CE 595:
Finite Elements in Elasticity

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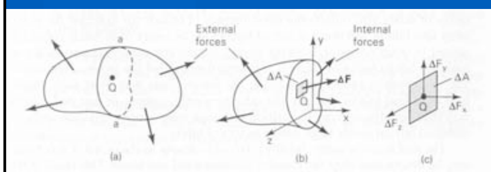
Section 1: Review of Elasticity

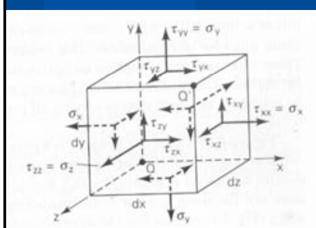
1. Stress & Strain
2. Constitutive Theory
3. Energy Methods

Section 1.1: Stress and Strain

- Stress at a point Q:



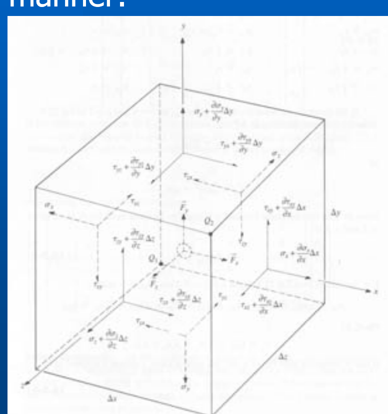
$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}; \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}; \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A}.$$



$$\text{Stress matrix } [\boldsymbol{\sigma}(Q)] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}; \text{ Stress vector } (\boldsymbol{\sigma}(Q)) = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{pmatrix}.$$

1.1: Stress and Strain (cont.)

- Stresses must satisfy equilibrium equations in pointwise manner:



$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \overline{F}_x = 0$$

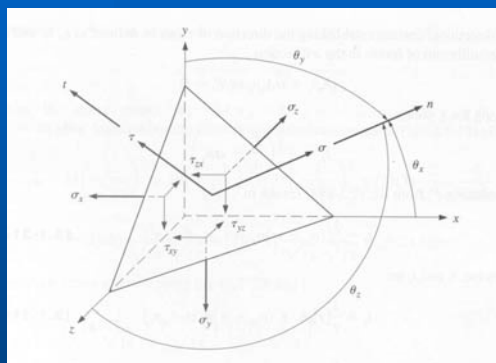
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \overline{F}_y = 0$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \overline{F}_z = 0$$

"Strong Form"

1.1: Stress and Strain (cont.)

- Stresses act on inclined surfaces as follows:



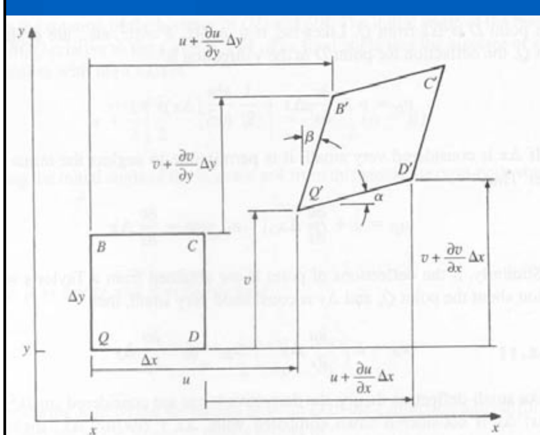
$$\mathbf{S}_{\hat{\mathbf{n}}}(Q) = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

$$= [\boldsymbol{\sigma}(Q)](\hat{\mathbf{n}}).$$

$$\sigma = \mathbf{S}_{\hat{\mathbf{n}}}(Q) \cdot \hat{\mathbf{n}}; \ \tau = \sqrt{|\mathbf{S}_{\hat{\mathbf{n}}}(Q)|^2 - \sigma^2}.$$

1.1: Stress and Strain (cont.)

- Strain at a pt. Q related to *displacements* :



$$Q:(x,y,z) \Rightarrow Q':(x',y',z')$$

Displacement functions

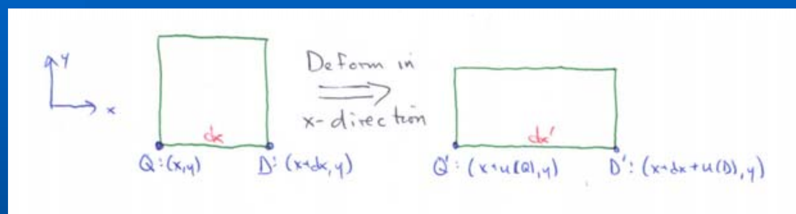
$$u(x,y,z), v(x,y,z), w(x,y,z)$$

defined by:

$$x' = x + u(x,y,z);$$
$$y' = y + v(x,y,z);$$
$$z' = z + w(x,y,z).$$

1.1: Stress and Strain (cont.)

- *Normal strain* relates to changes in *size* :



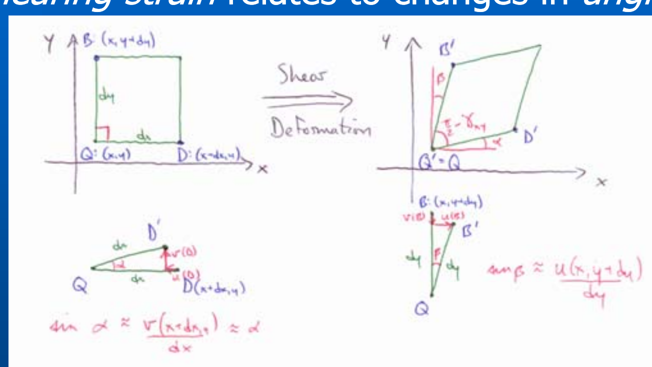
$$\varepsilon_x \equiv \frac{Q'D' - QD}{QD} = \frac{Q'D' - dx}{dx};$$

$$Q'D' = x_{D'} - x_{Q'} = x + dx + u(x+dx,y) - \left[x + u(x,y)\right] = dx + u(x+dx,y) - u(x,y).$$

$$\therefore \varepsilon_x = \frac{u(x+dx,y) - u(x,y)}{dx} \approx \frac{\partial u}{\partial x}(Q). \text{ Also, } \varepsilon_y = \frac{\partial v}{\partial y}(Q); \varepsilon_z = \frac{\partial w}{\partial z}(Q).$$

1.1: Stress and Strain (cont.)

- *Shearing strain* relates to changes in *angle* :



$$\gamma_{xy} = \alpha + \beta = \frac{v(x+dx,y)}{dx} + \frac{u(x,y+dy)}{dy} = \frac{\partial v}{\partial x}(Q) + \frac{\partial u}{\partial y}(Q). \gamma_{xz} = \frac{\partial w}{\partial x}(Q) + \frac{\partial u}{\partial z}(Q). \gamma_{yz} = \frac{\partial w}{\partial y}(Q) + \frac{\partial v}{\partial z}(Q).$$

1.1: Stress and Strain (cont.)

- Sometimes FEA programs use *elasticity shearing strains* :

$$\varepsilon_{xy} = \tfrac{1}{2}\gamma_{xy}.\ \varepsilon_{xz} = \tfrac{1}{2}\gamma_{xz}.\ \varepsilon_{yz} = \tfrac{1}{2}\gamma_{yz}.$$

- Strains must satisfy 6 *compatibility equations*:

$$\text{E.g.: } \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}.$$

(usually automatic for most formulations)

Section 1.2 : Constitutive Theory

- For linear elastic materials, stresses and strains are related by the *Generalized Hooke's Law* :

$$(\boldsymbol{\sigma}) = [\mathbf{C}]\{(\boldsymbol{\varepsilon}) - (\boldsymbol{\varepsilon})_o\} + (\boldsymbol{\sigma})_o .$$

$$(\boldsymbol{\sigma}) = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}; (\boldsymbol{\varepsilon}) = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}; [\mathbf{C}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \textit{ Elasticity matrix;}$$

$$(\boldsymbol{\sigma})_o = \textit{residual stresses}; \ (\boldsymbol{\varepsilon})_o = \textit{residual strains}.$$

1.2 : Constitutive Theory (cont.)

- For isotropic linear elastic materials, elasticity matrix takes special form:

$$[\mathbf{C}] = \frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}.$$

$$E = \text{Young's modulus, } \nu = \text{Poisson's ratio.}$$

1.2 : Constitutive Theory (cont.)

- Special cases of GHL:
  - *Plane Stress* : all "out-of-plane" stresses assumed zero.

$$(\boldsymbol{\sigma}) = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}; (\boldsymbol{\varepsilon}) = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}; [\mathbf{C}] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}.$$

$$\text{Note: } \varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right) \text{ required.}$$

  - *Plane Strain* : all "out-of-plane" strains assumed zero.

$$(\boldsymbol{\sigma}) = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}; (\boldsymbol{\varepsilon}) = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}; [\mathbf{C}]^{-1} = \frac{1-\nu^2}{E}\begin{bmatrix} 1 & -\frac{\nu}{1-\nu} & 0 \\ -\frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{2}{1-\nu} \end{bmatrix}.$$

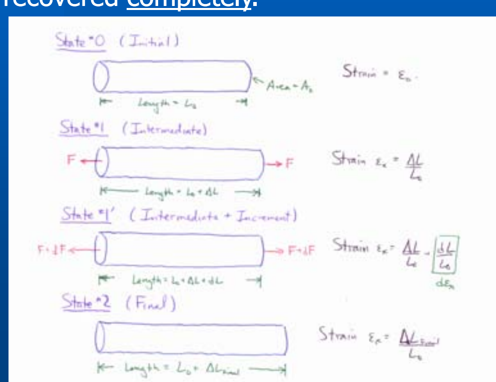
$$\text{Note: } \sigma_z = \nu\left(\sigma_x + \sigma_y\right) \text{ required.}$$

1.2 : Constitutive Theory (cont.)

- Other constitutive relations:
  - *Orthotropic* : material has "less" symmetry than isotropic case.
    FRP, wood, reinforced concrete, …
  - *Viscoelastic* : stresses in material depend on both strain and strain rate.
    Asphalt, soils, concrete (creep), …
  - *Nonlinear* : stresses not proportional to strains.
    Elastomers, ductile yielding, cracking, …

1.2 : Constitutive Theory (cont.)

- *Strain Energy*
  - Energy stored in an elastic material during deformation; can be recovered <u>completely</u>.



Work done during $1 \rightarrow 1'$:

$dW = (F + dF)(dL) \approx FdL.$

$F = \sigma_x A_o ; dL = d\varepsilon_x L_o .$

$\therefore dW = (\sigma_x d\varepsilon_x)(A_o L_o).$

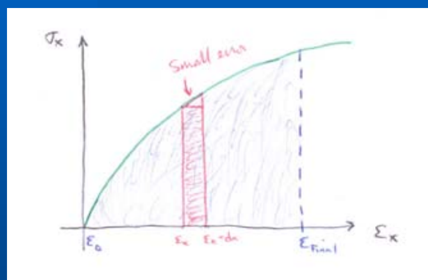
$$\Rightarrow W = (A_o L_o) \int_{\varepsilon_o}^{\varepsilon_{final}} \sigma_x d\varepsilon_x .$$

If all external work is stored,

$$U = W = (V_o) \int_{\varepsilon_o}^{\varepsilon_{final}} \sigma_x d\varepsilon_x .$$

1.2 : Constitutive Theory (cont.)

