Informal Notes on Inertial Navigation

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AAE 565

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Navigation

Object: Determine where the aircraft (vehicle) is located and perhaps its velocity and acceleration as well.

Two main Types of Navigators used in aerospace today

Inertial Navigation

Based on accelerometers to measure acceleration and then integration of acceleration to get velocity and integration again to get position.

Satellite Based Navigation e.g. G.P.S.

Global Positioning System

We will first study inertial navigation.

Inertial Navigation

General Idea: Use linear accelerometers to measure acceleration. Integrate once to get velocity. Integrate again to get position

Concept 1: The linear accelerometer has its own mathematical model that often must be incorporated in a system analysis

From Homework 8 we learned

$$\underset{ft/sec^2}{\ddot{x} - g_x} \longrightarrow \boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \xrightarrow{a(t)} ft/sec^2$$

$\boxed{\omega_n = \sqrt{k/m}}$ $\quad 2\zeta\omega_n = \frac{c}{m} \quad \cancel{2}\zeta\omega_n = \frac{c}{2m} \quad \boxed{\zeta = \frac{c}{2m\omega_n}}$

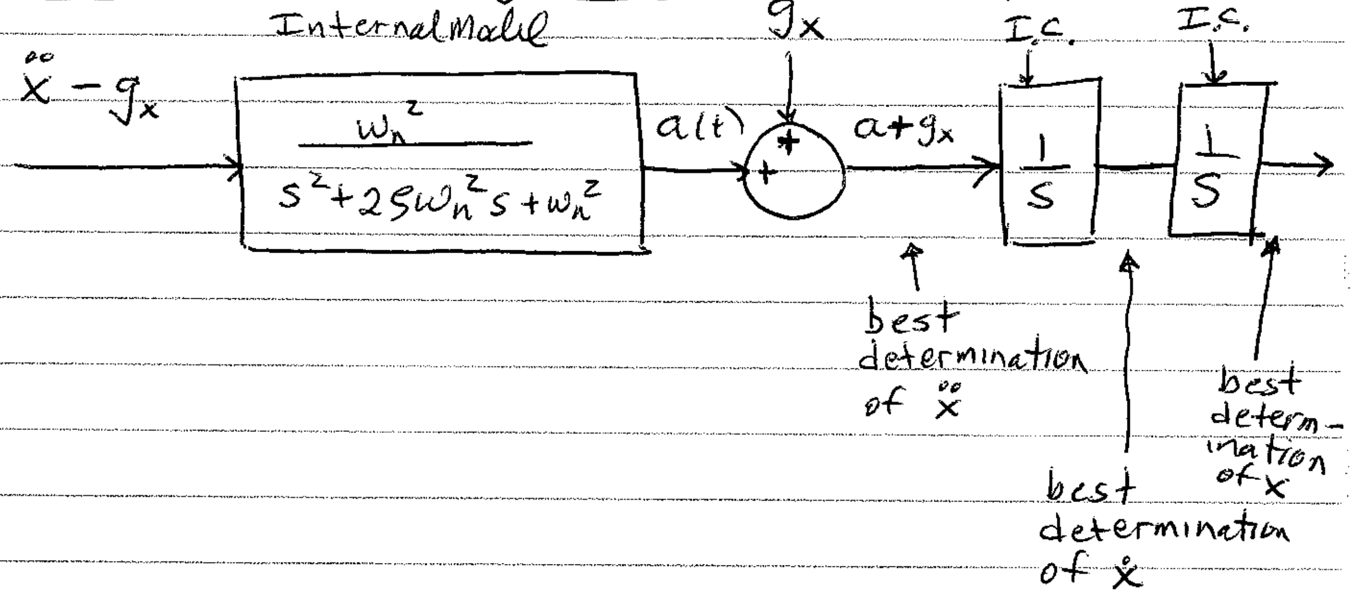
$\boxed{\zeta = \frac{c}{2m}\sqrt{\frac{m}{k}}}$ where g_x is the component of gravity in the x-direction.

In this example the accelerometer has a complex pair of poles.

Concept 2: We must add in g_x.

Concept 3: To determine velocity and position from the accelerometer signal we must perform 2 integrations. This adds two more poles to the system and two important IC's.

Accelerometer used to find Position and velocity (1DOF)



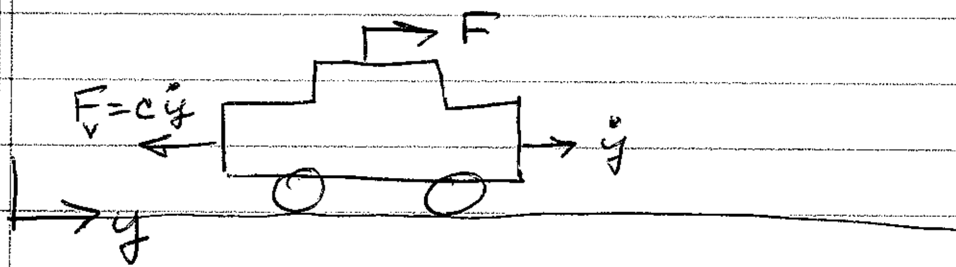
Observations

1. The internal model of of the accelerometer introduces errors in determining $\ddot{x}$
2. Any error in our determination of g_x will introduce errors in determining $\ddot{x}$
3. Any error in our determination of the initial condition on $\dot{x}$ and on x will introduce errors in determination of $\ddot{x}$.
4. To determine x with an accelerometer introduces 4 poles to our math model. Even if we can ignore the internal model of the accelerometer we still have two new poles (at $s=0$). These last two poles are often called OBSERVER poles.

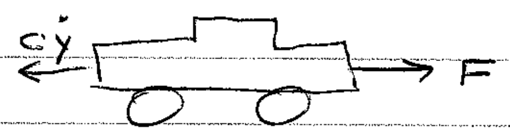
Example 1 1DOF Horizontal

Given

Cart in Viscous Fluid controlled by thruster. Use an accelerometer to position the cart at a given location



FBD

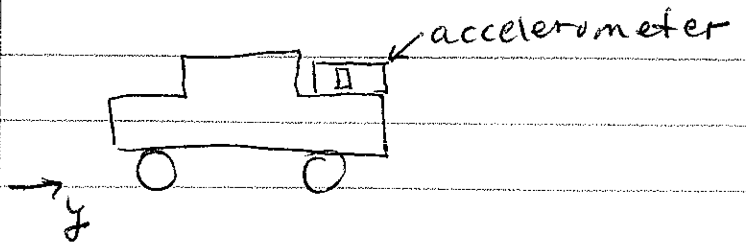


Eoms $\quad m\ddot{y} = F - c\dot{y} \quad \Rightarrow \quad \ddot{y} = \frac{F}{m} - \frac{c\dot{y}}{m}$

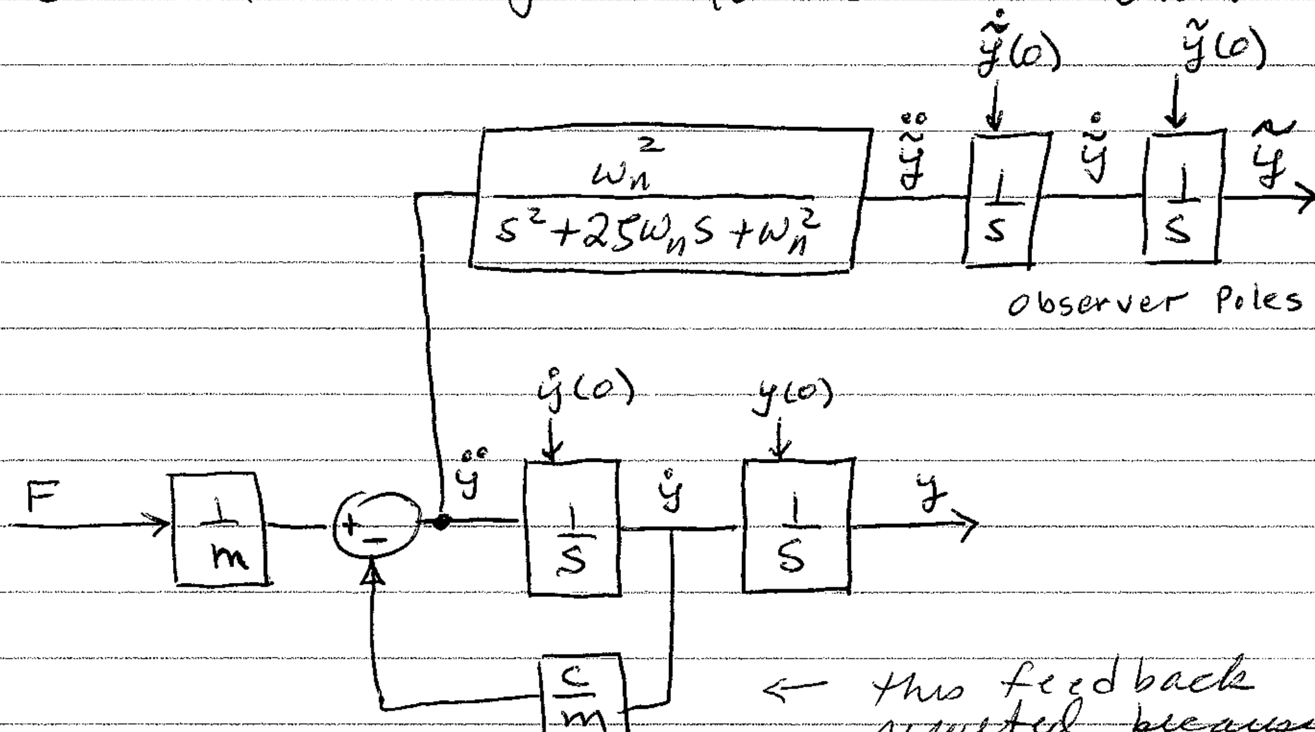
$$\ddot{y} + \frac{c}{m}\dot{y} = \frac{F}{m}$$

$$\frac{y(s)}{F(s)} = \frac{1/m}{s^2 + \frac{c}{m}s}$$

An Accelerometer is assumed to be perfectly aligned in the y-direction. There is no component of gravity in this direction



Our block diagram looks like this.

