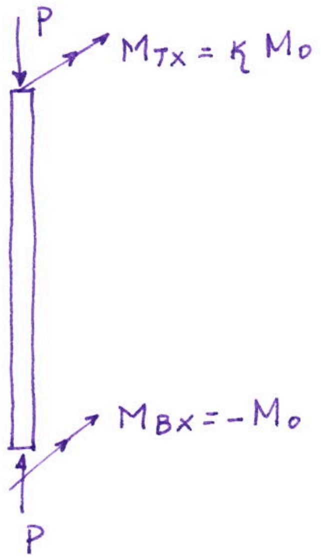
SLOPE-DEFLECTION FOR BEAM-COLUMNS



$$EI\, v^{IV} + P v'' = 0$$

$$v^{IV} + F_v^2\, v'' = 0 .$$

$$\therefore v = \frac{M_0}{P} \times \left[\frac{\eta - \cos F_v L}{\sin F_v L} \cdot \sin F_v z + \cos F_v z + \frac{z}{L}(1-\eta) - 1 \right]$$

$$\therefore v = \frac{M_0}{P}\left[\frac{\eta}{\sin F_v L} \cdot \sin(F_v z) - \frac{\eta z}{L} \right] - \frac{M_0}{P}\left[\frac{\sin F_v z}{\tan F_v L} - \cos F_v z - \frac{z}{L} + 1 \right]$$

$$\underbrace{\text{Let } M_A = M_0 \text{ (bottom)} \quad \& \quad M_B = -\eta M_0 \text{ (top)} \qquad F_v \longrightarrow k = \sqrt{P/EI} \qquad F_v L = kL = \pi\sqrt{\frac{P}{P_x}}}_{\text{transformation of variables}}$$

$$\therefore v = -\frac{M_B}{EI k^2}\left[\frac{\sin(kz)}{\sin(kL)} - \frac{z}{L} \right] - \frac{M_A}{EI k^2}\left[\frac{\sin kz}{\tan kL} - \cos kz - \frac{z}{L} + 1 \right]$$

Take one derivative:

$$v' = -\frac{M_B}{EI k^2}\left[\frac{\cos(kz)}{\sin(kL)} - \frac{1}{kL} \right] - \frac{M_A}{EI k}\left[\frac{\cos kz}{\tan kL} + \sin kz - \frac{1}{kL} \right]$$

We realize that:

$\theta_A = v'$ @ $z = 0$ and $\theta_B = v'$ @ $z = L$

$$\therefore \theta_A = -\frac{M_B}{EIk} \cdot \left[\frac{1}{\sin kL} - \frac{1}{kL}\right] - \frac{M_A}{EIk} \cdot \left[\frac{1}{\tan kL} - \frac{1}{kL}\right]$$

$$\theta_A = \frac{M_B \cdot L}{EI}\left[\frac{\sin kL - kL}{(kL)^2 \sin kL}\right] + \frac{M_A L}{EI}\left[\frac{\sin kL - kL\cos kL}{(kL)^2 \sin kL}\right]$$

Similarly,

$$\theta_B = \frac{M_B \cdot L}{EI}\left[\frac{\sin kL - kL\cos kL}{(kL)^2 \sin kL}\right] + \frac{M_A L}{EI}\left[\frac{\sin kL - kL}{(kL)^2 \sin kL}\right]$$

$$\therefore \begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{Bmatrix} M_A \\ M_B \end{Bmatrix}$$

$$f_{11} = f_{22} = \frac{L}{EI}\left[\frac{\sin kL - kL\cos kL}{(kL)^2 \sin kL}\right]$$

$$f_{12} = f_{21} = \frac{L}{EI}\left[\frac{\sin kL - kL}{(kL)^2 \sin kL}\right]$$

$$\therefore \begin{Bmatrix} M_A \\ M_B \end{Bmatrix} = \underbrace{\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}^{-1}}_{\text{stiffness matrix}} \begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix}$$

$$\begin{Bmatrix} M_A \\ M_B \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix}$$

where $$C_{11} = C_{22} = \frac{EI}{L}\left[\frac{kL\cos kL - (kL)^2\cos kL}{2 - 2\cos(kL) - kL\sin kL}\right]$$

$$C_{21} = C_{12} = \frac{EI}{L}\left[\frac{(kL)^2 - kL\sin kL}{2 - 2\cos kL - kL\sin kL}\right]$$

$$\therefore\ M_A = \frac{EI}{L}\left(S_{ii}\,\theta_A + S_{ij}\,\theta_B\right)$$

