LINEARIZATION OF NONLINEAR EQUATIONS
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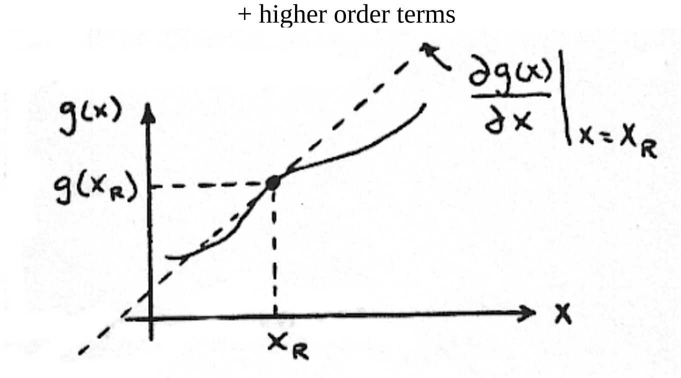
A. Linearization of Nonlinear Functions

A.1 <u>Scalar functions of one variable</u>.

We are given the nonlinear function g(x). We assume that g(x) can be represented using a Taylor series expansion about some point $\mathbf{x_R}$ as follows

$$\mathbf{g(x) = g(x)|_{x=x_R} + \frac{dg(x)}{dx}|_{x=x_R} (x - x_R) + \frac{1}{2!}\frac{d^2g(x)}{dx^2}|_{x=x_R} (x - x_R)^2}$$

+ higher order terms

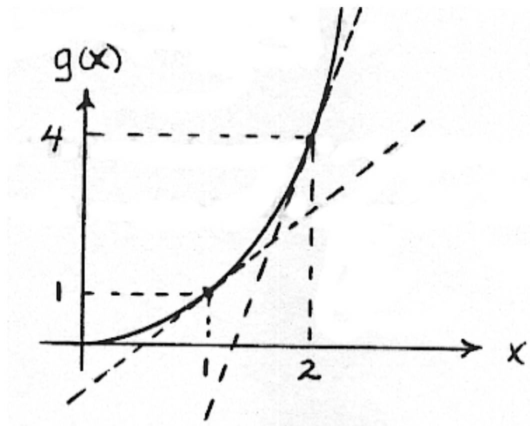


A linear approximation for g(x) involves taking only the first two terms

$$\mathbf{g(x) \approx g(x)|_{x=x_R} + \frac{dg(x)}{dx}\bigg|_{x=x_R} (x - x_R)}$$

This approximation is most accurate if $\mathbf{(x - x_R)}$ is small so that the neglected higher order terms are negligible.

Example: $\mathbf{g(x) = x^2}$



Expanding g(x) about $\mathbf{x_R = 2}$ gives

$$\mathbf{g(x) \approx g(x)|_{x=x_R} + \frac{dg}{dx}|_{x=x_R}\ (x - x_R)}$$

$$\mathbf{\approx 2^2 + 2x|_{x=x_R}\ (x-2)}$$

$$\mathbf{= 4 + 4(x-2) = -4 + 4x}$$

Notice that this is a linear function of x. The simplification resulted because we evaluated all nonlinear terms at the number $\mathbf{x = x_R = 2}$. Because we evaluated the terms on the right hand side of the equation above at $\mathbf{x = x_R = 2}$, the only term that depends on **x** is the **x-2** term, and this is a linear term. This approximation to **g(x)** is valid near **x=2**.

We can generate another approximation to **g(x)** by expanding **g(x)** about $\mathbf{x_R}$ =1 gives

$$\mathbf{g(x) = 1^2 + 2x|_{x=1}\ (x-1) = 1 + 2(x-1)}$$

$$\mathbf{= -1 + 2x}$$

This second approximation is valid near **x=1**. Clearly the linear approximation depends on the choice of reference point $\mathbf{x_R}$.

