DIMENSIONAL ANALYSIS a.k.a. UNIT CHECKING

SYNOPSIS In this lesson the four fundamental quantities of mechanics are defined. Newtons Second law of motion provides a relationship between these four fundamental quantities. Since there are four fundamental quantities and one rule relating them, any three fundamental quantities are independent while the fourth can be derived from the other three using the rule. The concept of dimension is introduced and two commonly used dimensional systems are discussed.

OBJECTIVES Upon completion of this lesson you should be able to:

1. Determine the dimensions of any answer from the dimensions of the quantities used to compute the answer and from the formula for computing the answer.

2. Determine when an equation or formula is dimensionally correct or incorrect and perhaps suggest how the equation must be modified.

3. Determine when the numerical value of an answer to a particular problem is "reasonable" by your physical understanding of the dimensions of the answer. For example, the length of this sheet of paper is a) 1, b) 1 foot, c) 1 light year, d) 1 second, e) 1 pound?

I. The Concept of Dimension

Suppose we are measuring the distance between two points. We say the dimension of distance is length.

Examples: dim[distance] = length

dim[area] = length squared

dim[speed] = length divided by time

There are four fundamental quantities used in the study of dynamics. They are listed below with their dimensional abbreviation

Force:	F
Mass:	M
Length:	L
Time:	T

We can use these abbreviations to write the dimensions of all other quantities in dynamics.

Examples: dim [position] = L

dim [velocity] = L/T

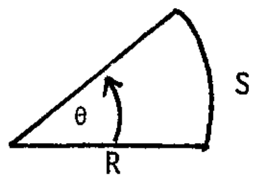
dim [acceleration] = L/T^2

dim [angle] = 1 (dimensionless)

dim [sine(angle)] = 1 (dimensionless)

dim [volume] = L^3

The dimensions of angular quantities are sometimes confusing. Recall from calculus that the arc length of a circle, S, is given in terms its radius times the angle, θ, defining the arc as follows.



$S = R\theta$

We note that dim [S] = L and dim [R] = L since they are both lengths.

If the equation above is to be a correct equality, θ must be dimensionless.

Recall that when you learned the above formula that θ was expressed in units of radians and typically varied from 0 to 2π radians. The radian must be a dimensionless quantity for our equation to be correct. i.e.

dim [angle in units of radians] = 1 (i.e. dimensionless)

We sometimes also express angles in units of degrees. Since all angles are dimensionless we can say

dim [angle in units of degrees] = 1 (i.e. dimensionless).

You note also that MATLAB trigonometric functions assume θ is expressed in radians.

The units of a quantity are related to their dimensions but can provide other useful information.

units[a length in units of feet]=feet
units[an angle in units of radian]=unit-less
units[an angle in units of degree]=degree

Checking units in an equation is helpful if an equation has mixed units. For example

$$\textbf{Propeller Efficiency} = \frac{\textbf{Power Output in horsepower}}{\textbf{Power Input in ft - lbf/sec}} \bullet \frac{\textbf{550 ft - lbf/sec}}{\textbf{horsepower}}$$

$$\textbf{units[Propeller Efficiency]} = \frac{\textbf{horsepower}}{\textbf{ft} - \textbf{lbf/sec}} \bullet \frac{\textbf{ft} - \textbf{lbf/sec}}{\textbf{horsepower}} = \textbf{unit} - \textbf{less}$$

II. Dimensional Systems

Newton has provided us with a relationship between the four fundamental quantities in mechanics. Loosely speaking Newton's Second Law of motion says "Force equals mass times acceleration". We can write this dimensionally as follows

$$F = ML/T^2$$

We can take any three of the four fundamental quantities as "basic dimensions" and consider the fourth as a "derived dimension". Two sets of "basic dimensions" are in common use today.

Gravitational Dimensions

The "basic dimensions" are F, L, T and the derived quantity is mass (M).

Absolute Dimensions

The "basic dimensions" are M, L, T and the derived quantity is force (F).

The table on the next page 2 summarizes the dimensional expressions for quantities which are commonly found in dynamics.

