Consider a satellite with a rectangular shape oriented as indicated in the figure below.

Assume speed ratio $s >> 1$ and the atmospheric properties are: temperature $T_\infty$, number density $n_\infty$ and molecular mass $m$. The flight speed is $U_\infty$ and the velocity is normal to the leading surface. Assume the fully diffuse gas interaction with the satellite surface which has a temperature $T_w$.

a) Develop an expression for the drag coefficient for the satellite, $C_d$.
b) Keeping the satellite volume constant, find the configurations that give minimum drag for two cases: $W=L$ and $W=H$.
c) Compare the minimum drag values to a cube ($W=L=H$) and a sphere of the same volume.

Solution:
For leading surface $\theta=0$. The contribution to the drag force is due to the normal momentum flux of incident and reflected molecules.

Drag force on leading surface = $(p_i + p_r) WH$.

$$p_i = \frac{\rho_\infty}{2 \sqrt{\pi} \beta^2} \left[ s e^{-s^2} + \sqrt{(\pi)} \left( 1 + \text{erf}(s) \right) \left( \frac{1}{2} + s^2 \right) \right]$$

For $s >> 1$:

$$p_i = \frac{\rho_\infty}{2 \sqrt{\pi} \beta^2} \left( \sqrt{(\pi)} 2 s^2 \right) = \frac{\rho_\infty s^2}{\beta^2} = \rho_\infty U_\infty^2$$

The normal momentum flux for diffusely reflected molecules is
\[ p_r = \frac{p_w}{2} = \rho_v \frac{U_v}{\sqrt{\frac{2}{\pi} R T_w}} \]

For the four side surfaces  \( \theta = \frac{\pi}{2} \). The contribution to the drag force is due to the parallel momentum flux of incident molecules.

Drag force on four side surfaces: \[ 2 \tau_i (W + H) \]

\[ \tau_i = \frac{\rho_v}{2 \sqrt{\pi \beta^2}} s = \frac{\rho_v U_v}{4} \sqrt{\frac{8}{\pi}} R T_w \]

For the trailing surface  \( \theta = \pi \). The normal momentum flux of incident molecules is

\[ p_{i, trailing} = \frac{\rho_v}{2 \sqrt{\pi \beta^2}} \left[ -s e^{-s^2} + \sqrt{\pi} (1 + erf (-s)) \left( \frac{1}{2} + s^2 \right) \right] \]

for \( s \gg 1 \) \( erf(-s) = -1 \) and the normal momentum flux approaches zero, therefore, contribution to the drag force is negligible in comparison to other surfaces.

Finally, the total drag on the rectangular satellite is equal to

\[ D = (p_i + p_r) WH + 2 \tau_i (W + H) L. \]

a) The drag coefficient for rectangular satellite in free molecular regime is

\[ c_d = \frac{(p_i + p_r) WH + 2 \tau_i (W + H) L}{\frac{1}{2} \rho_v U_v^2 WH} = 2 + \frac{2}{s} \sqrt{\frac{\pi}{4} \frac{T_w}{T_v}} + \frac{1}{s} \sqrt{\frac{4}{\pi} \left( \frac{L}{H} + \frac{L}{W} \right)}. \]

b) Keeping the satellite volume constant, let's find configurations that give minimum drag for two cases: \( L=W \) and \( W=H \).

For \( L=W \) case, the volume of the satellite is equal to \( V=L^2 W \) and \( H = \sqrt{\frac{V}{L}} \). Substituting this to the formula for the drag, we have

\[ D = (p_i + p_r) L \frac{V}{L^2} + 2 \tau_i (L + \frac{V}{L^2}) L = (p_i + p_r) \frac{V}{L} + 2 \tau_i (L^2 + \frac{V}{L}) \]

To find \( L \) that corresponds to the minimum drag we solve \( \frac{dD}{dL} = 0 \) making sure that

\[ \frac{d^2D}{dL^2} > 0 \]

\[ \frac{dD}{dL} = -(p_i + p_r) \frac{V}{L^2} + 2 \tau_i (2L - \frac{V}{L^2}) = 0 \] and

\[ L_{\text{min drag, } W=L} = \left[ \frac{V}{4} \left( 2 + \frac{p_i + p_r}{\tau_i} \right) \right]^{1/3} \]

and the corresponding minimum drag is

\[ D_{\text{min drag, } W=L} = \left( \frac{1}{4^{1/3}} + \frac{2}{4^{2/3}} \right) \tau_i \left( 2 + \frac{p_i + p_r}{\tau_i} \right)^{2/3} V^{2/3} \]