- *Strain Energy Density* : strain energy <u>per unit volume</u>.



$$\bar{U} = U/V_o = \int_{\varepsilon_o}^{\varepsilon_{final}} \sigma_x d\varepsilon_x .$$

$$\Rightarrow U = \int_{Volume} \bar{U} dV$$

- In general,

$$\bar{U} = \int_{\varepsilon_o}^{\varepsilon_{final}} \sigma_x d\varepsilon_x + \int_{\varepsilon_o}^{\varepsilon_{final}} \sigma_y d\varepsilon_y + \int_{\varepsilon_o}^{\varepsilon_{final}} \sigma_z d\varepsilon_z + \int_{\gamma_o}^{\gamma_{final}} \tau_{xy} d\gamma_{xy} + \int_{\gamma_o}^{\gamma_{final}} \tau_{yz} d\gamma_{yz} + \int_{\gamma_o}^{\gamma_{final}} \tau_{xz} d\gamma_{xz} .$$

Section 1.3 : Energy Methods

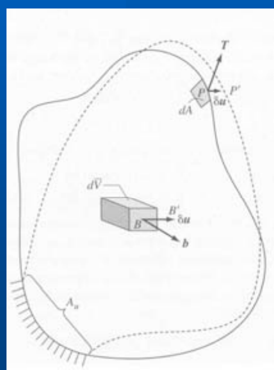
- Energy methods are techniques for satisfying equilibrium or compatibility on a <u>global</u> level rather than pointwise.

- Two general types can be identified:
  - Methods that assume equilibrium and enforce displacement compatibility.
    (Virtual force principle, complementary strain energy theorem, …)
  - Methods that assume displacement compatibility and enforce equilibrium.
    (Virtual displacement principle, Castigliano's 1st theorem, …)

Most important for FEA!

1.3 : Energy Methods (cont.)

- *Principle of Virtual Displacements* (Elastic case):
  (aka Principle of Virtual Work, Principle of Minimum Potential Energy)



- Elastic body under the action of body force **b** and surface stresses **T**.
- Apply an <u>admissible</u> virtual displacement $\delta\mathbf{u}$
  - Infinitesimal in size and speed
  - Consistent with constraints
  - Has appropriate continuity
  - Otherwise arbitrary
- PVD states that $\delta W_e = \delta W_i$ for <u>any</u> admissible $\delta\mathbf{u}$ is equivalent to static equilibrium.

1.3 : Energy Methods (cont.)

- External and Internal Work:

$$\delta W_e = \int_{volume} \mathbf{b}\cdot\boldsymbol{\delta}\mathbf{u}\,dV + \int_{surface} \mathbf{T}\cdot\boldsymbol{\delta}\mathbf{u}\,dA = \int_{volume} \mathbf{b}\cdot\boldsymbol{\delta}\mathbf{u}\,dV + \int_{surface} [\boldsymbol{\sigma}](\hat{\mathbf{n}})\cdot\boldsymbol{\delta}\mathbf{u}\,dA.$$

$$\delta W_i = \delta U = \int_{volume} \bar{U}(\boldsymbol{\delta}\mathbf{u})\,dV$$

$$= \int_{volume} \left\{\sigma_x\cdot\delta\varepsilon_x + \sigma_y\cdot\delta\varepsilon_y + \sigma_z\cdot\delta\varepsilon_z + \tau_{xy}\cdot\delta\gamma_{xy} + \tau_{yz}\cdot\delta\gamma_{yz} + \tau_{xz}\cdot\delta\gamma_{xz}\right\} dV.$$

- So, PVD for an elastic body takes the form

$$\int_{volume} \mathbf{b}\cdot\boldsymbol{\delta}\mathbf{u}\,dV + \int_{surface} [\boldsymbol{\sigma}](\hat{\mathbf{n}})\cdot\boldsymbol{\delta}\mathbf{u}\,dA = \int_{volume} (\boldsymbol{\sigma})\cdot(\boldsymbol{\delta\varepsilon})\,dV.$$

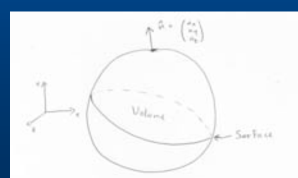
1.3 : Energy Methods (cont.)

- Recall: *Integration by Parts*

$$\int_a^b f(x)g'(x)dx = \left[f(x)g(x)\right]_a^b - \int_a^b g(x)f'(x)dx.$$

- In 3D, the corresponding rule is:

$$\int_{volume} f(x,y,z)\frac{\partial g}{\partial x}(x,y,z)dV = \int_{surface} f(x,y,z)g(x,y,z)n_x dA - \int_{volume} g(x,y,z)\frac{\partial f}{\partial x}(x,y,z)dV.$$



1.3 : Energy Methods (cont.)

- Take a closer look at internal work:

$$\delta\varepsilon_x = \frac{\partial(\delta u)}{\partial x} \Rightarrow \int_{volume}\{\sigma_x\cdot\delta\varepsilon_x\}dV = \boxed{\int_{surface}(\sigma_x\cdot\delta u)n_x dA - \int_{volume}\left\{\delta u\cdot\frac{\partial\sigma_x}{\partial x}\right\}dV}.$$

$$\delta\varepsilon_y = \frac{\partial(\delta v)}{\partial y} \Rightarrow \int_{volume}\{\sigma_y\cdot\delta\varepsilon_y\}dV = \boxed{\int_{surface}(\sigma_y\cdot\delta v)n_y dA - \int_{volume}\left\{\delta v\cdot\frac{\partial\sigma_y}{\partial y}\right\}dV}$$

$$\delta\varepsilon_z = \frac{\partial(\delta w)}{\partial z} \Rightarrow \int_{volume}\{\sigma_z\cdot\delta\varepsilon_z\}dV = \boxed{\int_{surface}(\sigma_z\cdot\delta w)n_z dA - \int_{volume}\left\{\delta w\cdot\frac{\partial\sigma_z}{\partial z}\right\}dV}$$

$$\delta\gamma_{xy} = \frac{\partial(\delta v)}{\partial x} + \frac{\partial(\delta u)}{\partial y} \Rightarrow$$

$$\int_{volume}\{\tau_{xy}\cdot\delta\gamma_{xy}\}dV = \boxed{\int_{surface}(\tau_{xy}\cdot\delta v)n_x dA - \int_{volume}\left\{\delta v\cdot\frac{\partial\tau_{xy}}{\partial x}\right\}dV} + \boxed{\int_{surface}(\tau_{xy}\cdot\delta u)n_y dA - \int_{volume}\left\{\delta u\cdot\frac{\partial\tau_{xy}}{\partial y}\right\}dV}.$$

$$\int_{volume}\{\tau_{yz}\cdot\delta\gamma_{yz}\}dV = \boxed{\int_{surface}(\tau_{yz}\cdot\delta w)n_y dA - \int_{volume}\left\{\delta w\cdot\frac{\partial\tau_{yz}}{\partial y}\right\}dV} + \boxed{\int_{surface}(\tau_{yz}\cdot\delta v)n_z dA - \int_{volume}\left\{\delta v\cdot\frac{\partial\tau_{yz}}{\partial z}\right\}dV}$$

$$\int_{volume}\{\tau_{xz}\cdot\delta\gamma_{xz}\}dV = \boxed{\int_{surface}(\tau_{xz}\cdot\delta w)n_x dA - \int_{volume}\left\{\delta w\cdot\frac{\partial\tau_{xz}}{\partial x}\right\}dV} + \boxed{\int_{surface}(\tau_{xz}\cdot\delta u)n_z dA - \int_{volume}\left\{\delta u\cdot\frac{\partial\tau_{xz}}{\partial z}\right\}dV}$$

1.3 : Energy Methods (cont.)

$$\therefore \delta W_i = \underbrace{\int_{surface} \left\{ \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \right\} \cdot \begin{pmatrix} \delta u \\ \delta v \\ \delta w \end{pmatrix} dA}_{\int_{surface} [\boldsymbol{\sigma}](\hat{\mathbf{n}})\cdot \boldsymbol{\delta}\mathbf{u} dA} - \int_{volume} \underbrace{\begin{pmatrix} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \end{pmatrix}}_{\mathbf{A}} \cdot \begin{pmatrix} \delta u \\ \delta v \\ \delta w \end{pmatrix} dV$$

$$\therefore \delta W_i = \delta W_e \Rightarrow \int_{surface} [\boldsymbol{\sigma}](\hat{\mathbf{n}}) \cdot \boldsymbol{\delta}\mathbf{u} dA - \int_{volume} \mathbf{A} \cdot \boldsymbol{\delta}\mathbf{u} dV = \int_{volume} \mathbf{b} \cdot \boldsymbol{\delta}\mathbf{u} dV + \int_{surface} [\boldsymbol{\sigma}](\hat{\mathbf{n}}) \cdot \boldsymbol{\delta}\mathbf{u} dA$$

$$\Rightarrow \int_{volume} \{\mathbf{A} + \mathbf{b}\} \cdot \boldsymbol{\delta}\mathbf{u} dV = \mathbf{0} \text{ for an } \textit{arbitrary } \boldsymbol{\delta}\mathbf{u}$$

$$\Rightarrow \mathbf{A} + \mathbf{b} = \mathbf{0} \Rightarrow \quad \begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \overline{F}_x = 0 \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \overline{F}_y = 0 \\ &\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \overline{F}_z = 0 \end{aligned}$$

- By reversing the steps, can show that the equilibrium equations imply $\delta W_i = \delta W_e$
- $\delta W_i = \delta W_e$ is called the weak form of static equilibrium.

1.3 : Energy Methods (cont.)

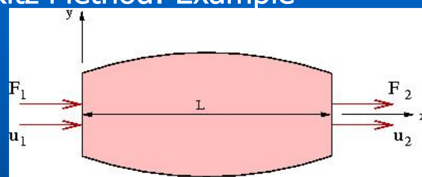
- *Rayleigh-Ritz Method* : a specific way of implementing the Principle of Virtual Displacements.
  - Define total potential energy $\Pi = W_i - W_e$; PVD is then stated as $\delta\Pi = \delta W_i - \delta W_e = 0$
  - Assume you can approximate the displacement functions as a sum of known functions with unknown coefficients.
  - Write everything in PVD in terms of virtual displacements and real displacements. (Note: stresses are real, not virtual!)
  - Using algebra, rewrite PVD in the form

$$\sum_{i=1}^{n} (\text{unknown virtual coefficient})_i * (\text{equation involving real coefficients})_i = 0.$$

  - Each unknown virtual coefficient generates one equation to solve for unknown real coefficients.

1.3 : Energy Methods (cont.)

- Rayleigh-Ritz Method: Example



Given: An axial bar has a length *L*, constant modulus of elasticity *E*, and a variable cross-sectional area given by the function $A(x) = A_o\left(1+\beta\sin\left(\frac{\pi x}{L}\right)\right)$, where β is a known parameter. Axial forces F_1 and F_2 act at $x = 0$ and $x= L$, respectively, and the corresponding displacements are u_1 and u_2 .

Required: Using the Rayleigh-Ritz method and the assumed displacement function $u(x) = u_1\left(1-\frac{x}{L}\right)+u_2\left(\frac{x}{L}\right)$, determine the equation that relates the axial forces to the axial displacements for this element.

1.3 : Energy Methods (cont.)

Solution :

1) Treat u_1 and u_2 as unknown parameters. Thus, the virtual displacement is given by

$$\delta u(x) = \delta u_1\left(1-\frac{x}{L}\right)+\delta u_2\left(\frac{x}{L}\right)$$

2) Calculate internal and external work:

$$\delta W_e = F_1\delta u_1 + F_2\delta u_2 \text{ (no body force terms).}$$

$$\delta W_i = \int_{bar} \sigma_x \delta\varepsilon_x dV = \int_{bar} \sigma_x \delta\varepsilon_x A(x)dx.$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = u_1 * \left(-\frac{1}{L}\right) + u_2 * \left(+\frac{1}{L}\right) = \frac{u_2-u_1}{L}.$$

$$\Rightarrow \delta\varepsilon_x = \frac{\delta u_2 - \delta u_1}{L} \text{ and } \sigma_x = E * \left(\frac{u_2-u_1}{L}\right).$$

1.3 : Energy Methods (cont.)