Note that any quantity such as length which has dimension L can be expressed in different <u>units</u>. For example length can be expressed in units of meters or feet.

dim [length in units of feet] = L

dim [length in units of meters] = L

Both the British system of units and the metric system of units have <u>both</u> gravitational and absolute dimensional forms. However, in 1960 delegates of the Eleventh General Conference in Weights and Measures

TABLE

Quantity	Gravitational dimension	Absolute dimension
Force	F	ML/T^2
Mass	FT^2/L	M
Length	L	L
Time	T	T
Position	L	L
Velocity	L/T	L/T
Acceleration	L/T^2	L/T^2
Angle	1	1
Angular velocity	1/T	1/T
Angular acceleration	$1/T^2$	$1/T^2$
Moment	FL	ML^2/T^2
Impulse	FT	ML/T
Angular impulse	FLT	ML^2/T
Momentum	FT	ML/T
Moment of momentum	FLT	ML^2/T
Areal velocity	L^2/T	L^2/T
Work (energy)	FL	ML^2/T^2
Frequency	1/T	1/T
Period	T	T
Moment of Inertia	FT^2L	ML^2

= rate of change of area wrt time $\frac{dA}{dt}$. For motion under a central force the area swept by a line segment from focus to particle is constant $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$.

This table is from Reference 2

defined and officially sanctioned an international system of units, the Systeme International d'Unites (designated SI in all languages). This absolute dimensional system is based upon the meter (L), kilogram(M), second (T), ampere, kelvin and candela. All other quantities such as force are derived from the basic quantities. In the United States both the Federal Bureau of Standards and the National Aeronautics and Space Administration have adopted the International System of Units as the preferred system.

We will give one example of an absolute dimensional system, the SI, and one example of a gravitational dimensional system, the British Gravitational Units (FPS). These are the two most commonly encountered systems of units.

The International System of Units (SI) also called the Metric Absolute System (MKS)

Fundamental Quantities of Mechanics

1) mass, kilogram, kg
2) length, meter, m
3) time, second, s

Derived Quantity

a) Force, newton, N
b) 1 newton = the force required to give 1 kg of mass an acceleration of 1 m/s^2
c) $1\ N = 1\ kg\ m/s^2$

British Gravitational Units (FPS)

Note abbreviation

Fundamental Quantities of Mechanics

1) Force, pound, lbf
2) length, foot, ft
3) time, second, sec

Derived Quantity

a) Mass, slug, abbreviated slug
b) 1 slug = the mass which has an acceleration of 1 ft/sec^2 due to a force of 1 lbf.
c) 1 slug = 1 lbf sec^2/ft.

The term weight is also commonly confused. The weight of a body is the force which gravity exerts upon the body. This force will vary for the same body depending upon the location of the body on the Earth, e.g. at higher altitudes the same body will weigh less because gravity is less strong at higher altitudes.

III. Dimensions and Functions

Common mathematical functions have dimensions which are obvious if the definition of the function is kept in mind. This is best seen by example.

1. If dim [X] = L^2 then dim [square root (X)] = L and dim [square of X] = L^4.

2. If θ is an angle dim [θ] = 1 and dim [sin θ] = 1, dim [cos θ] = 1 etc. This is true because sin θ is the ratio of two lengths, the opposite side of a right triangle divided by the hypotenuse.

3. Recall from calculus that the <u>dimensionless</u> number e = 2.72 was defined (dim [e] = 1). The number e is commonly raised to a power X. Powers are themselves dimensionless therefore dim [X] must be 1. A dimensionless number raised to any power is dimensionless therefore dim [e^X] = 1.

4. If dim [X] = L, dim [y] = L, dim [t] = T and if

$$\frac{dX(t_1)}{dt} = \lim_{t_1 - t_2 \to 0} \frac{X(t_1) - X(t_2)}{t_1 - t_2}$$

X is a function of time, X(t) means X at time t_1

$$\frac{d^2X(t_1)}{dt^2} = \lim_{t_1 - t_2 \to 0} \frac{\frac{dX(t_1)}{dt} - \frac{dX(t_2)}{dt}}{t_1 - t_2}$$

$$\frac{dX(y_1)}{dy} = \lim_{y_1 - y_2 \to 0} \frac{X(y_1) - X(y_2)}{y_1 - y_2}$$

