Given:

Approximate Solution

```
>> U=a1*sin(pi*x/(2*L))+a2*sin(3*pi*x/(2*L))+a3*sin(5*pi*x/(2*L));
```

First, Second, third, and forth derivatives of U(x) with respect to x

```
>> U1=simple(diff(U,x))

U1 =

(pi*(a1*cos((pi*x)/(2*L)) + 3*a2*cos((3*pi*x)/(2*L)) + 5*a3*cos((5*pi*x)/(2*L))))/(2*L)

>> U2=simple(diff(U1,x))

U2 =

-(pi^2*(a1*sin((pi*x)/(2*L)) + 9*a2*sin((3*pi*x)/(2*L)) + 25*a3*sin((5*pi*x)/(2*L))))/(4*L^2)

>> U3=simple(diff(U2,x))

U3 =

-(pi^3*(a1*cos((pi*x)/(2*L)) + 27*a2*cos((3*pi*x)/(2*L)) + 125*a3*cos((5*pi*x)/(2*L))))/(8*L^3)

>> U4=simple(diff(U3,x))

U4 =

(pi^4*(a1*sin((pi*x)/(2*L)) + 81*a2*sin((3*pi*x)/(2*L)) + 625*a3*sin((5*pi*x)/(2*L))))/(16*L^4)
```

Natural boundary conditions

```
>> RB1=simple(subs(K*U1-E*I*U2,{x},{0}))        @ x = 0

RB1 =

(pi*K*(a1 + 3*a2 + 5*a3))/(2*L)

>> RB2=simple(subs(E*I*U3,{x},{L}))        @x = L

RB2 =

0
```

Governing Equation

```
>> RL=simple(E*I*U4+P*U2-q);
```

Collocation method

Solve three simultaneous equations for a1, a2, and a3

$$K\frac{dU}{dx} - EI\frac{d^2U}{dx^2} = 0 \ @ \ x = 0$$

$$EI\frac{d^4x}{dx^4} + P\frac{d^2x}{dx^2} - q = 0 \ @ \ x = \frac{L}{3} \ and \ \frac{2L}{3}$$

Substitute x = L/3 and 2L/3 into RB1 equation

```
>> Eq_1 = simple(subs(RL,{x},{L/3}));

>>Eq_2 = simple(subs(RL,{x},{2*L/3}));
```

Find a1, a2, and a3

```
>> a1_new=simple(Collocation.a1)

 a1_new =

 -(16*3^(1/2)*L^4*q*(4*L^2*P + 71*E*I*pi^2 - 20*3^(1/2)*L^2*P + 125*3^(1/2)*E*I*pi^2))/(9831*pi^6*E^2*I^2 - 4920*pi^4*E*I*L^2*P + 624*pi^2*L^4*P^2)
```

-0.0154

```
 >> a2_new=simple(Collocation.a2)

 a2_new =

 -(32*3^(1/2)*L^4*q*(8*L^2*P - 62*E*I*pi^2 + 12*3^(1/2)*L^2*P - 63*3^(1/2)*E*I*pi^2))/(29493*pi^6*E^2*I^2 - 14760*pi^4*E*I*L^2*P + 1872*pi^2*L^4*P^2)
```

0.0061

```
 >> a3_new = simple(Collocation.a3)

 a3_new =

 (16*3^(1/2)*L^4*q*(20*L^2*P - 53*E*I*pi^2 + 4*3^(1/2)*L^2*P - 3^(1/2)*E*I*pi^2))/(49155*pi^6*E^2*I^2 - 24600*pi^4*E*I*L^2*P + 3120*pi^2*L^4*P^2)
```

-5.7504e-004

Least square method

$$I_{LS}(a) = \int_V R_L(x,a) \cdot R_L(x,a) dV + W_B \int_A R_B(x,a) \cdot R_B(x,a) dA \geq 0$$

Find minimum by setting derivatives to zero

$$I_k(a) = \frac{\partial I_{LS}}{\partial a_k} = \int_V \frac{\partial R_L(x,a)}{\partial a_k} \cdot R_L(x,a) dV + W_B \int_A \frac{\partial R_B(x,a)}{\partial a_k} \cdot R_B(x,a) dA = 0$$

Assume weighting function

>> WB=1/L;

First derivative of R_L with respect to a1, a2, and a3

>> RL_a1=simple(diff(RL,a1));

>> RL_a2=simple(diff(RL,a2));

>> RL_a3=simple(diff(RL,a3));

First derivative of R_B with respect to a1, a2, and a3

>> RB_a1=simple(diff(RB1,a1));