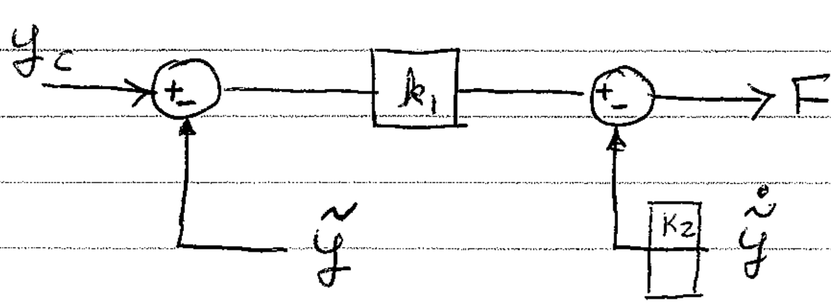


← this feedback resulted because the specific force was a function of $\dot{y}$

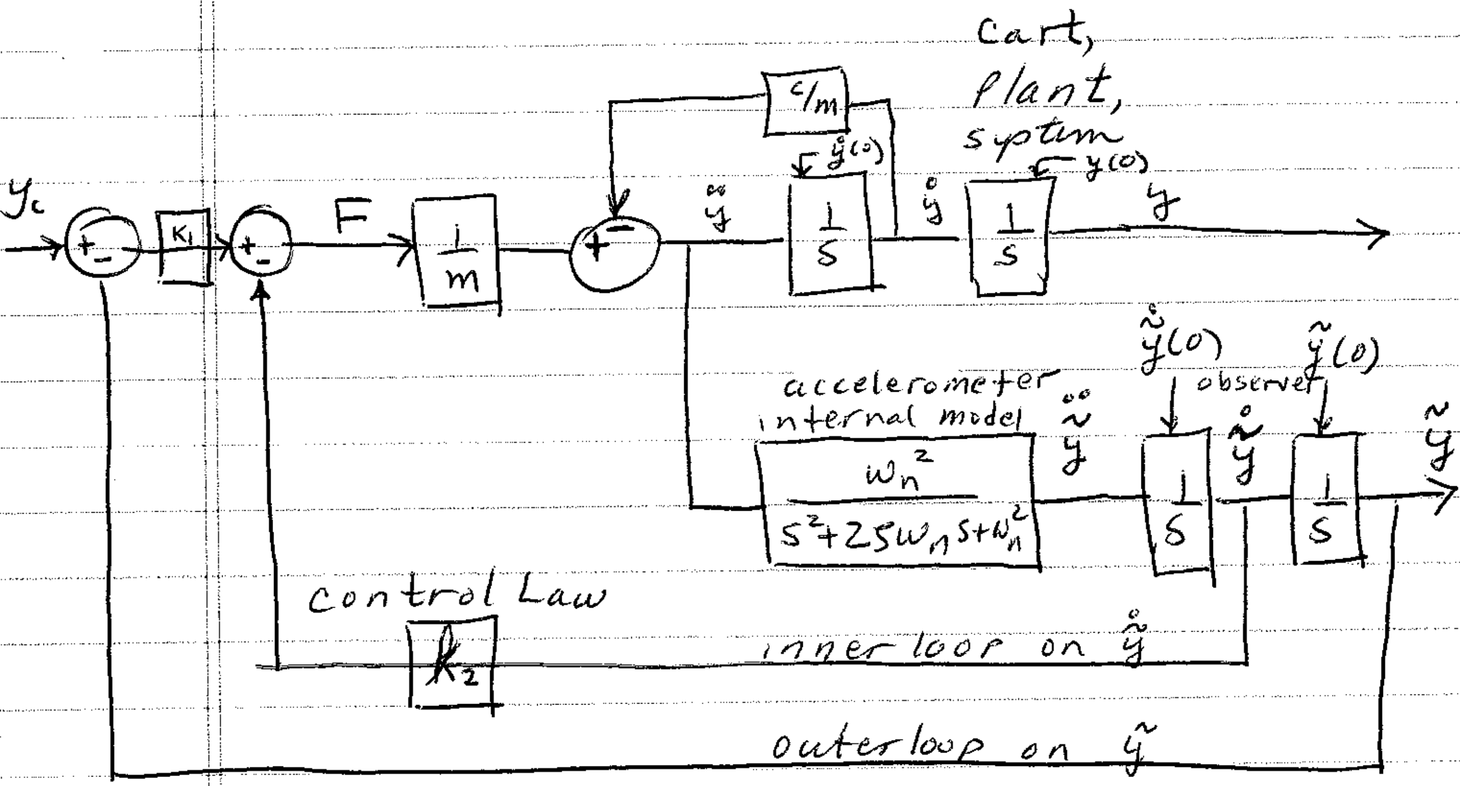
Notice that we have 6 poles in this system.

Now we want to use the accelerometer output to control $y(t)$. A typical control law would look like this

$$F = k_1(y_c - \tilde{y}) - k_2\dot{\tilde{y}}$$



Our control law must be based on $\tilde{y}$ and $\dot{\tilde{y}}$ because these are our measurements.



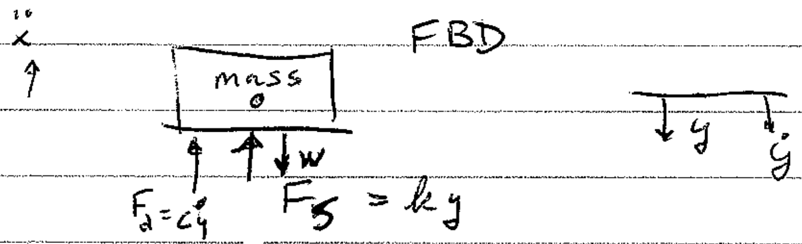
Observations

1. The loops are closed around the observer, not around the plant.
2. open loop Poles of the observer $(s=0,0)$ are not the same as the open loop poles of the plant $(s=0, -\frac{c}{m})$
3. Errors in the initial conditions on the observer will create errors in the positioning of the cart.
4. Important errors are

$$e(t) = y(t) - \tilde{y}(t)$$
$$\dot{e}(t) = \dot{y}(t) - \dot{\tilde{y}}(t)$$

These errors need to be studied for different parameters $(\omega_n, \zeta, \tilde{y}(0), \dot{\tilde{y}}(0))$.

Accelerometer Review



The acceleration of the mass in vertical direction will be

$$\ddot{x} - \ddot{y}$$

$$m(\ddot{x} - \ddot{y}) = +ky + c\dot{y} - \underset{mg}{W}$$

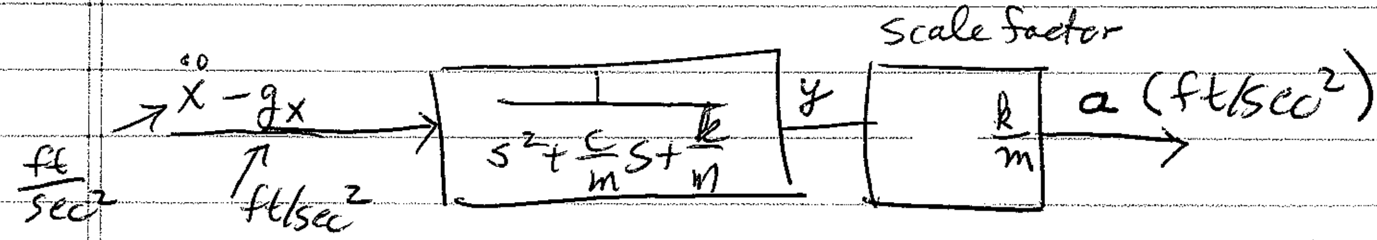
$$\ddot{x} + g = \ddot{y} + \underbrace{\frac{c}{m}}_{2\zeta\omega_n}\dot{y} + \underbrace{\frac{k}{m}}_{\omega_n^2} y$$

