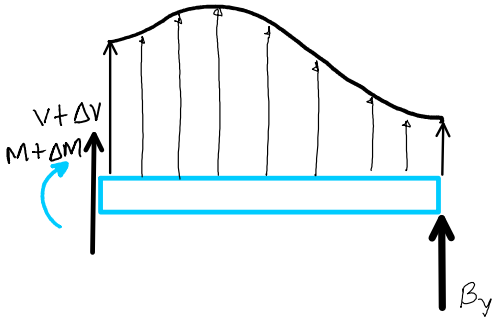
$M_B = M(L) = -\frac{\omega L^2}{6}$

Graphical Method for SFD and BMD

FBD of Δx element:







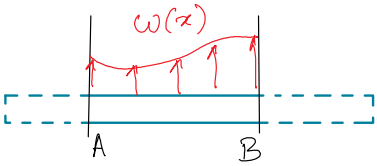
$$\sum F_y = 0 \Rightarrow V - (V+\Delta V) + w(x)\,\Delta x = 0 \Rightarrow \boxed{w(x) = \frac{dV}{dx}} \Rightarrow w_{DN} = -\frac{dV}{dx}$$

$$\sum M_O = 0 \Rightarrow -(V\Delta x) - (w\Delta x)\cdot\frac{\Delta x}{2} - M + (M+\Delta M) \Rightarrow \boxed{V(x) = \frac{dM}{dx}}$$

($(w\Delta x)\cdot\frac{\Delta x}{2} \approx 0$)

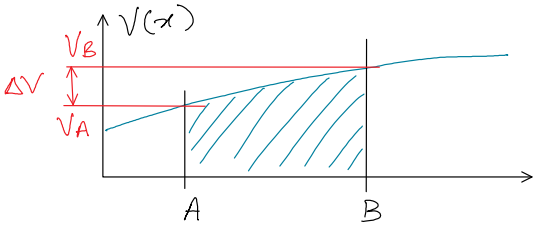
Note:

- $\Delta V_{AB} = V_B - V_A = \int_A^B w_{UP}\,dx$

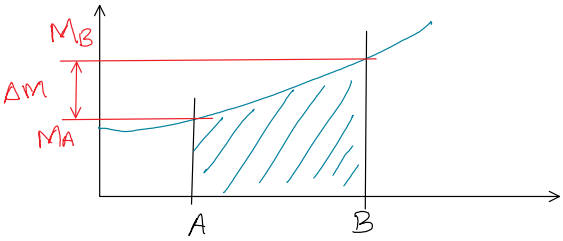


(change in shear is equal to area under the load)

- $\Delta M = \int_A^B V\,dx$

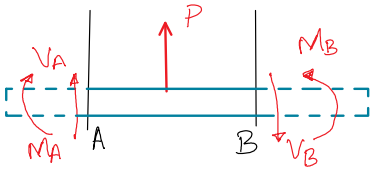


(change in bending-moment is equal to area under the shear)



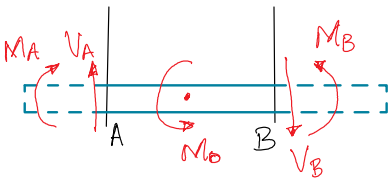
- Concentrated Load

$$\Delta V_{AB} = V_B - V_A = P$$
$$\Delta M_{AB} = M_B - M_A = 0$$



- Concentrated Moment

$$\Delta V_{AB} = V_B - V_A = 0$$
$$\Delta M_{AB} = M_B - M_A = -M_0$$



Example

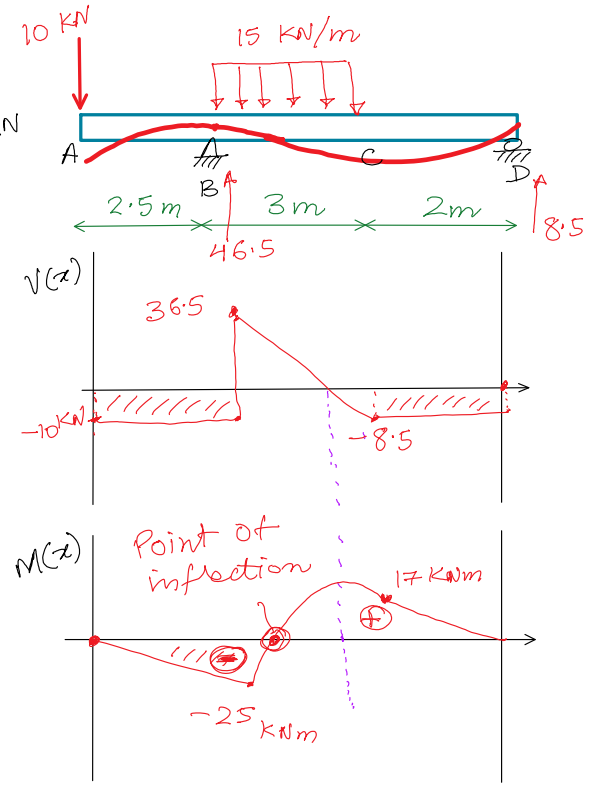
Reactions:

$\sum M_B = 0 \Rightarrow (10\times 2.5) - (15\times 3\times 1.5) + D_y \times 5 = 0$

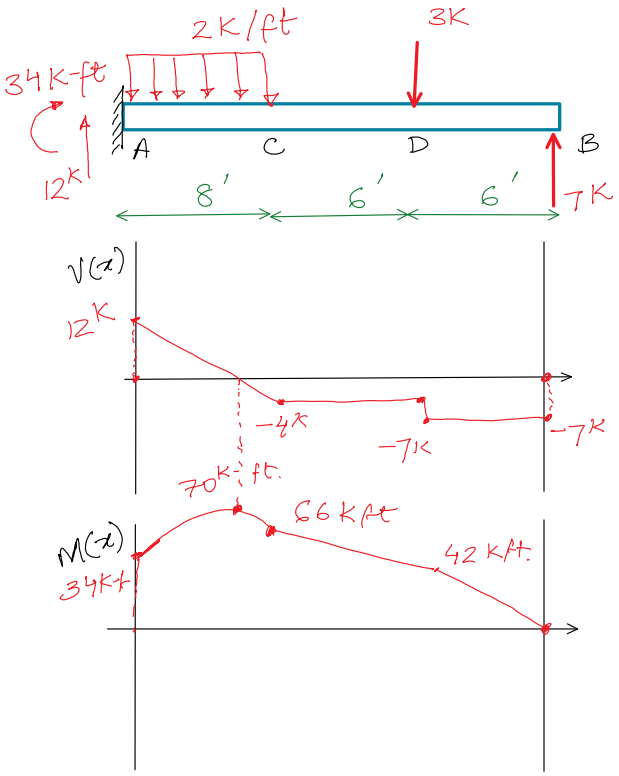
$\Rightarrow 5D_y = -25 + 67.5 \Rightarrow D_y = 8.5\ KN$

$\sum F_y = 0 \Rightarrow -10 - 15\times 3 + B_y + D_y = 0$

$\Rightarrow B_y = 55 - D_y = 46.5\ KN$



Example



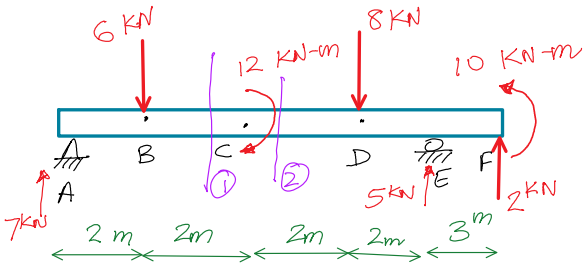
Example

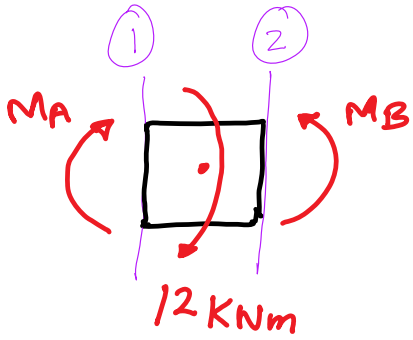
Reactions:

$$\sum M_A = 0 \Rightarrow -6\times 2 - 12 - 8\times 6 + E_y \times 8 + 2\times 11 + 10 = 0$$

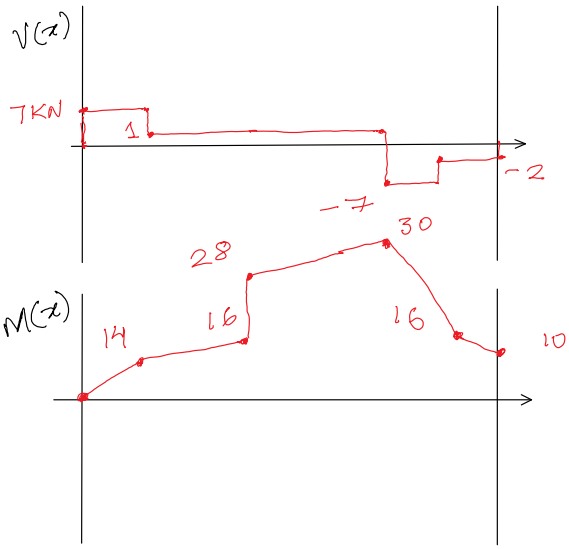
$$\Rightarrow E_y = \frac{12+12+48-22-10}{8} = 5\ \text{kN}$$

$$\sum F_y = 0 \Rightarrow A_y - 6 - 8 + 5 + 2 = 0 \Rightarrow A_y = 7\ \text{kN}$$



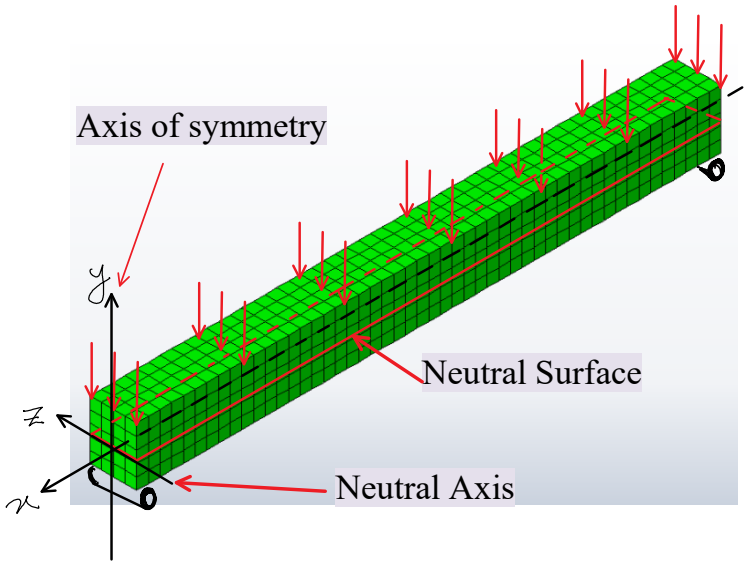


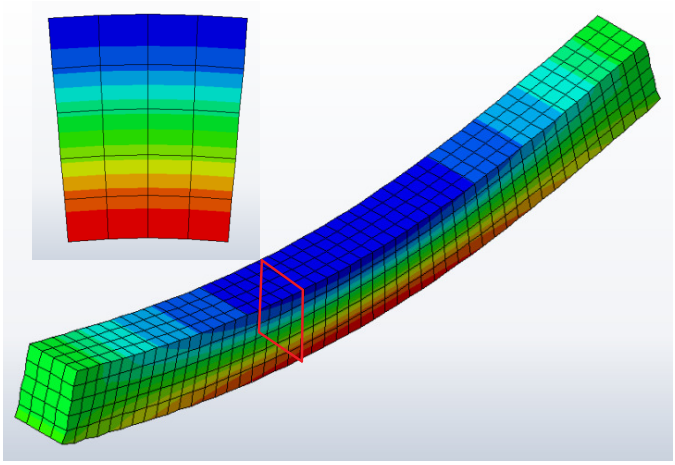
$$M_A + 12\ \text{kNm} = M_B$$



Bending Behavior of Beams

For straight beams of a homogeneous material, with constant cross-sections that are symmetric with respect to the *y*-axis and undergoing bending in the *x*-*y* plane:

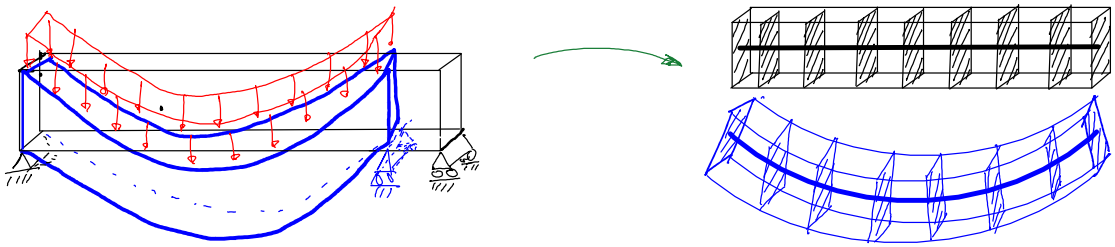




Observations:

- Horizontal lines along the longitudinal *x*-axis become curved after deformation
  - Lines along the *x*-*z* plane don't elongate / contract => Neutral Surface
  - Lines parallel to the *z*-axis and on the neutral surface at any cross-section is called Neutral Axis
  - Shorter curves in top-half (above the neutral surface) => Compression
  - Longer curves in bottom-half (below the neutral surface) => Tension
- Vertical lines (and cross-sections) remain straight (but rotate)
  - Cross-section does deform (but deformation is negligible)

Assumptions of Beam Theory:

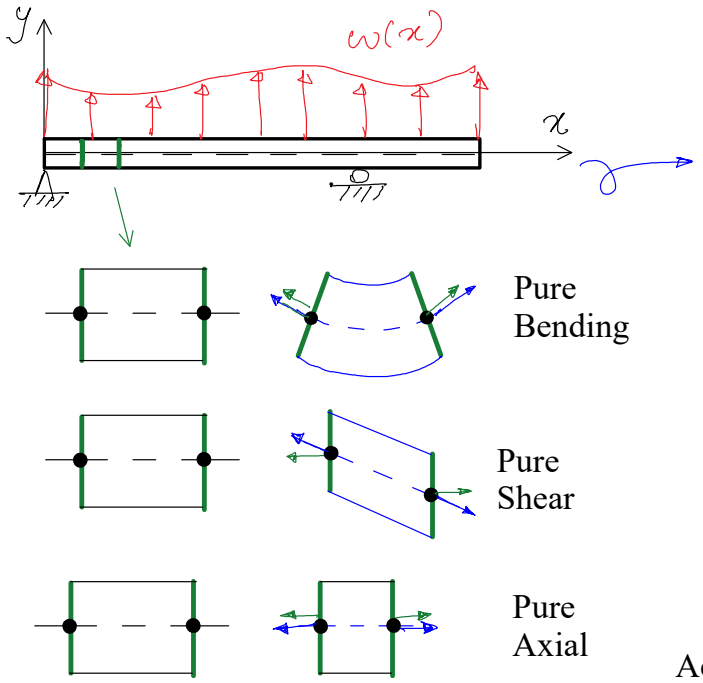


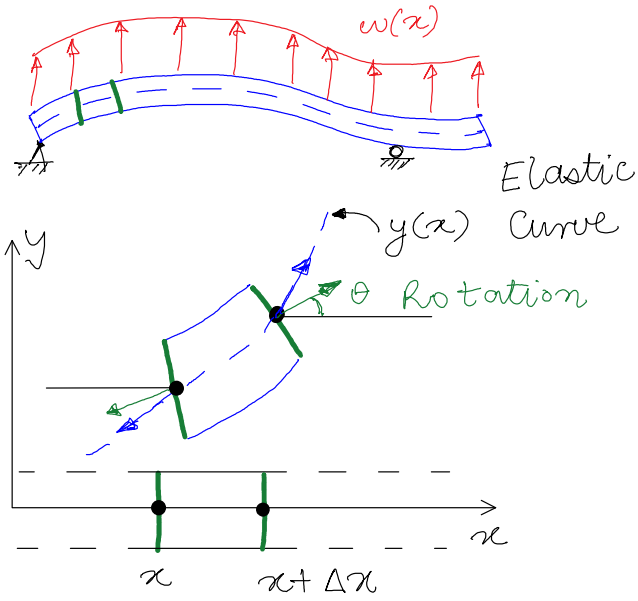
Beam theory is based on a kinematic assumption:

- Assume that a beam consists of infinitely many PLANE CROSS- SECTIONS that are connected with a FLEXIBLE STRING (at their centroids)
- The cross-sections remain PLANE before and after the deformation
- The deformation of the cross-sections within their own plane will be neglected i.e. cross-sections will treated as rigid cross-sections

Beams Theory

Under the assumptions of the beam theory, consider a Δx length of the beam:





Actual response is some combination of all 3 "modes."