A.2 Scalar function of 2 variables

Given the nonlinear function $\mathbf{g(x_1, x_2)}$. This function can be represented by a Taylor series expansion about $\mathbf{x_{1_R}, x_{2_R}}$ as follows

$$g(x_1, x_2) = \underline{g(x_{1_R}, x_{2_R})} + \frac{\partial g}{\partial x_1}\Big|_{x_1 = x_{1_R}, x_2 = x_{2_R}} (x_1 - x_{1_R}) + \frac{\partial g}{\partial x_2}\Big|_{x_1 = x_{1_R}, x_2 = x_{2_R}} \underline{(x_2}$$

$$\frac{1}{2!}\left[\frac{\partial}{\partial x_1}\frac{\partial g}{\partial x_1}\Big|_{x_1 = x_{1_R}, x_2 = x_{2_R}} (x_1 - x_{1_R})^2 + \frac{\partial}{\partial x_2}\frac{\partial g}{\partial x_2}\Big|_{x_1 = x_{1_R}, x_2 = x_{2_R}} (x_2 - x_{2_R})^2\right]$$

$$+\frac{\partial}{\partial x_1}\frac{\partial g}{\partial x_2}\Big|_{x_1 = x_{1_R}, x_2 = x_{2_R}} (x_1 - x_{1_R})(x_2 - x_{2_R}) + h.o.t.$$

A linear approximation of g can be obtained by retaining the first three terms above (underlined). The two variables in this problem can be associated together in a vector $\bar{x}$ as follows

$$g(\bar{x}) \text{ where } \bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example:

$$g(x_1, x_2) = x_1^2 \cos x_2$$

can be approximated about $x_{1_R} = 2, \ \ x_{2_R} = 0$ as follows

$$g(x_1, x_2) = (x_1^2 \cos x_2)\Big|_{x_{1_R} = 2,\ x_{2_R} = 0} + (2x_1 \cos x_2)\Big|_{x_{1_R} = 2,\ x_{2_R} = 0} (x_1 - 2)$$

$$-(x_1^2 \sin x_2)\Big|_{x_{1_R} = 2,\ x_{2_R} = 0} (x_2 - 0)$$

$$= 4 + 4(x_1 - 2) + 0 = -4 + 4x_1$$

The same function can be approximated about $x_{1_R} = 2 \quad x_{2_R} = \pi / 4$

$$g(x_1, x_2) = 2^2 \cos(\frac{\pi}{4}) + (2x_1 \cos x_2)\Big|_{x_{1_R} = 2,\ x_{2_R} = \frac{\pi}{4}} (x_1 - 2)$$

$$-(x_1^2 \sin x_2)\Big|_{x_{1_R} = 2,\ x_{2_R} = \frac{\pi}{4}} (x_2 - \frac{\pi}{4})$$

$$= 0 + 0 - 4(x_2 - \frac{\pi}{4}) = \pi - 4x_2$$

Notice again how important the linearization point or reference point is to the linearized result.

A.3 Vector function of a vector of variables.

Let $\bar{\mathbf{g}}(\bar{\mathbf{x}})$ be an nx1 vector of nonlinear functions. Let $\bar{\mathbf{x}}$ be an nx1 vector of variables

$$\bar{\mathbf{g}} = \begin{bmatrix} \mathbf{g_1} \\ \mathbf{g_2} \\ \vdots \\ \mathbf{g_n} \end{bmatrix}, \quad \bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \vdots \\ \mathbf{x_n} \end{bmatrix}$$

