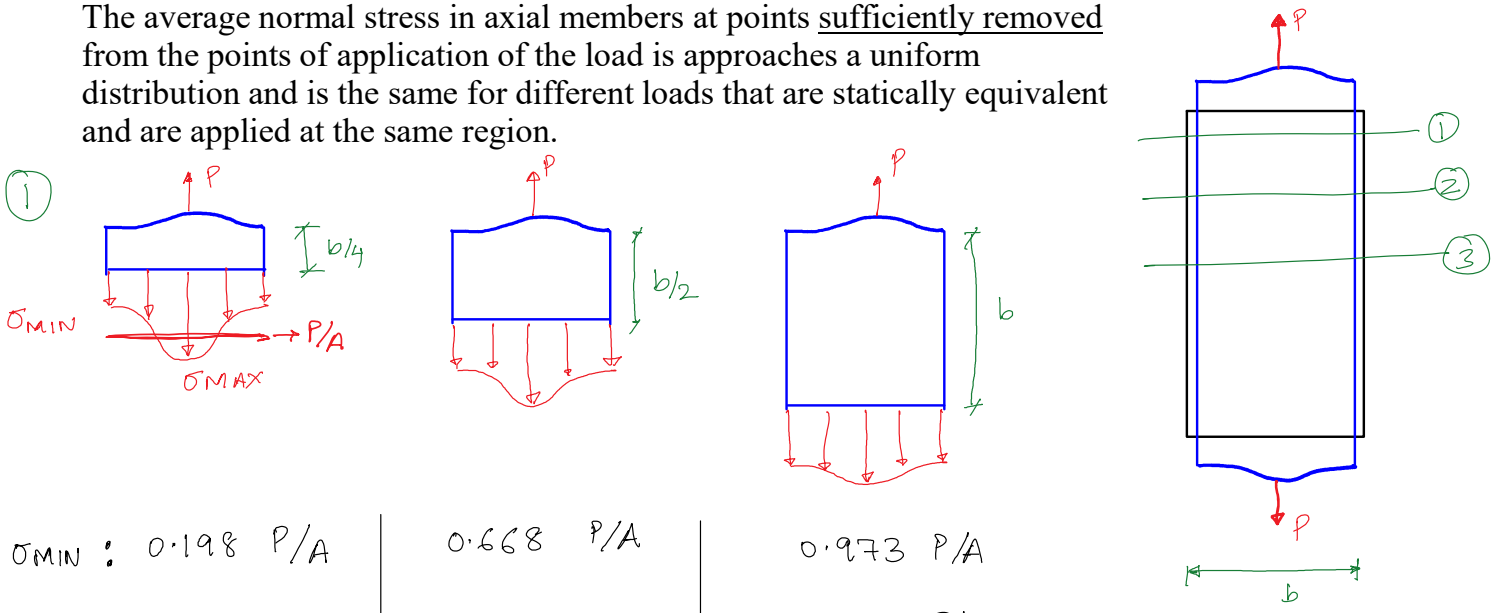
Axial Members - 1D Behavior

Ref: Ch-4 Hibbeler, 10th Ed.

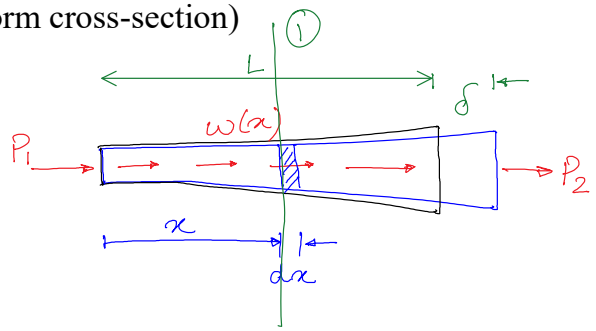
St. Venant's Principle:

The average normal stress in axial members at points sufficiently removed from the points of application of the load is approaches a uniform distribution and is the same for different loads that are statically equivalent and are applied at the same region.