Definitions:

- Elastic Curve: $y(x)$: The function describing the deformed position of the neutral surface.
  Note: Tangent to the elastic curve is the slope of the elastic curve → $\text{Tangent} = \frac{dy}{dx}$
- Cross-section Rotation: $\theta(x)$: The function describing the rotation of the cross-section measured from the positive x-axis
  Note: $\theta(x) \neq \frac{dy}{dx}$
- Curvature: $\kappa(x)$: The rate of change of cross-section rotation with "x" → $\kappa(x) \equiv \frac{d\theta}{dx}$

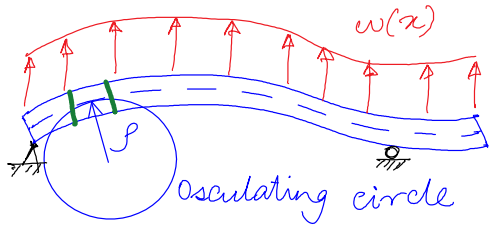
Note: There are two types of Beam Theories that are commonly used:

- **Timoshenko Beam Theory** accounts for all 3 modes of deformation above.
  - This theory is more general and is suitable for all beams. It should be employed especially for "deep" / "short" beams i.e. beams whose length/depth ratio < 5.
  - This is because for such beams bending deformations are small and shear deformations cannot be neglected because they contribute significantly to the response.

- **Bernoulli-Euler Beam theory** neglects the shear deformation.
  - This theory is applicable to "long" / "slender" beams whose length/depth ratio > 10.
  - For such beams, bending deformations are much larger than shear deformations.
  - Thus, one can neglect shear deformations in comparison to the bending deformations.
  - Under this theory: $\theta(x) = \frac{dy}{dx}$; $\kappa(x) = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$

Strains and Curvature in Bernoulli-Euler Beams: (Negligible Shear Deformations)

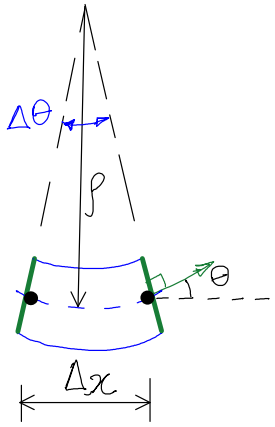
Under negligible shear deformations, every Δx length of the beam is an arc of an osculating circle with radius ρ.

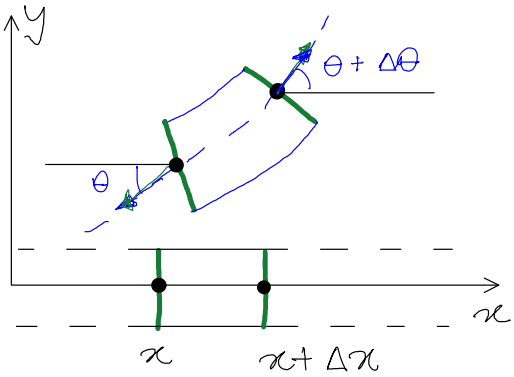


Using calculus, one can show:

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \approx \frac{1}{\left(\frac{d^2y}{dx^2}\right)}$$

($\left(\frac{dy}{dx}\right)^2 \approx 0$)





Note: ρ is also called radius of curvature $\Rightarrow \rho\,\Delta\theta \approx \Delta x$

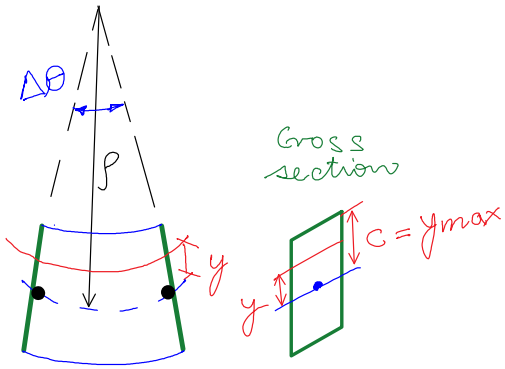
Curvature

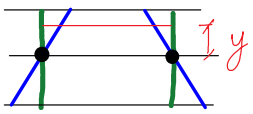
$$\frac{1}{\rho} = \lim_{\Delta x \to 0} \frac{\Delta\theta}{\Delta x} = \frac{d\theta}{dx} \equiv \kappa$$

Since θ is small

$$\Rightarrow \frac{dy}{dx} = \tan\theta \approx \theta \quad \Rightarrow \quad \kappa = \frac{1}{\rho} = \frac{d\theta}{dx} \approx \frac{d^2y}{dx^2}$$

Furthermore:





Strain distribution

$\rho\,\Delta\theta$

Average Normal Strain in fiber at distance "y":

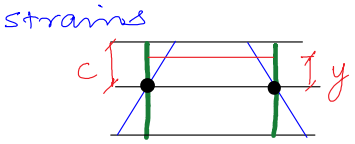
$$\epsilon = \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta}$$

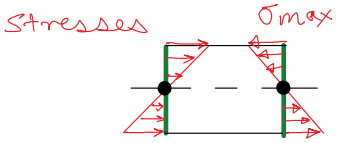
$$\Rightarrow \quad \epsilon = \frac{-y}{\rho} = -\kappa y$$

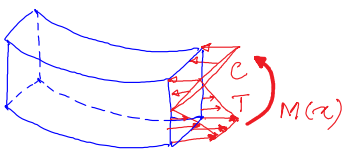
Thus: $\epsilon_{max} = \frac{-c}{\rho} \Rightarrow \frac{\epsilon}{\epsilon_{max}} = \frac{y}{c}$

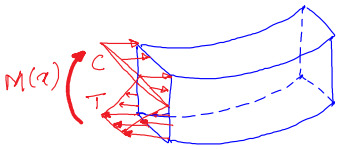
where ε_{max} is the strain at c (with the correct sign)

The flexure formulas





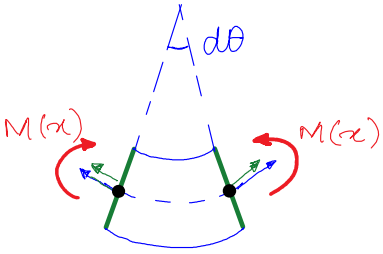




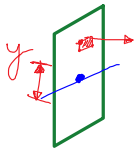
Since $\frac{\epsilon}{\epsilon_{max}} = \frac{y}{c}$

Hooke's Model: $\sigma = E\epsilon \Rightarrow \frac{\sigma}{\sigma_{max}} = \frac{y}{c}$

Location of Neutral Axis:



Gross section



$\epsilon = -\kappa y$

$\sigma = \frac{y}{c}\sigma_{max}$

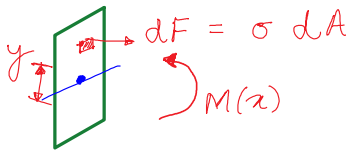
Resultant Axial Force on Cross-section $N(x) = 0$

$$N(x) = \int_A dF = \int_A \sigma\, dA = \int_A \frac{y}{c}\sigma_{max}\, dA = \frac{\sigma_{max}}{c}\left(\int_A y\, dA\right) = 0$$

Thus $\int_A y\, dA = 0 \Rightarrow$ Neutral axis coincides with the Horizontal centroidal axis!
(only for homogeneous cross-section material)

Stress and Bending Moment relationship:

Gross section



Bending moment:

$$M = \int_A -y\, dF = \int_A -\sigma y\, dA$$

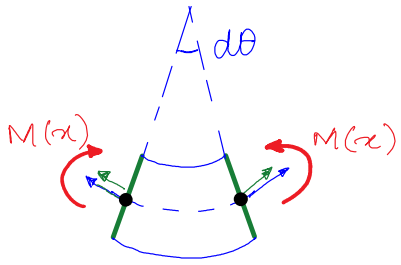
$$= \int_A -\left(\sigma_{max}\frac{y}{c}\right) y\, dA = -\frac{\sigma_{max}}{c}\underbrace{\int_A y^2\, dA}_{I}$$

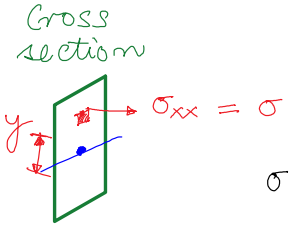
$\Rightarrow M = -\frac{\sigma_{max}}{c} I \Rightarrow \boxed{\sigma_{max} = -\frac{Mc}{I}}$

and $\sigma = -\frac{My}{I}$

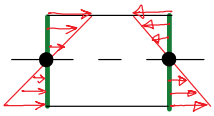
I: Second Moment of Area / Moment of Inertia (about the neutral axis)

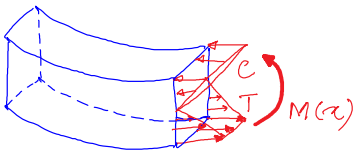
Bending-moment - Curvature relationship:





$\epsilon = -\kappa y$



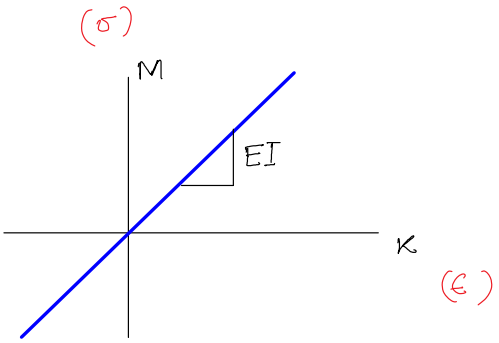


$\sigma = E\epsilon = -E\kappa y$

Bending moment:

$$M = \int_A -\sigma y \, dA = \int_A (E\kappa y) \cdot y \cdot dA = EI\kappa \qquad I = \int y^2 dA$$

$$\Rightarrow \boxed{M = EI\kappa} \quad \text{or} \quad \boxed{\kappa = \frac{M}{EI} = \frac{d^2y}{dx^2}}$$



Moment - curvature relationship

Governing equations for Bernoulli-Euler Beams

Equilibrium: $\frac{dM}{dx} = V \; ; \; \frac{dV}{dx} = w \quad \Rightarrow \quad \frac{d^2M}{dx^2} = \boxed{\frac{d^2}{dx^2}\left(EI \frac{d^2y}{dx^2}\right) = w}$

Strain: $\epsilon = -\kappa y = -\frac{y}{\rho} \; ; \; \kappa = \frac{d^2y}{dx^2}$

Stress: $\sigma = E\epsilon = -\frac{E}{\rho} y$

Material: $M = EI\kappa = \frac{EI}{\rho}$

Combining these equations:

$$\boxed{-\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{\rho}}$$

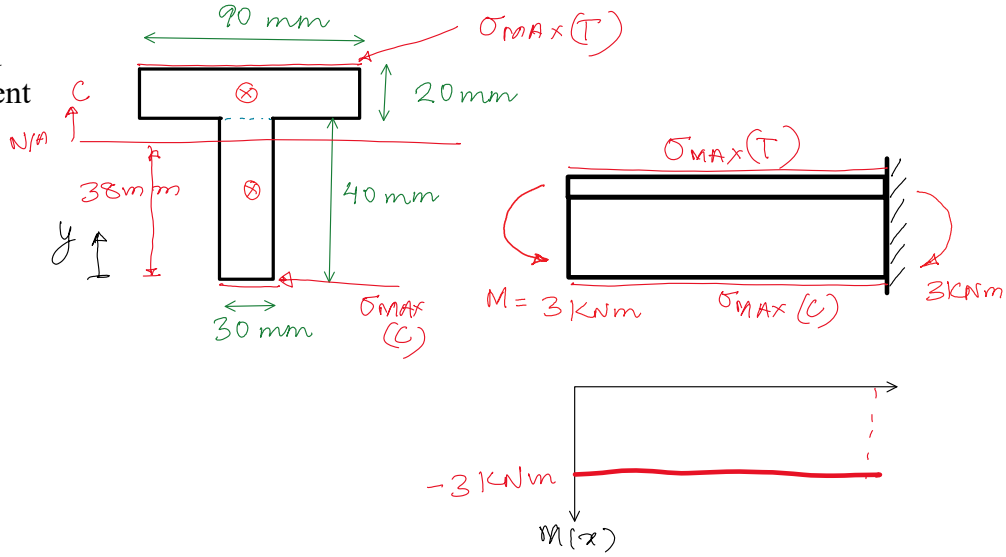
Example

Cantilever beam with a T-shaped cross-section is under a tip moment M = 3 kN-m.