(Cont) :

2) $$\therefore \delta W_i = \int_{x=0}^{x=L} \left\{ E * \left(\tfrac{u_2-u_1}{L}\right) * \tfrac{\delta u_2-\delta u_1}{L} * A_o \left\{1+\beta\sin\left(\tfrac{\pi x}{L}\right)\right\}\right\} dx$$

$$= E * \left(\tfrac{u_2-u_1}{L}\right) * \tfrac{\delta u_2-\delta u_1}{L} * A_o * \left\{L + \tfrac{2\beta L}{\pi}\right\}$$

$$\Rightarrow \delta W_i = \delta u_2 * \left\{EA_o\left(\tfrac{u_2-u_1}{L}\right)\left(1+\tfrac{2\beta}{\pi}\right)\right\} - \delta u_1 * \left\{EA_o\left(\tfrac{u_2-u_1}{L}\right)\left(1+\tfrac{2\beta}{\pi}\right)\right\}.$$

3) Equate internal and external work:

$$F_1\delta u_1 + F_2\delta u_2 = \delta u_2 * \left\{EA_o\left(\tfrac{u_2-u_1}{L}\right)\left(1+\tfrac{2\beta}{\pi}\right)\right\} - \delta u_1 * \left\{EA_o\left(\tfrac{u_2-u_1}{L}\right)\left(1+\tfrac{2\beta}{\pi}\right)\right\}.$$

$$\left.\begin{array}{l}\text{For } \delta u_1 : F_1 = EA_o\left(\tfrac{u_1-u_2}{L}\right)\left(1+\tfrac{2\beta}{\pi}\right) \\ \text{For } \delta u_2 : F_2 = EA_o\left(\tfrac{u_2-u_1}{L}\right)\left(1+\tfrac{2\beta}{\pi}\right)\end{array}\right\} \Rightarrow \boxed{\frac{EA_o}{L}\left(1+\tfrac{2\beta}{\pi}\right)\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}\begin{pmatrix}u_1 \\ u_2\end{pmatrix} = \begin{pmatrix}F_1 \\ F_2\end{pmatrix}}$$

Section 2: Finite Element Analysis Theory

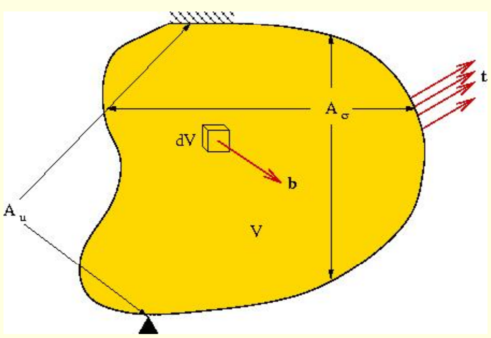
1. Method of Weighted Residuals

2. Calculus of Variations

Two distinct ways to develop the underlying equations of FEA!

Section 2: FEA Theory

- Some definitions:



- *V = volume of object*
- *A = surface area = $A_u + A_\sigma$*
- *A_u = surface of known displacements*
- *A_σ = surface of known stresses*
- ***b** = body force*
- ***t** = surface stresses (tractions)*

$$\mathbf{x} = (x, y, z); \mathbf{u}(\mathbf{x}) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}.$$

Section 2.1: Weighted Residual Methods

- A group of methods that take governing equations in the strong form and turn them into (related) statements in the weak form.
- Applicable to a wide class of problems (elasticity, heat conduction, mass flow, …).
- A "purely mathematical" concept.

2.1: Weighted residual methods (cont.)

- Need to write the equilibrium equations and boundary conditions in an abstract form as follows:

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+b_x&=0\\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{y}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+b_y&=0\\ \frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_{z}}{\partial z}+b_z&=0\end{aligned}\right\}\Rightarrow \mathbf{E}\left(\mathbf{u}\left(\mathbf{x}\right)\right)=\mathbf{0},$$

$$\left.\begin{aligned}\mathbf{u}-\mathbf{u}_0=\mathbf{0}\text{ on }A_u\\ [\boldsymbol{\sigma}](\hat{\mathbf{n}})-\mathbf{t}=\mathbf{0}\text{ on }A_\sigma\end{aligned}\right\}\Rightarrow \mathbf{B}\left(\mathbf{u}\left(\mathbf{x}\right)\right)=\mathbf{0}.$$

Solve these for $\mathbf{u}(\mathbf{x})$!

2.1: Weighted residual methods (cont.)

- Let $\mathbf{u}_{exact}(\mathbf{x})$ be the <u>exact</u> solution to the problem (differential equation <u>and</u> boundary conditions)

$$\therefore \mathbf{E}\left(\mathbf{u}_{exact}(\mathbf{x})\right) = \mathbf{0} \textit{ everywhere} \text{ in } V!$$

$$\mathbf{B}\left(\mathbf{u}_{exact}(\mathbf{x})\right) = \mathbf{0} \textit{ everywhere} \text{ on } A!$$

- Then, for <u>any</u> choice of vectors **W** and **W'**:

$$\mathbf{W} \cdot \mathbf{E}\left(\mathbf{u}_{exact}(\mathbf{x})\right) = 0 \textit{ everywhere} \text{ in } V!$$

$$\mathbf{W}' \cdot \mathbf{B}\left(\mathbf{u}_{exact}(\mathbf{x})\right) = 0 \textit{ everywhere} \text{ on } A!$$

2.1: Weighted residual methods (cont.)

- Integrate these "results" over the entire volume and surface: $\int_V \mathbf{W} \cdot \mathbf{E}\left(\mathbf{u}_{exact}(\mathbf{x})\right) dV + \int_A \mathbf{W}' \cdot \mathbf{B}\left(\mathbf{u}_{exact}(\mathbf{x})\right) dA = 0$

- Previous expression is still true if **W** and **W'** are functions of **x** (called *weighting functions):*

$$\mathbf{W}(\mathbf{x}) = \begin{pmatrix} W_1(x,y,z) \\ W_2(x,y,z) \\ W_3(x,y,z) \end{pmatrix}, \mathbf{W}'(\mathbf{x}) = \begin{pmatrix} W_1'(x,y,z) \\ W_2'(x,y,z) \\ W_3'(x,y,z) \end{pmatrix}$$

$$\Rightarrow \int_V \mathbf{W}(\mathbf{x}) \cdot \mathbf{E}\left(\mathbf{u}_{exact}(\mathbf{x})\right) dV + \int_A \mathbf{W}'(\mathbf{x}) \cdot \mathbf{B}\left(\mathbf{u}_{exact}(\mathbf{x})\right) dA = 0$$

2.1: Weighted residual methods (cont.)

- Now, consider an <u>approximate</u> solution to the same problem:

$$\begin{pmatrix} u_{approx}(x,y,z) \\ v_{approx}(x,y,z) \\ w_{approx}(x,y,z) \end{pmatrix} = \mathrm{a}_1 * \begin{pmatrix} N_{11}(x,y,z) \\ N_{21}(x,y,z) \\ N_{31}(x,y,z) \end{pmatrix} + \mathrm{a}_2 * \begin{pmatrix} N_{12}(x,y,z) \\ N_{22}(x,y,z) \\ N_{32}(x,y,z) \end{pmatrix} + \ldots + \mathrm{a}_n * \begin{pmatrix} N_{1n}(x,y,z) \\ N_{2n}(x,y,z) \\ N_{3n}(x,y,z) \end{pmatrix} \approx \begin{pmatrix} u_{exact}(x,y,z) \\ v_{exact}(x,y,z) \\ w_{exact}(x,y,z) \end{pmatrix}.$$

$$\Rightarrow \mathbf{u}_{approx}(\mathbf{x}) = \sum_{k=1}^{n} \boxed{\mathrm{a}_k} \boxed{\mathbf{N}_k(\mathbf{x})} \approx \mathbf{u}_{exact}(\mathbf{x})$$

Known functions

Unknown constants

- Matrix/vector form of this:

$$\begin{pmatrix} u_{approx}(x,y,z) \\ v_{approx}(x,y,z) \\ w_{approx}(x,y,z) \end{pmatrix} = \underbrace{\begin{bmatrix} N_{11}(x,y,z) & N_{12}(x,y,z) & \cdots & N_{1n}(x,y,z) \\ N_{21}(x,y,z) & N_{22}(x,y,z) & \cdots & N_{2n}(x,y,z) \\ N_{31}(x,y,z) & N_{32}(x,y,z) & \cdots & N_{3n}(x,y,z) \end{bmatrix}}_{\text{known functions}} \underbrace{\begin{pmatrix} \mathrm{a}_1 \\ \mathrm{a}_2 \\ \vdots \\ \mathrm{a}_n \end{pmatrix}}_{\text{unknowns}} \approx \begin{pmatrix} u_{exact}(x,y,z) \\ v_{exact}(x,y,z) \\ w_{exact}(x,y,z) \end{pmatrix}.$$

$$\Rightarrow \mathbf{u}_{approx}(\mathbf{x}) = [\mathbf{N}(\mathbf{x})](\mathbf{a}) \approx \mathbf{u}_{exact}(\mathbf{x})$$

2.1: Weighted residual methods (cont.)

- Plugging this approximate solution into the differential equation and boundary conditions results in some errors, called the *residuals.*

$$\mathbf{R}_E(\mathbf{x},\mathbf{a}) = \mathbf{E}\big(\mathbf{u}_{approx}(\mathbf{x})\big) \neq \mathbf{0} \text{ in } V!$$

$$\mathbf{R}_B(\mathbf{x},\mathbf{a}) = \mathbf{B}\big(\mathbf{u}_{approx}(\mathbf{x})\big) \neq \mathbf{0} \text{ on } A!$$

- Repeating the previous process now gives us an integral <u>close to but not exactly equal to zero</u>!

$$I(\mathbf{a}) = \int_V \mathbf{W}(\mathbf{x}) \cdot \mathbf{R}_E(\mathbf{x},\mathbf{a})\, dV + \int_A \mathbf{W}'(\mathbf{x}) \cdot \mathbf{R}_B(\mathbf{x},\mathbf{a})\, dA \neq 0 \,.$$

2.1: Weighted residual methods (cont.)

- Goal: Find the value of **a** that makes this integral as close as possible to zero – “best approximation”
- Idea: for *n* different choices of the weighting functions, derive an equation for **a** by requiring that the above integral equal zero:

$$\text{Equation \#1: } I_1(\mathbf{a}) = \int_V \mathbf{W}_1(\mathbf{x})\cdot\mathbf{R}_E(\mathbf{x},\mathbf{a})\,dV + \int_A \mathbf{W}_1'(\mathbf{x})\cdot\mathbf{R}_B(\mathbf{x},\mathbf{a})\,dA = 0\ .$$

$$\text{Equation \#2: } I_2(\mathbf{a}) = \int_V \mathbf{W}_2(\mathbf{x})\cdot\mathbf{R}_E(\mathbf{x},\mathbf{a})\,dV + \int_A \mathbf{W}_2'(\mathbf{x})\cdot\mathbf{R}_B(\mathbf{x},\mathbf{a})\,dA = 0\ .$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\text{Equation \#n: } I_n(\mathbf{a}) = \int_V \mathbf{W}_n(\mathbf{x})\cdot\mathbf{R}_E(\mathbf{x},\mathbf{a})\,dV + \int_A \mathbf{W}_n'(\mathbf{x})\cdot\mathbf{R}_B(\mathbf{x},\mathbf{a})\,dA = 0\ .$$

- Solve these equations for **a**!

2.1: Weighted residual methods (cont.)

