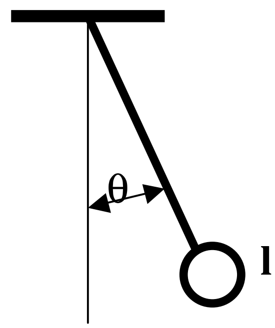
A&AE 421 Dynamic Analysis of a Simple Pendulum

Assume that a simple pendulum consists of a ball on a string. Gravity tends to make the pendulum (ball) return to the vertical position. If the ball is given an initial angle, θ, and let go it will oscillate. The aerodynamic drag on the ball tends to make the ball slow down and eventually stop.



Pendulum differential equation

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - .5\rho S C_D l\dot{\theta}^2[\mathrm{sign}(\dot{\theta})]/m$$

where **g** is the acceleration of gravity**,** **l** is the length of the pendulum, ρ is the air density, **S** is the cross-sectional area of the ball of the pendulum, **m** is the mass of the ball, and $\mathbf{C_D}$ is the drag coefficient of the ball. Motion variable θ is the angle that the pendulum makes with a vertical line. The sign function is required to insure that the drag force always resists the motion due to the rate of change of angle θ.

Pendulum state space model

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\dot{\bar{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - .5\rho S C_D l x_2{}^2 \mathrm{sign}(x_2)/m \end{bmatrix} = \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix}$$

AAE421 HW-2 (Due Friday 8/31/07)

a) From your physical knowledge of this dynamical situation, determine an equilibrium solution ($\theta(t)=\boldsymbol{\theta}_{\mathbf{trim}}$=constant) for the pendulum that is STABLE. Note that if you place the pendulum at a stable equilibrium it will tend stay there naturally.

b) From your physical knowledge of this dynamical situation, determine an equilibrium solution ($\theta(t)=\boldsymbol{\theta}_{\mathbf{trim}}$=constant) for the pendulum that is UNSTABLE. Note that if you place the pendulum at an unstable equilibrium it will NOT tend stay there naturally.

c) Define small perturbation variables as follows

$$\Delta x = \begin{bmatrix} x_1 - x_{1trim} \\ x_2 - x_{2trim} \end{bmatrix} = \begin{bmatrix} \theta - \theta_{trim} \\ \dot{\theta} - \dot{\theta}_{trim} \end{bmatrix}$$

Determine the linearized small perturbation differential equations for the pendulum of the form $\mathbf{\Delta\dot{x} = A \cdot \Delta x}$ where

$$A = \left.\frac{\partial \overline{\mathbf{g}}}{\partial \overline{\mathbf{x}}}\right|_{\mathbf{trim}} = \begin{bmatrix} \left.\frac{\partial \mathbf{g}_1}{\partial \mathbf{x}_1}\right|_{\mathbf{trim}} & \left.\frac{\partial \mathbf{g}_1}{\partial \mathbf{x}_2}\right|_{\mathbf{trim}} \\ \left.\frac{\partial \mathbf{g}_2}{\partial \mathbf{x}_1}\right|_{\mathbf{trim}} & \left.\frac{\partial \mathbf{g}_2}{\partial \mathbf{x}_2}\right|_{\mathbf{trim}} \end{bmatrix}$$

where g_1, g_2, x_1 and x_2 were defined earlier for the pendulum. Express your answer in terms of $\boldsymbol{\theta}_{\mathbf{trim}}$.

d) Referring to the stable equilibrium you found in part a, intuitively you know that if you give the pendulum an initial condition away from this equilibrium condition and let it start swinging that the pendulum should oscillate for a while and eventually stop oscillating. This suggests that the second order differential equation has two complex poles (i.e., a complex conjugate pair of poles) and that they are in the left half plane (i.e., are stable, negative real parts).

Assume

Air density $=\rho=0.002378$ slugs/ft^3,
The diameter of the ball is 1.25 inches,
The length of the pendulum is 2 feet,
$C_D=2.0$,
Mass of the ball M=0.00289 slugs,
Acceleration of gravity = 32.2 ft/sec^2.

Find numerical values of the four elements of the A matrix. Find numerical values of the two poles of the A matrix (i.e., find the eigenvalues of A). Do the values of the poles agree with your intuition? If not, find out what is wrong.