E = 165 GPa (cast iron)

Find

- σ_{max} (tensile) ;
- σ_{max} (compressive) ; and
- Radius of curvature ρ



$$\sigma_{MAX} = -\frac{Mc}{I}$$

$$\bar{y} = \frac{(20\times 90)\times 50 + (30\times 40)\times 20}{(20\times 90) + (30\times 40)} = 38\text{ mm}$$

$c_{tension} = 22$ mm

$c_{comp} = 38$ mm

$$I_{N/A} = \bar{I} + Ad^2$$

$$= \frac{1}{12}(90)(20)^3 + (90\times 20)\cdot(12)^2 + \frac{1}{12}\times(30)(40)^3 + (30\times 40)\times(18)^2$$

$$= 319200\ mm^4 + 548800\ mm^4 = 868000\ mm^4$$

$$\sigma_{MAX}(T) = -\frac{Mc_T}{I} = +\frac{(+3\text{ KNm})\times 22\text{ mm}}{868000\ mm^4} = 76\text{ MPa}$$

(1000 N · 1000 mm)

$$\sigma_{MAX}(C) = -\frac{Mc_C}{I} = -\frac{(-3\text{ KNm})\times(-38\text{ mm})}{868000\ mm^4} = -131.3\text{ MPa}$$

$$\rho = \frac{1}{\kappa} = \frac{EI}{M} = \frac{165\text{ GPa}\times 868000\ mm^4}{-3\text{ KNm}} = -47.74\text{ m}$$

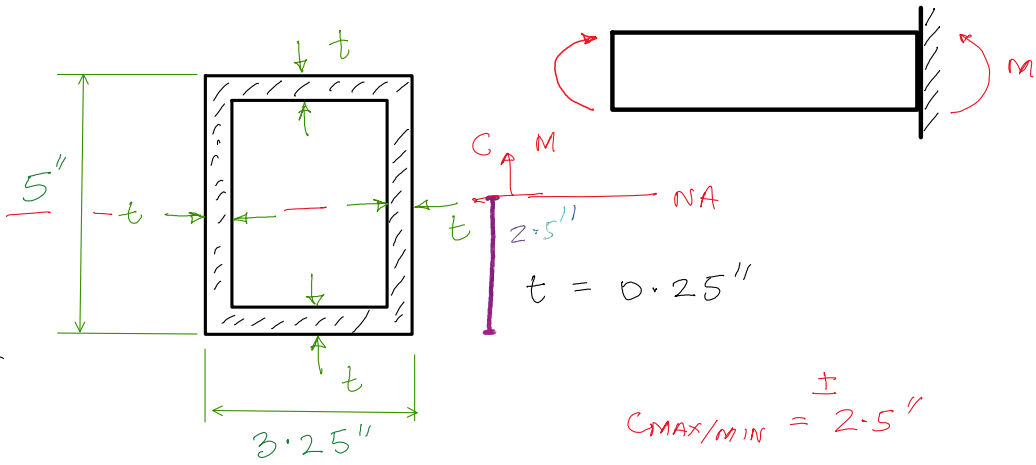
Example

Given $\sigma_{FAIL} = 60$ ksi

$FS = 3$

$E = 10600$ ksi (Aluminum)

Find M_{MAX}, ρ



$$\sigma_{ALLOW} = \frac{60}{3} = 20 \text{ ksi}$$

$$\sigma_{ALLOW} = -\frac{M_{MAX}\, c}{I}$$

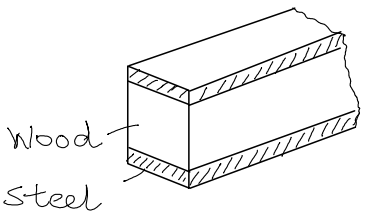
$$I = I_{OUTER} - I_{INNER} = \frac{1}{12}(3.25")(5")^3 - \frac{1}{12}(2.75")(4.5") = 12.97 \text{ in}^4$$

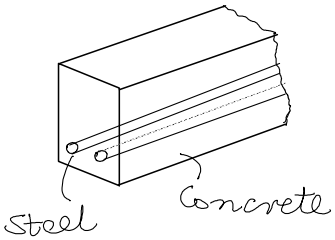
$$M_{MAX} = -\frac{\sigma_{ALLOW}\, I}{c} = \frac{20 \text{ ksi} \times 12.97 \text{ in}^4}{(\pm 2.5")} = \pm 103.8 \text{ K-in}$$

$$\rho = \frac{1}{\kappa} = \frac{EI}{M} = \frac{10{,}600 \text{ ksi} \times 12.97 \text{ in}^4}{\pm 103.8 \text{ K-in}} = \pm 1325 \text{ in}$$

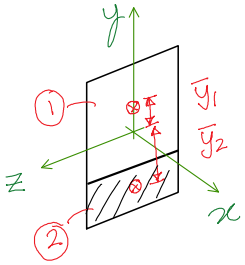
Composite Beams (2 or more materials)

Composite beams have cross-sections that are "built-up" by combining two or more materials for strengthening.

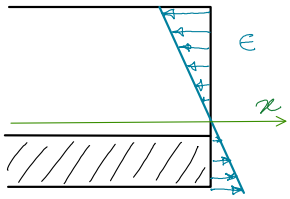




Kinematic Assumption (Plane sections remain plane) still holds.

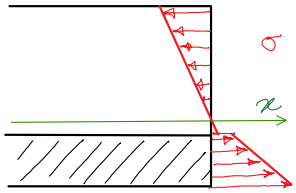


Strain:



$$\epsilon = -\frac{y}{\rho}$$

Stress



$$\sigma_1 = E_1 \epsilon = -E_1 y/\rho$$
$$\sigma_2 = E_2 \epsilon = -E_2 y/\rho$$

Neutral Axis:

$$N(x) = -\int_{A_1} E_1 \frac{y}{\rho}\, dA - \int_{A_2} E_2 \frac{y}{\rho}\, dA = -\frac{1}{\rho}\left(E_1 \bar{y}_1 A_1 + E_2 \bar{y}_2 A_2\right) = 0$$

$E_2 = n E_1$

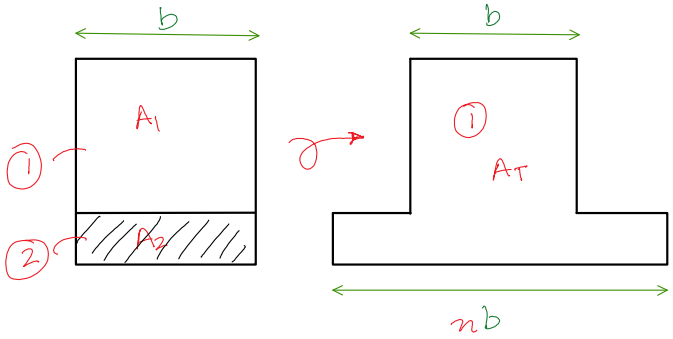
$$\Rightarrow \text{solve for } \bar{y}_1 ; \bar{y}_2 \qquad = -\frac{E_1}{\rho}\left(\bar{y}_1 A_1 + n \bar{y}_2 A_2\right) = 0$$

Bending Moment:

$$M(x) = -\int_{A_1} \frac{E_1}{\rho} y^2\, dA - \int_{A_2} \frac{E_2}{\rho} y^2\, dA = -\frac{1}{\rho}\left(E_1 I_1 + E_2 I_2\right) = -\frac{E_1}{\rho}\underbrace{\left(I_1 + n I_2\right)}_{I_T}$$

$E_2 = n E_1$

Transformed Section Approach:



$$dF_1 = \sigma_1\, dA = \left(-E_1 \frac{y}{\rho}\right)(b\, dy)$$

$$dF_2 = \sigma_2\, dA = \left(-E_2 \frac{y}{\rho}\right)(b\, dy) = \left(-n E_1 \frac{y}{\rho}\right)(b\, dy)$$

$$\Rightarrow dF_2 = \left(-E_1 \frac{y}{\rho}\right)(n b\, dy)$$

Note:

- $\int_{A_T} y\, dA = \left(\bar{y}_1 A_1 + n \bar{y}_2 A_2\right)$
- $\sigma_{max} = -\frac{Mc}{I_T}$; $\sigma_1 = -\frac{My}{I_T}$; $\sigma_2 = n\left(-\frac{My}{I_T}\right)$

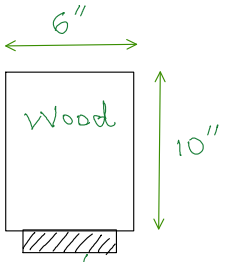
Example

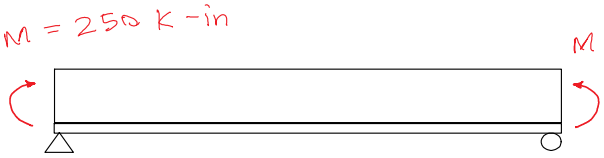
$E_{WOOD} = 1500$ ksi

$E_{STEEL} = 30000$ ksi

Find

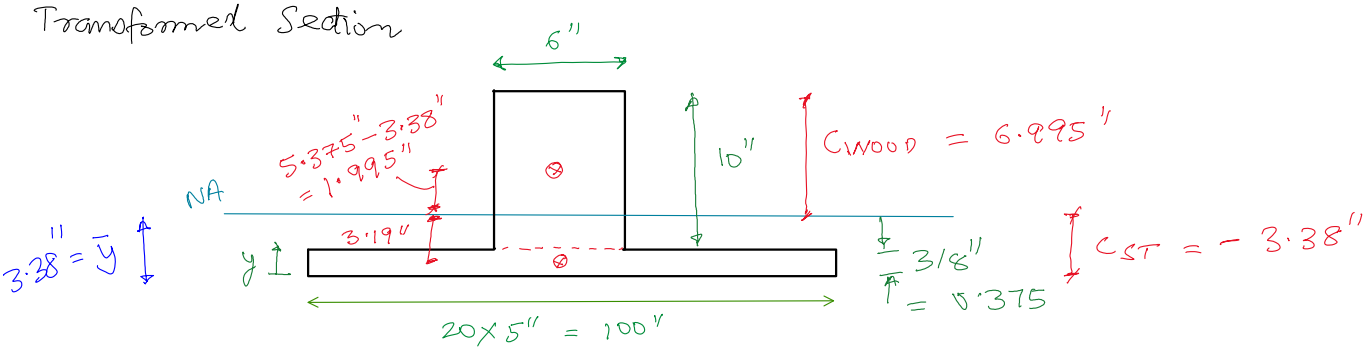
$(\sigma_{WOOD})_{MAX}$; $(\sigma_{ST})_{MAX}$





$n = \frac{E_{ST}}{E_W} = 20$

Transformed Section



$$\bar{y} = \frac{(6\times10)\times(5+0.375) + (100\times0.375)\times(3/16")}{(6\times10) + (100\times0.375)} = 3.38" \text{ from bottom}$$

$$I_T = \left(\bar{I}_1 + A_1 d_1^2\right) + \left(\bar{I}_2 + A_2 d_2^2\right)$$

$$= \left[\frac{1}{12}\cdot 6\times 10^3 + (6\times10)\times(1.995)^2\right]$$

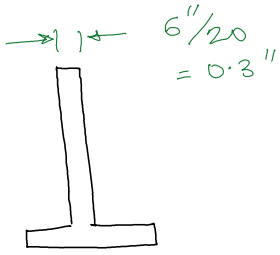
$$+ \left[\frac{1}{12}\times 100\times 0.375^3 + (100\times0.375)\times(3.19)^2\right]$$

$I_T = 1121.4$ in^4

$$\sigma_{WOOD} = -\frac{M\,C_{WOOD}}{I_T} = -\frac{(250 \text{ K-in})\cdot(6.995) \text{ in}}{1121.4 \text{ in}^4} = \boxed{-1.559 \frac{K}{in^2}} \text{ (ksi) (C)}$$

$$\sigma_{ST} = -n\frac{M\,C_{ST}}{I_T} = \frac{+(20)\cdot(250 \text{ K-in})(\overset{\oplus}{3.38"})}{1121.4 \text{ in}} = \boxed{15.07 \text{ ksi}} \text{ (T)}$$

Alternatively, convert to a steel section:



$$\bar{y} = \frac{(0.375+5)\times(6\times10)/20 + (0.375/2)(5\times0.375)}{(6\times10)/20 + (5\times3/8)} = 3.38''$$

$$I_T = \frac{1121.4}{20} = 56.07 in^4$$

$$\sigma_{WOOD} = \frac{1}{20}\left(\frac{Mc}{I_T}\right) = 1.559\ ksi$$

$$\sigma_{ST} = \frac{-Mc}{I_T} = 15.07\ ksi$$

Example: RC Beam

E_{STEEL} = 30000 ksi

E_{CONC} = 3750 ksi

M = 1200 kip-in

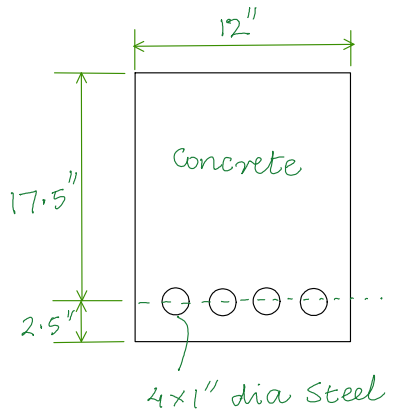
Find

- σ_{max} (steel) ;
- σ_{max} (concrete)

Note:

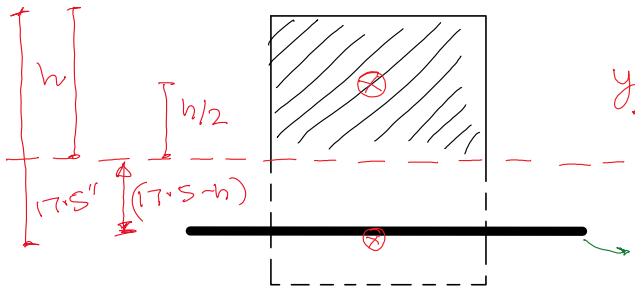
concrete : compression ONLY

$$n = \frac{E_{ST}}{E_{CONC}} = 8$$



$$A_{ST} = \frac{4 \times \pi \, 1^2}{4} = 3.14 \text{ in}^2$$

Transformed section



$n A_{ST} = 8 \times 3.14 \text{ in}^2 = 25.13 \text{ in}^2$ (Equivalent area in concrete)

$$\bar{y} = \frac{(h \times 12)\times(h/2) + (25.13 \times (17.5-h))}{(h \times 12) + 25.13} = 0 \quad \Rightarrow \quad 6h^2 + 25.13h - 439.8 = 0$$

$$h = \frac{-25.13 \pm \sqrt{(25.13)^2 - 4\times 6 \times 439.8}}{2 \times 6}$$

$\Rightarrow h = 6.72$ in (or ~~-10.91 in~~)

$$I_T = \underbrace{\frac{1}{12}(12")(6.72)^3 + (12" \times 6.72)(6.72/2)^2}_{I_{CONC} = 1213.9 \text{ in}^4} + \left[8 \times 4\left[\frac{\pi}{4}(0.5)^4\right]\right] + \underbrace{(25.13) \times (17.5-6.72)^2}_{2920.35 \text{ in}^4}$$