>> RB_a2=simple(diff(RB1,a2));

>> RB_a3=simple(diff(RB1,a3));

Combine above equations to find I_k(a) where k = 1, 2, and 3

>> LeastSQ_a1=simple(int(RL_a1*RL,x,0,L)+int(WB*RB1*RB_a1,x,0,L));

LeastSQ_a1 =

(K^2*pi^2*(a1 + 3*a2 + 5*a3))/(4*L^2) + (pi*(4*L^2*P - E*I*pi^2)*(64*q*L^4 + 4*P*pi^3*a1*L^2 - E*I*pi^5*a1))/(512*L^7)

>> LeastSQ_a2=simple(int(RL_a2*RL,x,0,L)+int(WB*RB1*RB_a2,x,0,L))

LeastSQ_a2 =

(3*K^2*pi^2*(a1 + 3*a2 + 5*a3))/(4*L^2) + (3*pi*(4*L^2*P - 9*E*I*pi^2)*(64*q*L^4 + 108*P*pi^3*a2*L^2 - 243*E*I*pi^5*a2))/(512*L^7)

>> LeastSQ_a3=simple(int(RL_a3*RL,x,0,L)+int(WB*RB1*RB_a3,x,0,L))

LeastSQ_a3 =

(5*K^2*pi^2*(a1 + 3*a2 + 5*a3))/(4*L^2) + (5*pi*(4*L^2*P - 25*E*I*pi^2)*(64*q*L^4 + 500*P*pi^3*a3*L^2 - 3125*E*I*pi^5*a3))/(512*L^7)

Find a1, a2, and a3

>> LeastSQ=solve(LeastSQ_a1,LeastSQ_a2,LeastSQ_a3,a1,a2,a3);

>> a1_new=subs(LeastSQ.a1,{E,I,P,K,L,q},{29000,394,350,285650,120,0.1})

a1_new =

-0.0049

>> a2_new=subs(LeastSQ.a2,{E,I,P,K,L,q},{29000,394,350,285650,120,0.1})

a2_new =

0.0014

>> a3_new=subs(LeastSQ.a3,{E,I,P,K,L,q},{29000,394,350,285650,120,0.1})

a3_new =

1.1824e-004

Galerkin's method

Weighting functions are:

>> W1=sin(pi*x/(2*L));

>> W2=sin(3*pi*x/(2*L));

>> W3=sin(5*pi*x/(2*L));

Integrate general expression for volume integral by parts first:

$$I_k(a) = \int_V W_k(x) \cdot R_L(x, a) dV = 0$$

where k = 1, 2, and 3.

$$I_k(a) = \int_V W_k(x) \cdot R_L(x,a)dV$$

$$= \left[W_k(x) \cdot \left(EI\frac{d^3U}{dx^3}\right)\right]_0^L - \int_0^L W_k'(x) * \langle EI\frac{d^3U}{dx^3}\rangle\, dx + \int_0^L W_k\ (x) * \langle P\frac{dU}{dx} - q\rangle\, dx$$

$$= \left[W_k(x) \cdot \left(EI\frac{d^3U}{dx^3}\right)\right]_0^L - \left[W_k'(x) \cdot \left(EI\frac{d^2U}{dx^2}\right)\right]_0^L + \int_0^L W_k''(x) * \langle EI\frac{d^2U}{dx^2}\rangle\, dx$$

$$+ \int_0^L W_k\ (x) * \langle P\frac{dU}{dx}\rangle\, dx + \int_0^L W_k\ (x) * \langle -q\rangle\, dx$$

By introducing boundary conditions,

$$I_k(a) = \int_V W_k(x) \cdot R_L(x,a)dV =$$

$$= -W_k(x=0) * EI\frac{d^3}{dx^3}U(x=0) - W_k'(x=L) * \langle EI\frac{d^2}{dx^2}U(x=L)\rangle + W_k'(x=0)$$

$$* \langle K\frac{d}{dx}U(x=0)\rangle + \int_0^L W_k''(x) * \langle EI\frac{d^2U}{dx^2}\rangle\, dx + \int_0^L W_k\ (x) * \langle P\frac{dU}{dx}\rangle\, dx$$