$$\&\ M_B = \frac{EI}{L}\left(S_{ij}\,\theta_A + S_{jj}\,\theta_B\right)$$

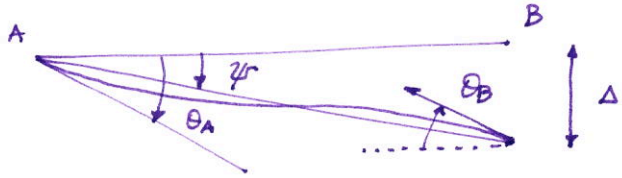
where, $$S_{ii} = S_{jj} = \frac{C_{11}L}{EI} = \frac{C_{22}L}{EI}$$

$$S_{ij} = S_{ji} = \frac{C_{12}L}{EI} = \frac{C_{21}L}{EI}$$

Stability coefficients S_{ii}, S_{ij} are $f(kL) \rightarrow f\left(\pi\sqrt{\frac{P}{P_x}}\right)$

See graph of <u>S vs. kL</u>

What happens when there is <u>SWAY</u>

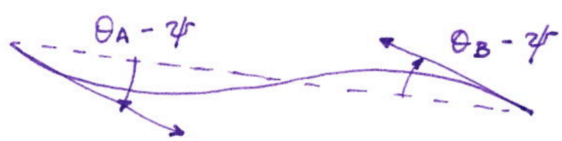


Chord rotation $= \psi = \dfrac{\Delta}{L}$

@ A, rotation w.r.t. chord $= \theta_A - \psi$

@ B, rotation w.r.t. chord $= \theta_B - \psi$

$\therefore$ We can treat the member with SWAY as a member w/o SWAY by replacing end rotations with $\theta_A - \psi$ and $\theta_B - \psi$



$$\therefore \quad M_A = \frac{EI}{L}\left[s_{ii}(\theta_A - \psi) + s_{ij}(\theta_B - \psi) \right]$$

$$M_B = \frac{EI}{L}\left[s_{ij}(\theta_A - \psi) + s_{jj}(\theta_B - \psi) \right]$$

$$\therefore \quad M_A = \frac{EI}{L}\left[s_{ii}\theta_A + s_{ij}\theta_B - (s_{ii} + s_{ij})\psi \right]$$

$$M_B = \frac{EI}{L}\left[s_{ij}\theta_A + s_{jj}\theta_B - (s_{ii} + s_{ij})\psi \right]$$

↳ Slope - deflection equations for members with <u>SWAY</u>

$r_i \longrightarrow$ Forces

$d_i \longrightarrow$ Deformations

$r_1 = P \qquad r_4 = -P$

$$r_2 = -\frac{M_{AB} + M_{BA} + P\Delta}{L} \qquad r_5 = \frac{M_{AB} + M_{BA} + P\Delta}{L}$$

$r_3 = M_{AB} \qquad r_6 = M_{BA}$

<u>global forces:</u>

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\Delta/L & -1/L & -1/L \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ \Delta/L & 1/L & 1/L \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} P \\ M_A \\ M_B \end{Bmatrix}$$

or

$$\{r\} = [T]_F \{F\}$$

<u>deformations</u>

$$\theta_A = d_3 + \frac{d_5 - d_2}{L} \qquad e = -(d_4 - d_1)$$

$$\theta_B = d_6 + \frac{d_5 - d_2}{L} \qquad \frac{\Delta}{L} = \frac{d_5 - d_2}{L}$$

$$\begin{Bmatrix} e \\ \theta_A \\ \theta_B \\ \Delta/L \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1/L & 1 & 0 & 1/L & 0 \\ 0 & -1/L & 0 & 0 & 1/L & 1 \\ 0 & -1/L & 0 & 0 & 1/L & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}$$

element deform.

global deform.

$$[D] = [T]_d \{d\}$$

$$\{F\} = [k]_e \{D\} \implies \{F\} = [k]_e [T]_d \{d\}$$

$$\therefore \{r\} = [T]_F \{F\} = \underbrace{[T]_F [k]_e [T]_d}_{\text{Element Stiffness Matrix}} \{d\}.$$

global

global.