The component of gravity in the sensitive direction is g_x

$$g_x = -g$$

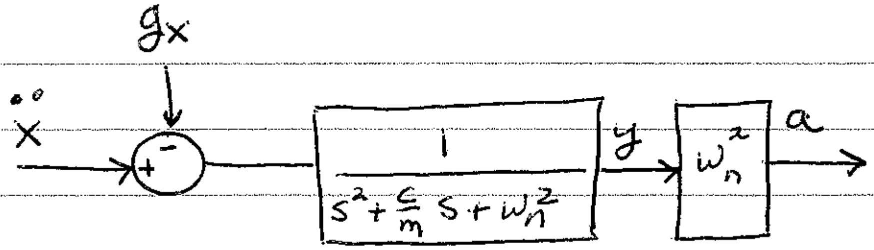
$$\ddot{x} - g_x = \ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y$$

$$\frac{y(s)}{\ddot{x} - g_x} = \frac{1}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

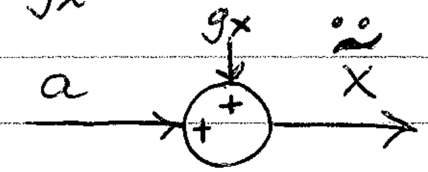


in static case $\quad ky = W = mg \Rightarrow \qquad g = \frac{k}{m}y$

Accelerometer Model (x is sensitive direction)



where y is the physical displacement of the mass and where a is the scaled output of the accelerometer. To use a to find $\ddot{x}$ we must add g_x back in.



Check units

$y = [ft]$

$F = -ky = \left(\frac{lbf}{ft}\right) ft$

$k = \left[\frac{lbf}{ft}\right]$

$$\frac{ky}{m} = \frac{lbf}{ft}\frac{ft}{slugs} = \frac{lbf}{slugs}$$

Recall $F = ma \Rightarrow \frac{F}{m} = a$ in $\frac{ft}{sec^2} = \frac{lbf}{slugs}$

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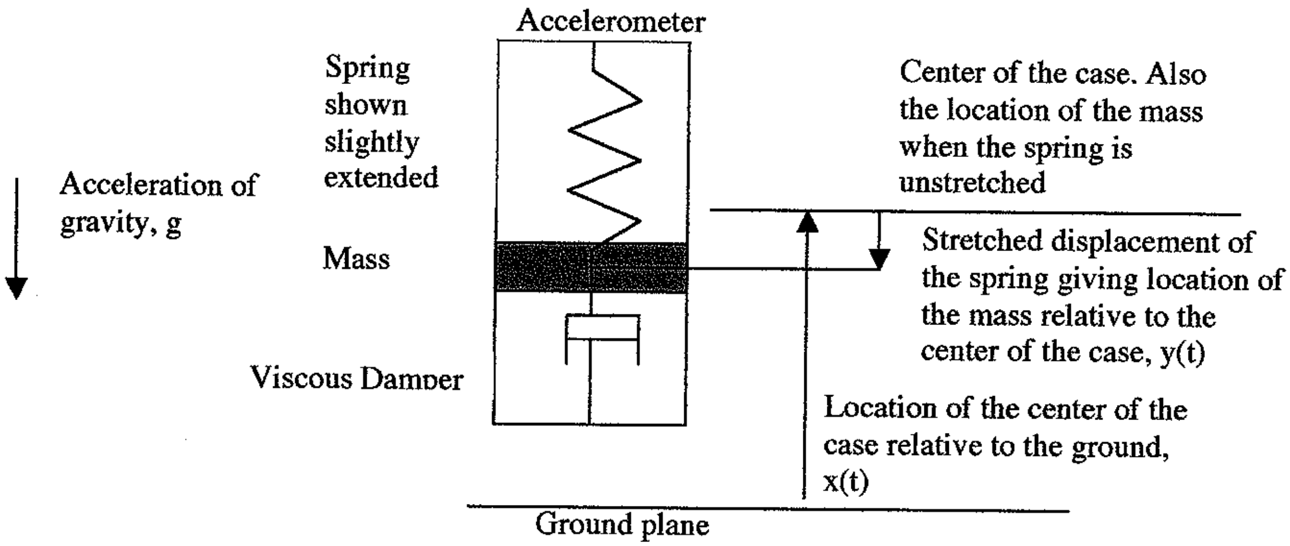
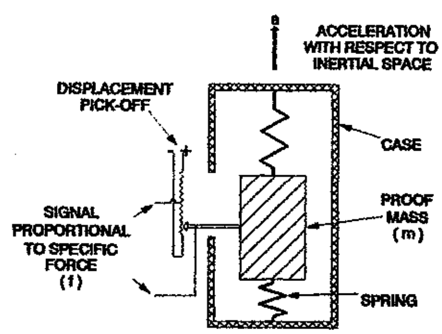
AAE565 Homework 8 Due Friday 3/13/09



An accelerometer is a transducer that consists of a case containing a mass (with mass **m**) connected to a spring with stiffness **k** and to a viscous damper with damping constant **c**. The mass is free to move up and down without friction with the case and is acted upon by the spring and the damper. Define **x(t)** to be the position of the center of the case of the accelerometer relative to the ground. The center of the case also corresponds to the point where the spring is unstretched. **y(t)** is the amount the spring is stretched and is also the location of the mass relative to the center of the case.

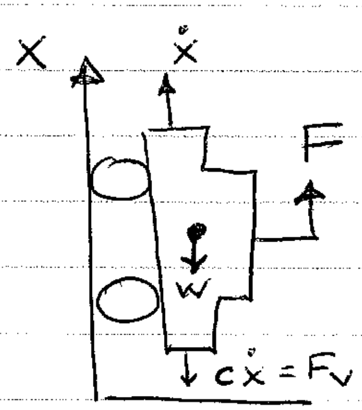
A. Find the differential equation that relates the accelerometer input $\ddot{\mathbf{x}}(\mathbf{t})$ to the accelerometer output **y(t)**. Note that the input is assumed to be the second derivative of **x(t)** with respect to time, **t**. Assume that the ground is an inertial frame and that the motion of the case is only in the x-direction (up and down).
B. What can you say about the ability of the accelerometer to separate (i.e., distinguish between) the influence of gravity (**g**) from the influence of input $\ddot{\mathbf{x}}(\mathbf{t})$ upon the accelerometer output **y(t)**?

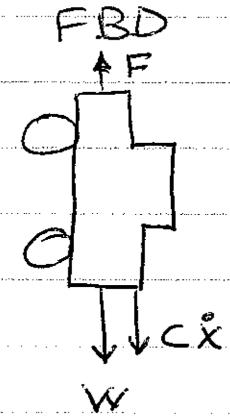


A simple accelerometer

Example 2 1DOF Vertical

Given: Cart in a viscous fluid on a vertical track so it functions as an elevator. The cart is driven by a force F.





EOM $\quad m\ddot{x} = F - c\dot{x} - W \quad \Rightarrow \quad \ddot{x} = \frac{F}{m} - \frac{c}{m}\dot{x} - g$

$$W = mg$$

$$g \cong g_o \left(\frac{R_o}{R_o + h}\right)^2 = g_o\left(\frac{R_o}{R_o + x}\right)^2$$