σ_{MIN} :	0.198 P/A	0.668 P/A	0.973 P/A
σ_{MAX} :	2.575 P/A	1.387 P/A	1.027 P/A

Elastic Deformation of Axial members (non-uniform cross-section)



$$\bar{\sigma}(x) = \frac{N(x)}{A(x)}$$

$$\bar{\epsilon}(x) = \frac{du}{dx} \Rightarrow \int du = \int \bar{\epsilon}(x)\,dx$$

Hooke's model: $\bar{\sigma}(x) = E\,\bar{\epsilon}(x) \Rightarrow \bar{\epsilon} = \frac{\bar{\sigma}(x)}{E} = \frac{1}{E}\frac{N(x)}{A(x)}$

$$\Rightarrow u(x) = \int_0^{u(x)} du = \int_0^{x} \frac{1}{E}\cdot\frac{N(x)}{A(x)}\cdot dx$$

Thus $$\delta = u(L) = \boxed{\int_0^{L} \frac{1}{E}\frac{N(x)}{A(x)}\cdot dx}$$

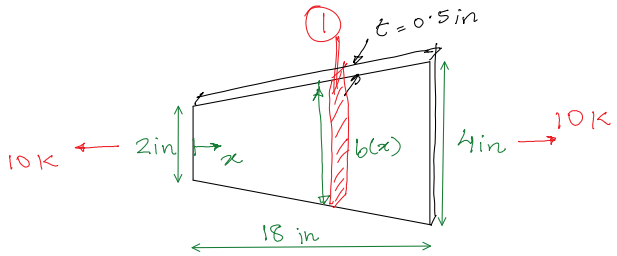
Note: For uniform N(x), A(x), E(x): $\delta = u(L) = \frac{1}{E}\frac{N}{A}\int_0^{L} dx = \boxed{\frac{NL}{AE}}$

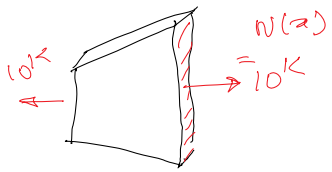
Example

Flat steel plate

$t = 0.5$ in

$E = 29000$ ksi





Find extension δ.

$A(x) = t \times b(x)$

$$\bar{\varepsilon}(x) = \frac{du}{dx} \Rightarrow \int_0^{18} du = \int_0^{18} \bar{\varepsilon}(x)\, dx$$

$$u(18) - u(0) = \int_0^{18} \bar{\varepsilon}(x)\, dx$$

$$b(x) = 2 + \left(\frac{x}{18}\right) \cdot 2 \cdot \frac{1}{2} = 2 + \frac{x}{9}$$

$$A(x) = (0.5)\left(2 + \frac{x}{9}\right) \text{ in}^2$$

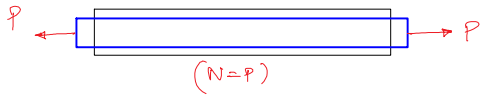
$$\bar{\varepsilon}(x) = \frac{\bar{\sigma}(x)}{E} = \frac{N(x)}{A(x)E} \Rightarrow u(18) - u(0) = \int_0^{18} \frac{10^K}{0.5\left(2 + \frac{x}{9}\right) \text{in}^2} \cdot \frac{1}{29000 \text{ ksi}} \cdot dx$$

$$\underbrace{u(18) - u(0)}_{\text{Change in length } \Delta L} = \frac{18}{2900} \left[\ln(x+18)\right]_0^{18} \text{ in}$$

$$\approx \boxed{0.0043 \text{ in}}$$

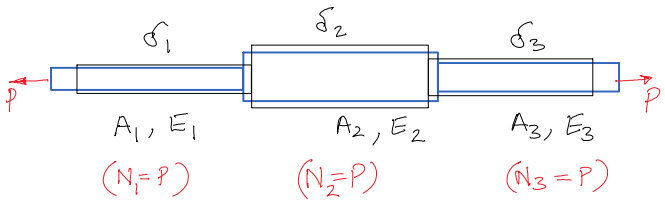
Elastic deformation of Axial members (uniform cross-section)

If $A(x)$ is constant:



$$\delta = u(L) = \frac{N}{AE}\int_0^L dx = \frac{NL}{AE}$$

Multiple sections:



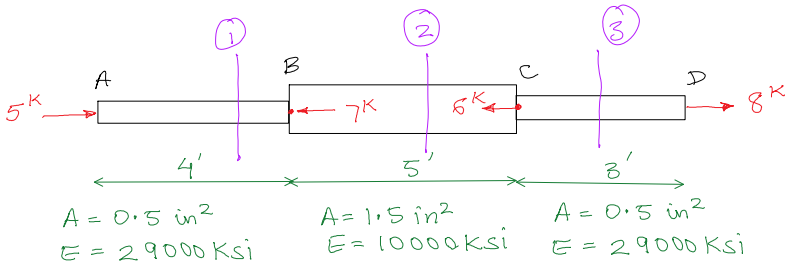
$$\delta = \delta_1 + \delta_2 + \delta_3$$

$$\boxed{\delta = \sum_{i=1}^{K} \frac{N_i L_i}{A_i E_i}}$$

$$\delta_1 = \frac{PL_1}{A_1E_1} \qquad \delta_2 = \frac{PL_2}{A_2E_2} \qquad \delta_3 = \frac{PL_3}{A_3E_3}$$

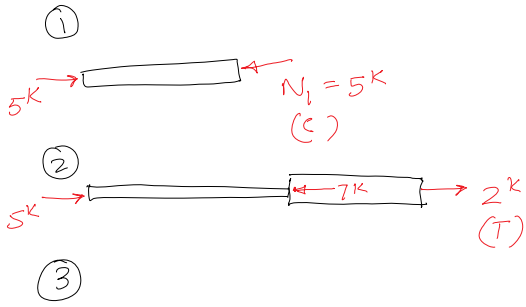
Example

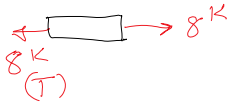
Find δ



$$\delta = \delta_1 + \delta_2 + \delta_3$$

$$= \frac{N_1L_1}{A_1E_1} + \frac{N_2L_2}{A_2E_2} + \frac{N_3L_3}{A_3E_3}$$



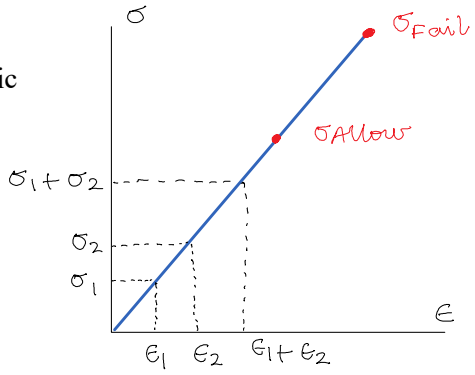


Member	N_i	L_i	A_i	E_i	$\delta_i = \frac{N_iL_i}{A_iE_i}$
AB	-5^K	$4\times12''$	0.5	29000	-0.0166 in
BC	2^K	$5\times12''$	1.5	10000	0.008 in
CD	8^K	$3\times12''$	0.5	29000	0.0199 in
					$\delta = 0.0113$ in

Principle of superposition

Total displacement d due to multiple loads P_i can be found by superposing (adding) the individual displacements d_i due to each of the loads, **if**:

1. Loads are small enough that material remains linear-elastic
2. Deformations are small enough that they don't cause changes in geometry



$$d = \sum_i d_i = \left(\sum_i P_i\right)\frac{L}{AE}$$

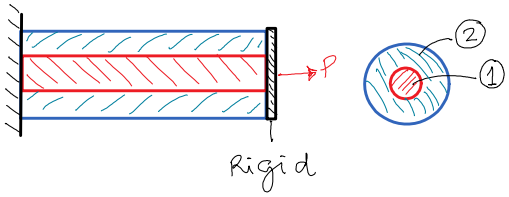
If load $P \rightarrow \delta$

then $\alpha P \rightarrow \alpha \delta$

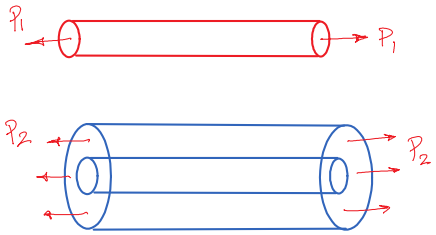
Statically Indeterminate Axial Problems:

When it is not possible to determine internal forces in members using equilibrium alone. Internal forces depend upon relative material stiffness of different components.