A linear approximation about $\bar{\mathbf{x}}_\mathbf{R}$ is

$$\bar{\mathbf{g}} \approx \bar{\mathbf{g}}(\bar{\mathbf{x}}_\mathbf{R}) + \frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{x}}}\Big|_{\bar{\mathbf{x}}=\bar{\mathbf{x}}_\mathbf{R}} (\bar{\mathbf{x}} - \bar{\mathbf{x}}_\mathbf{R})$$

where

$$\bar{\mathbf{g}}(\bar{\mathbf{x}}_\mathbf{R}) = \begin{bmatrix} \mathbf{g_1}(\mathbf{x}_{\mathbf{1_R}}, & \cdots & \mathbf{x}_{\mathbf{n_R}}) \\ \vdots & & \\ \mathbf{g_n}(\mathbf{x}_{\mathbf{1_R}}, & \cdots & \mathbf{x}_{\mathbf{n_R}}) \end{bmatrix} = \text{nx1 vector}$$

$$\frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \mathbf{g_1}}{\partial \mathbf{x_1}} & \frac{\partial \mathbf{g_1}}{\partial \mathbf{x_2}} & \cdots & \frac{\partial \mathbf{g_1}}{\partial \mathbf{x_n}} \\ \vdots & & \cdots & \\ \frac{\partial \mathbf{g_n}}{\partial \mathbf{x_1}} & \frac{\partial \mathbf{g_n}}{\partial \mathbf{x_2}} & \cdots & \frac{\partial \mathbf{g_n}}{\partial \mathbf{x_n}} \end{bmatrix} = \text{Jacobian Matrix} = \text{nxn matrix}$$

A.4 Accuracy of linearized solution.

When we approximate $\bar{\mathbf{g}}(\bar{\mathbf{x}})$ by retaining only the linear terms, we must guarantee that the deleted terms, i.e., the h.o.t. are negligible. This is true only when $\bar{\mathbf{x}} - \bar{\mathbf{x}}_\mathbf{R}$ is small, i.e. when the perturbations from the reference point are small.

B. Linearization on Nonlinear Differential Equations in First Order Form

B.1 First order form

Nonlinear differential equations in first order form can be written as

$$\dot{\bar{x}} = \bar{g}(\bar{x}, \bar{u}), \qquad \bar{x}(0)$$

where

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \bar{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}, \quad \dot{\bar{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

Note that $\bar{u}$ represents specified forcing functions and $\bar{x}(0)$ is a specified initial condition vector.

<u>Example B.1a</u>

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \bar{g} = \begin{bmatrix} x_2{}^2 - u^2 \\ -x_1{}^2 + 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2{}^2 - u^2 \\ -x_1{}^2 + 1 \end{bmatrix}$$

B.2 <u>The reference or trim solution</u>

When we were linearizing nonlinear functions, we saw how important the choice of reference point was. In linearizing nonlinear differential equations, we are also concerned with the reference about which we linearize. However, we are now interested in obtaining a linearized solution *valid for all time*. This requires that we linearize around a reference solution, which is valid for all time.

Let $\bar{x}_R(t)$ be a known solution to the nonlinear differential equation with specified forcing function $\bar{u}_R(t)$ and specified initial condition $\bar{x}_R(0)$. i.e.,

$$\dot{\bar{x}}(t) = \bar{g}(\bar{x}_R(t), u_R(t)) \qquad \bar{x}_R(0)$$

$x_R(t)$ is said to be the *reference solution to the nonlinear differential equation*.

<u>Example B.1b</u>

For the differential equations given in Example B.1a

$$\bar{x}_R(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u_R(t) = 1, \quad \dot{\bar{x}}_R(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is a constant solution to the nonlinear differential equation. Verify this fact for yourself by substituting this solution into the differential equation given in Example B.1a. Please keep straight in your mind the difference between a differential equation (e.g. $\dot{\mathbf{x}} = \mathbf{x}$) and a solution to a differential equation (e.g. $\mathbf{x} = \mathbf{0}$ **for** $\dot{\mathbf{x}} = \mathbf{x}$).

Example B.1c

For the differential equations given in Example B.1a

$$\mathbf{x_R(t)} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \mathbf{u_R(t)} = -1 \qquad \dot{\mathbf{x}}_\mathbf{R} = [\begin{matrix} 0 \\ 0 \end{matrix}]$$

is another constant solution to the nonlinear differential equations.