$\approx 0.05 \text{ in}^4$

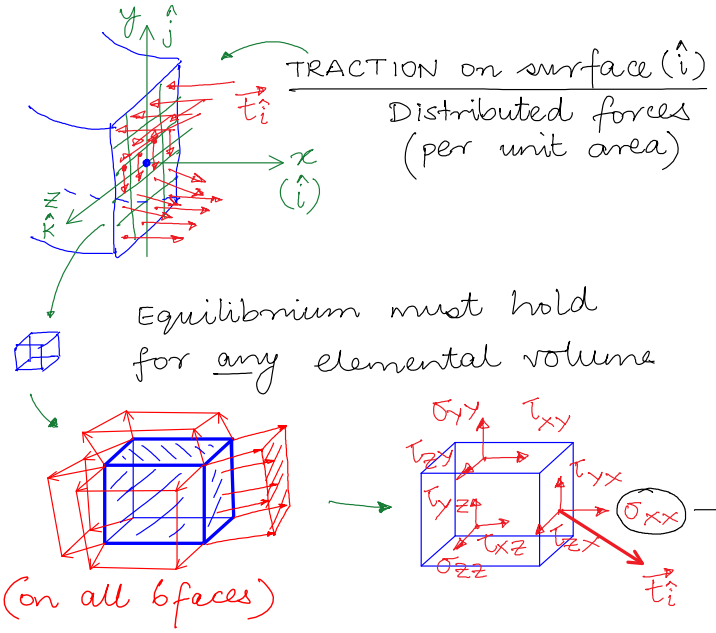
$\Rightarrow I_T = 4134 \text{ in}^4$

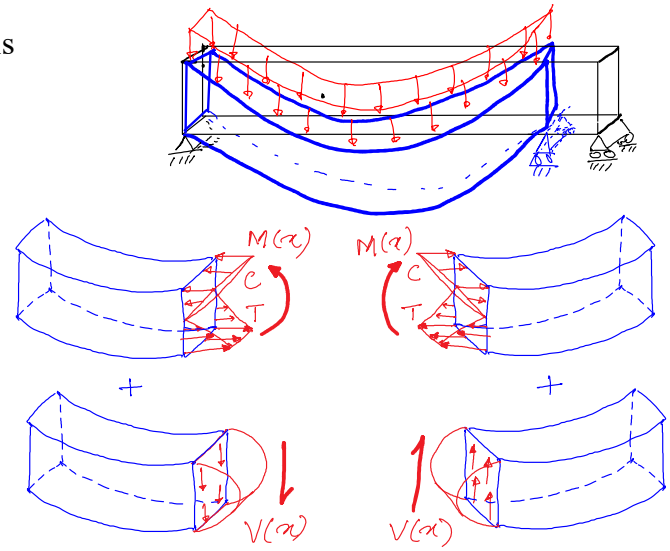
$$\sigma_{MAX(CONC)} = -\frac{MC}{I_T} = -\frac{(1200 \text{ K-in})(6.72 \text{ in})}{4134 \text{ in}^4} = -1.95 \frac{K}{in^2} = \underline{-1.95 \text{ ksi}} \ (C)$$

$$\sigma_{MAX(ST)} = -n\frac{MC}{I_T} = -\frac{8 \times (1200 \text{ Kin})(17.5-h)}{4134 \text{ in}^4} = \underline{25 \text{ ksi}} \ (T)$$

Transverse Shear in Beams

In the previous chapter we saw how the bending moment M is related to the flexural stresses (in the longitudinal direction):



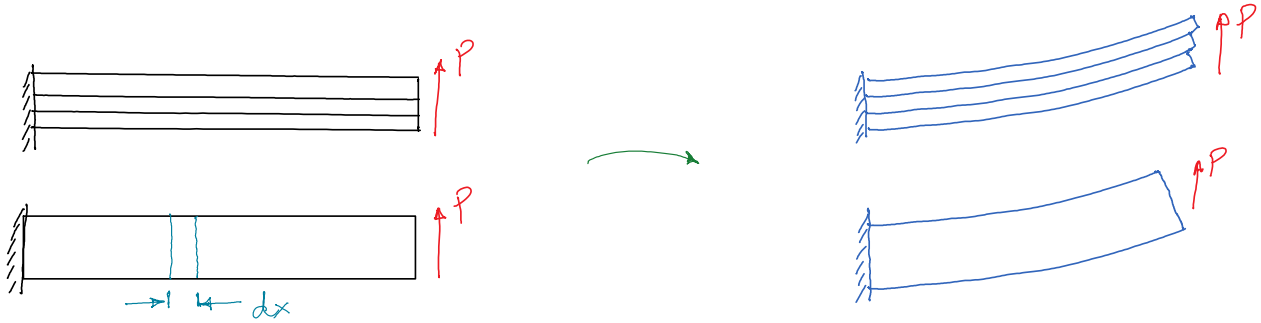


$$\sigma = -\frac{My}{I} \quad ; \quad M = -\int_A \sigma_{xx}\, y\, dA$$

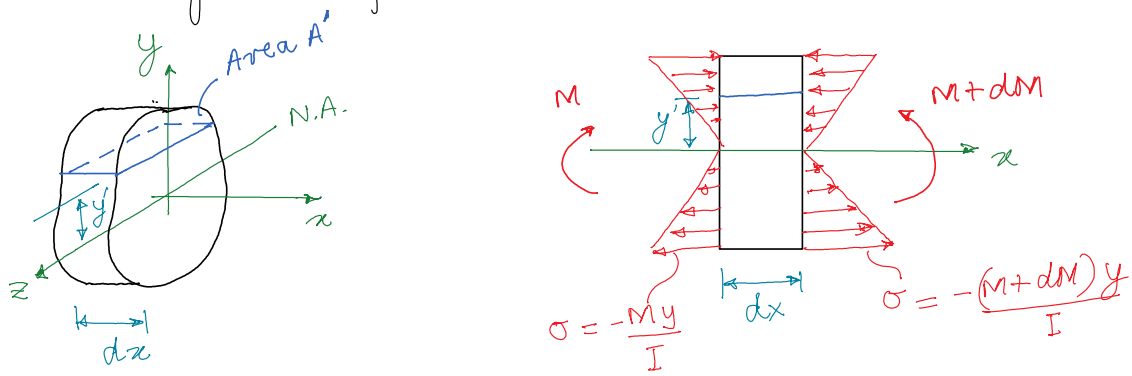
Now, let's see how shear force V is related to transverse shear stresses (τ_{yx} and τ_{xy}) in the cross-section.

$$V = -\int_A \tau_{yx} \cdot dA \quad ; \quad \text{Find} \quad \tau_{yx}$$

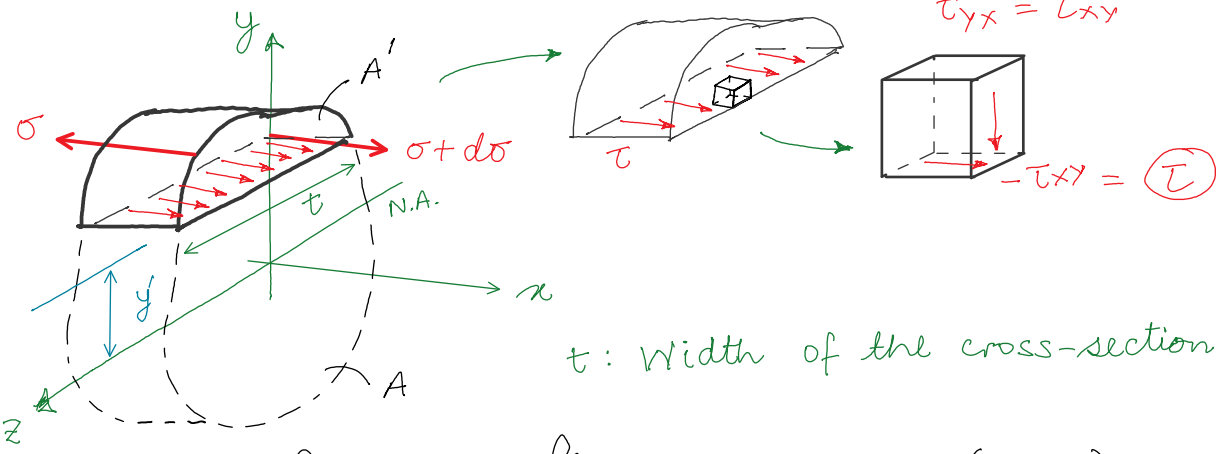
Note that shear stresses develop within beams to maintain compatibility.



Consider a segment of the beam:



Free body diagram for shear stresses:



t : Width of the cross-section at y'

$$\sum F_x = 0 \Rightarrow -\int_{A'} \sigma \, dA + \int_{A'} (\sigma + d\sigma) \, dA + \tau (t \, dx) = 0$$

$$\Rightarrow \int_{A'} \frac{My}{I} dA - \int_{A'} \frac{(M + dM)}{I} y \, dA + \tau (t \, dx) = 0$$

$$\Rightarrow \tau = \left(\frac{dM}{dx}\right) \cdot \frac{\int_{A'} y \, dA}{t \, I} \Rightarrow \boxed{\tau = \frac{V Q}{I t}}$$

($\frac{dM}{dx} = V$, $\int_{A'} y \, dA = Q(y')$)

Note:

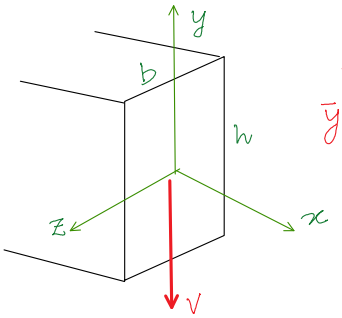
- τ : shear stress at height y' in the cross-section
- V : shear force over the cross-section
- Q: First moment of area A' (that is above y')
- I : Second moment of the entire area of cross-section
- t : thickness (width) of the cross-section at height y'

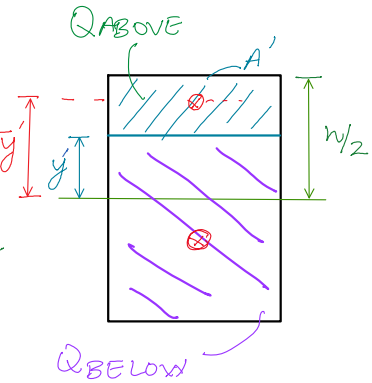
$$Q(y') = \int_{A'} y \, dA$$

Example: Shear over a rectangular cross-section

$$Q = \bar{y}'A' = \underbrace{b\left(\tfrac{h}{2}-y'\right)}_{A'} \cdot \underbrace{\left(y' + \frac{\left(\tfrac{h}{2}-y'\right)}{2}\right)}_{\bar{y}'}$$

$$\Rightarrow Q = \frac{1}{2}\left(\frac{h^2}{4} - y'^2\right) b$$





Note: $Q_{ABOVE} = -Q_{BELOW}$

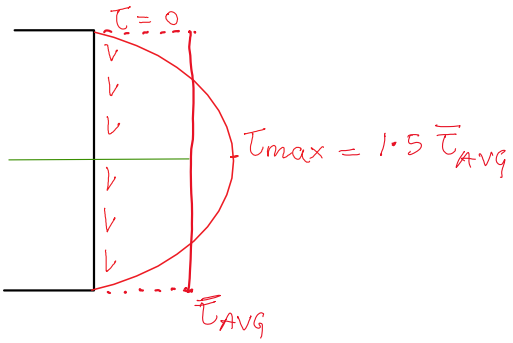
$$I = \frac{1}{12} b h^3$$

$$\Rightarrow \tau = \frac{VQ}{It} = \frac{6V \cdot \frac{1}{2}\left(\frac{h^2}{4} - y'^2\right) b}{\frac{1}{12} b h^3 \cdot b} \Rightarrow \boxed{\tau = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y'^2\right)}$$

Note:

- @ $y' = \pm h/2$: $\tau = 0$
- @ $y' = 0$: $\tau = \boxed{\tau_{max} = \frac{3}{2}\frac{Vh^2}{bh^3}}$

$$= \frac{3}{2}\boxed{\frac{V}{bh}} = \bar{\tau}_{AVG}$$

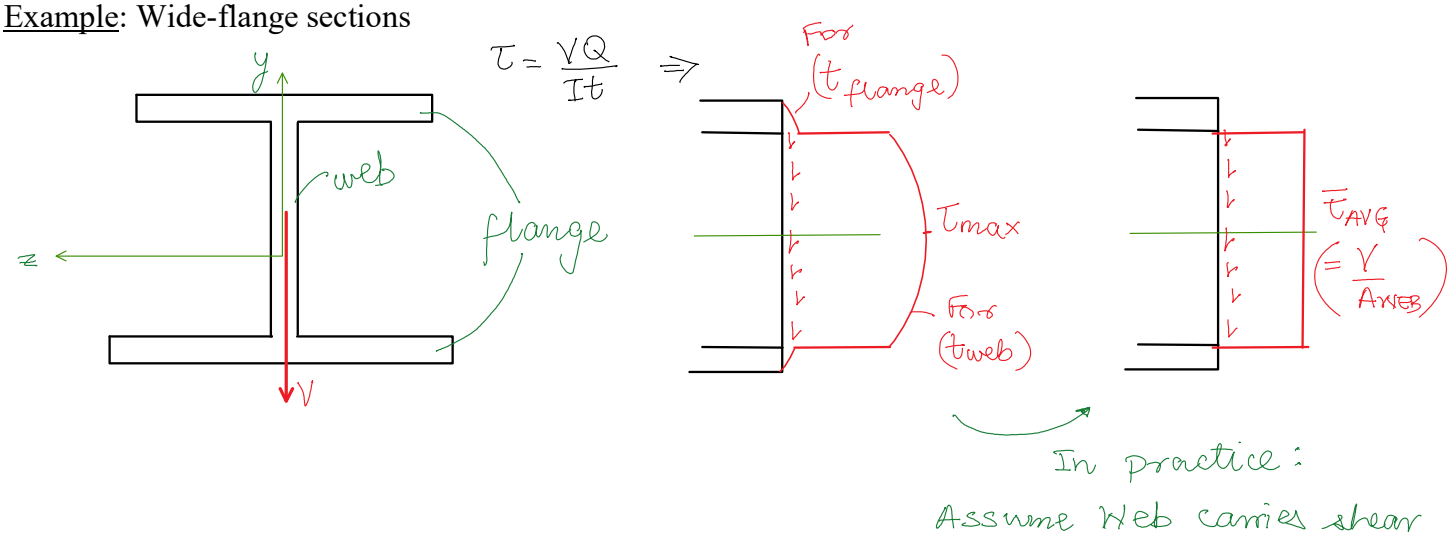


- $$\int_A \tau\, dA = \int_{-h/2}^{h/2} \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right) b\, dy$$

$$= \frac{6V}{bh^3} \cdot \frac{h^2}{4} \cdot b \int_{-h/2}^{h/2} dy - \frac{6Vb}{bh^3} \int_{-h/2}^{h/2} y^2 \cdot dy$$