- Notes on weighted residual methods:
  - It is typical (but not required) to assume that the known functions satisfy the displacement boundary conditions exactly on A_u. (*Essential conditions*)

$$I_k(\mathbf{a}) = \int_V \mathbf{W}_k(\mathbf{x})\cdot\mathbf{R}_E(\mathbf{x},\mathbf{a})\,dV + \int_{A_\sigma} \mathbf{W}_k'(\mathbf{x})\cdot\mathbf{R}_B(\mathbf{x},\mathbf{a})\,dA = 0\ ,\ k = 1,2,\ldots,n.$$

  - In some methods, one must integrate the volume integral by parts to get “appropriate” equations.
  - Different methods result from different ideas about how to choose the weighting functions.

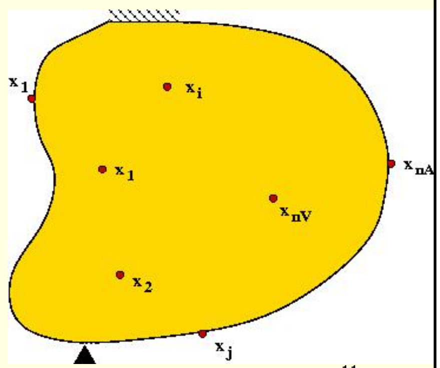
2.1: Weighted residual methods (cont.)

1. Collocation Method:
   - Assume only <u>one</u> PDE and <u>one</u> BC to solve!

$$\mathbf{R}_E(\mathbf{x},\mathbf{a}) = R_E(\mathbf{x},\mathbf{a}); \mathbf{R}_B(\mathbf{x},\mathbf{a}) = R_B(\mathbf{x},\mathbf{a}).$$

   - Idea: pick *n* points in object (at least one in *V* and one on *A)* and <u>require</u> residual to be zero at each point!

$$R_E(\mathbf{x}_i,\mathbf{a})=0,\ i=1,2,\ldots,n_V.$$
$$R_B(\mathbf{x}_j,\mathbf{a})=0,\ j=1,2,\ldots,n_A.$$
$$(n_V + n_A = n.)$$



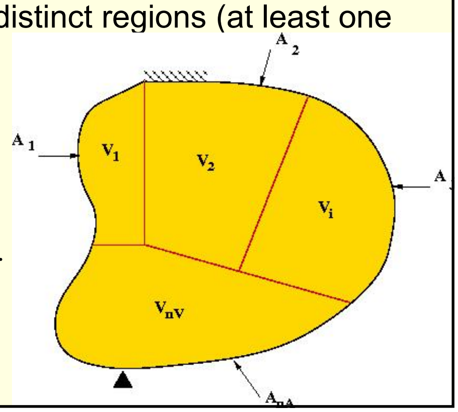
2.1: Weighted residual methods (cont.)

2. Subdomain Method:
   - Assume only <u>one</u> PDE and <u>one</u> BC to solve!

$$\mathbf{R}_E(\mathbf{x},\mathbf{a}) = R_E(\mathbf{x},\mathbf{a}); \mathbf{R}_B(\mathbf{x},\mathbf{a}) = R_B(\mathbf{x},\mathbf{a}).$$

   - Divide object up into *n* distinct regions (at least one in *V* and one on *A).*
   - Require integral over <u>each region</u> to be zero.

$$\tilde{I}_i(\mathbf{a}) = \int_{V_i} R_E(\mathbf{x},\mathbf{a})\,dV = 0,\ i=1,2,\ldots,n_V$$
$$\tilde{I}_j(\mathbf{a}) = \int_{A_j} R_B(\mathbf{x},\mathbf{a})\,dA = 0,\ j=1,2,\ldots,n_A.$$
$$(n_V + n_A = n.)$$



2.1: Weighted residual methods (cont.)

- Notes on collocation and subdomain methods:
  - Weighting functions for collocation method are the Dirac delta functions:

$$\mathbf{W}_i(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}_i),\ i=1,2,\ldots,n_V.\ \mathbf{W}'_j(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}_j),\ j=1,2,\ldots,n_A.$$

  - Weighting functions for subdomain method are the indicator functions:

$$\mathbf{W}_i(\mathbf{x}) = \begin{cases} 1 \text{ if } \mathbf{x}_i \in V_i \\ 0 \text{ if } \mathbf{x}_i \notin V_i \end{cases},\ i=1,2,\ldots,n_V.\ \mathbf{W}'_j(\mathbf{x}) = \begin{cases} 1 \text{ if } \mathbf{x}_j \in A_j \\ 0 \text{ if } \mathbf{x}_i \notin A_j \end{cases},\ j=1,2,\ldots,n_A.$$

  - Advantage: Simple to formulate.
  - Disadvantage: Used mostly for problems with only one governing equation (axial bar, beam, heat,…).

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2.1: Weighted residual methods (cont.)

3. Least Squares Method:
   - Considers magnitude of residual over the object.

$$I_{LS}(\mathbf{a}) = \int_V \mathbf{R}_E(\mathbf{x},\mathbf{a}) \cdot \mathbf{R}_E(\mathbf{x},\mathbf{a})\, dV + \alpha \int_{A_\sigma} \mathbf{R}_B(\mathbf{x},\mathbf{a}) \cdot \mathbf{R}_B(\mathbf{x},\mathbf{a})\, dA \geq 0.$$

   - Finds minimum by setting derivatives to zero

$$I_k(\mathbf{a}) = \frac{\partial I_{LS}(\mathbf{a})}{\partial \mathrm{a}_\mathrm{k}} = \int_V \underbrace{\boxed{\frac{\partial \mathbf{R}_E}{\partial \mathrm{a}_\mathrm{k}}(\mathbf{x},\mathbf{a})}}_{=\mathbf{W}(\mathbf{x})} \cdot \mathbf{R}_E(\mathbf{x},\mathbf{a})\, dV + \alpha \int_{A_\sigma} \underbrace{\boxed{\frac{\partial \mathbf{R}_B}{\partial \mathrm{a}_\mathrm{k}}(\mathbf{x},\mathbf{a})}}_{=\mathbf{W}'(\mathbf{x})} \cdot \mathbf{R}_B(\mathbf{x},\mathbf{a})\, dA = 0\ .$$

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2.1: Weighted residual methods (cont.)

4. Galerkin's Method:
   - Idea: Project residual of differential equation onto original approximating functions!

$$\mathbf{W}_k\left(\mathbf{x}\right)=\frac{\partial \mathbf{u}_{approx}\left(\mathbf{x}\right)}{\partial \mathrm{a}_\mathrm{k}}=\mathbf{N}_k\left(\mathbf{x}\right),\ k=1,2,\ldots,n.$$

   - To get **W'**, must integrate any derivatives in volume integral by parts!

$$\text{Let } \mathbf{R}_E\left(\mathbf{x},\mathbf{a}\right)=\mathbf{R}_{E,deriv}\left(\mathbf{x},\mathbf{a}\right)+\mathbf{R}_{E,noderiv}\left(\mathbf{x},\mathbf{a}\right);$$

$$\tilde{\mathbf{R}}_{E,deriv}\left(\mathbf{x},\mathbf{a}\right)=\int \mathbf{R}_{E,deriv}\left(\mathbf{x},\mathbf{a}\right)dx.\ \text{(Assume derivative involves } x.)$$

2.1: Weighted residual methods (cont.)

$$\therefore \int_V \mathbf{N}_k\left(\mathbf{x}\right)\cdot\mathbf{R}_E\left(\mathbf{x},\mathbf{a}\right)dV=\underbrace{\int_A \mathbf{N}_k\left(\mathbf{x}\right)\cdot\tilde{\mathbf{R}}_{E,deriv}\left(\mathbf{x},\mathbf{a}\right)n_x dA}_{\text{Introduces the boundary conditions!}}$$

$$-\int_V \frac{\partial \mathbf{N}_k\left(\mathbf{x}\right)}{\partial x}\cdot\tilde{\mathbf{R}}_{E,deriv}\left(\mathbf{x},\mathbf{a}\right)dV$$

$$+\int_V \mathbf{N}_k\left(\mathbf{x}\right)\cdot\mathbf{R}_{E,noderiv}\left(\mathbf{x},\mathbf{a}\right)dV$$

$$\therefore I_k\left(\mathbf{a}\right)=\int_V \mathbf{N}_k\left(\mathbf{x}\right)\cdot\mathbf{R}_E\left(\mathbf{x},\mathbf{a}\right)dV=0\ ,\ k=1,2,\ldots,n.$$

- *Must use integrated-by-parts version of* $I_k\left(\mathbf{a}\right)$ *!*

2.1: Weighted residual methods (cont.)

- Notes on least square and Galerkin methods:
  - More widely used than collocation and subdomain, since they are truly global methods.
  - For least squares method:
    - Equations to solve for **a** are always symmetric but tend to be ill-conditioned.
    - Approximate solution needs to be very smooth.
  - For Galerkin's method:
    - Equations to solve for **a** are usually symmetric but much more "robust".
    - Integrating by parts produces "less smooth" version of approximate solution; more useful for FEA!

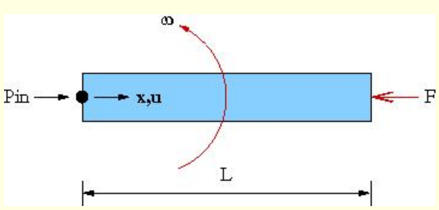
2.1: Weighted residual methods (cont.)

- Example: 1D Axial Rod "dynamics"
  - Given: Axial rod has constant density ρ, area *A*, length *L*, and spins at constant rate ω. It is pinned at *x = 0* and has applied force *-F* at *x = L*. The governing equation and boundary conditions for the steady-state rotation of the rod are:

$$E\frac{d^2u}{dx^2}+\rho\omega^2 x=0 \text{ for } 0<x<L;$$

$$u(x=0)=0; E\frac{du}{dx}(x=L)=-\frac{F}{A}.$$

  - Required: Using each of the four weighted residual methods and the approximate solution $u(x)=a_1x+a_2x^2$ estimate the displacement of the rod.



2.1: Weighted residual methods (cont.)

- Some preliminaries:
  - Problem has an exact solution given by

$$u(x) = u_o * \left\{ \lambda \left[\tfrac{1}{2}\left(\tfrac{x}{L}\right) - \tfrac{1}{6}\left(\tfrac{x}{L}\right)^3 \right] - \left(\tfrac{x}{L}\right) \right\}; u_o = \frac{FL}{EA} \lambda = \frac{\rho A \omega^2 L^2}{F}.$$

  - Approximate solution satisfies *essential* boundary condition *u(x = 0) = 0.*
  - Two unknown constants → *n* = 2.
  - Notation:

$$V = \{0 < x < L\}; A_u = \{x = 0\}; A_\sigma = \{x = L\}.$$

$$\mathbf{E}(\mathbf{u}) = E\frac{d^2u}{dx^2} + \rho\omega^2 x; \mathbf{B}(\mathbf{u}) = E\frac{du}{dx} + \frac{F}{A}.$$

2.1: Weighted residual methods (cont.)

- Solution:
  1. Collocation Method --
     - Since n=2, have two collocation points. One must be at $x = L$ (must have one on A_σ). Assume other at $x = L/3$.
     - Equation #1: evaluate residual of **E**(**u**) at $x = L/3$:

$$R_E(x, \mathbf{a}) = \mathbf{E}(\mathbf{u}_{approx}) = E\frac{d^2}{dx^2}\left[a_1 x + a_2 x^2\right] + \rho\omega^2 x = 2E a_2 + \rho\omega^2 x.$$

$$\therefore \text{Equation \#1 is: } 2E a_2 + \tfrac{1}{3}\rho\omega^2 L = 0.$$

     - Equation #2: evaluate residual of **B**(**u**) at $x = L$:

$$R_B(x, \mathbf{a}) = \mathbf{B}(\mathbf{u}_{approx}) = E\frac{d}{dx}\left[a_1 x + a_2 x^2\right] + \frac{F}{A} = E a_1 + 2E a_2 x + \frac{F}{A}.$$

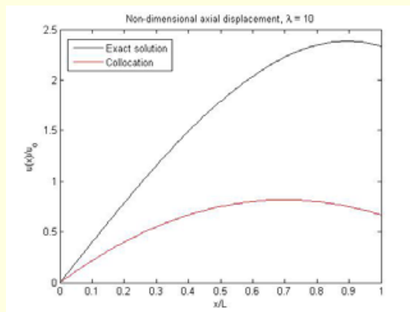
$$\therefore \text{Equation \#1 is: } E a_1 + 2E a_2 L + \frac{F}{A} = 0.$$

2.1: Weighted residual methods (cont.)