Example B.1d

For the differential equations given in Example B.1a

$$\bar{\mathbf{x}}_\mathbf{R} = \begin{bmatrix} \mathbf{x}_1 = \pm 1 \\ \mathbf{x}_2 = \pm \mathbf{u_R} = \mathbf{const} \end{bmatrix} \quad \mathbf{u_R} = \mathbf{const} \qquad \dot{\mathbf{x}}_\mathbf{R} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is a constant solution to the nonlinear differential equations for any constant.

B.3 Linearization about a reference solution

Let $\bar{\mathbf{x}}_\mathbf{R}(\mathbf{t}),\ \bar{\mathbf{u}}_\mathbf{R}(\mathbf{t})$ be a reference solution. We now want to find a linearized solution to the nonlinear differential equation about this reference solution.

We again expand $\bar{\mathbf{g}}(\bar{\mathbf{x}})$ in a Taylor series expansion about $\bar{\mathbf{x}}_\mathbf{R}$ and $\bar{\mathbf{u}}_\mathbf{R}$ i.e.,

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{g}}(\bar{\mathbf{x}}_\mathbf{R}, \bar{\mathbf{u}}_\mathbf{R}) + \frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{x}}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}_\mathbf{R},\ \mathbf{u}=\bar{\mathbf{u}}_\mathbf{R}} (\bar{\mathbf{x}} - \bar{\mathbf{x}}_\mathbf{R}) + \frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{u}}} \Big|_{\mathbf{x}=\bar{\mathbf{x}}_\mathbf{R},\ \mathbf{u}=\bar{\mathbf{u}}_\mathbf{R}} (\bar{\mathbf{u}} - \bar{\mathbf{u}}_\mathbf{R})$$

+h.o.t.

The linear approximation is obtained by assuring that $\bar{\mathbf{x}} - \bar{\mathbf{x}}_\mathbf{R}$ and $\bar{\mathbf{u}} - \bar{\mathbf{u}}_\mathbf{R}$ are small enough that the h.o.t. can be neglected.

B.4 Definition of small distribution variables

Define

$$\delta\bar{\mathbf{x}} = \bar{\mathbf{x}} - \mathbf{x_R}$$
$$\delta\bar{\mathbf{u}} = \bar{\mathbf{u}} - \bar{\mathbf{u}}_\mathbf{R}$$
$$\delta\dot{\bar{\mathbf{x}}} = \dot{\bar{\mathbf{x}}} - \dot{\bar{\mathbf{x}}}_\mathbf{R}$$

For the linearized solution to be valid, these perturbations must be “small.”

B.5 Separation of the linearized differential equations into two parts

Assuming that the perturbations are small, we can write the approximation to the differential equations as

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{g}}(\bar{\mathbf{x}}_R, \bar{\mathbf{u}}_R) + \frac{\partial \bar{\mathbf{g}}}{\partial \mathbf{x}}\Big|_R (\bar{\mathbf{x}} - \bar{\mathbf{x}}_R) + \frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{u}}}\Big|_R (\bar{\mathbf{u}} - \bar{\mathbf{u}}_R)$$

we can now substitute the small perturbation variables

$$\underline{\dot{\bar{\mathbf{x}}}_R} + \delta\dot{\bar{\mathbf{x}}} = \underline{\mathbf{g}(\bar{\mathbf{x}}_R, \bar{\mathbf{u}}_R)} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\Big|_R \delta\bar{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\Big|_R \delta\bar{\mathbf{u}}$$

In the equation above we have simplified the notation with $|_R$ to denote $|_{\bar{\mathbf{x}}=\bar{\mathbf{x}}_R,\ \bar{\mathbf{u}}=\bar{\mathbf{u}}_R}$.

