Moment Equation

Moment of the Boltzmann equation (also called transfer or transport equation) may be obtained by multiplying both sides of the Boltzmann equation by a molecular quantity $Q$ and integrating the resulting equation over the whole velocity space:

$$
\int_{-\infty}^{\infty} Q \frac{\partial}{\partial t} (nf) \, dv + \int_{-\infty}^{\infty} Q \mathbf{v} \cdot \frac{\partial}{\partial r} (nf) \, dv + \int_{-\infty}^{\infty} Q F \cdot \frac{\partial}{\partial \mathbf{v}} (nf) \, dv =
$$

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{4\pi} Q n^2 (f^* f_1^* - ff_1) v_r \frac{d\sigma}{d\Omega} \, d\Omega \, d\mathbf{v}_1 \, dv
$$

Remember $\int_{-\infty}^{\infty} Q f \, dv = \bar{Q}$

If $Q$ is independent of time, it can be taken inside time differentiation in the first term.

Also, since $\mathbf{v}$ is an independent variable, time differentiation may be taken out of the first integral, so it becomes:
\[
\int_{-\infty}^{\infty} Q \frac{\partial}{\partial t} (nf) \, dv = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} Qnf \, dv = \frac{\partial}{\partial t} (n \overline{Q})
\]

\(Q\) can be taken inside the derivative in the second term, and, because \(v\) is an independent variable,

\[
v \cdot \frac{\partial}{\partial t} (Qnf) = \nabla \cdot (n \, v \, Q \, f)
\]

Since \(r\) and \(v\) are independent variables, divergence can be taken out of integral sign, and the second term becomes

\[
\int_{-\infty}^{\infty} \nabla \cdot (n \, v \, Qf) \, dv = \nabla \cdot (n \, \overline{vQ})
\]

As we have seen, averages of \(v\) and \(Q\) are established through distribution function \(f\) and therefore must be treated as functions of \(r\) and \(t\).
The third term can be written as

\[ \int_{-\infty}^{\infty} Q F \cdot \frac{\partial}{\partial v} (nf) \, dv = \int_{-\infty}^{\infty} F \cdot \frac{\partial}{\partial v} (nQf) \, dv - \int_{-\infty}^{\infty} F \frac{\partial Q}{\partial v} nf \, dv \]

If \( F \) is independent of \( v \) and, since \( f = 0 \) or \( f \to 0 \) as \( c \to \infty \), the first integral vanishes and the second becomes

\[ -\int_{-\infty}^{\infty} F \frac{\partial Q}{\partial v} nf \, dv = -n F \frac{\partial Q}{\partial v} \]

The right-hand side of the moment of Boltzmann equation is called the collision integral \( \Delta[Q] \):

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{4\pi} Q n^2 (f^* f_1^* - ff_1) v_r \frac{d\sigma}{d\Omega} d\Omega \, dv_1 \, dv = J[Q] \]
We will now write down the alternative form of the collision integral to better understand its physical meaning.

Two symmetries are associated with the collision integral.

• $J[Q]$ is unchanged if collision partners (or after-collision partners) are interchanged. Mathematically, interchanging collision partners involves interchanging $v$ and $v_1$ and $Q$ and $Q_1$ (or $v^*$ and $v_1^*$ and $Q^*$ and $Q_1^*$ for after-collision partners).

• $J[Q]$ is unchanged for inverse collisions. This means that we can interchange $Q$ and $Q^*$ (or $Q_1$ and $Q_1^*$) as long as $v_1, v, f_1$, and $f$ are interchanged with $v_1^*, v^*, f_1^*$, and $f^*$.

The use of these symmetries allows one to write the collision integral in the form:

$$J[Q] = \frac{1}{2} \int \int \int n^2 (Q^* + Q_1^* - Q - Q_1) \frac{d\sigma}{d\Omega} v_r d\Omega d v_1 d v$$
The moment of the Boltzmann equation can be written as:

\[
\frac{\partial}{\partial t} (nQ) + \nabla \cdot (n\mathbf{v}Q) - nF \cdot \frac{\partial Q}{\partial \mathbf{v}} = J[Q]
\]

where

\[
J[Q] = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{4\pi} n^2 (Q^* + Q_1^* - Q - Q_1) f f_1 f_1 \nu_r \frac{d\sigma}{d\Omega} d\Omega d\mathbf{v}_1 d\mathbf{v}
\]

\(Q^* + Q_1^* - Q - Q_1\) represents the change of quantity \(Q\) as the result of a collision. The change is summed over all classes of collision and halved to account for the double counting of collisions over the summation. If the quantity \(Q\) is

- Mass \(m\)
- Momentum \(mv\)
- Energy \(1/2mv^2\)
- or linear combination of those then the conservation laws of binary elastic collision require that \(Q^* + Q_1^* - Q - Q_1 = 0\). The collision integral is then equal to 0.
Any molecular quantity $Q$ that satisfies the condition $Q + Q_1 + Q^* - Q_1^* = 0$ is called *summational invariant*. The summational invariant thus is a function of molecular velocity that satisfies generalized conservation equation. It can be proved (see, for example, Haris, “An Introduction to the Theory of the Boltzmann Equation”) that the most general form for any summational invariant is a linear combination of mass, linear momentum and energy:

$$Q = A \frac{1}{2} m \vec{v}^2 + \vec{B} \cdot m \vec{v} + C$$