- Solution:
  1. Collocation Method --
     - Solve simultaneous equations:

$$a_1 = -\frac{F}{EA} + \frac{\rho\omega^2 L^2}{3E};\ a_2 = -\frac{\rho\omega^2 L}{6E} \Rightarrow u_{approx}(x) = u_o * \left\{ \lambda\left[\tfrac{1}{3}\left(\tfrac{x}{L}\right) - \tfrac{1}{6}\left(\tfrac{x}{L}\right)^2 \right] - \left(\tfrac{x}{L}\right) \right\}.$$

     - Plot results:



2.1: Weighted residual methods (cont.)

- Solution:
  2. Subdomain Method --
     - Since n=2, have two subdomains. One must be at $x = L$ (= A_σ). Other must be $0 < x < L$ (= V).
     - Equation #1: integrate residual of **E**(**u**) over *V*:

$$R_E(x, \mathbf{a}) = 2Ea_2 + \rho\omega^2 x \Rightarrow \tilde{I}_1(\mathbf{a}) = \int_0^L \left\{ 2Ea_2 + \rho\omega^2 x \right\} dx = 2ELa_2 + \tfrac{1}{2}\rho\omega^2 L^2.$$

$$\therefore \text{Equation \#1 is: } 2ELa_2 + \tfrac{1}{2}\rho\omega^2 L^2 = 0.$$

     - Equation #2: evaluate residual of **B**(**u**) at $x = L$:

$$R_B(x, \mathbf{a}) = \mathbf{B}(\mathbf{u}_{approx}) = E\frac{d}{dx}\left[a_1 x + a_2 x^2 \right] + \frac{F}{A} = Ea_1 + 2Ea_2 x + \frac{F}{A}.$$

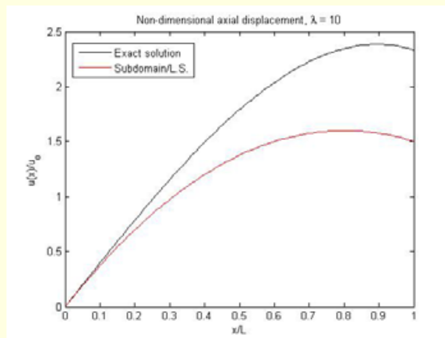
$$\therefore \text{Equation \#1 is: } Ea_1 + 2Ea_2 L + \frac{F}{A} = 0.$$

2.1: Weighted residual methods (cont.)

- Solution:
  2. Subdomain Method --
     - Solve simultaneous equations:

$$a_1 = -\frac{F}{EA} + \frac{\rho\omega^2 L^2}{2E};\ a_2 = -\frac{\rho\omega^2 L}{4E} \Rightarrow u_{approx}(x) = u_o * \left\{\lambda\left[\tfrac{1}{2}\left(\tfrac{x}{L}\right) - \tfrac{1}{4}\left(\tfrac{x}{L}\right)^2\right] - \left(\tfrac{x}{L}\right)\right\}.$$

     - Plot results:



2.1: Weighted residual methods (cont.)

- Solution:
  3. Least Squares Method --
     - For dimensional equality, take $\alpha = 1/L$ in $I_{LS}(\mathbf{a})$. Once again, the "integral" over A_σ is just evaluation at $x = L$.

$$I_k(\mathbf{a}) = \frac{\partial I_{LS}(\mathbf{a})}{\partial \mathrm{a_k}} = \int_V \frac{\partial \mathbf{R}_E}{\partial \mathrm{a_k}}(\mathbf{x},\mathbf{a}) \cdot \mathbf{R}_E(\mathbf{x},\mathbf{a})\,dV + \frac{1}{L}\frac{\partial \mathbf{R}_B}{\partial \mathrm{a_k}}(x = L,\mathbf{a}) \cdot \mathbf{R}_B(x = L,\mathbf{a}) = 0.$$

     - Equation #1: take derivative with respect to a_1:

$$\frac{\partial R_E}{\partial \mathrm{a}_1}(x,\mathbf{a}) = \frac{\partial}{\partial \mathrm{a}_1}\left\{2E\mathrm{a}_2 + \rho\omega^2 x\right\} = 0;\ \frac{\partial R_B}{\partial \mathrm{a}_1}(x,\mathbf{a}) = \frac{\partial}{\partial \mathrm{a}_1}\left\{E\mathrm{a}_1 + 2E\mathrm{a}_2 x + \frac{F}{A}\right\} = E.$$

$$\text{Equation \#1 is: } \frac{E}{L}\left\{E\mathrm{a}_1 + 2E\mathrm{a}_2 x + \frac{F}{A}\right\}_{x=L} = \frac{E}{L}\left\{E\mathrm{a}_1 + 2E\mathrm{a}_2 L + \frac{F}{A}\right\} = 0.$$

2.1: Weighted residual methods (cont.)

- Solution:
  3. Least Squares Method --
     - Equation #2: take derivative with respect to a_2:

$$\frac{\partial R_E}{\partial a_2}(x,\mathbf{a}) = \frac{\partial}{\partial a_2}\left\{2Ea_2 + \rho\omega^2 x\right\} = 2E; \frac{\partial R_B}{\partial a_2}(x,\mathbf{a}) = \frac{\partial}{\partial a_2}\left\{Ea_1 + 2Ea_2 x + \frac{F}{A}\right\} = 2Ex.$$

$$I_2(\mathbf{a}) = \int_0^L 2E\left\{2Ea_2 + \rho\omega^2 x\right\}dx + \left[\frac{2Ex}{L}\left\{Ea_1 + 2Ea_2 x + \frac{F}{A}\right\}\right]_{x=L} = 2E^2 a_1 + 8E^2 a_2 L + E\rho\omega^2 L^2 + \frac{2EF}{A}.$$

So Equation #2 is $2E^2 a_1 + 8E^2 a_2 L + E\rho\omega^2 L^2 + \frac{2EF}{A} = 0.$

- Solve equations:

$$a_1 = -\frac{F}{EA} + \frac{\rho\omega^2 L^2}{2E};\ a_2 = -\frac{\rho\omega^2 L}{4E} \Rightarrow u_{approx}(x) = u_o * \left\{\lambda\left[\tfrac{1}{2}\left(\tfrac{x}{L}\right) - \tfrac{1}{4}\left(\tfrac{x}{L}\right)^2\right] - \left(\tfrac{x}{L}\right)\right\}.$$

Same as subdomain method!

2.1: Weighted residual methods (cont.)

- Solution:
  4. Galerkin's Method --
     - Weighting functions are

$$\mathbf{W}_1(\mathbf{x}) = N_1(x) = x;\ \mathbf{W}_2(\mathbf{x}) = N_2(x) = x^2.$$

- Integrate general expression for volume integral by parts <u>first</u>:

$$\int_V \mathbf{N}_k(\mathbf{x})\cdot\mathbf{R}_E(\mathbf{x},\mathbf{a})\,dV = \int_0^L N_k(x) * \left\{Eu''_{approx}(x) + \rho\omega^2 x\right\}dx$$

$$\boxed{\begin{aligned} &= \left[N_k(x)Eu'_{approx}(x)\right]_{x=0}^{x=L} \\ &- \int_0^L N'_k(x) * \left\{Eu'_{approx}(x)\right\}dx \\ &+ \int_0^L N_k(x) * \left\{\rho\omega^2 x\right\}dx. \end{aligned}}$$

Set this equal to zero for k = 1,2!

2.1: Weighted residual methods (cont.)

- Solution:
  4. Galerkin's Method --
     - Equation #1 uses $N_1(x)=x$ in previous:

$$I_1(\mathbf{a}) = \left[x * \left\{Eu'_{approx}(x)\right\}\right]_{x=0}^{x=L} - \int_0^L 1 * \left\{E(a_1 + 2a_2 x)\right\} dx + \int_0^L x * \left\{\rho\omega^2 x\right\} dx = 0.$$

$$\Rightarrow \text{ Equation \#1 is } \; -Ea_1 L - Ea_2 L^2 + \tfrac{1}{3}\rho\omega^2 L^3 - \frac{FL}{A} = 0.$$

     - Equation #2 uses $N_2(x)=x^2$ in previous:

$$I_2(\mathbf{a}) = \left[x^2 * \left\{Eu'_{approx}(x)\right\}\right]_{x=0}^{x=L} - \int_0^L 2x * \left\{E(a_1 + 2a_2 x)\right\} dx + \int_0^L x^2 * \left\{\rho\omega^2 x\right\} dx = 0.$$

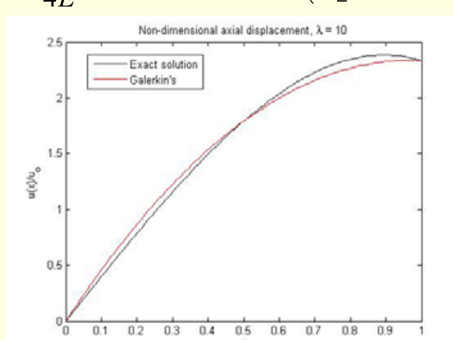
$$\Rightarrow \text{ Equation \#2 is } \; -Ea_1 L^2 - \tfrac{4}{3}Ea_2 L^3 + \tfrac{1}{4}\rho\omega^2 L^4 - \frac{FL^2}{A} = 0.$$

2.1: Weighted residual methods (cont.)

- Solution:
  4. Galerkin's Method --
     - Solve simultaneous equations:

$$a_1 = -\frac{F}{EA} + \frac{7\rho\omega^2 L^2}{12E};\; a_2 = -\frac{\rho\omega^2 L}{4E} \Rightarrow u_{approx}(x) = u_o * \left\{\lambda\left[\tfrac{7}{12}\left(\tfrac{x}{L}\right) - \tfrac{1}{4}\left(\tfrac{x}{L}\right)^2\right] - \left(\tfrac{x}{L}\right)\right\}.$$

     - Plot results:



Section 2: Finite Element Analysis Theory

1. Method of Weighted Residuals

2. Calculus of Variations

Two distinct ways to develop the underlying equations of FEA!

Section 2.2: Calculus of Variations

- A formal technique for associating minimum or maximum principles with weak form equations that can be solved approximately.
- A more physically motivated approach than weighted residuals.
- Not all problems amenable to this technique.

2.2: Calculus of Variations (cont.)

- Minimum/Maximum Principles *(Variational Principles)* involve the following:
  - A set of equations $\mathbf{E}(\mathbf{u}(\mathbf{x}))=\mathbf{0}$ and boundary conditions $\mathbf{B}(\mathbf{u}(\mathbf{x}))=\mathbf{0}$ to solve for $\mathbf{u}(\mathbf{x})$.
  - A scalar quantity $J(\mathbf{u}(\mathbf{x}))$ "related" to **E** and **B** (called a *functional*).
- A variational principle states that solving $\mathbf{E}(\mathbf{u}(\mathbf{x}))=\mathbf{0}$ and $\mathbf{B}(\mathbf{u}(\mathbf{x}))=\mathbf{0}$ is equivalent to finding the function that gives $J(\mathbf{u}(\mathbf{x}))$ a maximum or minimum value.
- Requires $\delta J(\mathbf{u}(\mathbf{x}))=0$ -- "First variation of $J(\mathbf{u}(\mathbf{x}))$ must be zero *(stationarity)"*.

FEA Theory -3-

2.2: Calculus of Variations (cont.)