$$\therefore \{r\} = [k]\{d\} \implies \underset{6\times6}{[k]} = \underset{6\times3}{[T]_F} \; \underset{3\times4}{[k]_e} \; \underset{4\times6}{[T]_d}$$

$$[k] = \frac{EI}{L} \begin{bmatrix} A/I & 0 & 0 & -A/I & 0 & 0 \\ 0 & \frac{2(S_{ii}+S_{ij})-(kL)^2}{L^2} & -\frac{(S_{ii}+S_{ij})}{L} & 0 & -2(S_{ii}+S_{ij})+(kL)^2 & -\frac{S_{ii}+S_{ij}}{L} \\ 0 & -\frac{S_{ii}+S_{ij}}{L} & S_{ii} & 0 & \frac{S_{ii}+S_{ij}}{L} & S_{ij} \\ -A/I & 0 & 0 & A/I & 0 & 0 \\ 0 & -\frac{2(S_{ii}+S_{ij})+(kL)^2}{L^2} & \frac{S_{ii}+S_{ij}}{L} & 0 & \frac{2(S_{ii}+S_{ij})-(kL)^2}{L^2} & \frac{S_{ii}+S_{ij}}{L} \\ 0 & -\left(\frac{S_{ii}+S_{ij}}{L}\right) & S_{ij} & 0 & \frac{S_{ii}+S_{ij}}{L} & S_{ii} \end{bmatrix}$$

Substituting the expressions for the stability functions s_{ii}, s_{ij}

$$[k] = \frac{EI}{L}\begin{bmatrix} A/I & 0 & 0 & -A/I & 0 & 0 \\ 0 & \frac{12}{L^2}\phi_1 & -\frac{6}{L}\phi_2 & 0 & -\frac{12}{L^2}\phi_1 & -\frac{6}{L}\phi_2 \\ 0 & -\frac{6}{L}\phi_2 & 4\phi_3 & 0 & \frac{6}{L}\phi_2 & 2\phi_4 \\ -A/I & 0 & 0 & A/I & 0 & 0 \\ 0 & -\frac{12}{L^2}\phi_1 & \frac{6}{L}\phi_2 & 0 & \frac{12}{L^2}\phi_1 & \frac{6}{L}\phi_2 \\ 0 & -\frac{6}{L}\phi_2 & 2\phi_4 & 0 & \frac{6}{L}\phi_2 & 4\phi_3 \end{bmatrix}$$

Symmetry

The expressions for ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are given as:

$$\phi_1 = \frac{1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}\left[\pm (kL)^2\right]^n}{12\,\phi}$$

$$\phi_2 = \frac{\frac{1}{2} + \sum_{1}^{\infty} \frac{1}{(2n+2)!}\left[\pm (kL)^2\right]^n}{6\,\phi}$$

$$\phi_3 = \frac{\frac{1}{3} + \sum_{1}^{\infty} \frac{2(n+1)}{(2n+3)!}\left[\pm (kL)^2\right]^n}{4\,\phi}$$

$$\phi_4 = \frac{\frac{1}{6} + \sum_{1}^{\infty} \frac{1}{(2n+3)!}\left[\pm (kL)^2\right]^n}{2\,\phi}$$

where,

$$\phi = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{2(n+1)}{(2n+4)!}\left[\pm (kL)^2\right]^n$$

- Chen has shown that these power series expressions are convenient & efficient to use in numerical analysis.
- No difficulties will arise if P is compressive, tensile, or zero
- The series will converge to a high degree of accuracy if $n=10$ is used.
- If the axial force is small, use a Taylor series expansion for the ϕ_i, and retain only 2 terms of the series.