Notice that the underlined terms are numerically equal from the definition of reference solution. Since they are equal, they can be cancelled out leaving

$$\delta\dot{\bar{\mathbf{x}}} = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\Big|_R \delta\bar{\mathbf{x}} \quad + \quad \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\Big|_R \delta\bar{\mathbf{u}}$$

This is a set of linear small perturbation differential equations. In summary, the original nonlinear problem

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{g}}(\bar{\mathbf{x}}, \bar{\mathbf{u}}), \qquad \bar{\mathbf{x}}(0)$$

with solution $\bar{\mathbf{x}}(\mathbf{t})$ for specified input u(t) has been decomposed into two separate problems.

- The reference problem

$$\dot{\bar{\mathbf{x}}}_R = \bar{\mathbf{g}}(\bar{\mathbf{x}}_R, \vec{\mathbf{u}}_R)$$

  with initial condition $\bar{\mathbf{x}}_R(\mathbf{0})$ with solution $\bar{\mathbf{x}}_R(\mathbf{t})$ to input $\bar{\mathbf{u}}_R(\mathbf{t})$

- The small perturbation problem

$$\delta\dot{\bar{\mathbf{x}}} = \frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{x}}}\Big|_R \delta\bar{\mathbf{x}} + \frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{u}}}\Big|_R \delta\bar{\mathbf{u}}$$

  with initial condition

$$\delta\bar{\mathbf{x}}(\mathbf{0}) = \bar{\mathbf{x}}(\mathbf{0}) - \bar{\mathbf{x}}_R(\mathbf{0})$$

  with solution $\delta\bar{\mathbf{x}}(\mathbf{t})$ to input $\delta\bar{\mathbf{u}}(\mathbf{t})$.

Finally the total approximate solution is given by the entire solution procedure is shown in Figure 1.

$$\bar{\mathbf{x}}(\mathbf{t}) = \bar{\mathbf{x}}_R(\mathbf{t}) + \delta\bar{\mathbf{x}}(\mathbf{t})$$

B.6 On picking a reference solution

Any solution to $\dot{\bar{\mathbf{x}}} = \bar{\mathbf{g}}(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ makes a good reference solution but these solutions can be hard to find. An easier set of solutions are constant solutions i.e., solutions so that $\dot{\bar{\mathbf{x}}}_{\mathbf{R}}(\mathbf{t}) = \bar{\mathbf{0}}$ and $\bar{\mathbf{x}}_{\mathbf{R}}(\mathbf{t}) =$ constant for $\bar{\mathbf{u}}_{\mathbf{R}}(\mathbf{t}) =$ constant. For constant reference solutions, finding the reference solution to a *nonlinear differential equation* becomes a problem of finding the solution to a *nonlinear algebraic equation*

$$\mathbf{g}(\bar{\mathbf{x}}_{\mathbf{R}}, \bar{\mathbf{u}}_{\mathbf{R}}) = \bar{\mathbf{0}}$$

B.7 Linearization Example

$$\dot{\bar{\mathbf{x}}}(\mathbf{t}) = \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x_2}^2 - \mathbf{u}^2 \\ -\mathbf{x_1}^2 + \mathbf{1} \end{bmatrix}$$

a) Choice of Reference Solution

To simplify our choice, assume that the reference solution is constant, i.e., $\dot{\mathbf{x}}_1 = \dot{\mathbf{x}}_2 = \mathbf{0}$. This requires that $\mathbf{x_2}^2 - \mathbf{u}^2 = \mathbf{0}$ and $-\mathbf{x_1}^2 + \mathbf{1} = \mathbf{0}$. These equations can be satisfied whenever

$$\mathbf{x_2}^2 = \mathbf{u}^2 \text{ and } \mathbf{x_1}^2 = \mathbf{1}$$

Values of $\mathbf{x_1}$ and $\mathbf{x_2}$ which satisfy these equations are

$$\mathbf{x_2} = \pm\mathbf{u} \text{ where } \mathbf{u} \text{ is any constant}$$