where $h = x$ in this problem

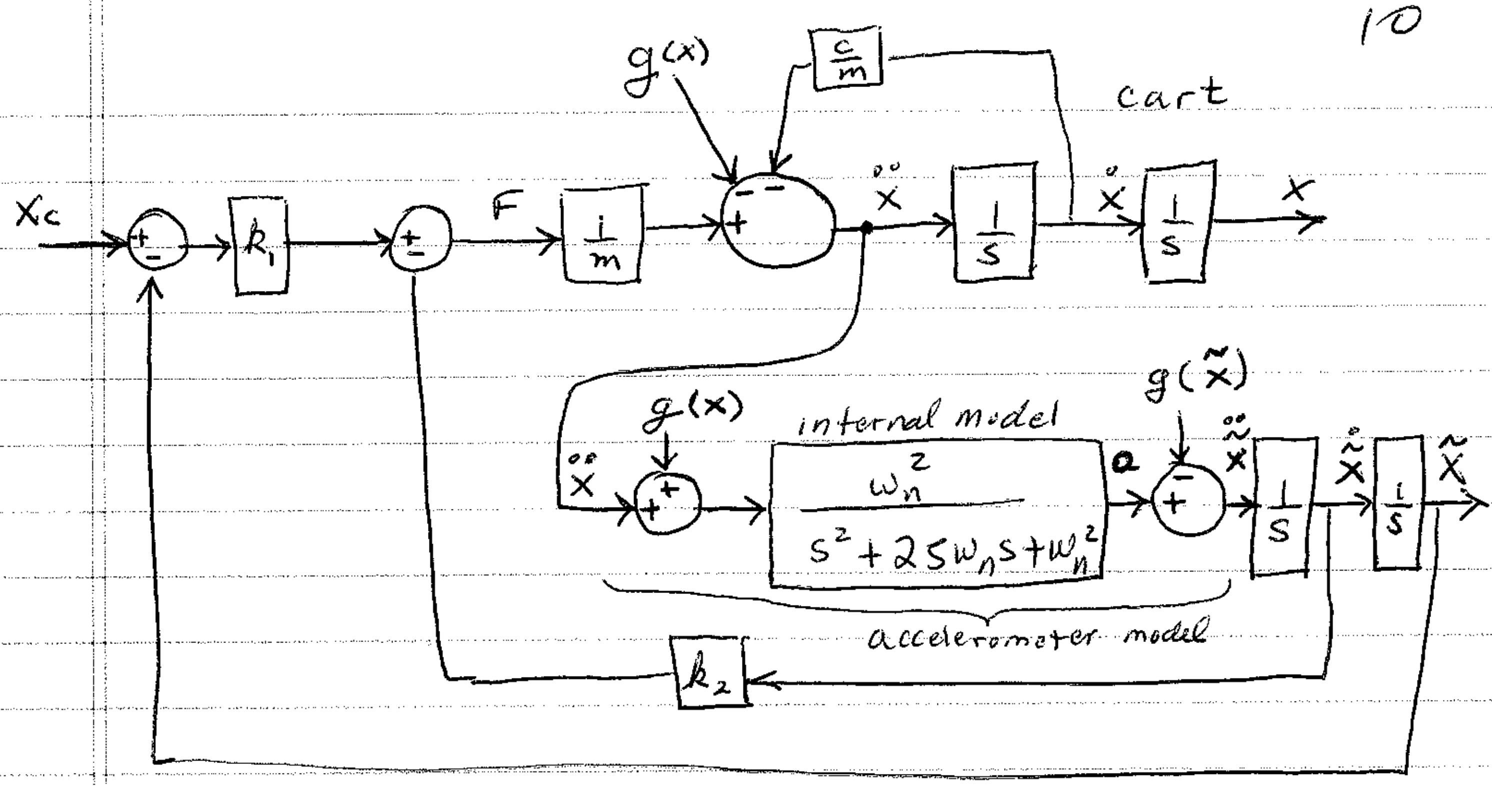
Linearize this around $x = 0$ gives

$$g \approx g_o - \frac{2g_o}{R_o}x$$

$g_o = 32.17\,ft/sec^2$
$R_o = 20{,}925{,}650\,ft$

An accelerometer is assumed to be perfectly aligned in the x-direction. Gravity g is aligned in this direction.

Again we want to use the accelerometer to control $x(t)$. A typical control law might be $\quad F = k_1(x_c - \tilde{x}) - k_2\tilde{\dot{x}}$



Observations

1. $g(x)$ is added into the input to the accelerometer
2. We must subtract this back out. But we don't know x. We only know $\tilde{x}$. So we subtract out $g(\tilde{x})$. This is a significant source of error.

Example Suppose $x = 10$, $\tilde{x} = 11$

Suppose $g(x) = 32$ ft/sec^2 $g(\tilde{x}) = 31$ ft/sec

Ignoring the internal transfer function

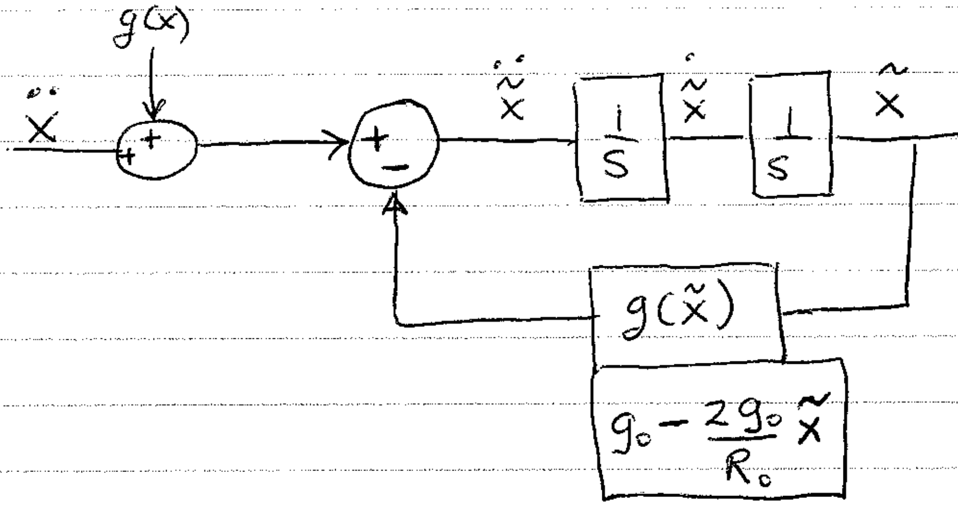
$$\ddot{\tilde{x}} = \ddot{x} + 32 - 31 = \ddot{x} + 1 \text{ ft/sec}^2$$

$\ddot{\tilde{x}}$ will be too large. As a result $\dot{\tilde{x}}$ will be too large and $\tilde{x}$ will get even larger than it already is

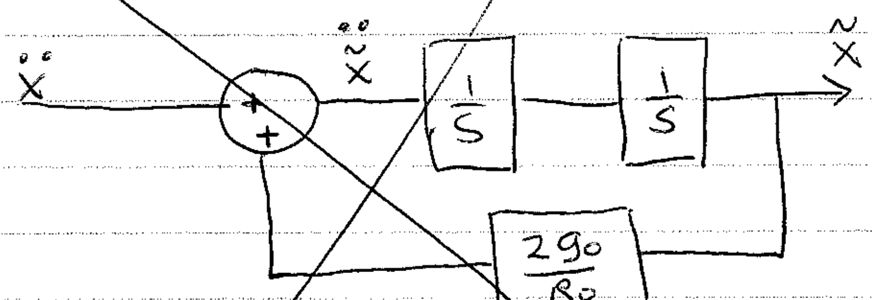
$\tilde{x} = 11$ was too big and it will get even bigger.

3. Because we subtract off $g(\tilde{x})$ rather than $g(x)$ there will be an altitude mode in the observer that is UNSTABLE!
The observer will <u>always</u> diverge

This can be further explored by explicitly showing the details in the block diagram. Ignoring the internal model of the accelerometer

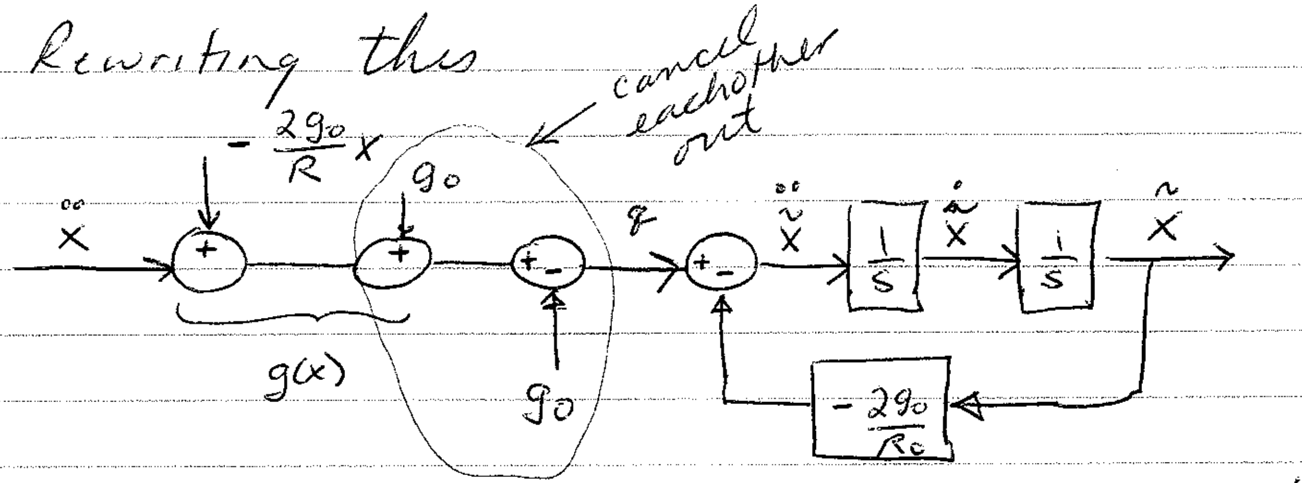


Suppose $x = 0$ so $g(x) = g_0$



This is an UNSTABLE second order system

Rewriting this



Unstable Loop!

$$\frac{\tilde{x}(s)}{q(s)} = \frac{\frac{1}{s^2}}{1 - \frac{2g_0}{R_0}\frac{1}{s^2}} = \frac{1}{s^2 - \frac{2g_0}{R_0}}$$

The poles of this unstable system are at

$$s_1 = -\sqrt{\frac{2g_0}{R_0}} \quad \text{stable} \quad = -.0017535/\text{sec}$$

$$s_2 = +\sqrt{\frac{2g_0}{R_0}} \quad \text{UNSTABLE!}$$

$$s_2 = +\sqrt{\frac{2 \cdot 32.17}{20,925\,650}} = \sqrt{3.0747 \times 10^{-6}} \text{ per sec} = +.0018/\text{sec} = .0017535/\text{s}$$

The Time to double amplitude t_2, is given by

$$e^{0.0018 t_2} = 2$$

$$\ln(e^{.0018 t_2}) = \ln 2$$

$$0.0018 t_2 = \ln 2$$

$$t_2 = \ln 2 / .0018 = 395 \text{ sec} = 6.59 \text{ min}$$

Summary

When using an accelerometer to reconstruct vertical position, the observer for vertical position has an unstable pole with a time to double amplitude of 395 seconds.