$$+ \int_0^L W_k\ (x) * \langle -q\rangle\, dx$$

Calculate I_1(a), I_2(a), and I_3(a)

```
>> I_a1=-subs(W1*(E*I*U3),{x},{0})-subs(diff(W1,x)*(E*I*U2),{x},{L})+subs(diff(W1,x)*K*U1,{x},{0})+int(diff(W1,2,x)*E*I*U2,x,0,L)+int(W1*P*U2,x,0,L)-int(W1*q,x,0,L)

I_a1 =

(K*pi^2*(a1 + 3*a2 + 5*a3))/(4*L^2) - (2*L*q)/pi - (P*pi^2*a1)/(8*L) + (E*I*pi^4*a1)/(32*L^3)


>> I_a2=-subs(W2*(E*I*U3),{x},{0})-subs(diff(W2,x)*(E*I*U2),{x},{L})+subs(diff(W2,x)*K*U1,{x},{0})+int(diff(W2,2,x)*E*I*U2,x,0,L)+int(W2*P*U2,x,0,L)-int(W2*q,x,0,L)

I_a2 =

(3*K*pi^2*(a1 + 3*a2 + 5*a3))/(4*L^2) - (2*L*q)/(3*pi) - (9*P*pi^2*a2)/(8*L) + (81*E*I*pi^4*a2)/(32*L^3)

>> I_a3=-subs(W3*(E*I*U3),{x},{0})-subs(diff(W3,x)*(E*I*U2),{x},{L})+subs(diff(W3,x)*K*U1,{x},{0})+int(diff(W3,2,x)*E*I*U2,x,0,L)+int(W3*P*U2,x,0,L)-int(W3*q,x,0,L)
```

```
I_a3 =

 (5*K*pi^2*(a1 + 3*a2 + 5*a3))/(4*L^2) - (2*L*q)/(5*pi) - (25*P*pi^2*a3)/(8*L) + (625*E*I*pi^4*a3)/(32*L^3)
```

Find a1, a2, and a3

```
>> Galerkin=solve(I_a1,I_a2,I_a3,a1,a2,a3);

>>a1_new = subs(Galerkin.a1,{E,P,I,K,L,q},{29000,350,394,285650,120,0.1})

a1_new =

    0.1428

>> a2_new = subs(Galerkin.a2,{E,P,I,K,L,q},{29000,350,394,285650,120,0.1})

a2_new =

   -0.0083

>> a3_new = subs(Galerkin.a3,{E,P,I,K,L,q},{29000,350,394,285650,120,0.1})

a3_new =

   -0.0020
```

Rayleigh-Riz Method

$$\Pi = U_i - W_e$$

$$U_i = \frac{1}{2}EI\int_0^L (U'')^2 dx + \frac{1}{2}K(U'(x=0))^2$$

$$W_e = \frac{1}{2}P\int_0^L (U')^2 dx + q\int_0^L U dx$$

Internal Energy

```
>> Ui = 1/2*E*I*int(U2^2,x,0,L)+1/2*K*(subs(U1,{x},{0}))^2;
```

External Work-done

```
>> We=1/2*P*int(U1^2,x,0,L)+q*int(U,x,0,L);

>> PI=Ui-We;
```

Solve a1, a2, and a3 by solving:

$$\frac{d\Pi}{da_1} = 0, \quad \frac{d\Pi}{da_2} = 0, \quad \frac{d\Pi}{da_3} = 0$$

```
>> Eqn_1=diff(PI,a1);

>> Eqn_2=diff(PI,a2);

>> Eqn_3=diff(PI,a3);

>> Rayleigh=solve(Eqn_1,Eqn_2,Eqn_3,a1,a2,a3);

>> a1_new = subs(Rayleigh.a1,{E,P,I,K,L,q},{29000,350,394,285650,120,0.1})

a1_new =

    0.1428

>> a2_new = subs(Rayleigh.a2,{E,P,I,K,L,q},{29000,350,394,285650,120,0.1})

a2_new =

   -0.0083

>>  a3_new = subs(Rayleigh.a3,{E,P,I,K,L,q},{29000,350,394,285650,120,0.1})

a3_new =

   -0.0020
```

Summary:

	a1	a2	a3
Collocation	-0.0154	0.0061	-5.7504e-004
Least square (WB=1)	-0.0049	0.0014	1.1824e-004
Galerkin	0.1428	-0.0083	-0.0020
Rayleigh-Riz	0.1428	-0.0083	-0.0020

The approximate solutions calculated by Galerkin's and Rayleigh-Riz's method turn out to be exactly same. Also, when a very small weighting function, which can be obtained by trials and errors, is used in Least square method, the solution is similar to Galerkin's and Rayleigh-Riz's method. As shown below, the curves generated by Collocation method and Least square method with WB=1 do not represent the boundary condition of rotational spring at x = 0. In contrary, the deflection curves obtained by Galerkin's, Rayleigh-Riz's, and Least square with a small WB are well representing the boundary conditions.