X is a function of position y. X(y) is X evaluated when $y = y_1$

$$\frac{d^2x(y_1)}{dy^2} = \lim_{y_1 - y_2 \to 0} \frac{\frac{dX(y_1)}{dy} - \frac{dX(y_2)}{dy}}{y_1 - y_2}$$

then

$$\dim \left[\frac{dX}{dt}\right] = \frac{L}{T}$$

$$\dim \left[\frac{d^2X}{dt^2}\right] = \frac{L}{T^2}$$

$$\dim \left[\frac{dX}{dy}\right] = \frac{L}{L} = 1$$

$$\dim \left[\frac{d^2X}{dy^2}\right] = \frac{L}{L^2} = \frac{1}{L}$$

Furthermore if

$$\int_{t_1}^{t_N} x(t)dt \approx \sum_{i=1}^{N-1} X(t_i)\ (t_{i+1}-t_i)$$

then

$$\dim \left[\int_{t_1}^{t_N} X(t)dt\right] = LT$$

5. If $\bar{r}$ and $\bar{F}$ are vectors, if $\dim [\bar{r}] = L$ and $\dim [\bar{F}] = F$, if

$$|\bar{r}x\bar{F}| = |\bar{r}| \cdot |\bar{F}| \sin \text{(angle between } \bar{r} \text{ and } \bar{F})$$

and if

$$\bar{r} \cdot \bar{F} = |\bar{r}| \cdot |\bar{F}| \cos \text{(angle between } \bar{r} \text{ and } \bar{F})$$

then

$$\dim [|\bar{r}x\bar{F}|] = LF$$

$$\dim [\bar{r} \cdot \bar{F}] = LF$$

IV. Dimensions and Equations

Equations are mathematical statements of equality, i.e. the right hand side of an equal sign equals the left side of an equal sign. This is also true for the dimensions and units of both sides of equal signs. Two useful axioms regarding dimensions and units are stated below.

AXIOM #1: The dimensions and units of both sides of an equality must be the same.

AXIOM #2. Only quantities with the same dimensions and units can be added or subtracted.

Axiom #3. Powers are dimensionless (unitless).

Examples: For the examples which follow assume

$\dim[N] = F = ML/T^2$

$\dim[\nu] = L/T$

$\dim[m] = M = FT^2/L$

$\dim[\omega] = \frac{1}{T}$

$\dim[t] = T$

$\dim[Q] = F = ML/T^2$

$\dim[h] = L$

Check the dimensions of the following equations.

a) $N^2 = m\nu^2/h + Q^2 \sin(wt)$

b) $N^2 = (m\nu^2/h)^2 + Q^2 \sin^2(wt)$

c) $N = m\nu^2/h + Q \sin(wt)$

Answers: Equation a) is not an equality because the first term on the right hand side has dimension F while the other two terms have dimension F^2. Equations b) and c) are dimensionally correct.

V. Closing Comments

1. All physical quanities have dimensions and units. All answers to all problems you will be asked to compute in your engineering career will have dimensions and units. For an answer to a problem to be correct you must specify the units as well as a numerical answer.

2. You can frequently tell if your answer to a problem is correct by noting whether or not the numerical value and the corresponding units make physical sense.

3. When you derive an equation, always check the units.

Examples:

i) The distance from the Earth to the nearest star is a) 12 light years or b) 12 feet?

ii) The wing span of a private aircraft is a) 12 meters b) 12 kilometers?

VI. References

1. E.A. Mechtley, "The International System of Units. Physical Constants and Conversion Factors, Revised", National Aeronautics and Space Administration publication NASA SP-7012, 1969.

2. B.H. Karnapp, Introduction to Dynamics, Addison-Wesley Publishing Company, 1974.

VII. Exercises

1. The numerical value 12 is the answer to many problems. You are to make up four different problems which have the numerical value 12 as its answer. Each answer should have different units. For example one of your four questions might be: "How many inches are there in one foot". The answer is obviously 12 inches.

The following three problems are taken from reference 2.

Problem 1.12

Determine whether the following equation is dimensionally homogeneous:

$$\frac{d}{dt}\int_0^x F\,dx = \frac{1}{2}\frac{dm}{dt}\,v^2 + mva\ ,$$

where F is a force, x is a distance, v is speed, a is acceleration, m is mass, and t is time.

Problem 1.13

Given that F is a force, x is displacement, θ is an angle, and v is a speed, determine the dimensions of the quantities I and k in order that the following equation is dimensionally homogeneous:

$$\int_0^x F\,dx = \frac{1}{2}\,I\,\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}\,kv^2\ .$$

Problem 1.15

The number of people p in a community as a function of time is given by the equation

$$p = k_o + k_1 t\,\sin\left(\frac{2\pi t}{\tau}\right) + k_2 e^{k_3 t}$$

The dimension of P is "people". Find the proper dimensions for the constants K_0, K_1, K_2, K_3. In this equation the variable t equals time.