For \( W=H \) case, the volume of the satellite is equal to \( V= LH^2 \) and \( H=(V/L)^{1/2} \). For the drag
we have:

\[ D = (p_i + p_r) \frac{V}{L} + 2 \tau_i (\sqrt{\frac{V}{L}} + \sqrt{\frac{V}{L}}) L = (p_i + p_r) \frac{V}{L} + 4 \tau_i \sqrt{V} L \]

To find \( L \) corresponding to the minimum drag we solve the equation

\[ \frac{dD}{dL} = -(p_i + p_r) \frac{V}{L^2} + 2 \tau_i \sqrt{\frac{V}{L}} = 0 \]

and the corresponding minimum drag is

\[ D_{\text{min drag, W-H}} = (2^{2/3} + \frac{4}{2^{1/3}}) \tau_i \left( \frac{p_i + p_r}{\tau_i} \right)^{1/3} V^{2/3} \]

Note that

\[ \frac{p_i + p_r}{\tau_i} = \rho_\infty U_\infty^2 + \rho_\infty U_\infty \sqrt{\frac{\pi}{2} RT_w} = 2 \sqrt{\pi} s + \pi \sqrt{\frac{T_w}{T_\infty}}. \]

If \( s \gg 1 \) and \( T_\infty \) is fixed and the second term can be neglected in comparison to the first one and

\[ \frac{p_i + p_r}{\tau_i} \approx 2 \sqrt{\pi} s. \]

Therefore,

\[ L_{\text{min drag, W-L}} = \frac{V^{1/3}}{4^{1/3}} \left( 2 + 2 \sqrt{\pi} s \right)^{1/3} = \left( \frac{V(1 + \sqrt{\pi}) s}{2} \right)^{1/3} \]

\[ L_{\text{min drag, W-H}} = \frac{V^{1/3}}{2^{2/3}} \left( \frac{p_i + p_r}{\tau_i} \right)^{2/3} = \sqrt{\frac{\pi s}{2}} \]

c) Drag for the cube satellite (\( L=W=H \)):

\[ D_{\text{cube}} = (p_i + p_r) V^{2/3} + \tau_i V^{2/3} = \tau_i \left( 4 + \frac{p_i + p_r}{\tau_i} \right) V^{2/3} \]

Let's compare the minimum drags for \( W=L \) and \( W=H \) and the cube satellite:

\[ \frac{D_{\text{min drag, W-L}}}{D_{\text{cube}}} = \frac{\left( \frac{1}{4^{1/3}} + \frac{2}{4^{2/3}} \right) \tau_i \left( 2 + \frac{p_i + p_r}{\tau_i} \right)^{2/3} V^{2/3}}{\tau_i \left( 4 + \frac{p_i + p_r}{\tau_i} \right) V^{2/3}} = \frac{\left( \frac{1}{4^{1/3}} + \frac{2}{4^{2/3}} \right) \left( 2 + 2 \sqrt{\pi} s \right)^{2/3}}{4 + 2 \sqrt{\pi} s} \]

\[ \frac{D_{\text{min drag, W-H}}}{D_{\text{cube}}} = \frac{\left( 2^{2/3} + \frac{4}{2^{1/3}} \right) \tau_i \left( \frac{p_i + p_r}{\tau_i} \right)^{1/3} V^{2/3}}{\tau_i \left( 4 + \frac{p_i + p_r}{\tau_i} \right) V^{2/3}} = \frac{\left( 2^{2/3} + \frac{4}{2^{1/3}} \right)^{1/3}}{4 + 2 \sqrt{\pi} s} \]

Hence, for \( s \gg 1 \) the minimum drag for \( W=H \) configuration is less than that for \( W=L \) case.
The ratios of drag forces are plotted in Fig. 1 for $s > 5$. Similarly we can compare the minimum drag values to that for a sphere (Eq. (7.71) of Bird).