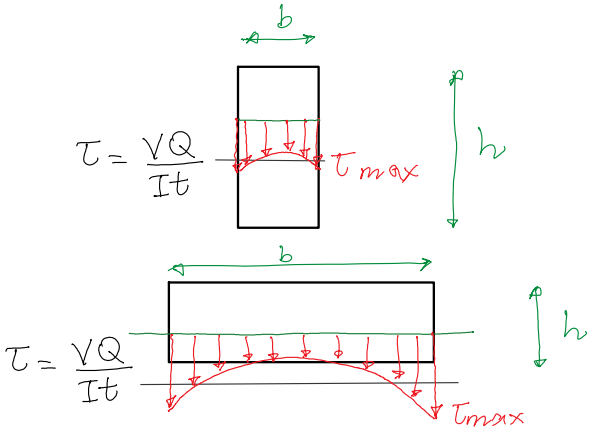
$$= \frac{6V}{4h} \cdot [y]_{-h/2}^{h/2} - \frac{6V}{h^3} \cdot \left[\frac{y^3}{3}\right]_{-h/2}^{h/2} \Rightarrow \boxed{V = \int_A \tau\, dA}$$

Example: Wide-flange sections

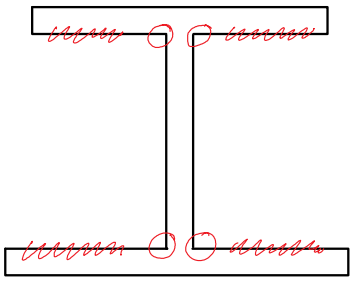


Limitations of the Shear formula:

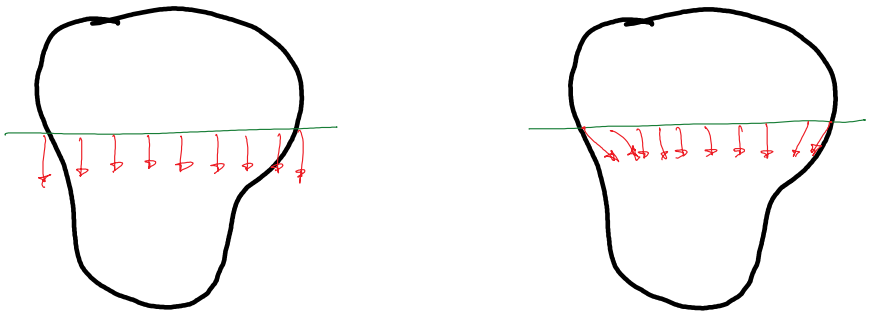
- Average shear stress: τ is the average shear stress across the width "t" of the cross-section.
  - Good approximation for small widths t.
  - Maximum τ occurs on the sides and minimum at mid-width:
    For $b/h = 0.5$ $\tau_{max} = 1.03\ \tau$ but for $b/h = 2$ $\tau_{max} = 1.4\ \tau$



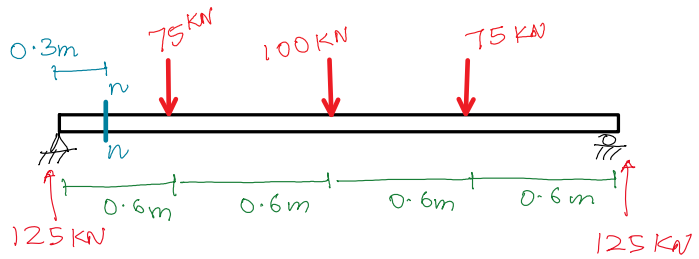
- Wide-flange:
  - Stress concentrations at corners
  - τ on inner surface of flange is zero

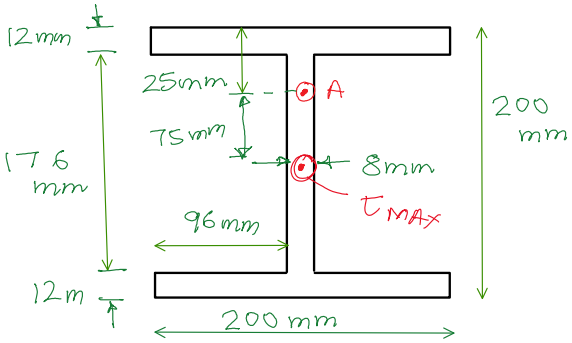


- Inclined faces of the cross-section: shear stress must not have component in a direction that is normal to the cross-section.



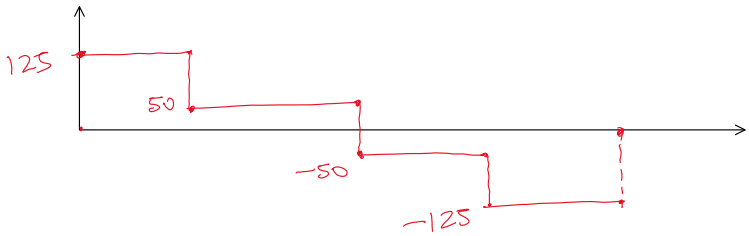
Example





Find @ n-n

- Maximum shear stress
- Shear stress at A



(CS @ n-n)

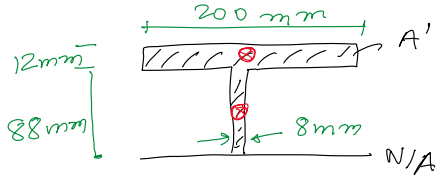
$$\bar{I} = \frac{1}{12}\cdot(200)\cdot(200)^3 - 2\left[\frac{1}{12}\cdot(96)\times(176)^3\right]$$

$$= 4.61\times10^{-5}\ m^4$$

$$\tau_{MAX} = \frac{VQ}{It} = \frac{125\,KN \times 2.56\times10^{-4}\,m^3}{4.61\times10^{-5}\,m^4 \times 8\,mm} = \boxed{86.78\,MPa}$$

$$Q = \sum \bar{y}_i A_i = (12\times200)\times(94) + (88\times8)\times44$$

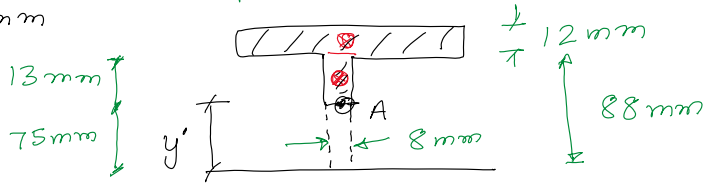
$$= 2.56\times10^{-4}\ m^3$$



$$\tau_A = \frac{VQ_A}{It} = \frac{125\,KN\cdot 2.34\times10^{-4}\,m^3}{4.61\times10^{-5}\,m^4\ \ 8\,mm} = \boxed{79.34\,MPa}$$

$$Q_A = (200\times12)\times94 + (13\times8)\times81.5$$

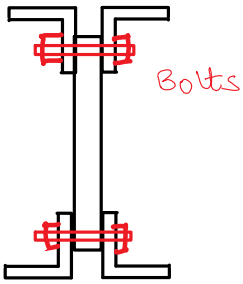
$$= 2.34\times10^{-4}\ m^3$$

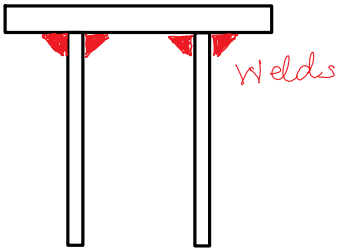


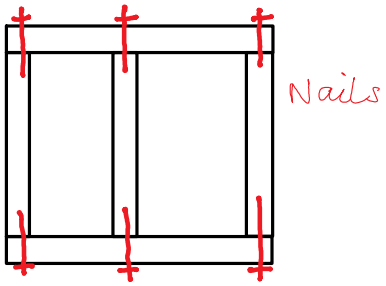
Shear flow in built-up beams

Beam cross-sections that are built-up by combining multiple elements have to be connected together to make sure that they bend together (compatibility).

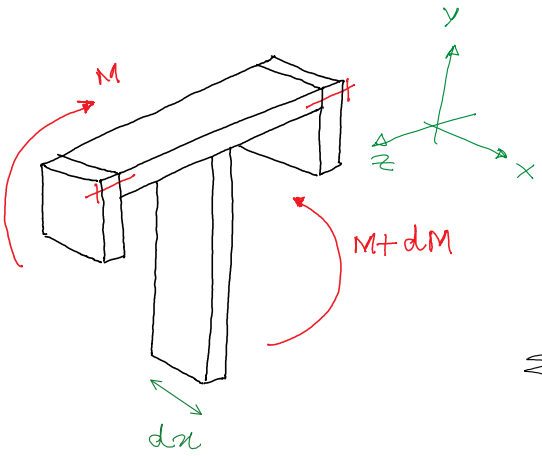
Different elements of the cross-section tend to shear against one another and an addition (distributed) force is needed to maintain the connectivity between them.
This force (measured per unit length of the beam) is called shear flow: q

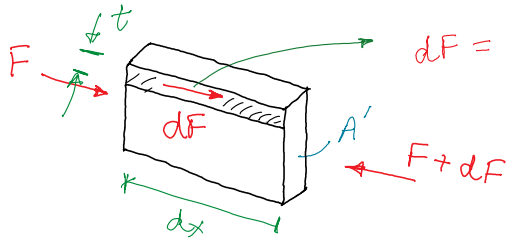




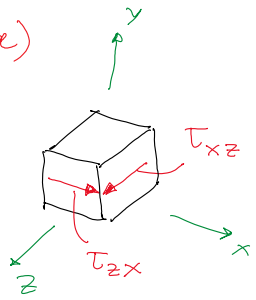


Shear Flow: $q = \dfrac{\text{Force for connectivity}}{\text{Length of beam}} \left(\dfrac{F}{L}\right)$





$dF = \tau_{xz}\,(t\,dx)$



$\Sigma F_x = 0 \Rightarrow (F) + dF - (F+dF) = 0$

$$\Rightarrow dF = (F+dF) - (F) = \int_{A'} \frac{(M+dM)}{I}\, y\, dA - \int_{A'} \frac{M\,y}{I}\, dA$$

$$\Rightarrow dF = \frac{dM}{I} \int_{A'} y\, dA = \frac{(dM)\,Q}{I}$$

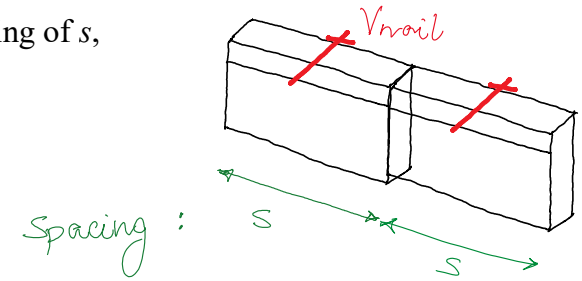
Thus $q = \dfrac{dF}{dx} = \left(\dfrac{dM}{dx}\right) \cdot \dfrac{Q}{I}$ i.e. $\boxed{q = \dfrac{VQ}{I}}$ (Shear Flow)

Note:

$$\tau_{xz} = \left(\frac{dF}{dx}\right)\frac{1}{t} = \frac{q}{t} = \frac{VQ}{It}$$

Connecting elements:

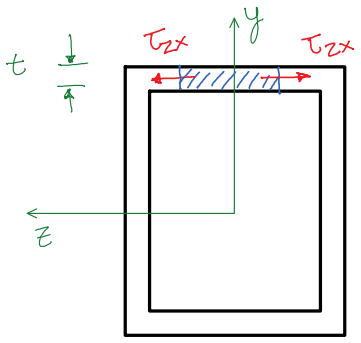
If connecting elements (e.g. nails / bolts) are placed at a spacing of *s*, then each nail / bolt will need to carry shear force:

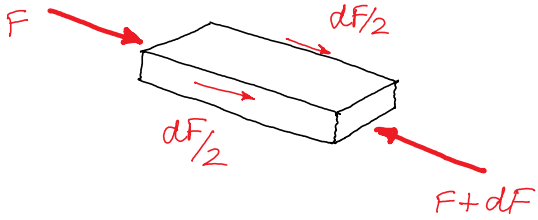


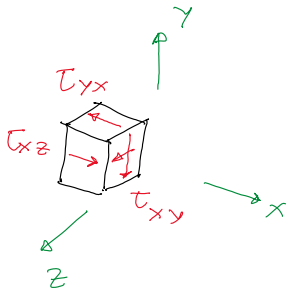
$$V_{nail\ (per\ nail)} = q\ s$$

$$\Rightarrow\ s_{ALLOW} = \frac{V_{ALLOW}}{q}$$

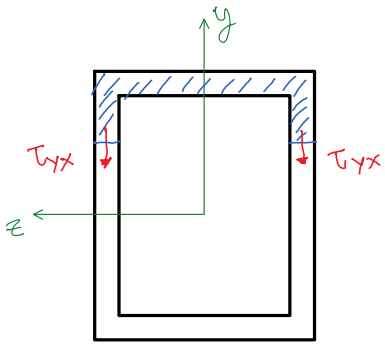
Example: Box beams





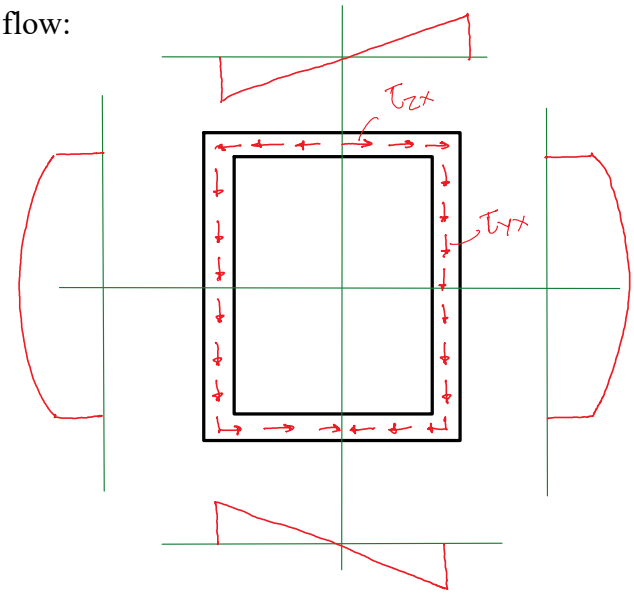


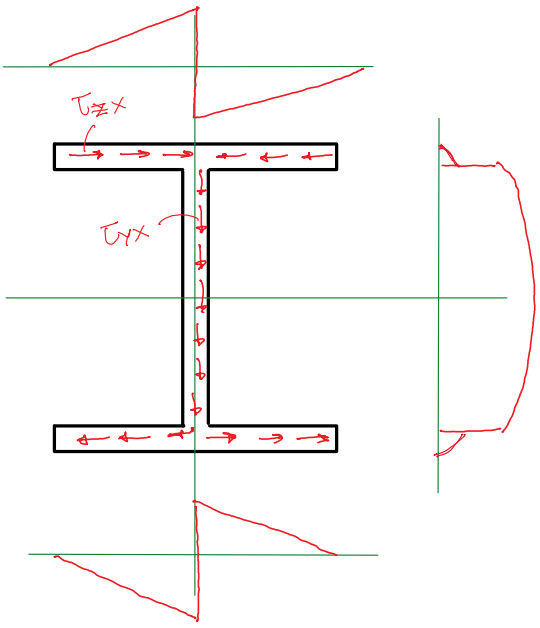
$$\tau_{zx} = \frac{q}{t_{TOTAL}} = \frac{VQ}{I\,t_{TOT}} = \frac{VQ}{I(2t)}$$



$$\tau_{yx} = \frac{q}{t_{TOT}} = \frac{VQ}{I\,t_{TOT}} = \frac{VQ}{I(2t)}$$

Shear flow:





Example

$I = 329.33\ in^4$

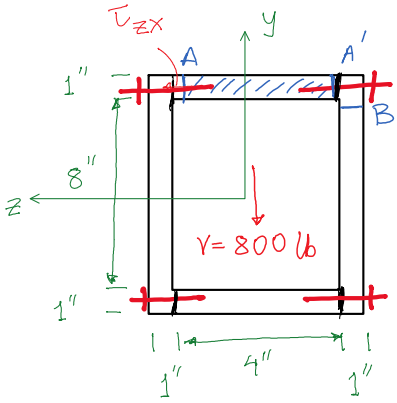
$V = 800\ lbs$

V_{allow} (per nail) = 300 lbs

Find shear stress at A and B

Find maximum spacing of nails if connected at A

Find maximum spacing of nails if connected at B



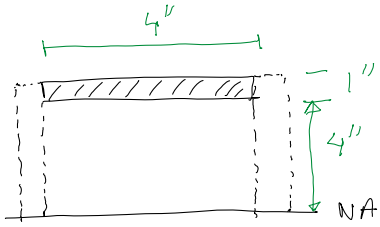
For Point A:

$$q_A = \frac{VQ}{I} = \frac{800\ lbs \quad \cancel{Q_A}\ (18\ in^3)}{329.33\ in^4} = 43.73\ \frac{lb}{in}$$

$$Q_A = (4" \times 1") \times 4.5" = 18\ in^3$$

$$V_{nail_A} = q_A \cdot S_A \Rightarrow (2 \times 300\ lbs) = 43.73\ \frac{lb}{in} \cdot S_A \quad (2\ Nails)$$

$$\Rightarrow S_A = \frac{300 \times 2}{43.73} = \boxed{13.7"}$$



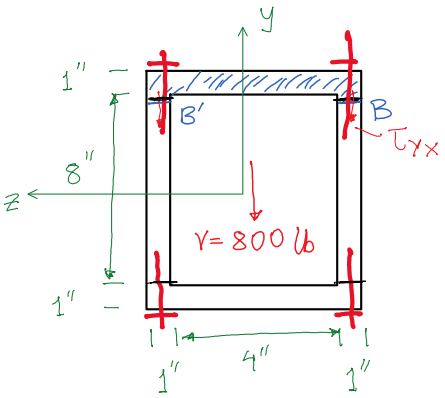
For Point B:

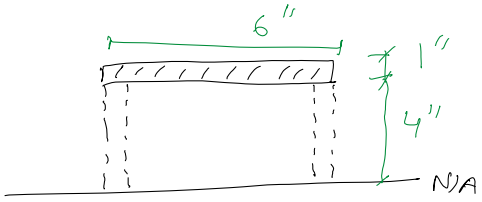
$$q_B = \frac{VQ_B}{I} = \frac{800\ lb \times \cancel{Q_B}\ (27\ in^3)}{329.33\ in^4} = 65.59\ \frac{lb}{in}$$

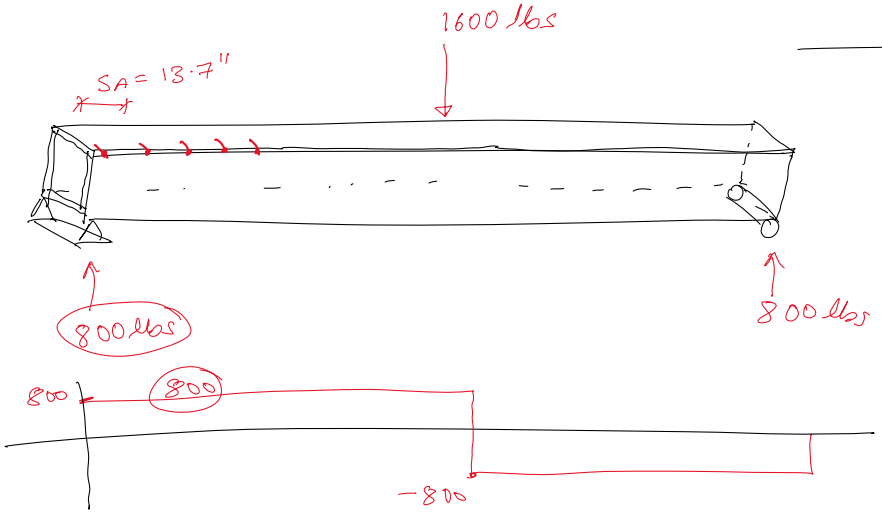
$$Q_B = (6" \times 1") \times 4.5" \Rightarrow 27\ in^3$$

$$V_{nail} = q_B\ S_B \Rightarrow (2 \times 300) = 65.59\ \frac{lb}{in} \cdot S_B \quad (2\ Nails)$$

$$\Rightarrow S_B = \frac{2 \times 300}{65.59} = \boxed{9.15\ in}$$

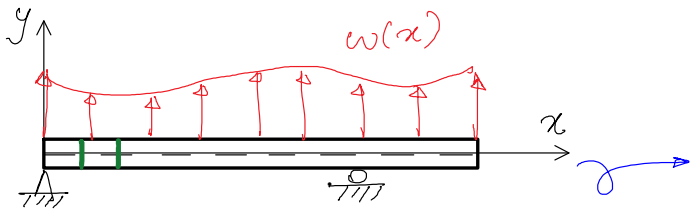


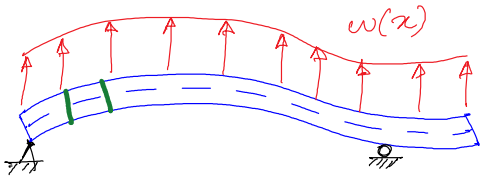




Deflection of Beams

Recall: Governing equations for Bernoulli-Euler Beams:





Equilibrium: $\frac{dM}{dx} = V \;;\; \frac{dV}{dx} = w_{up} \Rightarrow \frac{d^2M}{dx^2} = \boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = w_{up}}$

Strain: $\epsilon = -\kappa y = -\frac{y}{\rho} \;;\; \kappa = \frac{d^2y}{dx^2}$

If $EI = const$ $\quad EI\cdot\frac{d^4y}{dx^4} = w_{up}$

Stress: $\sigma = E\epsilon = -\frac{E}{\rho}y$

Material: $M = EI\kappa = \frac{EI}{\rho}$

Combining these equations:

$$\boxed{-\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{\rho}}$$

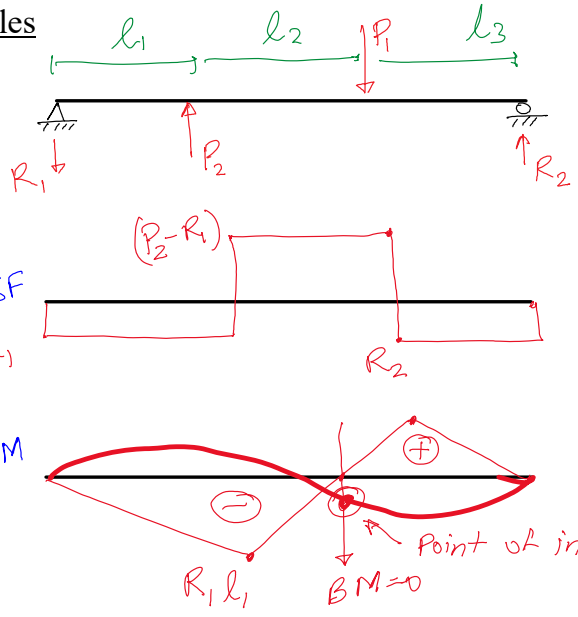
Sketching Approximate Deflected Shape:

Deflected shape (elastic curve) can be sketched by:

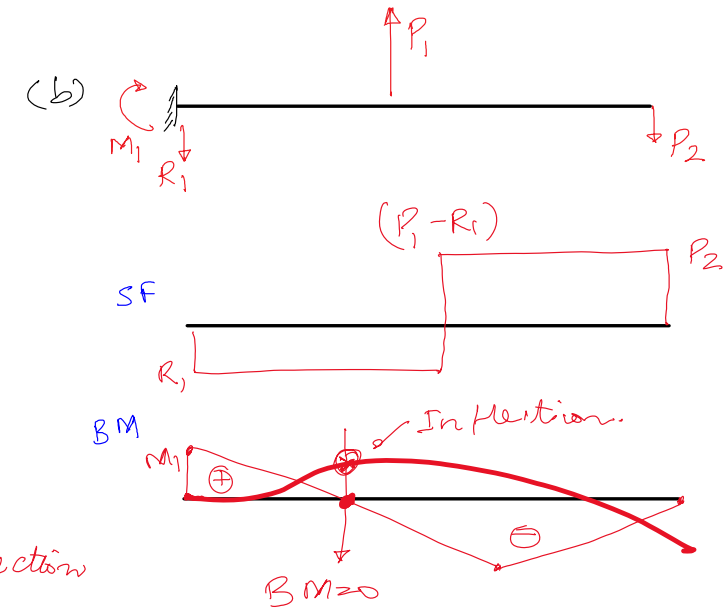
- Analyzing the loads and support conditions
- Bending moment diagram (using +/- curvatures and inflection points)

Examples

(a)

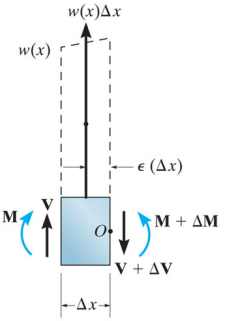


(b)



Computing Beam Deflections: Double Integration

Deflected shapes of beams can also be calculated precisely by integrating the governing equations of beam equilibrium:



$$\frac{dV}{dx} = w_{up} \qquad \frac{dM}{dx} = V \qquad \frac{d^2M}{dx^2} = \frac{d^2}{dx^2}\left(EI\,\frac{d^2y}{dx^2}\right) = w$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{M}{EI}(x)}$$

$$M = EI\,\frac{d^2y}{dx^2}$$

Rotation $\Rightarrow$ $\quad \frac{dy}{dx} = \int \frac{M}{EI}\,dx + C_1$

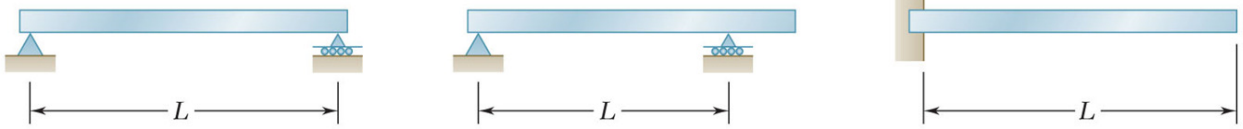
Displacement $\Rightarrow$ $\quad y(x) = \int \left[\int \frac{M}{EI}\,dx\right] dx + C_1 x + C_2$

$K = \frac{d\theta}{dx}$ $\qquad \theta = \frac{dy}{dx}$

Note:

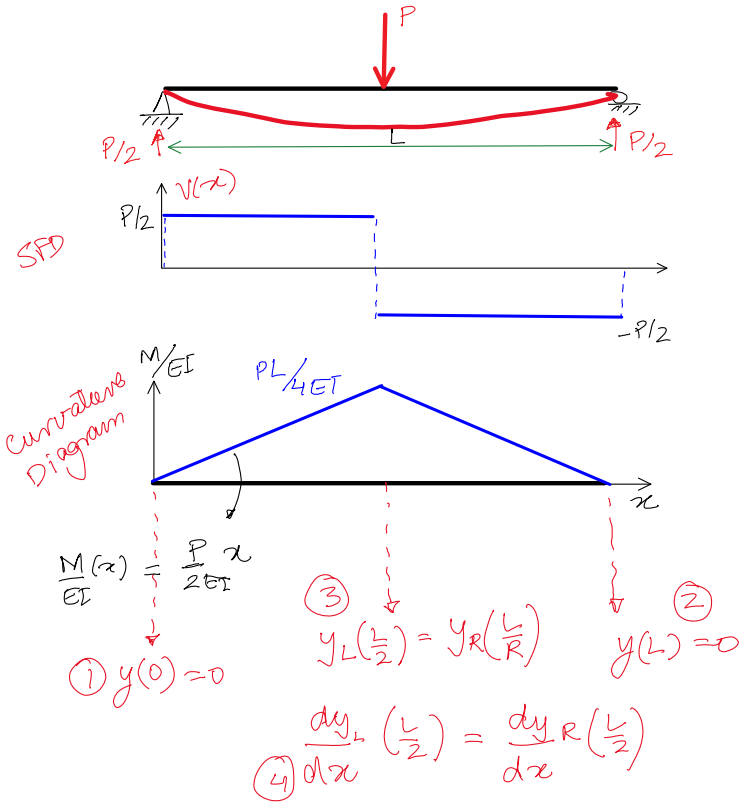
- Constants are evaluated using boundary and compatibility conditions at supports or interior points.
- If $M(x)$ diagram is discontinuous or has discontinuous changes in slope, then a single equation will not be possible for deflections and all segments would have to be integrated separately.