- What is a functional?
  - A *function* takes a point in space as input and returns a scalar number as output.

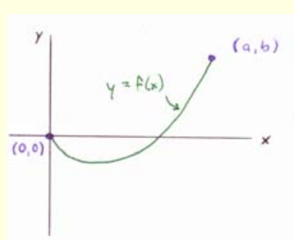
$$\text{E.g., } u(x,y,z)=x+2y-3z.$$

(Vector-valued function $\mathbf{u}(\mathbf{x})$ gives vector as output.)

  - A *functional* takes a function as input and returns a scalar number as output.

$$\text{E.g., } \ell(f(x))=\int_0^a\sqrt{1+(f'(x))^2}\,dx$$

Arc-length of *f(x)* from *x=0* to *x=a*.



FEA Theory -4-

2.2: Calculus of Variations (cont.)

- A few examples:

$$(a,b)=(4,2); f_1(x)=\tfrac{1}{2}x \Rightarrow \ \ell(f_1(x))=\int_0^4\sqrt{1+\left(\tfrac{1}{2}\right)^2}dx=4.4721.$$

$$(a,b)=(4,2); f_2(x)=\tfrac{1}{2}x^2-6 \Rightarrow \ \ell(f_2(x))=\int_0^4\sqrt{1+(x)^2}dx=9.2936.$$

- Recall that straight line is shortest distance between two points. How do we prove that?

Let α= scalar #, $g(x)=$ any function such that $g(0)=0=g(4)$.

$\therefore$ Should have $\ell(f_1(x)+\alpha g(x))\geq \ell(f_1(x))$ for all α and $g(x)$.

2.2: Calculus of Variations (cont.)

- For a given function $g(x)$, consider a Taylor series expansion of arc-length formula in terms of α:

$$\ell(f_1(x)+\alpha g(x))\approx \ell(f_1(x))+\alpha*\left[\frac{d}{d\alpha}\ell(f_1(x)+\alpha g(x))\right]_{\alpha=0}+\frac{1}{2}\alpha^2*\left[\frac{d^2}{d\alpha^2}\ell(f_1(x)+\alpha g(x))\right]_{\alpha=0}+\ldots$$

$$\left[\frac{d}{d\alpha}\ell(f_1(x)+\alpha g(x))\right]_{\alpha=0}=\left[\frac{d}{d\alpha}\int_0^4\sqrt{1+\{f_1'(x)+\alpha g'(x)\}^2}dx\right]_{\alpha=0}$$

$$=\left[\int_0^4\frac{\{f_1'(x)+\alpha g'(x)\}g'(x)}{\sqrt{1+\{f_1'(x)+\alpha g'(x)\}^2}}dx\right]_{\alpha=0}$$

$$\boxed{=\int_0^4\frac{f_1'(x)g'(x)}{\sqrt{1+\{f_1'(x)\}^2}}dx}$$

= some number β;
Assume β > 0.

2.2: Calculus of Variations (cont.)

- Suppose that α is small and negative:

$$\ell\left(f_1(x)+\alpha g(x)\right)\approx\ell\left(f_1(x)\right)+\alpha*\left[\frac{d}{d\alpha}\ell\left(f_1(x)+\alpha g(x)\right)\right]_{\alpha=0}\underbrace{+\frac{1}{2}\alpha^2*\left[\frac{d^2}{d\alpha^2}\ell\left(f_1(x)+\alpha g(x)\right)\right]_{\alpha=0}+\ldots}_{Negligible!}$$

$$\therefore\ell\left(f_1(x)+\alpha g(x)\right)\approx\ell\left(f_1(x)\right)+\alpha*\beta<\ell\left(f_1(x)\right)!$$ Can't happen!!!

- Same problem if $\beta < 0$ and α small and positive. So, must have $\beta = 0$!

$$\therefore\int_0^4\frac{f_1'(x)g'(x)}{\sqrt{1+\left\{f_1'(x)\right\}^2}}dx=0.$$

2.2: Calculus of Variations (cont.)

- Integrate by parts:

$$\therefore\int_0^4\frac{f_1'(x)g'(x)}{\sqrt{1+\left\{f_1'(x)\right\}^2}}dx=\left[\frac{f_1'(x)}{\sqrt{1+\left\{f_1'(x)\right\}^2}}*g(x)\right]_{x=0}^{x=4}-\int_0^4\frac{d}{dx}\left\{\frac{f_1'(x)}{\sqrt{1+\left\{f_1'(x)\right\}^2}}\right\}g(x)dx=0.$$

- But $g(x=0)=0$ and $g(x=4)=0$.

← Must equal zero!!!

$$\therefore\int_0^4\frac{f_1'(x)g'(x)}{\sqrt{1+\left\{f_1'(x)\right\}^2}}dx=-\int_0^4\boxed{\frac{d}{dx}\left\{\frac{f_1'(x)}{\sqrt{1+\left\{f_1'(x)\right\}^2}}\right\}}g(x)\,dx=0 \text{ for any choice of } g(x).$$

$$\therefore\frac{f_1'(x)}{\sqrt{1+\left\{f_1'(x)\right\}^2}}=\text{constant, or } f_1'(x)=\text{some other constant.}$$

$$\Rightarrow f_1(x)=mx+b \text{ and must pass through } (0,0) \text{ and } (4,2)\Rightarrow f_1(x)=\tfrac{1}{2}x!$$

2.2: Calculus of Variations (cont.)

- Key ideas in this "proof":
  - Considered an arbitrary increment of the input function.
  - Derivative of the functional forced to be zero.
  - This implies a certain equation must equal zero.
- Calculus of Variations gives you a "direct" way of performing these calculations!

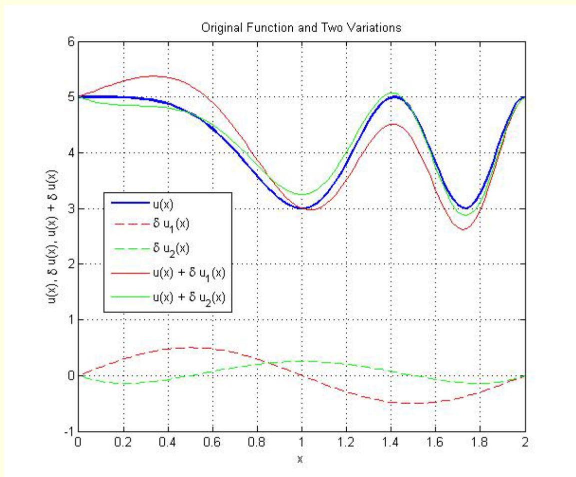
2.2: Calculus of Variations (cont.)

- Some definitions:
  - General form of a functional is

$$J(\mathbf{u}(\mathbf{x})) = \int_V \tilde{E}\left(\mathbf{x}, \mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial x}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial y}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial z}(\mathbf{x}), \frac{\partial^2 \mathbf{u}}{\partial x^2}(\mathbf{x}), \cdots, \frac{\partial^n \mathbf{u}}{\partial z^n}(\mathbf{x})\right) dV$$
$$+\int_A \tilde{B}\left(\mathbf{x}, \mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial x}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial y}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial z}(\mathbf{x}), \frac{\partial^2 \mathbf{u}}{\partial x^2}(\mathbf{x}), \cdots, \frac{\partial^m \mathbf{u}}{\partial z^m}(\mathbf{x})\right) dA.$$

  - A variation of $\mathbf{u}(\mathbf{x})$ is $\delta \mathbf{u}(\mathbf{x}) = \alpha \mathbf{v}(\mathbf{x}), \alpha \ll 1.$
  - Note: if $\mathbf{u}(\mathbf{x})$ must satisfy some boundary conditions, so must $\mathbf{u}(\mathbf{x}) + \delta \mathbf{u}(\mathbf{x})$.

2.2: Calculus of Variations (cont.)



2.2: Calculus of Variations (cont.)

- Some properties of the variation of $\mathbf{u}(\mathbf{x})$:
  - Derivatives and variations can interchange.

$$\frac{\partial}{\partial z}\left(\delta \mathbf{u}(\mathbf{x})\right)=\frac{\partial}{\partial z}\left(\alpha \mathbf{v}(\mathbf{x})\right)=\alpha\frac{\partial \mathbf{v}(\mathbf{x})}{\partial z}=\delta\left(\frac{\partial \mathbf{u}(\mathbf{x})}{\partial z}\right).$$

  - Integrals and variations can interchange.

$$\delta\int_V \mathbf{u}(\mathbf{x})\,dV=\int_V \delta\mathbf{u}(\mathbf{x})\,dV.$$

  - Variation of sum is sum of variations.

$$\delta\left(\mathbf{u}(\mathbf{x})+\tilde{\mathbf{u}}(\mathbf{x})\right)=\delta\mathbf{u}(\mathbf{x})+\delta\tilde{\mathbf{u}}(\mathbf{x}).$$

  - Variation of product obeys “product rule”.

$$\delta\left(\mathbf{u}(\mathbf{x})\cdot\tilde{\mathbf{u}}(\mathbf{x})\right)=\mathbf{u}(\mathbf{x})\cdot\delta\tilde{\mathbf{u}}(\mathbf{x})+\delta\mathbf{u}(\mathbf{x})\cdot\tilde{\mathbf{u}}(\mathbf{x}).$$

2.2: Calculus of Variations (cont.)

- Some properties of the variation of $\mathbf{u}(\mathbf{x})$:
  - "Chain rule" applies to <u>dependent variables only</u>!

$$\begin{aligned}\delta\tilde{E}\left(\mathbf{x},\mathbf{u}(\mathbf{x}),\frac{\partial\mathbf{u}}{\partial x}(\mathbf{x}),\cdots,\frac{\partial^n\mathbf{u}}{\partial z^n}(\mathbf{x})\right) &= \frac{\partial\tilde{E}}{\partial\mathbf{u}}\left(\mathbf{x},\mathbf{u}(\mathbf{x}),\frac{\partial\mathbf{u}}{\partial x}(\mathbf{x}),\cdots,\frac{\partial^n\mathbf{u}}{\partial z^n}(\mathbf{x})\right)\cdot\delta\mathbf{u} \\ &+ \frac{\partial\tilde{E}}{\partial\left(\frac{\partial\mathbf{u}}{\partial x}\right)}\left(\mathbf{x},\mathbf{u}(\mathbf{x}),\frac{\partial\mathbf{u}}{\partial x}(\mathbf{x}),\cdots,\frac{\partial^n\mathbf{u}}{\partial z^n}(\mathbf{x})\right)\cdot\delta\left(\frac{\partial\mathbf{u}}{\partial x}\right) \\ &+ \cdots \\ &+ \frac{\partial\tilde{E}}{\partial\left(\frac{\partial^n\mathbf{u}}{\partial z^n}\right)}\left(\mathbf{x},\mathbf{u}(\mathbf{x}),\frac{\partial\mathbf{u}}{\partial x}(\mathbf{x}),\cdots,\frac{\partial^n\mathbf{u}}{\partial z^n}(\mathbf{x})\right)\cdot\delta\left(\frac{\partial^n\mathbf{u}}{\partial z^n}\right).\end{aligned}$$

2.2: Calculus of Variations (cont.)