$$\mathbf{x_1} = \pm\mathbf{1}$$

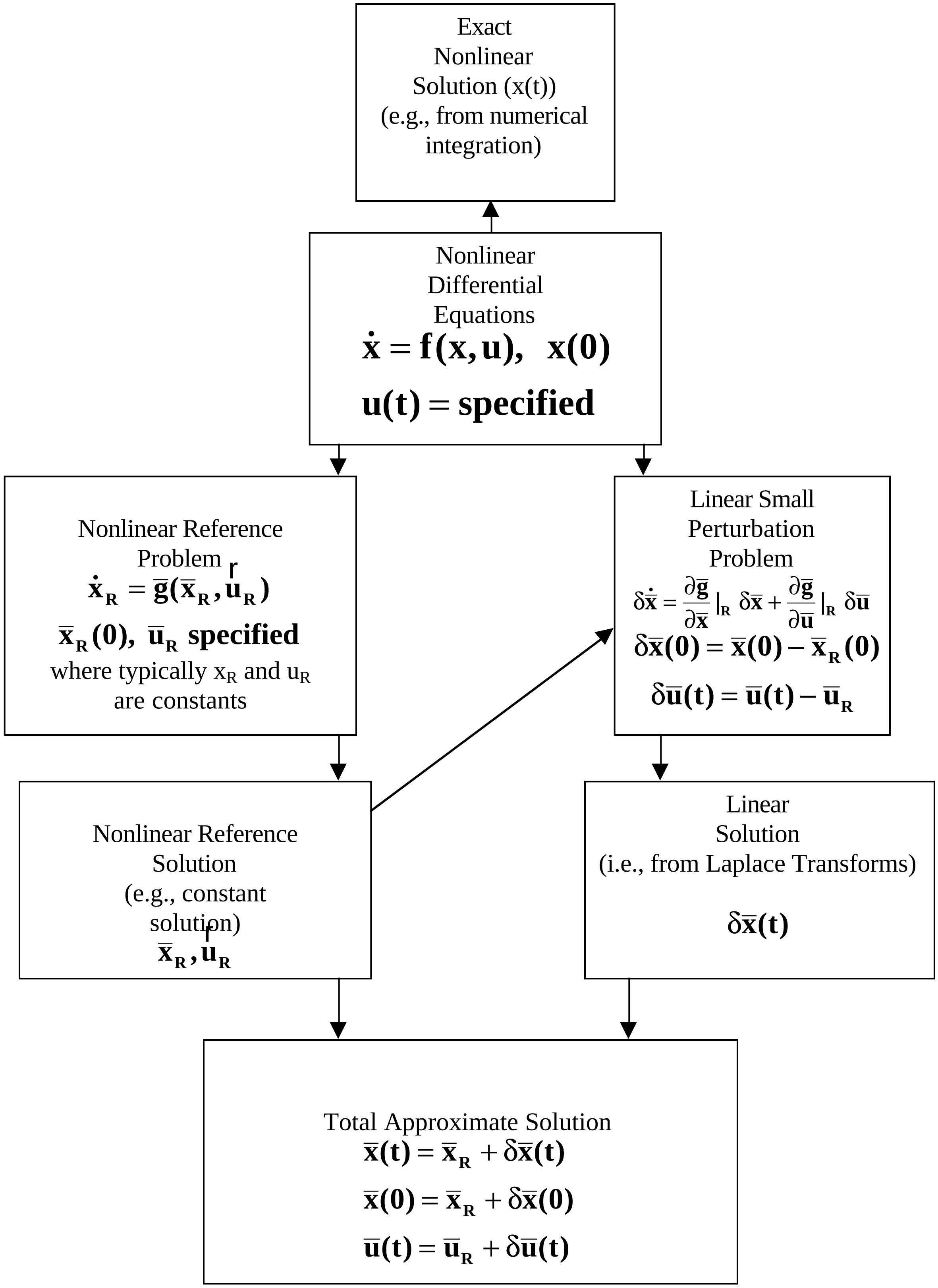


Figure 1 Solution Procedures for Nonlinear Differential Equations

We will consider two different reference solutions

$$\underline{\textbf{Ref.\#1}} \qquad\qquad \underline{\textbf{Ref.\#2}}$$

$$\bar{\mathbf{x}}_{\mathbf{R}}(\mathbf{t}) = \begin{bmatrix} +1 \\ +1 \end{bmatrix}, \quad \mathbf{u}_{\mathbf{R}}(\mathbf{t}) = +1 \qquad \bar{\mathbf{x}}_{\mathbf{R}}(\mathbf{t}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_{\mathbf{R}}(\mathbf{t}) = -1$$

$$\bar{\mathbf{x}}_{\mathbf{R}}(\mathbf{0}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{x}_{\mathbf{R}}(\mathbf{0}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

b) Small Perturbation Equations of Motion

$$\delta\dot{\bar{\mathbf{x}}} = \frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{x}}}\Big|_{\mathbf{R}} \delta\bar{\mathbf{x}} \quad + \quad \frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{u}}}\Big|_{\mathbf{R}} \delta\mathbf{u}$$

where $\delta\bar{\mathbf{x}} = \bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mathbf{R}} \qquad \delta\mathbf{u} = \mathbf{u} - \mathbf{u}_{\mathbf{R}}$

$$\frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{x}}} = \begin{bmatrix} \frac{\partial \mathbf{g}_1}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{g}_1}{\partial \mathbf{x}_2} \\ \frac{\partial \mathbf{g}_2}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{g}_2}{\partial \mathbf{x}_2} \end{bmatrix}, \quad \frac{\partial \bar{\mathbf{g}}}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial \mathbf{g}_1}{\partial \mathbf{u}} \\ \frac{\partial \mathbf{g}_2}{\partial \mathbf{u}} \end{bmatrix}$$