We need to use **Compatibility** to solve such problems:



Consider the displacements of individual parts:



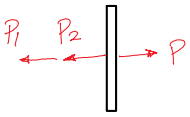
$$\delta_1 = \frac{P_1 L}{A_1 E_1}$$

$$\delta_2 = \frac{P_2 L}{A_2 E_2}$$

Compatibility: $\boxed{\delta_1 = \delta_2}$

$$\frac{P_1 L}{A_1 E_1} = \frac{P_2 L}{A_2 E_2} \quad \text{---(1)}$$

FBD of end plate:

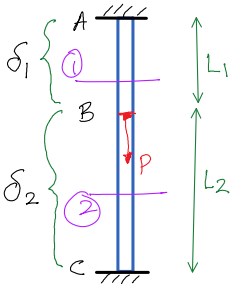


Equilibrium:

$$\sum F_x = 0 \Rightarrow P = P_1 + P_2 \quad \text{---(2)}$$

Two equations, two unknowns P_1 P_2. Solve for it!

Another example is an over-constrained system:
In this case, we can consider the displacements of the individual parts

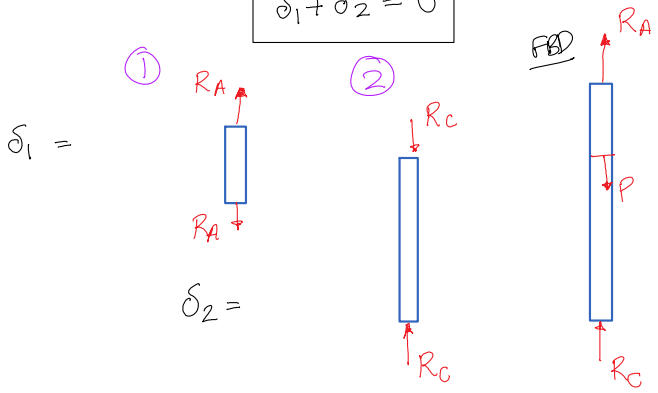


$\delta_1 \quad \delta_2$

$\Rightarrow \delta_1 = -\delta_2$

Note: Compatibility

$$\delta_1 + \delta_2 = 0$$

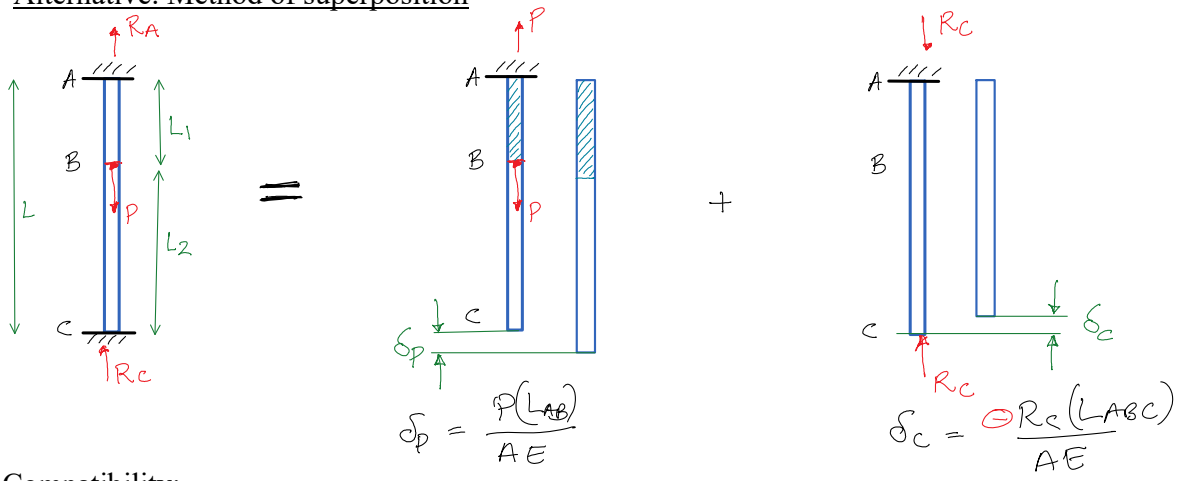


Equilibrium:

$$P = R_A + R_C$$

Two equations, two unknowns: $R_A \quad R_C$. Solve for them!