This makes the use of accelerometers for vertical positioning useful only for short periods of time
$t \ll 395$ seconds.

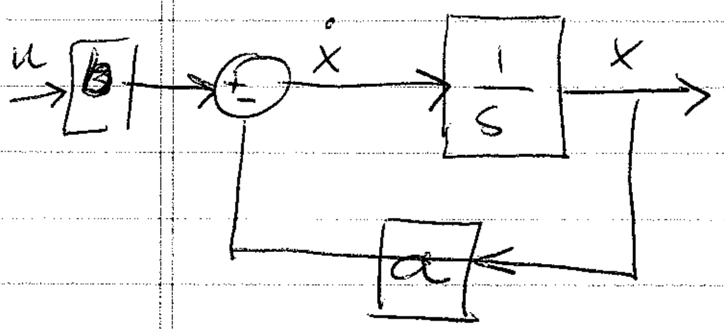
Because of this sophisticated inertial navigation systems always have altitude aiding to stabilize this unstable mode.

Loop Stability

First order system

$\dot{x} = -|a|x + bu$

pole $s = -|a|$ is stable

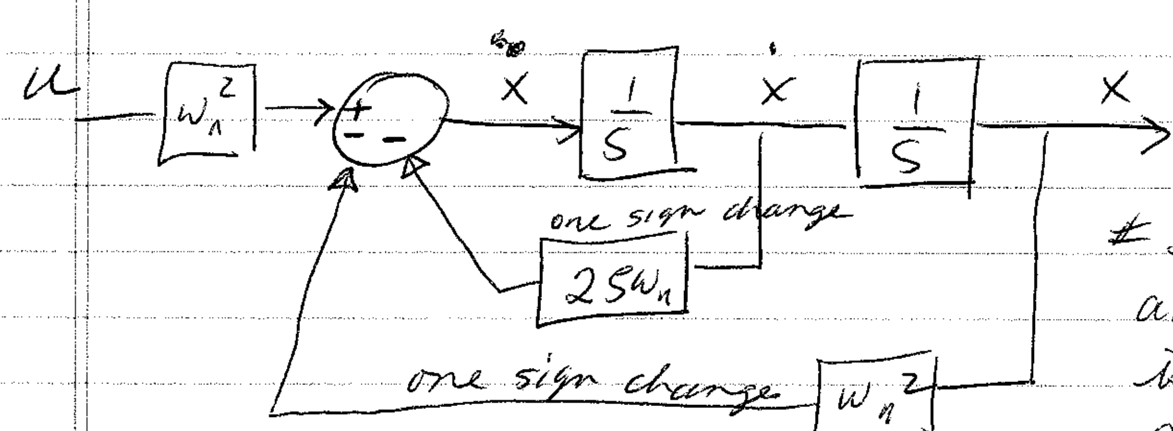


sign changes around the loop is 1 (ODD)

Second Order System

$+\ddot{x} + \underbrace{2\zeta\omega_n}_{+\text{ for stability}}\dot{x} + \underbrace{\omega_n^2}_{+\text{ for stability}}x = \omega_n^2 u$

$\ddot{x} = -2\zeta\omega_n\dot{x} - \omega_n^2 x + \omega_n^2 u$



sign changes around loops is 1 ODD

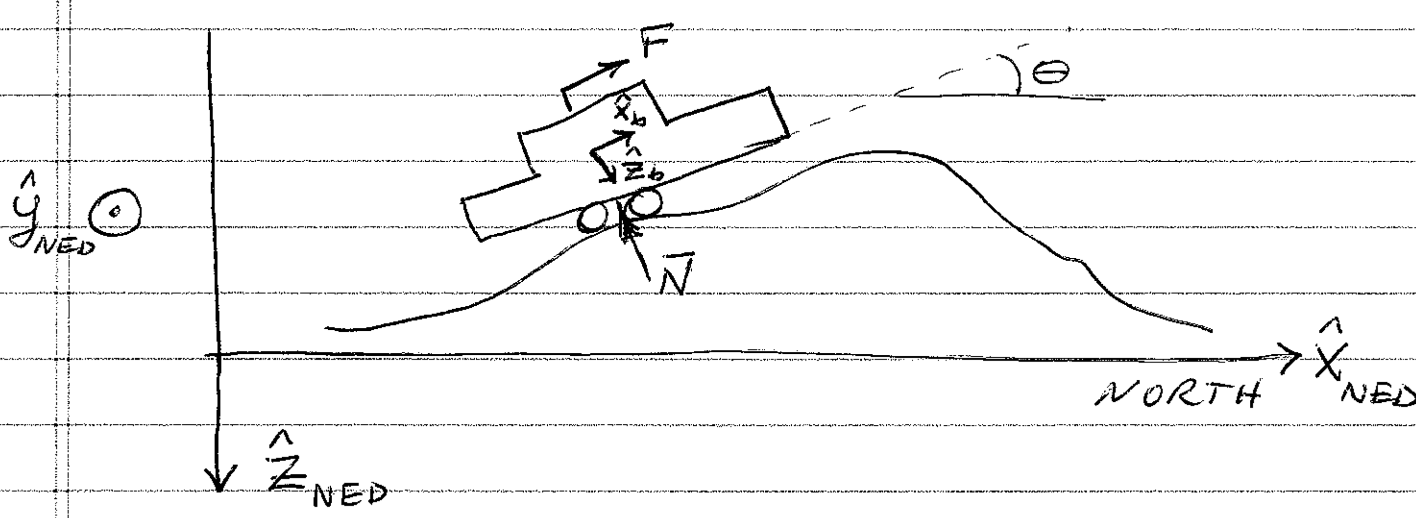
Rule: For first & 2^{nd} order systems the number of sign changes around a loop must be odd for stability.

Guidance For higher order systems the connection is less definite but a good believability check.

Example #3

Cart on an uneven surface traveling north and free to rotate about angle θ.

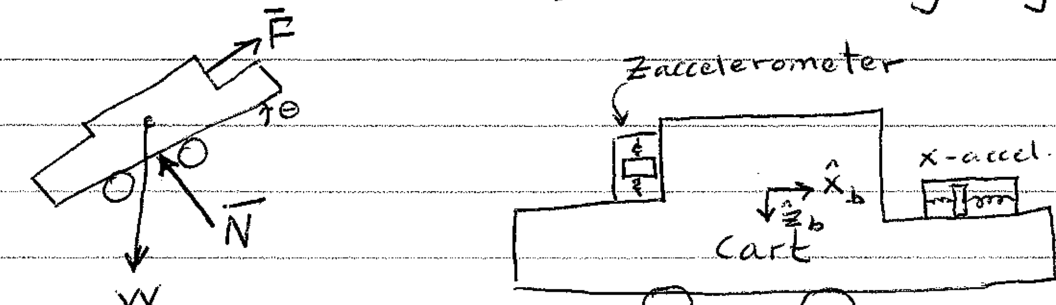
Two accelerometers are rigidly mounted to the body of the cart. A rate gyro is mounted on the cart to measure $\dot{\theta}$.



A control force $\bar{F}$ is used to move the cart. Normal Force $\bar{N}$ is perpendicular to the ground. Weight is in the $\hat{z}_{NED}$ direction.

$$\bar{F} = F\,\hat{x}_b \qquad \bar{N} = -N\,\hat{z}_b \qquad \bar{W} = m\bar{g} = mg\,\hat{z}_{NED}$$

FBD



Note: Two body fixed accelerometers mounted mutually perpendicular to measure in x & z directions.

Simplified Accelerometer Model

Ignore the internal model.

In vector form equations of motion for a vehicle are written as

$$\bar{F} + \bar{W} = m\,{}^{i}\frac{d\bar{V}}{dt}$$

Where $\bar{F}$ is the resultant of all forces except force due to gravity.

$$\frac{\bar{F}}{m} + \bar{g} = {}^{i}\frac{d\bar{V}}{dt}$$

Recall that accelerometers measure inertial acceleration minus the acceleration of gravity in the sensitive direction of the instrument.

In vector form this is equal to

$${}^{i}\frac{d\bar{V}}{dt} - \bar{g} = \frac{\bar{F}}{m} = \bar{f}$$

where $\bar{f}$ are also called specific forces. Three mutually perpendicular body fixed accelerometers will measure 3 body axis components of the specific force $\bar{f}$.

Assume that the NED frame is an inertial frame.