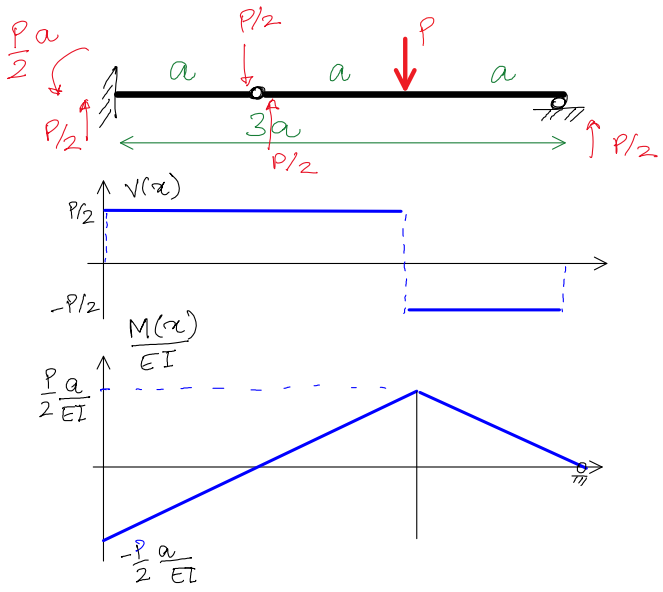
Boundary Conditions



(a) Simply supported beam (b) Overhanging beam (c) Cantilever beam

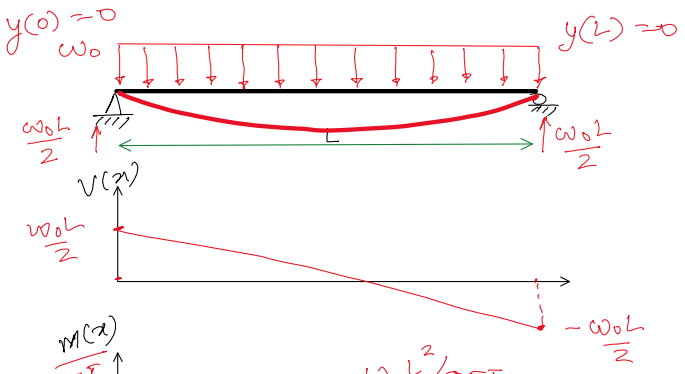
Interface Conditions





Example

Find $y(x)$?



$$\frac{dV}{dx} = -w_0$$

$$\Rightarrow V(x) = -w_0 x + \frac{w_0 L}{2}$$

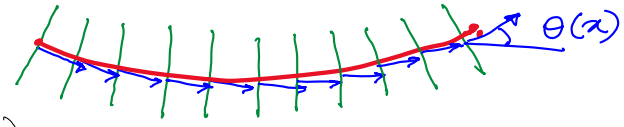
$$\frac{dM}{dx} = V(x)$$

$$M(x) = \int V\,dx + C_3 \qquad M(0) = 0$$

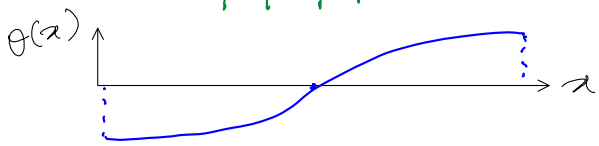
$$\frac{M(x)}{EI} = -\frac{w_0 x^2}{EI}\frac{1}{2} + \frac{w_0 L}{2EI}x$$

BMD → Curvature

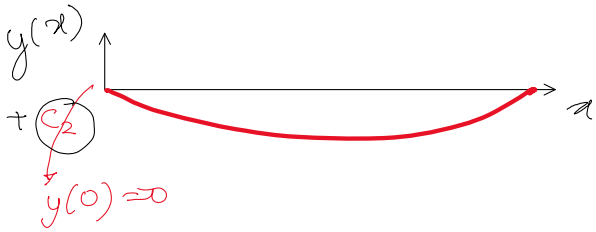




$$\frac{d^2y}{dx^2} = \frac{M}{EI}(x) = -\frac{w_0 x^2}{2EI} + \frac{w_0 L x}{2EI}$$



Rotation: $\theta(x)$ $\boxed{\frac{dy}{dx}} = -\frac{w_0 x^3}{6EI} + \frac{w_0 L x^2}{4EI} + C_1$



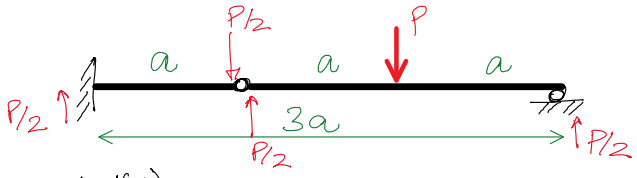
Displacement: $y(x) = -\frac{w_0 x^4}{24EI} + \frac{w_0 L x^3}{12EI} + C_1 x + C_2$

$y(L) = 0 \Rightarrow$

$$0 = -\frac{w_0 L^4}{24EI} + \frac{w_0 L^3}{12EI} + C_1 L \Rightarrow C_1 = -\frac{w_0 L^3}{24EI}$$

$$\boxed{y(x) = \frac{1}{EI}\left[-\frac{w_0 x^4}{24} + \frac{w_0 L x^3}{12} - \frac{w_0 L^3 x}{24}\right]}$$

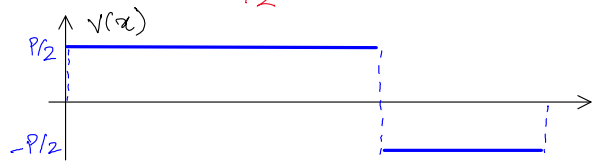
Example:
Find the displacement along the length of the beam:

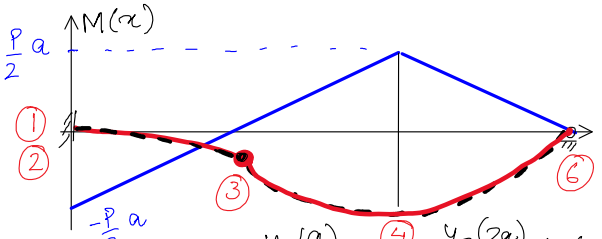


Consider 3 segments :

⇒ 6 constants

Need 6 boundary conditions





$y_1(0) = 0$

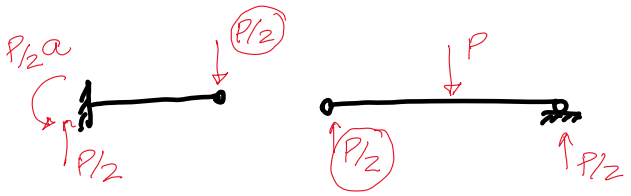
$\frac{dy_1}{dx_1}(0) = 0$

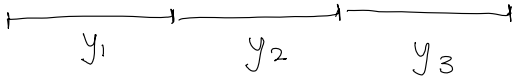
$y_1(a) = y_2(a)$

$y_2(2a) = y_3(2a)$

$y_3(3a) = 0$

$\frac{dy_2}{dx}(2a) = \frac{dy_3}{dx}(2a)$





Example (continued)

For $0 < x < a$:

$\frac{M}{EI} = K(x)$

$\Rightarrow = \frac{d^2y}{dx^2} = \frac{M(x)}{EI} = \frac{P}{2EI}(x-a)$

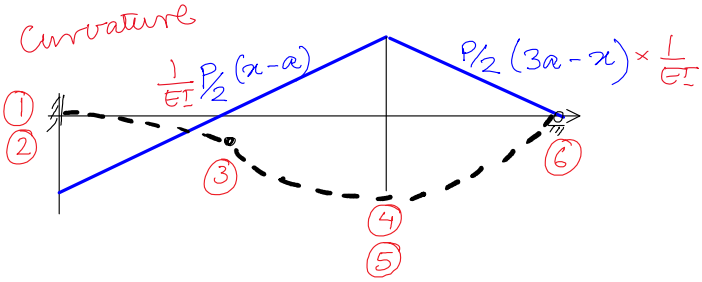
Rotation

$\Rightarrow \frac{dy(x)}{dx} = \frac{P}{2EI}\left(\frac{x^2}{2} - ax\right) + C_1$ ② 0

Displacement

$\Rightarrow y(x) = \frac{P}{2EI}\left(\frac{x^3}{6} - a\frac{x^2}{2}\right) + C_1 x + C_2$

Curvature



① $y(0) = 0$

② $\frac{dy}{dx}(0) = 0$

③ $y(a) = -\frac{Pa^3}{6EI}$

For $a < x < 2a$: $\frac{M(x)}{EI} = \frac{P}{2}(x-a)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{M(x)}{EI} = \frac{P}{2EI}(x-a)$

$\Rightarrow \frac{dy(x)}{dx} = \frac{P}{2EI}\left(\frac{x^2}{2} - ax\right) + C_3$

$\Rightarrow y(x) = \frac{P}{2EI}\left(\frac{x^3}{6} - a\frac{x^2}{2}\right) + C_3 x + C_4$

④ $\left.\frac{dy}{dx}\right|_{2a} = C_3$

③ $y(a) = \frac{-Pa^3}{6EI} + C_3 a + C_4$

$\Rightarrow \boxed{C_3 a + C_4 = 0}$

For $2a < x < 3a$: $M(x) = \frac{P}{2}(3a - x)$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{M(x)}{EI} = \frac{P}{2EI}(3a - x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{P}{2EI}\left(3ax - \frac{x^2}{2}\right) + C_5$$

$$\Rightarrow y(x) = \frac{P}{2EI}\left(3a\frac{x^2}{2} - \frac{x^3}{6}\right) + C_5 x + C_6$$

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⑥ $\Rightarrow y(3a) = 0$

$$\Rightarrow 0 = \frac{-P}{2EI}\left(3a\frac{(3a)^2}{2} - \frac{(3a)^3}{6}\right) + C_5 3a + C_6$$

$$\Rightarrow \boxed{C_5 3a + C_6 = \frac{9}{2}\frac{Pa^3}{EI}}$$

4 equations, 4 unknown constants => solve!

⑤ $y(2a) = \frac{P}{2EI}\left(\frac{8a^3}{6} - 2a^3\right) + C_3 2a + C_4$

④ $\left.\frac{dy}{dx}\right|_{2a} = \frac{2Pa^2}{EI} + C_5$

$$\Rightarrow \boxed{C_3 - C_5 = 2\frac{Pa^2}{EI}}$$

⑤ $y(2a) = \frac{P}{2EI}\left(\frac{12a^3}{2} - \frac{8a^3}{6}\right) + C_5 2a + C_6$

$$= \frac{7}{3}\frac{Pa^3}{EI} + C_5 2a + C_6$$

⑤ $\Rightarrow \boxed{C_3 2a + C_4 - C_5 2a - C_6 = \frac{8}{3}\frac{Pa^3}{EI}}$

$\Rightarrow C_3 = \frac{55}{12}\frac{Pa^2}{EI}$ $\quad C_5 = \frac{31}{12}\frac{Pa^2}{EI}$

$C_4 = \frac{-55}{12}\frac{P}{EI}a^3$ $\quad C_6 = \frac{-39}{12}\frac{Pa^3}{EI}$

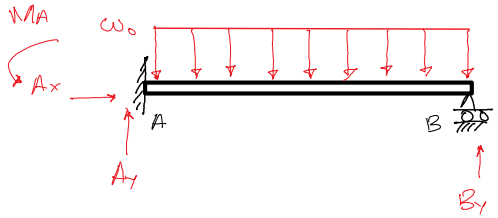
Statically Indeterminate Beams:

The process of double integration can be extended as "quadruple" integration to solve statically indeterminate problems as well!

Example

4 unknown Reactions ; 3 equations

$\Rightarrow$ DOI = 1



$$EI\frac{d^4y}{dx^4} = -w_0$$

$$V(x)\quad EI\frac{d^3y}{dx^3} = -w_0x + C_1 \quad (C_1 = \tfrac{5}{8}w_0L)$$

$$M(x)\quad EI\cdot\frac{d^2y}{dx^2} = -w_0\frac{x^2}{2} + C_1x + C_2 \quad (C_2 = -\tfrac{w_0L^2}{8})$$

$$EI\,\theta(x) = EI\frac{dy}{dx} = -w_0\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2x + C_3 \quad (C_3 = \tfrac{dy}{dx}(0) = 0)$$

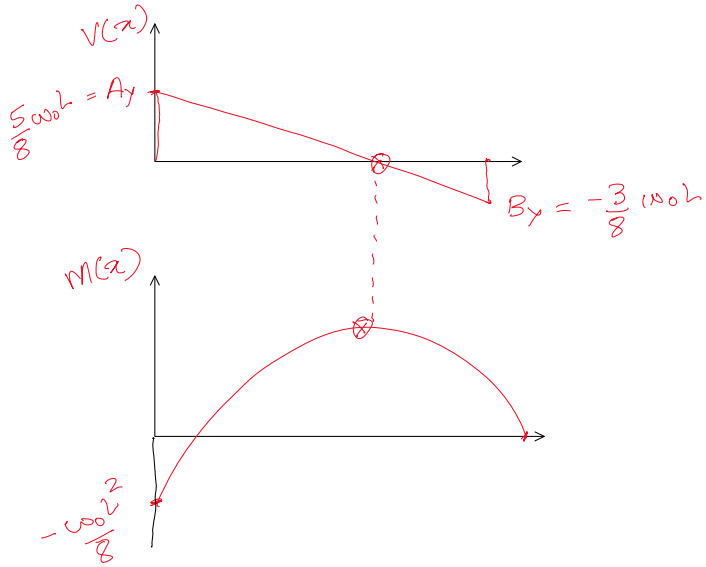
$$EI\,y(x) = EI\,y(x) = -w_0\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4 \quad (C_4 = y(0) = 0)$$

① $y(0) = 0$ ② $\frac{dy}{dx}(0) = 0$ ③ $y(L) = 0$ ④ $M(L) = 0$

Using ③ & ④ $\Rightarrow$

$$C_1 = \tfrac{5}{8}w_0L$$

$$C_2 = -\frac{w_0L^2}{8}$$



Example

Draw SFD, BMD, and deflected shape

DOI = 3

6 unknowns ; 3 eqns ⇒ DOI = 3

Symmetry ⇒ $A_y = C_y = P/2$

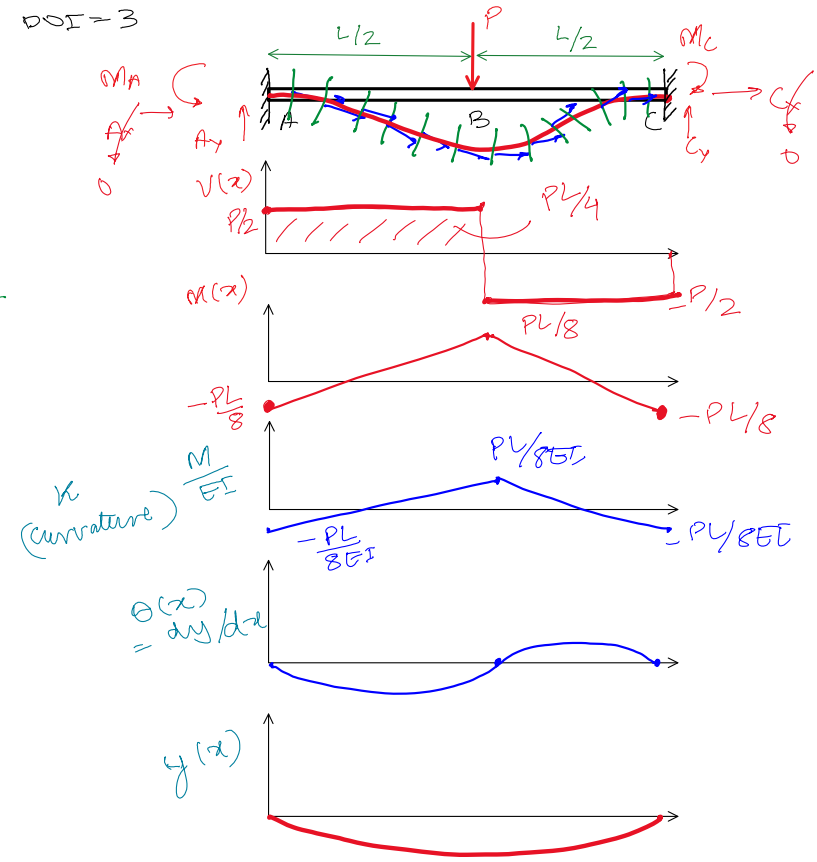
$M_A = M_C$

$0 < x < L/2$ | $L/2 < x < L$

$EI \frac{d^4y}{dx^4} = 0$ | $EI \frac{d^4y}{dx^4} = 0$

4 const | 4 const.