Alternative: Method of superposition



$$\delta_P = \frac{P(L_{AB})}{AE}$$

$$\delta_C = \frac{-R_C(L_{ABC})}{AE}$$

Compatibility:

$$\delta = \delta_P + \delta_C = 0 \Rightarrow \frac{P L_{AB}}{AE} - \frac{R_C L_{ABC}}{AE} = 0 \Rightarrow \boxed{R_C = \frac{P L_{AB}}{L_{ABC}}} \Rightarrow R_A = P - R_C$$

Equilibrium:

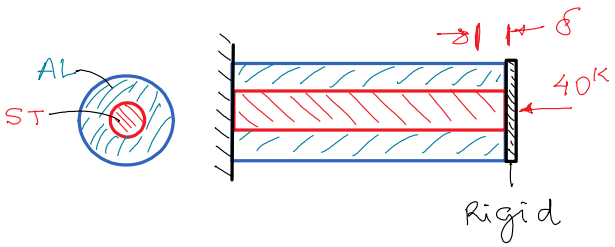
$$\stackrel{+}{\Sigma} F_y = 0$$

$\Rightarrow \quad P =$ $\Rightarrow R_A = P - R_C = P\left(1 - \frac{L_1}{L}\right) \Rightarrow \boxed{R_A =}$

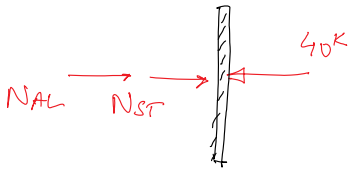
Example

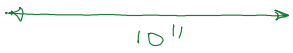
AL : $E_{AL} = 10000$ ksi ; 2.5" dia

ST : $E_{ST} = 29000$ ksi ; 1" dia



Find $\bar{\sigma}_{ST}$, $\bar{\sigma}_{AL}$, δ





$$A_{AL} = \frac{\pi}{4}2.5^2 - \frac{\pi \cdot 1^2}{4}$$

$$A_{ST} = \frac{\pi 1^2}{4}$$

$\sum F_x \to 0 \Rightarrow N_{AL} + N_{ST} = 40^K$ — (1) $\Rightarrow$ Indeterminate

$\Rightarrow$ Compatibility $\Rightarrow \delta_{AL} = \delta_{ST}$

$$\frac{N_{AL}}{A_{AL}E_{AL}} = \frac{N_{ST}}{A_{ST}E_{ST}} \Rightarrow N_{AL} = N_{ST}\cdot\left(\frac{A_{AL}\cdot E_{AL}}{A_{ST}\,E_{ST}}\right)$$ — (2)

From (2) to (1)

$$\underbrace{N_{ST}\left(\frac{(AE)_{AL}}{(AE)_{ST}}\right)}_{N_{AL}} + N_{ST} = 40^K \Rightarrow \boxed{N_{ST} = 14.23^K}\ (C)$$

$$\Rightarrow \boxed{N_{AL} = 25.76^K}\ (C)$$

$$\bar{\sigma}_{ST} = \frac{-N_{ST}}{A_{ST}} = \frac{-14.23^K}{(\pi 1^2)/4\ in^2} = \boxed{-18.12\ ksi}$$

$$\bar{\sigma}_{AL} = \frac{-N_{AL}}{A_{AL}} = \frac{-25.76^K}{\left(\frac{\pi}{4}2.5^2 - \frac{\pi}{4}1^2\right)in^2} = \boxed{-6.25\ ksi}$$

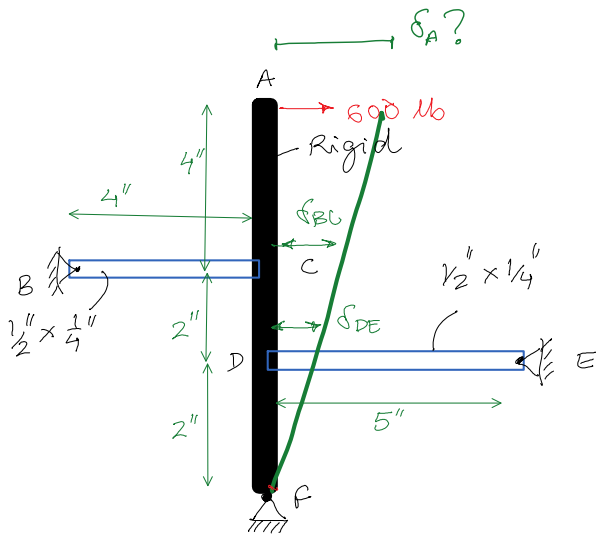
$$\delta = \delta_{ST} = \delta_{AL} = \frac{-N_{ST}\cdot L}{A_{ST}\cdot E_{ST}} = \frac{-14.23^K \times 10\,in}{\frac{\pi}{4}1^2\,in^2 \cdot 29000^K/in^2} = \boxed{-0.00625\ in}$$