Inertial Velocity of Cart $\bar{V} = V_x \hat{x}_{NED} + V_z \hat{z}_{NED}$

Inertial Acceleration $\frac{{}^{NED}d\bar{V}}{dt} = \dot{V}_x \hat{x}_{NED} + \dot{V}_z \hat{z}_{NED}$

$$\Sigma \text{Forces} = \bar{F} + \bar{N} + \bar{W}$$

$$\frac{\Sigma \text{Forces}}{m} = \underbrace{\frac{\bar{F}}{m} + \frac{\bar{N}}{m}}_{\text{specific forces } \bar{f}} + \bar{g} \qquad = \bar{f} + \bar{g}$$

(never include gravity)

Specific Forces $\bar{f}$

x-accelerometer measures this

z-accelerometer measures this

$$\bar{f} = f_{x_b} \hat{x}_b + f_{z_b} \hat{z}_b = f_x \hat{x}_{NED} + f_z \hat{z}_{NED}$$

where
$$f_x = f_{x_b} \cos\theta + f_{z_b} \sin\theta$$
$$f_z = -f_{x_b} \sin\theta + f_{z_b} \cos\theta$$

For Cart $\quad \bar{F} + \bar{N} + \bar{W} = m \frac{{}^{NED}d\bar{V}}{dt}$

$$\frac{\bar{F}}{m} + \frac{\bar{N}}{m} + \bar{g} = \frac{{}^{NED}d\bar{V}}{dt}$$

Body fixed Accelerometers measure body axis components of

$$\frac{{}^{NED}d\bar{V}}{dt} - \bar{g} = \frac{\bar{F}}{m} + \frac{\bar{N}}{m} = \bar{f} = f_{x_b}\hat{x}_b + f_{z_b}\hat{z}_b$$

For the body fixed accelerometers but resolving them into NED

$$\dot{V}_x \hat{x}_{NED} + \dot{V}_z \hat{z}_{NED} - g \hat{z}_{NED} = f_x \hat{x}_{NED} + f_z \hat{z}_{NED}$$

in $\hat{x}_{NED}$ direction

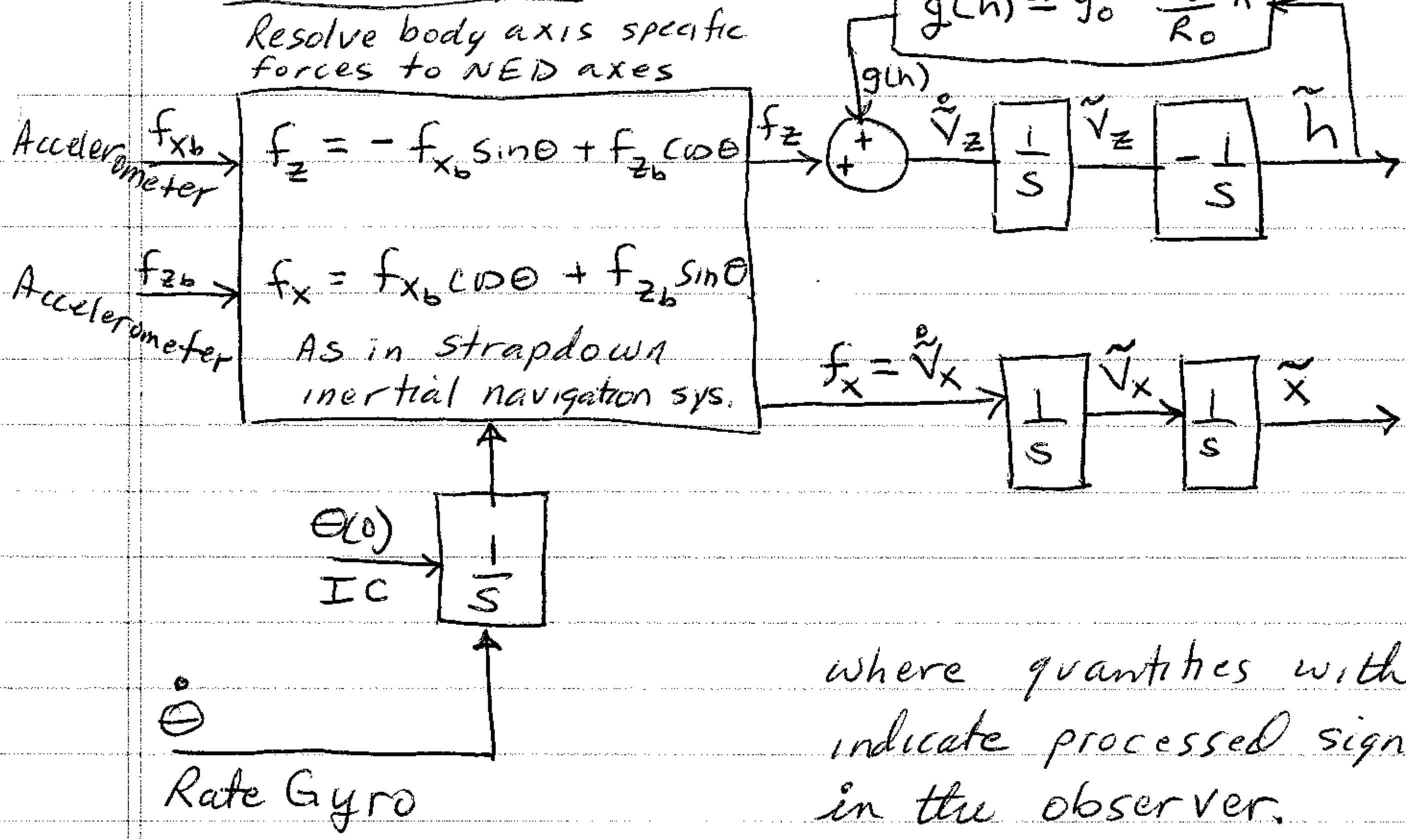
$$\dot{V}_x = f_x$$

In $\hat{z}_{NED}$ direction

$$\dot{V}_z - g = f_z$$

$$\Rightarrow \dot{V}_z = f_z + g$$

Processing of Accelerometers and Rate Gyros



where quantities with ~ indicate processed signal in the observer.

Observations

1. This system also has the unstable altitude mode.
2. Each accelerometer contributes two poles.
3. The rate gyro integration to get angle θ contributes 1 more pole.
4. This is similar to a STRAPDOWN inertial Navigator.

In summary, there are 3 poles at $s = 0$ and the two altitude poles are at $s = \pm\sqrt{\frac{2g_0}{R_0}}$

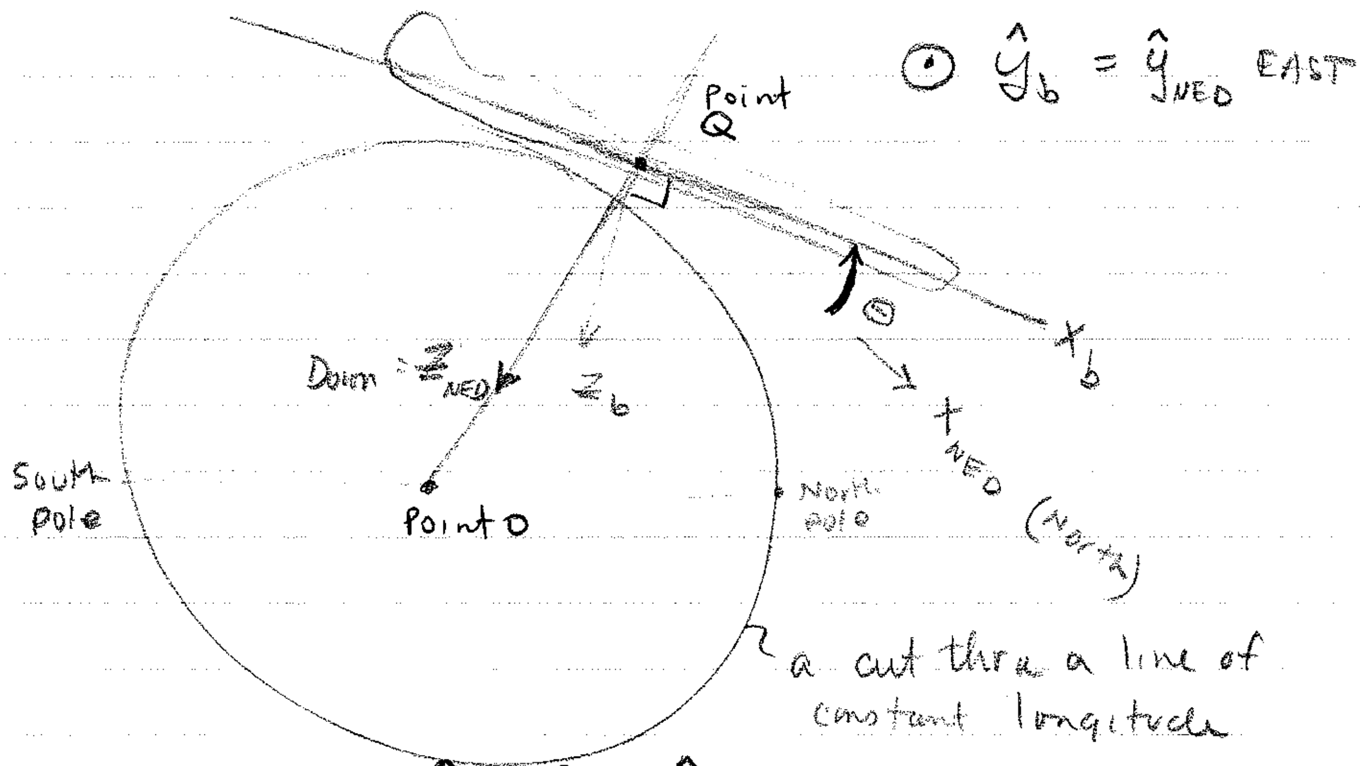
5. We do not need to know the details of the EOM of the vehicle. We only need to build the EOM's of the observer. This is quite different.