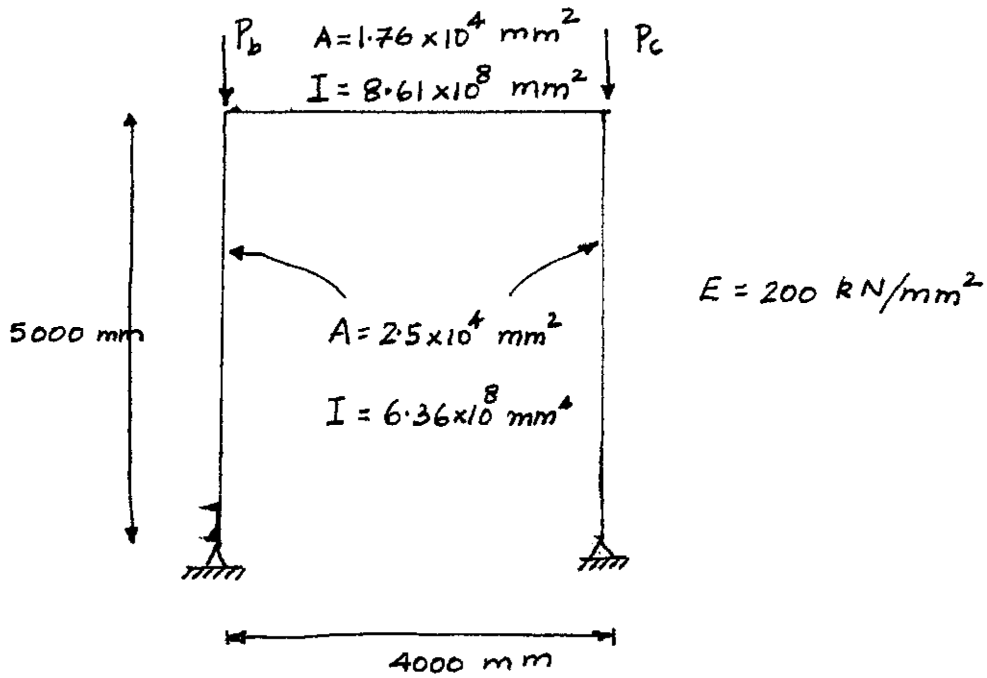
$$
[K] = \begin{bmatrix}
A/I & 0 & 0 & -A/I & 0 & 0 \\
0 & 12/L^2 & -6/L & 0 & -12/L^2 & -6/L \\
0 & -6/L & 4 & 0 & 6/L & 2 \\
-A/I & 0 & 0 & A/I & 0 & 0 \\
0 & -12/L^2 & 6/L & 0 & 12/L^2 & 6/L \\
0 & -6/L & 2 & 0 & 6/L & 4
\end{bmatrix} \cdot \frac{EI}{L}
$$

$$
\pm \frac{P}{L} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6/5 & -L/10 & 0 & -6/5 & -L/10 \\
0 & -L/10 & 2L^2/15 & 0 & L/10 & -L^2/30 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -6/5 & L/10 & 0 & 6/5 & L/10 \\
0 & -L/10 & -L^2/30 & 0 & L/10 & 2L^2/15
\end{bmatrix}
$$

$$[K] = [K]_o + [K]_g$$

original stiffness matrix + geometric stiffness matrix

- The geometric stiffness matrix accounts for the effects of axial force on bending stiffness of the member.
- You can also use it for nonlinear (elastic) force - deformation analysis of frames
- <u>Example:</u>



If $P_b = P_c$, find the critical buckling load $[P_{cr}]$ for the structure.

- Solve the eigenvalue problem to get $P = P_{cr}$ that makes $[K]_G$ so large that it offsets $[K]_0$ and makes the stiffness matrix singular.
- We need to perform eigenvalue analysis on 2 matrices $[K]_0$ and $[K]_G$. The eigenvalue will be the coefficient of $[K]_G$.

Important

- Normalize ALL THE LOADS ACTING ON THE STRUCTURE

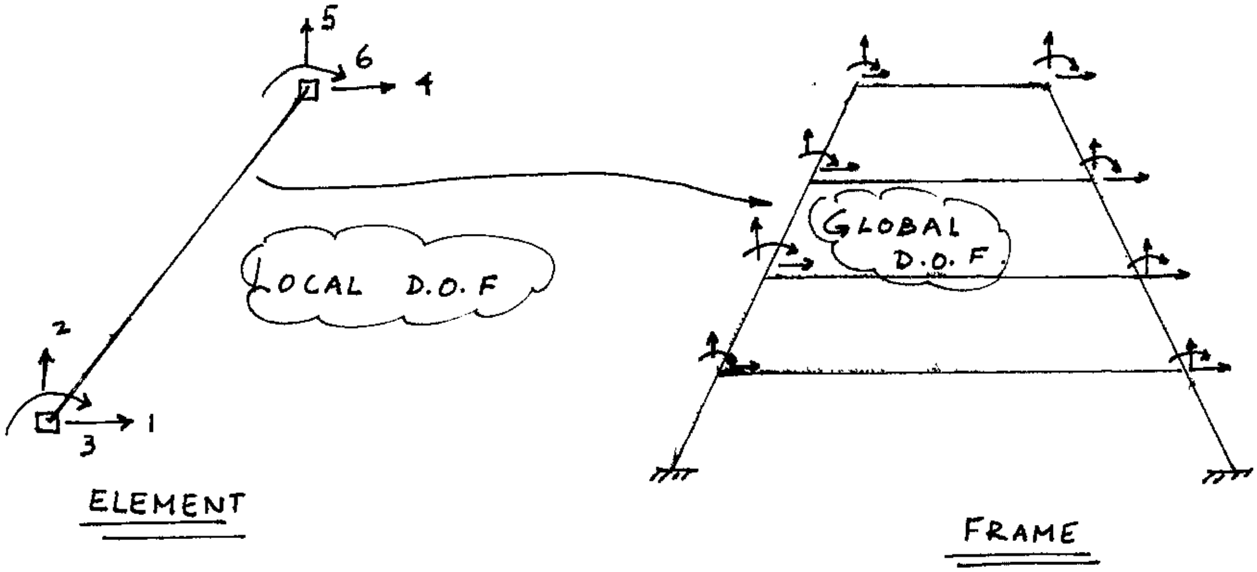
(1) Proportion all the loads acting on the structure so that the max. value is equal to unity in the units (lb, kip, N, kN, etc.) you are using.