$$\frac{\partial \bar{\mathbf{9}}}{\partial \bar{\mathbf{x}}} = \begin{bmatrix} \mathbf{0} & \mathbf{2x}_2 \\ -\mathbf{2x}_1 & \mathbf{0} \end{bmatrix}, \quad \frac{\partial \bar{\mathbf{g}}}{\partial \mathbf{u}} = \begin{bmatrix} -\mathbf{2u} \\ \mathbf{0} \end{bmatrix}$$

Using Ref. #1 $\bar{\mathbf{x}}_{\mathbf{R}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}, \quad \mathbf{u}_{\mathbf{R}} = \mathbf{1}$

$$\begin{bmatrix} \delta\dot{\mathbf{x}}_1 \\ \delta\dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & +\mathbf{2} \\ -\mathbf{2} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_1 \\ \delta\mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} -\mathbf{2} \\ \mathbf{0} \end{bmatrix} \delta\mathbf{u}$$

Using Ref. #2 $\bar{\mathbf{x}}_{\mathbf{R}} = \begin{bmatrix} -\mathbf{1} \\ -\mathbf{1} \end{bmatrix}, \quad \mathbf{u}_{\mathbf{R}} = \mathbf{1}$

$$\begin{bmatrix} \delta\dot{\mathbf{x}}_1 \\ \delta\dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{2} \\ +\mathbf{2} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_1 \\ \delta\mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} +\mathbf{2} \\ \mathbf{0} \end{bmatrix} \delta\mathbf{u}$$

c. The Linear Solution for Reference #1

(1) $\delta\dot{\bar{\mathbf{x}}}_1 = \mathbf{2}\delta\mathbf{x}_2 - \mathbf{2}\delta\mathbf{u}$