Example

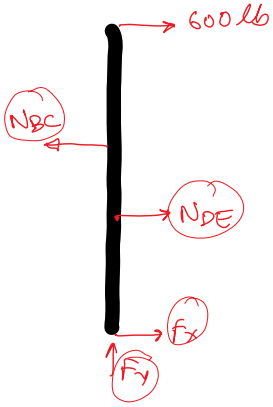
Find: N_{BC}, N_{DE}, δ_A

Given: $E = 29000$ ksi

$(AE)_{BC}$ $(AE)_{DE}$



FBD



⇒ Indeterminate

$\sum F_x = 0 \Rightarrow -N_{BC} + N_{DE} + 600 + F_x = 0$ — (1)

$\sum F_y = 0 \Rightarrow F_y = 0$ (2)

$\sum M_F = 0 \Rightarrow -600 \times 8'' + N_{BC} \times 4'' - N_{DE} \times 2'' = 0$ (3)

Compatibility: $\frac{\delta_{BC}}{2\ 4''} = \frac{\delta_{DE}}{2''} \Rightarrow \frac{N_{BC} L_{BC}}{(AE)_{BC}} = 2\frac{N_{DE} L_{DE}}{(AE)_{DE}}$ — (4)

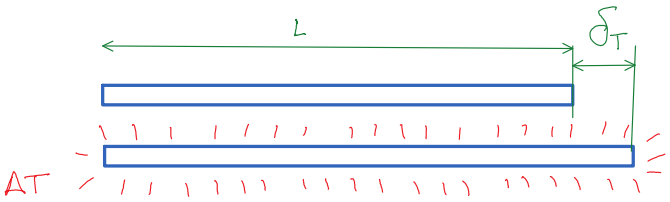
4 eqns + 4 unknowns $\Rightarrow$ $\delta_{BC} = 0.0011$ in $\Rightarrow \delta_A = 2\delta_{BC} = 0.0022$ in.

$\delta_{DE} = 0.00055$ in

$P_{BC} = 1000$ lbs (T)

$P_{DE} = -400$ lbs (C)

Thermal Strains and Stresses



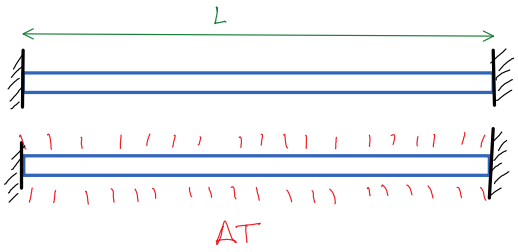
Coeff. of Therm. Exp.

Average Thermal strain $\frac{\delta_T}{L} = \bar{\epsilon}_T = \alpha \Delta T$

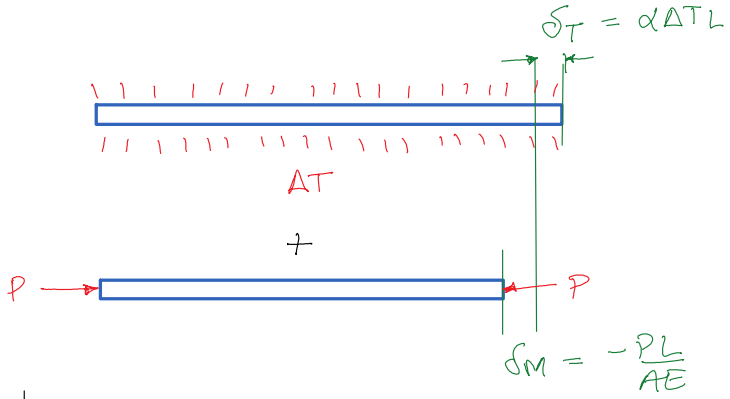
Elongation $\delta_T = \alpha \Delta T L$