(2) All the length based and force based units must be consistent

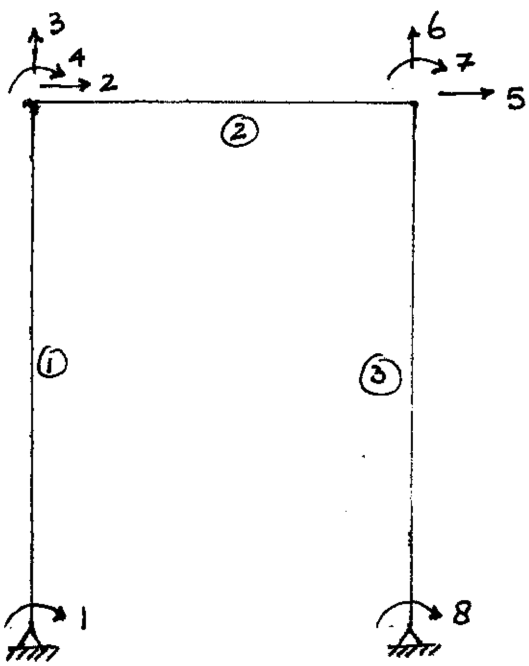
$$E, I, A, L, x_i, y_i \longrightarrow \text{CONSISTENT UNITS.}$$

(3) The critical buckling load will be the smallest value from the eigenvalue analysis.

- The corresponding eigenvector will be the mode



The local d.o.f. are related the global d.o.f. through id array.



Frame with global d.o.f.

ID ARRAY

LOCAL D.O.F.	ELEMENT ①	ELEMENT ②	ELEMENT ③
1	0	2	5
2	0	3	6
3	1	4	7
4	2	5	0
5	3	6	0
6	4	7	8

Develop a computer program to

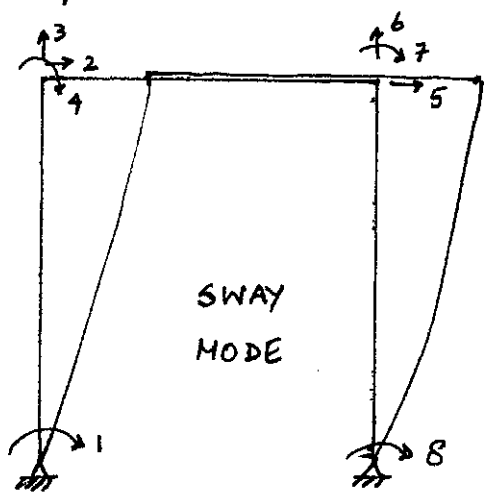
(1) ASSEMBLE structure stiffness matrix $[K]_0$ using id array and element stiffness matrix

(2) ASSEMBLE geometric stiffness matrix $[K]_G$ using id array and element geometric stiffness matrix

(3) Solve eigenvalue problem

(4) Note first two eigenvalues and sketch corresponding buckling modes:

- $P_{cr})_1 = \underline{10.34 \text{ kN}} \equiv \left\{ \begin{array}{l} 0.000287 \\ 1 \\ 0.00517 \\ 0 \\ 1 \\ -0.00517 \\ 0 \\ 0.000287 \end{array} \right\} \begin{array}{l} \rightarrow 1 \\ \rightarrow 2 \\ \rightarrow 3 \\ \rightarrow 4 \\ \rightarrow 5 \\ \rightarrow 6 \\ \rightarrow 7 \\ \rightarrow 8 \end{array}$



- $P_{cr})_2 = \underline{97 \text{ kN}} \equiv \begin{array}{l} 1 \leftarrow \\ 2 \leftarrow \\ 3 \leftarrow \\ 4 \leftarrow \\ 5 \leftarrow \\ 6 \leftarrow \\ 7 \leftarrow \\ 8 \leftarrow \end{array} \left\{ \begin{array}{l} 0.185 \\ 1 \\ 0 \\ -0.1 \\ -1 \\ 0 \\ 0.1 \\ -0.185 \end{array} \right\}$