Strapdown Inertial Navigation:

Notes by Dominick Andrisani, II
March 2002

Part I Derivation of Equations of Motion of an aircraft in planar (2-dimensional) motion

① Simulation of 2-Dimensional Planar Motion



Body Frame $\hat{x}_b$ $\hat{y}_b$ $\hat{z}_b$ origin is at point Q

NED Frame $\hat{x}_{NED}$ (North) $\hat{y}_{NED}$ (East) $\hat{z}_{NED}$ (Down) origin is at Pt Q

Aircraft is heading due North. Motion is in the x-z plane.

Assume the inertial velocity of point Q is given by

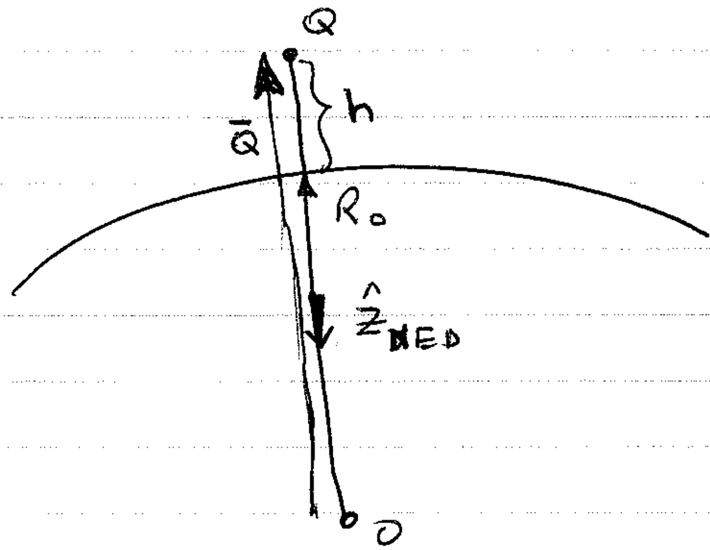
$$\frac{{}^{i}d\bar{Q}}{dt} = v_x \hat{x}_{NED} + v_z \hat{z}_{NED}$$

Where $\bar{Q}$ is a vector from O to Q.

$$^{NED}\bar{\omega}^{b} = \dot{\theta}\,\hat{y}_b = \dot{\theta}\,\hat{y}_{NED}$$

(2)

Assume that the aircraft is at a certain altitude above the surface of the earth



$$\bar{Q} = -(R_o + h)\,\hat{z}_{NED}$$

$$\frac{{}^{i}d\bar{Q}}{dt} = \frac{{}^{NED}d\bar{Q}}{dt} + {}^{i}\omega^{NED} \times \bar{Q}$$

$${}^{i}\bar{\omega}^{NED} = \omega\,\hat{y}_{NED} \qquad \text{Find } \omega$$

$$v_x\,\hat{x}_{NED} + v_z\,\hat{z}_{NED} = -\dot{h}\,\hat{z}_{NED} + \omega\,\hat{y}_{NED} \times \left(-(R_o+h)\,\hat{z}_{NED}\right)$$

$$= -\dot{h}\,z_{NED} - \hat{x}_{NED}\,\omega\,(R_o+h)$$

Equating components gives

$$v_x = -\omega(R_o+h) \qquad v_z = -\dot{h}$$

$$\omega = -\frac{v_x}{R_o+h}$$

$$\boxed{{}^{i}\bar{\omega}^{NED} = -\frac{v_x}{R_o+h}\,\hat{y}_{NED}} \qquad \boxed{v_z = -\dot{h}}$$

(3) Note also that

~~Rate gyros mounted fixed wrt the body will measure~~ $\cancel{{}^{i}\bar{\omega}^{b} = \omega_y \hat{y}_{NED}}$

$${}^{i}\bar{\omega}^{b} = {}^{i}\bar{\omega}^{NED} + {}^{NED}\bar{\omega}^{b} = \underbrace{\left(\dot{\theta} - \frac{v_x^e}{R_0+h}\right)}_{\omega_y} \hat{y}_{NED}$$

$$\omega_y = \dot{\theta} - \frac{v_x^e}{R_0+h}$$

$$\boxed{\dot{\theta} = \omega_y + \frac{v_x^e}{R_0+h}}$$

~~This will be one of the equations used by the inertial navigator. It relates the sensor signal~~ $\cancel{\omega_x}$ ~~to states of the aircraft~~ $\cancel{(\theta, v_x^e, h)}$.

We will now derive the equations of motion of the aircraft using Newtons Second Law. We start this by finding the acceleration of point Q (assumed to be the center of mass of the aircraft) with respect to an inertial observer

$$\frac{{}^{i}d}{dt}\frac{{}^{i}d\bar{Q}}{dt} = \frac{{}^{NED}d}{dt}\frac{{}^{i}d\bar{Q}}{dt} + {}^{i}\bar{\omega}^{NED} \times \frac{{}^{i}d\bar{Q}}{dt}$$

4

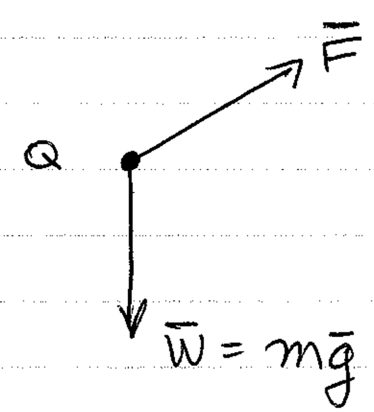
$$\frac{{}^{NED}d}{dt}\frac{{}^{i}d\bar{Q}}{dt} = \dot{v}_x \, \hat{x}_{NED} + \dot{v}_z \, \hat{z}_{NED}$$

$${}^{i}\bar{\omega}^{NED} \times \frac{{}^{i}d\bar{Q}}{dt} = \begin{vmatrix} \hat{x}_{NED} & \hat{y}_{NED} & \hat{z}_{NED} \\ 0 & -\frac{v_x}{R_0+h} & 0 \\ v_x & 0 & v_z \end{vmatrix}$$

$$= -\frac{v_x v_z}{R_0+h}\, \hat{x}_{NED} + \hat{z}_{NED}\, \frac{v_x^2}{R_0+h}$$

$$\frac{{}^{i}d}{dt}\frac{{}^{i}d}{dt}\bar{Q} = \hat{x}_{NED}\left[\dot{v}_x - \frac{v_x v_z}{R_0+h}\right] + \hat{z}_{NED}\left[\dot{v}_z + \frac{v_x^2}{R_0+h}\right]$$

Now we examine the Forces acting on the aircraft so that we can use them in Newton's Second Law



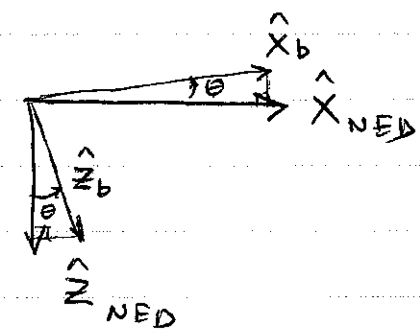
$$\bar{W} = mg(h)\, \hat{z}_{NED}$$

$$\frac{\bar{F}}{m} = f_x \, \hat{x}_{NED} + f_z \, \hat{z}_{NED} \quad \Leftarrow \text{specific forces}$$

(5) Often it is easier to specify specific forces in the X_b Z_b directions

~~Linear accelerometers aligned in the $\hat{X}_b$ and $\hat{Z}_b$ body fixed directions will measure body-axis components of $\frac{\bar{F}}{m}$~~

$$\frac{\bar{F}}{m} = f_{Xb}\,\hat{X}_b + f_{Zb}\,\hat{Z}_b = f_X\,\hat{X}_{NED} + f_Z\,\hat{Z}_{NED}$$



$$\hat{X}_b = \hat{X}_{NED}\cos\theta - \hat{Z}_{NED}\sin\theta$$
$$\hat{Z}_b = \hat{X}_{NED}\sin\theta + \hat{Z}_{NED}\cos\theta$$

$$\frac{\bar{F}}{m} = f_{Xb}\left(\hat{X}_{NED}\cos\theta - \hat{Z}_{NED}\sin\theta\right)$$
$$+ f_{Zb}\left(\hat{X}_{NED}\sin\theta + Z_{NED}\cos\theta\right)$$
$$= \hat{X}_{NED}\left(f_{Xb}\cos\theta + f_{Zb}\sin\theta\right)$$
$$+ \hat{Z}_{NED}\left(-f_{Xb}\sin\theta + f_{Zb}\cos\theta\right)$$

$$\therefore \boxed{\begin{aligned} f_X &= f_{Xb}\cos\theta + f_{Zb}\sin\theta \\ f_Z &= -f_{Xb}\sin\theta + f_{Zb}\cos\theta \end{aligned}}$$

5

Force of gravity = Weight

$$W = F_g = \frac{G M_E m}{(R_o + h)^2}$$

where R_o is the radius of the Earth and h is the altitude above the surface

G is the gravitational constant and M_E is the mass of the Earth.