$\Rightarrow \boxed{\delta_T = \alpha \Delta T L}$

Statically Indeterminate



=



Total Strain = Mechanical Strain + Thermal Strain

$$\bar{\epsilon}_{TOTAL} = \bar{\epsilon}_{MECH} + \bar{\epsilon}_{THERM}$$

Stress

$\bar{\sigma} = E \bar{\epsilon}_{MECH}$ ONLY

$= E (\bar{\epsilon}_{TOTAL} - \bar{\epsilon}_{THERMAL})$

$\Rightarrow \bar{\sigma} = E (0 - \alpha \Delta T)$

$\Rightarrow \boxed{\text{Force} = -AE \alpha \Delta T}$

or: $\delta_M + \delta_T = 0$

$-\frac{PL}{AE} + \alpha \Delta T L = 0$

$\Rightarrow \boxed{P = \alpha \Delta T AE}$ (C)

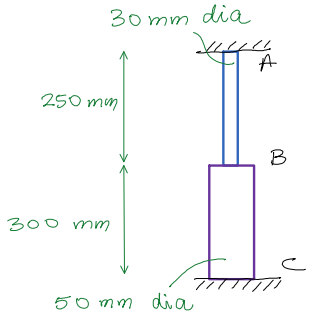
Example

$\Delta T = 50^\circ C$

Find $\bar{\sigma}^{AB}$, $\bar{\sigma}^{BC}$, δ_B

(AB)
$E_{ST} = 200\ GPa$
$\alpha_{ST} = 11.7 \times 10^{-6}/^\circ C$

(BC)
$E_{BR} = 105\ GPa$
$\alpha_{BR} = 20.19 \times 10^{-6}/^\circ C$

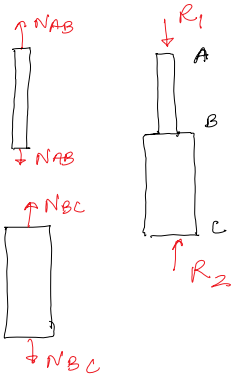


$A_{AB} = \frac{\pi (30)^2}{4}\ mm^2$

$A_{BC} = \frac{\pi (50)^2}{4}\ mm^2$

Compatibility: $\delta_{AB} = -\delta_{BC}$

$$\frac{(-R)\, L_{AB}}{A_{AB} E_{AB}} + \alpha_{AB} \Delta T \cdot L_{AB} = \frac{-(-R) L_{BC}}{A_{BC} E_{BC}} + \alpha_{BC} \cdot \Delta T\, L_{BC}$$



$\sum F_y = 0 \Rightarrow R_1 = R_2$

$R_1 = R_2 = R$

From this equation:

$\Rightarrow R = 142623\ N$ (C)

$$\delta_{AB} = \frac{\ominus R \cdot L_{AB}}{A_{AB} E_{AB}} + \alpha_{AB} \Delta T\, L_{AB} = \boxed{-0.106\ mm}$$

$$\delta_{BC} = \boxed{0.106\ mm}$$

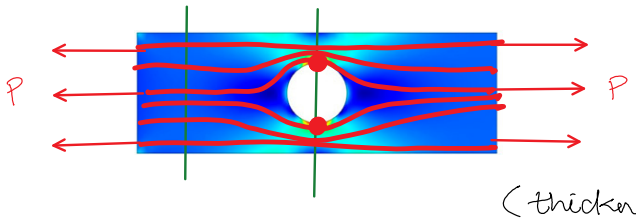
$$\bar{\sigma}_{AB} = \frac{\ominus R}{A_{AB}} = \frac{-142623}{A_{AB}} = -202\ MPa\ (C)$$

$$\bar{\sigma}_{BC} = \frac{\ominus R}{A_{BC}} = \frac{-142623}{A_{BC}} = -72.6\ MPa\ (C)$$

Stress Concentrations

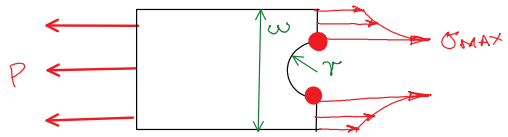
Stress is higher in members that have discontinuities or changes in cross-section.