- Let's go back to arc-length example:

$$\ell\left(f_1(x)+\boxed{\alpha g(x)}\right)\approx\ell\left(f_1(x)\right)+\alpha*\boxed{\left[\frac{d}{d\alpha}\ell\left(f_1(x)+\alpha g(x)\right)\right]_{\alpha=0}}+\ldots$$

$= \delta f_1(x)$ $\quad$ $= \delta\ell\left(f_1(x)\right).$

$$\delta\ell\left(f_1(x)\right)=\delta\int_0^4\sqrt{1+\left\{f_1'(x)\right\}^2}\,dx=\int_0^4\delta\sqrt{1+\left\{f_1'(x)\right\}^2}\,dx$$

$$=\int_0^4\frac{\frac{1}{2}*\delta\left(\left\{f_1'(x)\right\}^2\right)}{\sqrt{1+\left\{f_1'(x)\right\}^2}}dx$$

$$=\int_0^4\frac{\frac{1}{2}*2f_1'(x)*\boxed{\delta f_1'(x)}}{\sqrt{1+\left\{f_1'(x)\right\}^2}}dx$$

$= \alpha g'(x)$

Thus, we see that

$$\delta\ell\left(f_1(x)\right)=\alpha\int_0^4\frac{f_1'(x)g'(x)}{\sqrt{1+\left\{f_1'(x)\right\}^2}}dx$$

Just like before!

2.2: Calculus of Variations (cont.)

- Minimum/Maximum Principles *(Variational Principles)* involve the following:
  - A set of equations $\mathbf{E}(\mathbf{u}(\mathbf{x})) = \mathbf{0}$ and boundary conditions $\mathbf{B}(\mathbf{u}(\mathbf{x})) = \mathbf{0}$ to solve for $\mathbf{u}(\mathbf{x})$.
  - A scalar quantity $J(\mathbf{u}(\mathbf{x}))$ "related" to **E** and **B** (called a *functional*).

What is the relation?

2.2: Calculus of Variations (cont.)

- Let's consider a 1D version of this:

$$\mathbf{u}(\mathbf{x}) = u(x); \mathbf{E}(\mathbf{u}(\mathbf{x})) = E(u(x)); J(u(x)) = \int_{x_a}^{x_b} \tilde{E}(x,u,u',u'')\,dx.$$

- Want to minimize *J(u)*, so require δ*J(u)* = *0*:

$$\delta J(u(x)) = \int_{x_a}^{x_b} \delta\tilde{E}(x,u,u',u'')\,dx$$

$$= \int_{x_a}^{x_b} \left\{ \frac{\partial\tilde{E}}{\partial u}\delta u + \frac{\partial\tilde{E}}{\partial u'}\delta u' + \frac{\partial\tilde{E}}{\partial u''}\delta u'' \right\} dx$$

$$= \int_{x_a}^{x_b} \left\{ \boxed{\frac{\partial\tilde{E}}{\partial u}\delta u} + \boxed{\frac{\partial\tilde{E}}{\partial u'}\frac{d}{dx}(\delta u)} + \boxed{\frac{\partial\tilde{E}}{\partial u''}\frac{d^2}{dx^2}(\delta u)} \right\} dx = 0.$$

2.2: Calculus of Variations (cont.)

- Integrate 2nd term by parts:

$$\int_{x_a}^{x_b} \frac{\partial \tilde{E}}{\partial u'} \frac{d}{dx}(\delta u)\,dx = \left[\frac{\partial \tilde{E}}{\partial u'} * \delta u\right]_{x=x_a}^{x=x_b} - \int_{x_a}^{x_b} \left\{\frac{d}{dx}\left(\frac{\partial \tilde{E}}{\partial u'}\right)\delta u\right\} dx$$

- $\left[\frac{\partial \tilde{E}}{\partial u'} * \delta u\right]_{x=x_a}^{x=x_b}$ involves the boundary conditions!
  - *Essential BC's:* E.g., $\delta u(x = x_a) = 0$ or $\delta u(x = x_b) = 0.$
  - *Natural BC's*: E.g, $\frac{\partial \tilde{E}}{\partial u'}(x = x_a) = 0$ or $\frac{\partial \tilde{E}}{\partial u'}(x = x_b) = 0.$
  - *Other BC's:* E.g., $\frac{\partial \tilde{E}}{\partial u'}(x = x_a) = \text{some number.}$

2.2: Calculus of Variations (cont.)

- Integrate 3rd term by parts twice:

$$\int_{x_a}^{x_b} \frac{\partial \tilde{E}}{\partial u''} \frac{d^2}{dx^2}(\delta u)\,dx = \left[\frac{\partial \tilde{E}}{\partial u''} * \frac{d}{dx}(\delta u)\right]_{x=x_a}^{x=x_b} - \int_{x_a}^{x_b} \left\{\frac{d}{dx}\left(\frac{\partial \tilde{E}}{\partial u''}\right) * \frac{d}{dx}(\delta u)\right\} dx$$

$$= \left[\frac{\partial \tilde{E}}{\partial u''} * \frac{d}{dx}(\delta u)\right]_{x=x_a}^{x=x_b} - \left[\frac{d}{dx}\left(\frac{\partial \tilde{E}}{\partial u''}\right) * \delta u\right]_{x=x_a}^{x=x_b} + \int_{x_a}^{x_b} \left\{\frac{d^2}{dx^2}\left(\frac{\partial \tilde{E}}{\partial u''}\right) * \delta u\right\} dx.$$

$$\left[\frac{\partial \tilde{E}}{\partial u''} * \frac{d}{dx}(\delta u)\right]_{x=x_a}^{x=x_b} \text{ and } \left[\frac{d}{dx}\left(\frac{\partial \tilde{E}}{\partial u''}\right) * \delta u\right]_{x=x_a}^{x=x_b} \text{ involve BC's!}$$

2.2: Calculus of Variations (cont.)

- Pull all of this together:

$$\delta J\left(u(x)\right)=0=\boxed{\int_{x_a}^{x_b}\left\{\frac{\partial\tilde{E}}{\partial u}\delta u\right\}dx}+\boxed{\left[\frac{\partial\tilde{E}}{\partial u'}*\delta u\right]_{x=x_a}^{x=x_b}-\int_{x_a}^{x_b}\left\{\frac{d}{dx}\left(\frac{\partial\tilde{E}}{\partial u'}\right)*\delta u\right\}dx}$$

$$+\boxed{\left[\frac{\partial\tilde{E}}{\partial u''}*\frac{d}{dx}(\delta u)\right]_{x=x_a}^{x=x_b}-\left[\frac{d}{dx}\left(\frac{\partial\tilde{E}}{\partial u''}\right)*\delta u\right]_{x=x_a}^{x=x_b}+\int_{x_a}^{x_b}\left\{\frac{d^2}{dx^2}\left(\frac{\partial\tilde{E}}{\partial u''}\right)*\delta u\right\}dx.}$$

$$\Rightarrow \delta J\left(u(x)\right)=0=\int_{x_a}^{x_b}\delta u*\left\{\frac{\partial\tilde{E}}{\partial u}-\frac{d}{dx}\left(\frac{\partial\tilde{E}}{\partial u'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial\tilde{E}}{\partial u''}\right)\right\}dx$$

$$+\text{ boundary condition terms.}$$

2.2: Calculus of Variations (cont.)

- Assuming all boundary conditions are either essential or natural, end up with:

$$0=\int_{x_a}^{x_b}\delta u*\left\{\frac{\partial\tilde{E}}{\partial u}-\frac{d}{dx}\left(\frac{\partial\tilde{E}}{\partial u'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial\tilde{E}}{\partial u''}\right)\right\}dx$$ for any choice of δu

$$\therefore \frac{\partial\tilde{E}}{\partial u}-\frac{d}{dx}\left(\frac{\partial\tilde{E}}{\partial u'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial\tilde{E}}{\partial u''}\right)=0!$$

The Euler equation for $J\left(u(x)\right)$

2.2: Calculus of Variations (cont.)

- The "relation" between $J\left(u(x)\right)$ being minimum and $E\left(u(x)\right)=0$ is as follows:

If you can find an operator $\tilde{E}\left(x,u,u',u''\right)$ such that

$$E\left(u(x)\right)=\frac{\partial \tilde{E}}{\partial u}-\frac{d}{dx}\left(\frac{\partial \tilde{E}}{\partial u'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial \tilde{E}}{\partial u''}\right)=0,$$

then solving $\delta J\left(u(x)\right)=\delta\int_{x_a}^{x_b}\tilde{E}\left(x,u,u',u''\right)dx=0$ is the same as solving $E\left(u(x)\right)=0$.

2.2: Calculus of Variations (cont.)

- Some notes:
  - If you have boundary conditions that neither essential nor natural, then must explicitly include a "boundary term" in the functional.

$$J\left(u(x)\right)=\int_{x_a}^{x_b}\tilde{E}\left(x,u,u',u''\right)dx+\left[\tilde{B}\left(x,u,u',u'',u'''\right)\right]_{x=x_a}^{x=x_b}.$$

(See Slide #10 for general statement of this idea.)

  - As number of dependent variables increases (e.g., 2D), one functional will produce multiple Euler equations:

$$J\left(u(x,y),v(x,y)\right)=\int_{Area}\tilde{E}\left(x,y,u,v,\tfrac{\partial u}{\partial x},\tfrac{\partial v}{\partial x},\tfrac{\partial u}{\partial y},\tfrac{\partial v}{\partial y}\right)dA$$

$$\Rightarrow\frac{\partial \tilde{E}}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial \tilde{E}}{\partial \frac{\partial u}{\partial x}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \tilde{E}}{\partial \frac{\partial u}{\partial y}}\right)=0 \text{ and } \frac{\partial \tilde{E}}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial \tilde{E}}{\partial \frac{\partial v}{\partial x}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \tilde{E}}{\partial \frac{\partial v}{\partial y}}\right)=0$$

2.2: Calculus of Variations (cont.)

- Notes:
  - There are no general procedures for finding the operator $\tilde{E}$ for a given set of equations $\mathbf{E}(\mathbf{u}(\mathbf{x})) = \mathbf{0}$.
  - However, $\tilde{E}$ is known for many of the more common finite element analysis problems.
  - Special case for which $\tilde{E}$ can always be found:

$$\mathbf{E}(\mathbf{u}(\mathbf{x})) = [\mathbf{M}(\mathbf{x})]\mathbf{u}(\mathbf{x}) + \mathbf{b} = \mathbf{0};$$

$[\mathbf{M}(\mathbf{x})]$ = matrix of derivative operators such that $[\mathbf{M}(\mathbf{x})]$ satisfies

$$\int_V \mathbf{u}_1(\mathbf{x}) \cdot [\mathbf{M}(\mathbf{x})]\mathbf{u}_2(\mathbf{x})\, dV = \int_V \mathbf{u}_2(\mathbf{x}) \cdot [\mathbf{M}(\mathbf{x})]\mathbf{u}_1(\mathbf{x})\, dV + \text{BC's}$$

for all possible choices of $\mathbf{u}_1(\mathbf{x})$ and $\mathbf{u}_2(\mathbf{x})$. *"Self-adjoint" equations*

2.2: Calculus of Variations (cont.)

- Notes:
  - For self-adjoint equations, $\tilde{E}$ and $J(\mathbf{u})$ can be shown to be:

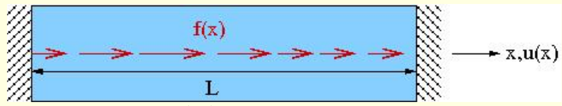
$$\tilde{E} = \frac{1}{2}\mathbf{u}(\mathbf{x}) \cdot [\mathbf{M}(\mathbf{x})]\mathbf{u}(\mathbf{x}) + \mathbf{u}(\mathbf{x}) \cdot \mathbf{b};$$

$$J(\mathbf{u}) = \int_V \left\{ \frac{1}{2}\mathbf{u}(\mathbf{x}) \cdot [\mathbf{M}(\mathbf{x})]\mathbf{u}(\mathbf{x}) + \mathbf{u}(\mathbf{x}) \cdot \mathbf{b} \right\} dV.$$

(Depending on problem details, may be necessary to integrate $J(\mathbf{u})$ by parts before taking variation.)

2.2: Calculus of Variations (cont.)