(2) $\delta\dot{x}_2 = -2\delta x_1$

take Laplace transforms

(1) $s\delta x_1(s) - \delta x_1(0) = 2\delta x_2(s) - 2\delta u(s)$

(2) $s\delta x_2(s) - \delta x_2(0) = -2\delta x_1(s)$

multiply (2) by s

$$s^2\delta x_2(s) - s\delta x_2(0) = -2s\delta x_1(s)$$

multiply (1) by –2

$$-2s\delta x_1(s) = -4\delta x_2(s) + 4\delta u(s) - 2\delta x_1(0)$$

set these equal

$$s^2\delta x_2(s) - s\delta x_2(0) = -4\delta x_2(s) + 4\delta u(s) - 2\delta x_1(0)$$
$$\delta x_2(s)\left[s^2 + 4\right] = s\delta x_2(0) - 2\delta x_1(0) + 4\delta u(s)$$
$$\delta x_2(s) = \frac{s\delta x_2(0) - 2\delta x_1(0)}{s^2 + 4} + \left(\frac{4}{s^2 + 4}\right)\delta u(s)$$

The first term on the right gives initial condition response. The second term on the right contains the transfer function $\frac{\delta x_2(s)}{\delta u(s)} = \frac{4}{s^2 + 4}$.

To find $\delta x_1(t)$ take the inverse Laplace transform. From (2)

$$s\delta x_2(s) - \delta x_2(0) = -2\delta x_1(s)$$
$$\delta x_1(s) = -\frac{1}{2}\left[s\delta x_2(s) - \delta x_2(0)\right]$$

To find the solutions $\delta x_1(t)$ and $\delta x_2(t)$ you must be given the input $\delta u(t)$ and the initial conditions $\left(\delta x_1(0), \delta x_2(0)\right)$. Then the solutions can be found using inverse Laplace transforms.

d) Total Solution for Reference #1

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_{1_R}(t) \\ x_{2_R}(t) \end{bmatrix} + \begin{bmatrix} \delta x_1(t) \\ \delta x_2(t) \end{bmatrix} = \begin{bmatrix} 1 + \delta x_1(t) \\ 1 + \delta x \ \ (t) \end{bmatrix}$$

$$u(t) = u_R(t) + \delta u(t) = 1 + \delta u(t)$$

$$\bar{x}_1(0) = \begin{bmatrix} 1 + \delta x_1(0) \\ 1 + \delta x_2(0) \end{bmatrix}$$

e) Comment:

For this procedure to be valid the perturbations must be small, i.e., all must be small.

Suppose we have the nonlinear problem with

$$\mathbf{x_1(0) = 1.01}$$

$$\mathbf{x_2(0) = .99}$$

$$\mathbf{u(t) = 1.0 + .01 \sin \omega t}$$

then we can use Ref.#1 and then we can have

$$\boldsymbol{\delta} \mathbf{x_1(0) = .01}$$

$$\boldsymbol{\delta} \mathbf{x_2(0) = -.01}$$

$$\boldsymbol{\delta} \mathbf{u(t) = .01 \sin \omega t}$$

On the other hand if for the nonlinear problem we have

$$\mathbf{x_1(0) = -1.01}$$

$$\mathbf{x_2(0) = -.99}$$

$$\mathbf{u(t) = -1 - .01 \sin \omega t}$$

We would use Ref. #2 with

$$\boldsymbol{\delta} \mathbf{x_1(0) = -.01}$$

$$\boldsymbol{\delta} \mathbf{x_2(0) = .01}$$

$$\boldsymbol{\delta} \mathbf{u(t) = -.01 \sin \omega t}$$

C. Concluding Comments

We have seen how the solution to nonlinear differential equations can be found by decomposing the problem into two simpler parts. The reference part is simpler because it is often a nonlinear algebraic problem. The second small perturbation part is simpler because it often involves solving linear differential equations with constant coefficients. The total approximate solution to the original nonlinear differential equation was shown to be the sum of the two simpler parts.