Example: Plate with a hole in tension:

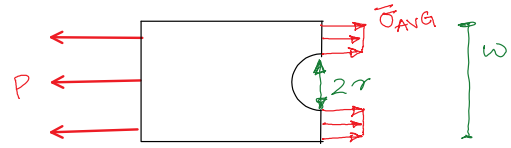




(thickness : t)

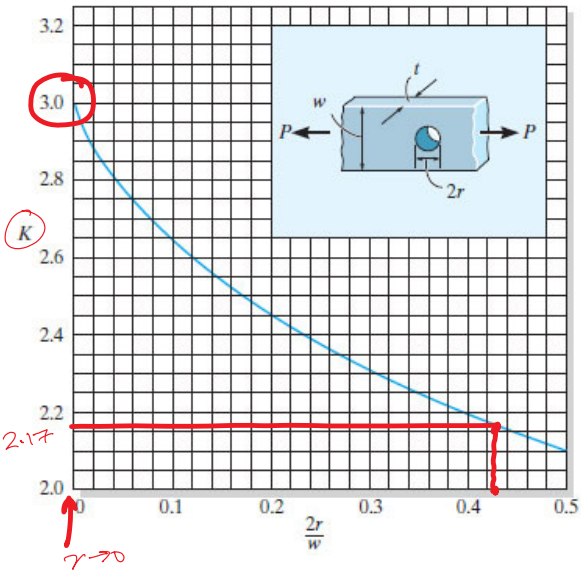


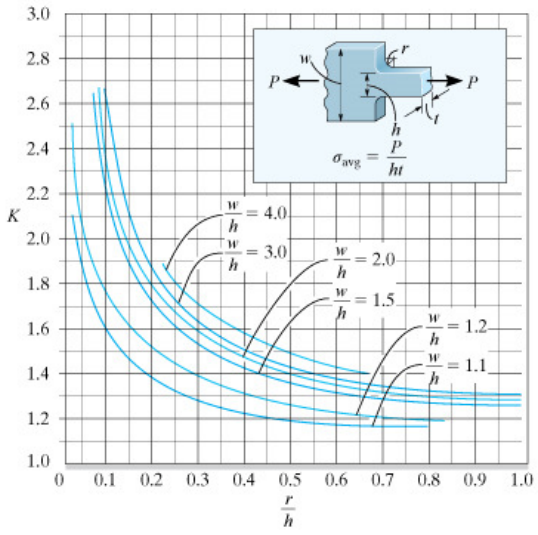
$$\sigma_{MAX} = K\,\bar{\sigma}_{AVG}$$



$$\bar{\sigma}_{AVG} = \frac{P}{A_{NET}} = \frac{P}{(w-2r)\,t}$$

K: Stress concentration factor

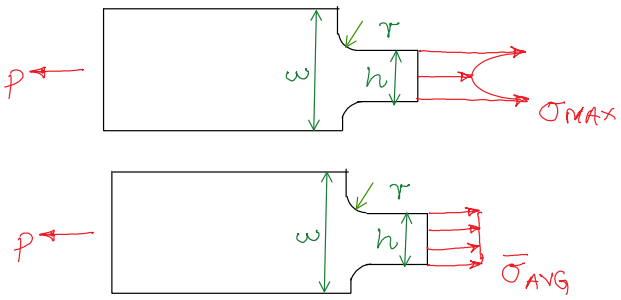




Change in cross-section:

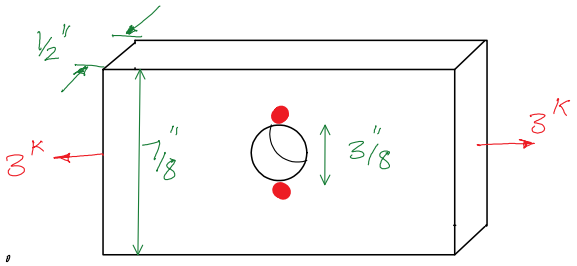
$$\sigma_{MAX} = K\,\bar{\sigma}_{AVG}$$

$$\bar{\sigma}_{AVG} = \frac{P}{ht}$$



Example

Find σ_{MAX}



$$\sigma_{MAX} = K\,\bar{\sigma}_{AVG}$$

$$\bar{\sigma}_{AVG} = \frac{P}{A_{NET}} = \frac{3K}{(1/2")(7/8" - 3/8")} = 12\,ksi$$

$$\frac{2r}{w} = \frac{3/8}{7/8} = \frac{3}{7} \approx 0.43$$

$$\sigma_{MAX} = 2.17\,\bar{\sigma}_{AVG} = \boxed{26\ ksi}$$

Example

σ_{MAX} ?

(A/B/C/D)

$r_f = 0.1$ in.