8 constants.



Example: Use of symmetry

$0 < x < L/2$

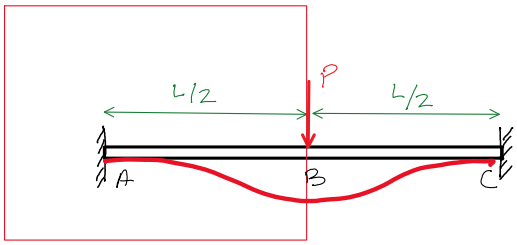
$EI \frac{d^4y}{dx^4} = 0$ → 4 const

$V(0) = A_y = P/2$

$y(0) = 0$

$\frac{dy}{dx}(0) = 0$

$\frac{dy}{dx}(L/2) = 0$



① $EI \frac{d^3y}{dx^3} = C_1 = P/2$ ($V(x)$)

② $EI \frac{d^2y}{dx^2} = C_1 x + C_2 = \frac{-PL}{8}$ ⇒ $M(0) = -\frac{PL}{8}$ ($M(x)$)

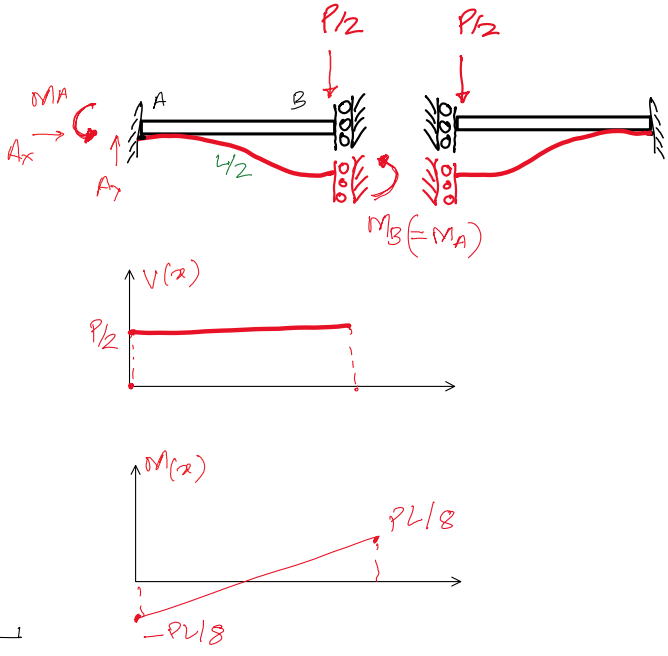
③ $EI \frac{dy}{dx} = C_1 \frac{x^2}{2} + C_2 x + C_3$ ($\theta(x)$), $C_3 = 0$

④ $EI\, y(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4$, $C_4 = 0$

From ③:

$EI \frac{dy}{dx}(L/2) = C_1 \frac{L}{8} + C_2 L/2 = 0$ ($C_1 = P/2$)

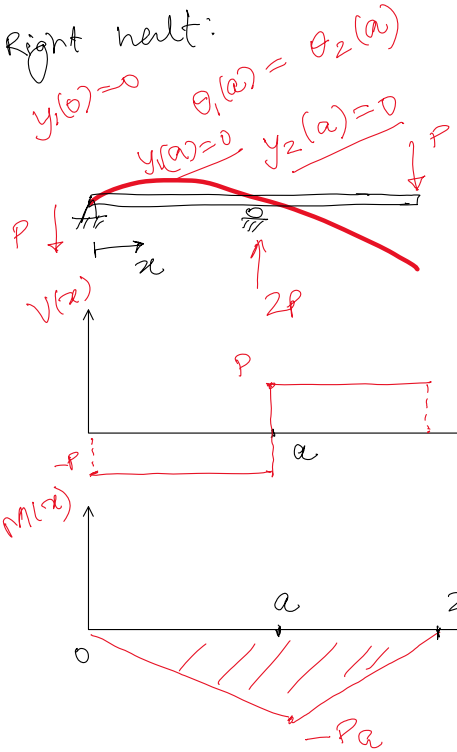
⇒ $C_2 = -\frac{PL}{8}$

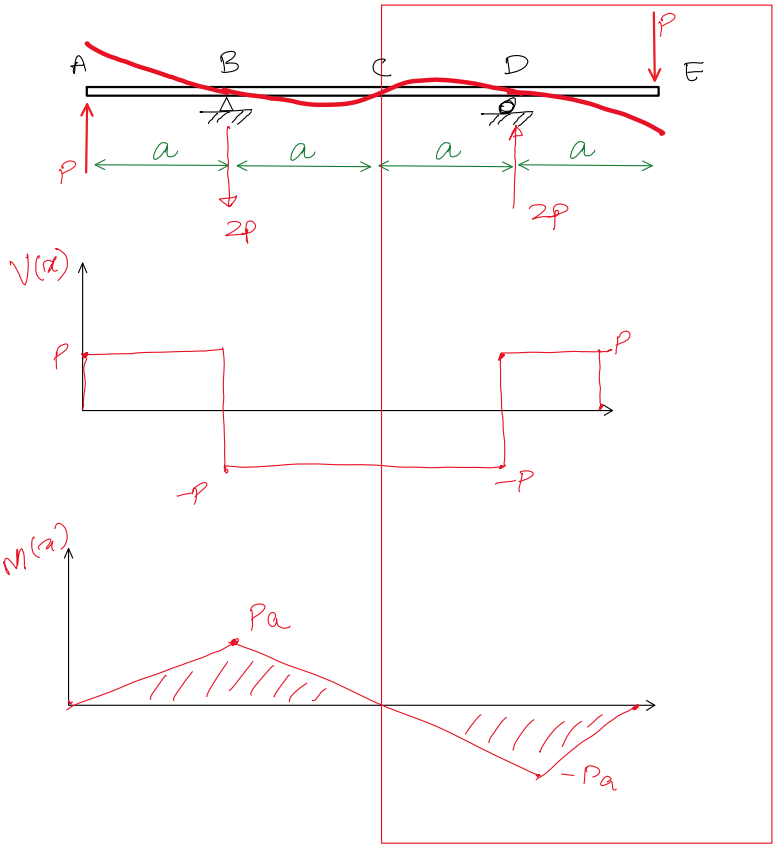


Example: Use of anti-symmetry

Draw SFD, BMD, approximate deflected shape

Find elastic curve y(x)





$0 < x < a$

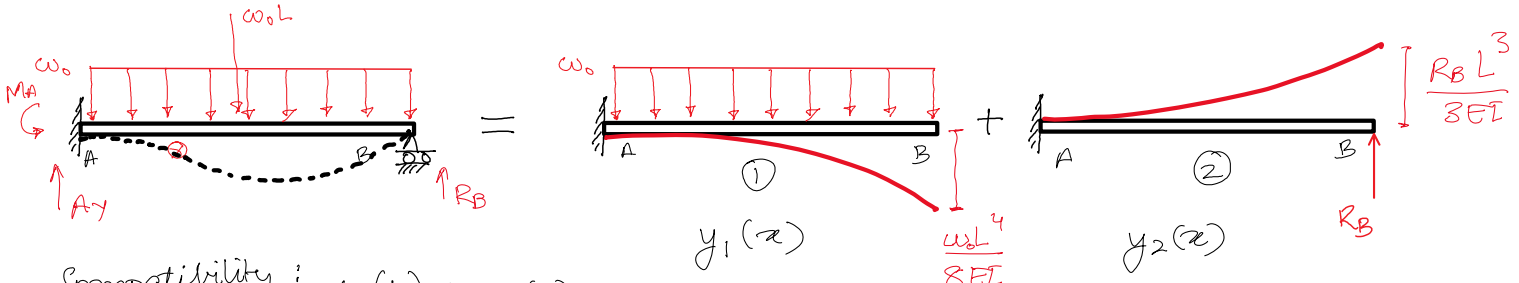
$y_1(x) = ---- + C_1 x + C_2$

$a < x < 2a$

$y_2(x) = ------ + C_3 x + C_4$

Method of Superposition for Indeterminate problems (using tables)

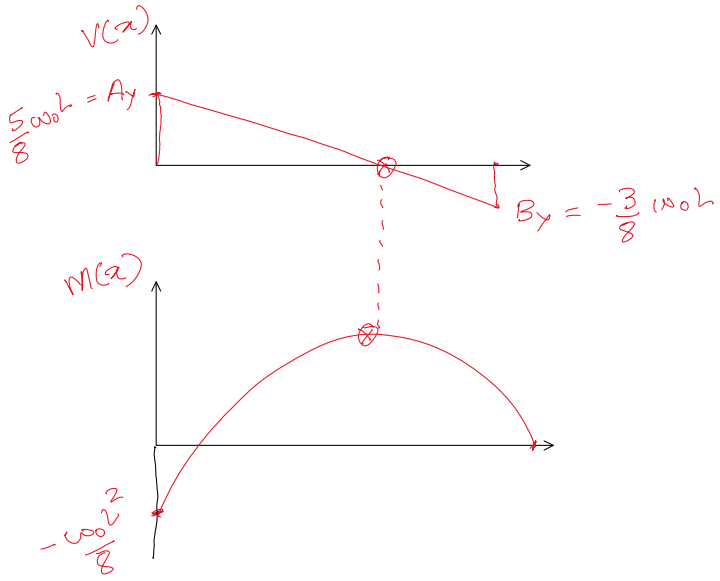
For multiple loads:	Slope and deflection due to all loads = Σ Slope and deflection due to individual loads



Compatibility: $y_1(L) + y_2(L) = 0$

(From Tables) $-\frac{w_0 L^4}{8EI} + \frac{R_B \cdot L^3}{3EI} = 0 \Rightarrow \boxed{R_B = \frac{3}{8} w_0 L}$

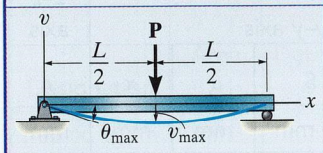
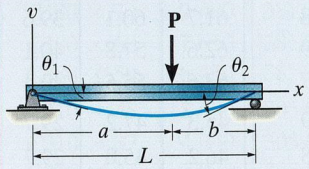
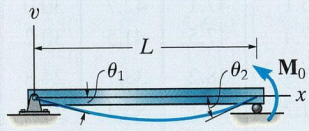
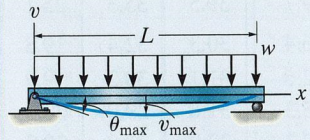
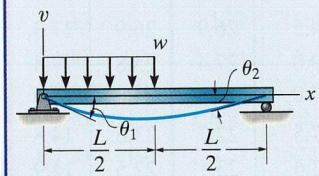
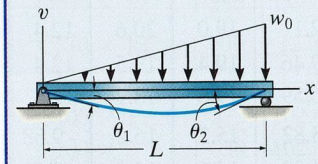
$\Rightarrow A_y = w_0 L - R_B = \boxed{\frac{5}{8} w_0 L}$



APPENDIX C

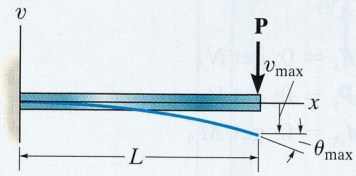
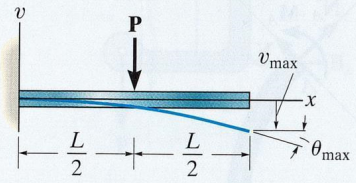
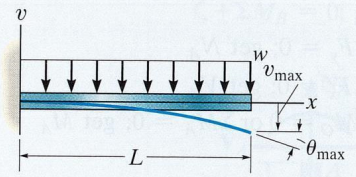
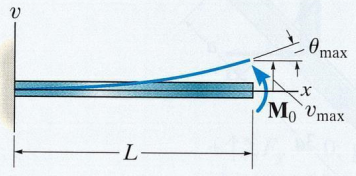
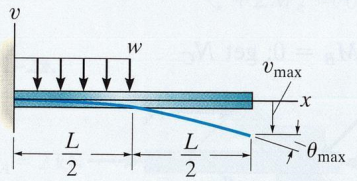
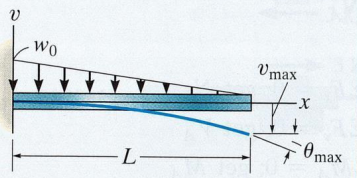
SLOPES AND DEFLECTIONS OF BEAMS

Simply Supported Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{16EI}$	$v_{max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	$v\Big\|_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \frac{-M_0 L}{6EI}$ $\theta_2 = \frac{M_0 L}{3EI}$	$v_{max} = \frac{-M_0 L^2}{9\sqrt{3}\,EI}$ at $x = 0.5774L$	$v = \frac{-M_0 x}{6EIL}(L^2 - x^2)$
	$\theta_{max} = \frac{-wL^3}{24EI}$	$v_{max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v\Big\|_{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{max} = -0.006563\frac{wL^4}{EI}$ at $x = 0.4598L$	$v = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \frac{-7w_0L^3}{360EI}$ $\theta_2 = \frac{w_0L^3}{45EI}$	$v_{max} = -0.00652\frac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \frac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

Deflections of beams for common loads (continued)

Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{2EI}$	$v_{max} = \frac{-PL^3}{3EI}$	$v = \frac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \frac{-PL^2}{8EI}$	$v_{max} = \frac{-5PL^3}{48EI}$	$v = \frac{-Px^2}{12EI}(3L - 2x) \quad 0 \le x \le L/2$ $v = \frac{-PL^2}{48EI}(6x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \frac{-wL^3}{6EI}$	$v_{max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \frac{M_0 L}{EI}$	$v_{max} = \frac{M_0 L^2}{2EI}$	$v = \frac{M_0 x^2}{2EI}$
	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right) \quad 0 \le x \le L/2$ $v = \frac{-wL^3}{384EI}(8x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \frac{-w_0 L^3}{24EI}$	$v_{max} = \frac{-w_0 L^4}{30EI}$	$v = \frac{-w_0 x^2}{120EIL}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$