Acceleration of gravity

$$g = \frac{W}{m} = \frac{G M_E}{(R_o + h)^2}$$

Acceleration of gravity at the surface

$$g_o = \frac{W}{m} = \frac{G M_E}{R_o^2} \qquad G M_E = g_o R_o^2$$

$$\therefore \; g(h) = g_o \frac{R_o^2}{(R_o + h)^2} = \boxed{g_o \frac{1}{\left(1 + \frac{h}{R_o}\right)^2} = g(h)}$$

$g_o = 32.0892233358$ ft/sec^2

$R_o = 2.092746409099$ E+7 ft

7

Now we are ready to complete Newtons 2nd Law

$$m \frac{{}^{i}d\,{}^{i}d\,\bar{Q}}{dt\,dt} = \bar{F} + \bar{W}$$

$$\frac{{}^{i}d}{dt}\frac{{}^{i}d}{dt}\bar{Q} = \frac{\bar{F}}{m} + \frac{\bar{W}}{m}$$

$$\hat{x}_{NED}\left[\dot{v}_x - \frac{v_x v_z}{R_o+h}\right] + \hat{z}_{NED}\left[\dot{v}_z + \frac{v_x^2}{R_o+h}\right]$$

$$= f_x\,\hat{x}_{NED} + f_z\,\hat{z}_{NED} + g(h)\,\hat{z}_{NED}$$

Equating components gives

$$\boxed{\begin{aligned} \dot{v}_x - \frac{v_x v_z}{R_o+h} &= f_x \\ \dot{v}_z + \frac{v_x^2}{R_o+h} &= f_z + g(h) \end{aligned}}$$

We can now collect all the equations to give a complete set of differential equations with which to <u>simulate</u> aircraft motion.

State Vector $\bar{X} = [\Theta, v_x, v_z, x, h]$
Input Vector $\bar{u} = [\omega_y, f_{xb}, f_{zb}]$

$$\dot{X} = f(X, u)$$

$$\boxed{\begin{aligned} \dot{\Theta} &= v_x/(R_o+h) + \omega_y \\ \dot{v}_x &= +v_x v_z/(R_o+h) + f_x \\ \dot{v}_z &= -v_x^2/(R_o+h) + f_z + g(h) \\ \dot{x} &= v_x \\ \dot{h} &= -v_z \end{aligned}}$$

where $f_x = f_{xb}\cos\Theta + f_{zb}\sin\Theta$
$f_z = -f_{xb}\sin\Theta + f_{zb}\cos\Theta$

To simulate these equations of motion we must have f_{xb} f_{zb} ω_y as functions of time from $t=0 \rightarrow t_{final}$. We must also have the initial conditions 8

$\Theta(0)$ $v_x(0)$ $v_z(0)$ $X(0)$ $h(0)$

Inputs: f_{xb} f_{zb} ω_y
Outputs: Θ v_x v_z x h

Note that when specific forces are a function of the motion variables then these equations of motion will include phugoid & shortperiod type motions. 8

Part II 2-Dimensional Strapdown Inertial Navigation

In inertial navigation we place instruments on board an aircraft and use them to reconstruct the trajectory of the aircraft. There are two approaches to this problem

1. the inertial platform
2. the strapdown inertial navigator

These notes will concentrate on the strapdown inertial navigator.

In strapdown inertial navigation inertial sensors are mounted fixed to the body of the aircraft. Two sensor types are used

1. angular rate gyros
2. linear accelerometers

In general 3-d motion 3 single degree of freedom rate gyros are used to measure the body axis roll rate P, ~~yaw~~ pitch rate Q, and yaw rate R. The angular velocity of the aircraft body with respect to the inertial frame is given by

$$ {}^{i}\bar{\omega}^{b} = P\,\hat{x}_b + Q\,\hat{y}_b + R\,\hat{z}_b $$

In our 2-dimensional example we need only be concerned with measuring the one rotational rate $\omega_y = Q$ (pitch rate)

A linear accelerometer is a device that measures a component of specific force $\frac{\bar{F}}{m}$ as described in Part I of these notes. We generally employ 3 linear accelerometers to measure the three components of specific force. In strapdown inertial navigation we mount the accelerometers fixed wrt the body so that they measure f_{xb} f_{yb} f_{zb} in the equation

$$ \frac{\bar{F}}{m} = f_{xb}\,\hat{x}_b + f_{yb}\,\hat{y}_b + f_{zb}\,\hat{z}_b . $$

In our 2-dimensional example we need only measure f_{xb} and f_{zb}.

11 Instrementation Errors

Rate gyros and accelerometers are never perfect instruments. They always measure with some degree of error. A simple error model for the rate gyro and linear accelerometers in our 2-dimensional problem are as follows

$$\underbrace{\omega_y'}_{\text{measurement}} = \underbrace{\omega_y}_{\text{exact}} + \underbrace{\epsilon_{\omega_y}}_{\text{error}}$$

$$f_{xb}' = f_{xb} + \epsilon_{fx}$$

$$f_{zb}' = f_{zb} + \epsilon_{fz}$$

where the primes indicate our flawed measurements

Initial Condition Errors

For the inertial navigator to work equations of motion of the navigator will need to be integrated. These equations of motion will need initial conditions. Determining ~~the~~ accurate initial conditions for an inertial navigator are critically important for an accurate navigation solution. Nonetheless we often have errors in ~~our~~ the initial conditions we give to the navigator.

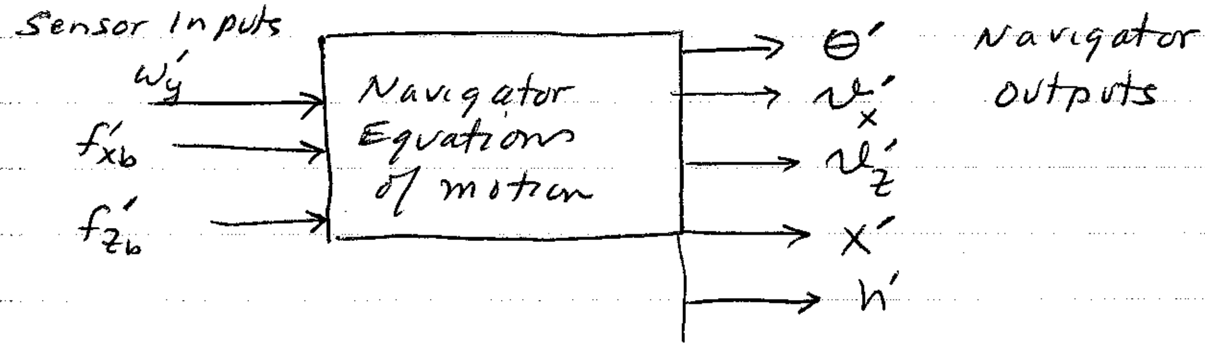
our best guess exact error

$$
\begin{aligned}
\theta'(0) &= \theta(0) + \epsilon_\theta \\
v_x'(0) &= v_x(0) + \epsilon_{v_x} \\
v_z'(0) &= v_z(0) + \epsilon_{v_z} \\
x'(0) &= x(0) + \epsilon_x \\
h'(0) &= h(0) + \epsilon_h
\end{aligned}
$$

where the primes indicate our flawed knowledge.

NAVIGATOR EQUATIONS OF MOTION

A block diagram of a 2-dimensional strapdown inertial navigator looks as follows



The equations of motion of the navigator look similar to the eom's of the aircraft simulator.

$$
\begin{aligned}
\dot{\theta}' &= v_x' / (R_0 + h') + \omega_y' \\
\dot{v}_x' &= v_x' v_z' / (R_0 + h') + f_x' \\
\dot{v}_z' &= -v_x'^2 / (R_0 + h') + f_z' + g(h') \\
\dot{x}' &= v_x' \\
\dot{h}' &= -v_z'
\end{aligned}
$$

Where
$$
\begin{aligned}
f_x' &= f_{xb}' \cos\theta' + f_{zb}' \sin\theta' \\
f_z' &= -f_{xb}' \sin\theta' + f_{zb}' \cos\theta'
\end{aligned}
$$

(13) The initial condition for the navigator equations of motion are $\theta'(0)$, $v_x'(0)$, $v_z'(0)$ $x'(0)$ and $h'(0)$.

Navigator state variables

$$\bar{x} = \begin{bmatrix} \dot{\theta}' \\ v_x' \\ v_z' \\ x' \\ h' \end{bmatrix}$$