- Example: axial deformation of fixed rod with axial load –



$$\frac{d\varepsilon}{dx}(x)+\frac{f(x)}{E}=0;\varepsilon(x)=\frac{du}{dx}.$$

$$u(x=0)=0=u(x=L).$$

  - Can re-write governing equations as:

$$\underbrace{\begin{bmatrix} 0 & \frac{d}{dx} \\ -\frac{d}{dx} & 1 \end{bmatrix}}_{[\mathbf{M}]}\underbrace{\begin{pmatrix} u(x) \\ \varepsilon(x) \end{pmatrix}}_{\mathbf{u}(\mathbf{x})}+\underbrace{\begin{pmatrix} \frac{f(x)}{E} \\ 0 \end{pmatrix}}_{\mathbf{b}}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

2.2: Calculus of Variations (cont.)

- Example:
  - Functional is then calculated as follows:

$$\tilde{E}=\frac{1}{2}\begin{pmatrix} u \\ \varepsilon \end{pmatrix}\cdot\begin{bmatrix} 0 & \frac{d}{dx} \\ -\frac{d}{dx} & 1 \end{bmatrix}\begin{pmatrix} u \\ \varepsilon \end{pmatrix}+\begin{pmatrix} u \\ \varepsilon \end{pmatrix}\cdot\begin{pmatrix} \frac{f(x)}{E} \\ 0 \end{pmatrix}=\tfrac{1}{2}u\tfrac{d\varepsilon}{dx}-\tfrac{1}{2}\varepsilon\tfrac{du}{dx}+\tfrac{1}{2}\varepsilon^2+\tfrac{uf(x)}{E};$$

$$J(\mathbf{u})=\int_0^L\left\{\tfrac{1}{2}u\tfrac{d\varepsilon}{dx}-\tfrac{1}{2}\varepsilon\tfrac{du}{dx}+\tfrac{1}{2}\varepsilon^2+\tfrac{uf(x)}{E}\right\}dx.$$

  - Euler equations for this functional:

$$\frac{\partial\tilde{E}}{\partial u}-\frac{d}{dx}\left(\frac{\partial\tilde{E}}{\partial\frac{du}{dx}}\right)=0\Rightarrow\tfrac{1}{2}\tfrac{d\varepsilon}{dx}+\tfrac{f(x)}{E}-\tfrac{d}{dx}\left(-\tfrac{1}{2}\varepsilon\right)=0,\text{ or }\tfrac{d\varepsilon}{dx}+\tfrac{f(x)}{E}=0.$$

$$\frac{\partial\tilde{E}}{\partial\varepsilon}-\frac{d}{dx}\left(\frac{\partial\tilde{E}}{\partial\frac{d\varepsilon}{dx}}\right)=0\Rightarrow\varepsilon-\tfrac{1}{2}\tfrac{du}{dx}-\tfrac{d}{dx}\left(\tfrac{1}{2}u\right)=0,\text{ or }\varepsilon-\tfrac{du}{dx}=0.$$

2.2: Calculus of Variations (cont.)

- So, what's all of this have to do with finite elements?
  - Have a set of equations $\mathbf{E}\left(\mathbf{u}\left(\mathbf{x}\right)\right)=\mathbf{0}$ and boundary conditions $\mathbf{B}\left(\mathbf{u}\left(\mathbf{x}\right)\right)=\mathbf{0}$ to solve for $\mathbf{u}\left(\mathbf{x}\right)$.
  - Have a functional $J\left(\mathbf{u}\left(\mathbf{x}\right)\right)=\int_V \tilde{E}\left(\mathbf{x},\mathbf{u},\frac{\partial \mathbf{u}}{\partial x},\frac{\partial \mathbf{u}}{\partial y},\frac{\partial \mathbf{u}}{\partial z}\right)dV$ related to $\mathbf{E}$ and $\mathbf{B}$ via the Euler equations on $\tilde{E}$.
- Finite element analysis attempts to find the best approximate solution to

$$\delta J\left(\mathbf{u}\left(\mathbf{x}\right)\right)=\int_V \delta\tilde{E}\left(\mathbf{x},\mathbf{u},\frac{\partial \mathbf{u}}{\partial x},\frac{\partial \mathbf{u}}{\partial y},\frac{\partial \mathbf{u}}{\partial z}\right)dV=0.$$

Weak form of governing equations!

2.2: Calculus of Variations (cont.)

- Look more closely at 1D version:

$$\int_{x_a}^{x_b}\delta u*\left\{\frac{\partial\tilde{E}(x,u,u')}{\partial u}-\frac{d}{dx}\left(\frac{\partial\tilde{E}(x,u,u')}{\partial(u')}\right)\right\}dx+\left[\text{boundary terms}\right]=0.$$

- Suppose we make "usual" approximation –

$$u\left(x\right)\approx u_{approx}\left(x\right)=\sum_{k=1}^{n}\mathrm{a}_k N_k\left(x\right)$$

$$\Rightarrow \delta u\left(x\right)\approx \delta u_{approx}\left(x\right)=\sum_{k=1}^{n}\delta\mathrm{a}_k N_k\left(x\right).$$

A "space" of *trial functions* Must belong to same "space"

E.g., if $\boxed{u_{approx}\left(x\right)=\mathrm{a}_0+\mathrm{a}_1 x+\mathrm{a}_2 x^2}$, then $\boxed{\delta u_{approx}\left(x\right)=\delta\mathrm{a}_0+\delta\mathrm{a}_1 x+\delta\mathrm{a}_2 x^2}$.

2.2: Calculus of Variations (cont.)

- Plug in approximations (ignoring boundary terms for now) –

$$\int_{x_a}^{x_b}\sum_{k=1}^{n}\left\{\delta a_k N_k(x)\right\}*\left\{\frac{\partial\tilde{E}(x,u=u_{approx},u'=u'_{approx})}{\partial u}-\frac{d}{dx}\left(\frac{\partial\tilde{E}(x,u=u_{approx},u'=u'_{approx})}{\partial(u')}\right)\right\}dx\approx 0,$$

$$\text{or }\sum_{k=1}^{n}\delta a_k*\int_{x_a}^{x_b}N_k(x)*\left\{\frac{\partial\tilde{E}(x,u=u_{approx},u'=u'_{approx})}{\partial u}-\frac{d}{dx}\left(\frac{\partial\tilde{E}(x,u=u_{approx},u'=u'_{approx})}{\partial(u')}\right)\right\}dx\approx 0.$$

- Since each δa_k is arbitrary, best approximation comes from

$$\int_{x_a}^{x_b}\boxed{N_k(x)*\left\{\frac{\partial\tilde{E}(x,u=u_{approx},u'=u'_{approx})}{\partial u}-\frac{d}{dx}\left(\frac{\partial\tilde{E}(x,u=u_{approx},u'=u'_{approx})}{\partial(u')}\right)\right\}}dx=0,k=1,2,\ldots,n$$

Function of a_1, a_2, …, a_n $\Rightarrow$ Get *n* equations for *n* constants!

2.2: Calculus of Variations (cont.)

- Notice the following:

$$\text{If }\frac{\partial\tilde{E}}{\partial u}-\frac{d}{dx}\left(\frac{\partial\tilde{E}}{\partial u'}\right)=E(u(x)),\text{ then}$$

$$\frac{\partial\tilde{E}(x,u=u_{approx},u'=u'_{approx})}{\partial u}-\frac{d}{dx}\left(\frac{\partial\tilde{E}(x,u=u_{approx},u'=u'_{approx})}{\partial(u')}\right)=E(u=u_{approx})=R_E(x,\mathbf{a}).$$

$$\therefore\boxed{\int_{x_a}^{x_b}N_k(x)*R_E(x,\mathbf{a})\,dx=0,k=1,2,\ldots,n.}$$

Galerkin's method!

Galerkin's Method and Calculus of Variations give <u>same</u> equations when "proper" $\tilde{E}$ is used!

2.2: Calculus of Variations (cont.)

- Notice something else:

$$\Pi(\mathbf{a}) = J\left(u = u_{approx}\right) - J\left(u = u_{exact}\right)$$

$$= \int_{x_a}^{x_b} \tilde{E}\left(x, u = u_{approx}, u' = u'_{approx}\right) dx - J\left(u = u_{exact}\right).$$

$$\therefore \frac{\partial \Pi}{\partial \mathrm{a}_k} = \int_{x_a}^{x_b} \frac{\partial \tilde{E}}{\partial \mathrm{a}_k}\left(x, u = u_{approx}, u' = u'_{approx}\right) dx$$

$$= \int_{x_a}^{x_b} \left\{ \frac{\partial \tilde{E}\left(x, u=u_{approx}, u'=u'_{approx}\right)}{\partial u} * \frac{\partial u_{approx}}{\partial \mathrm{a}_k} + \frac{\partial \tilde{E}\left(x, u=u_{approx}, u'=u'_{approx}\right)}{\partial u'} * \frac{\partial u'_{approx}}{\partial \mathrm{a}_k} \right\} dx$$

$$= \int_{x_a}^{x_b} \left\{ \frac{\partial \tilde{E}\left(x, u=u_{approx}, u'=u'_{approx}\right)}{\partial u} * N_k(x) + \frac{\partial \tilde{E}\left(x, u=u_{approx}, u'=u'_{approx}\right)}{\partial u'} * N'_k(x) \right\} dx$$

2.2: Calculus of Variations (cont.)

- Integrate 2nd term by parts (and ignore boundary terms again):

$$\frac{\partial \Pi}{\partial \mathrm{a}_k} = \int_{x_a}^{x_b} \left\{ \frac{\partial \tilde{E}\left(x, u=u_{approx}, u'=u'_{approx}\right)}{\partial u} * N_k(x) - \frac{d}{dx}\left\{ \frac{\partial \tilde{E}\left(x, u=u_{approx}, u'=u'_{approx}\right)}{\partial u'} \right\} * N_k(x) \right\} dx$$

$$= \int_{x_a}^{x_b} N_k(x) * \left\{ \frac{\partial \tilde{E}\left(x, u=u_{approx}, u'=u'_{approx}\right)}{\partial u} - \frac{d}{dx}\left\{ \frac{\partial \tilde{E}\left(x, u=u_{approx}, u'=u'_{approx}\right)}{\partial u'} \right\} \right\} dx.$$

$$\therefore \frac{\partial \Pi}{\partial \mathrm{a}_k} = 0 \Rightarrow \int_{x_a}^{x_b} N_k(x) * \left\{ \frac{\partial \tilde{E}\left(x, u=u_{approx}, u'=u'_{approx}\right)}{\partial u} - \frac{d}{dx}\left\{ \frac{\partial \tilde{E}\left(x, u=u_{approx}, u'=u'_{approx}\right)}{\partial u'} \right\} \right\} dx = 0.$$

Rayleigh-Ritz Method on Π gives same equations as $\delta J = 0$!

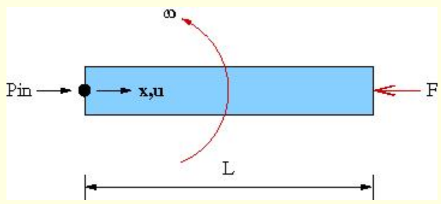
2.2: Calculus of Variations (cont.)

- Example: 1D Axial Rod "dynamics"
  - Given: Axial rod has constant density ρ, area A, length L, and spins at constant rate ω. It is pinned at $x = 0$ and has applied force $-F$ at $x = L$. The governing equation and boundary conditions for the steady-state rotation of the rod are:

$$E\frac{d^2u}{dx^2} + \rho\omega^2 x = 0 \text{ for } 0 < x < L;$$

$$u(x=0) = 0; E\frac{du}{dx}(x=L) = -\frac{F}{A}.$$

  - Required: Using the calculus of variations on an appropriate variational principle along with the approximate solution $u(x) = a_1 x + a_2 x^2$, estimate the displacement of the rod.



2.2: Calculus of Variations (cont.)

- Solution:
  - Find appropriate variational principle:

$$\mathbf{E}(\mathbf{u}) = E\frac{d^2u}{dx^2} + \rho\omega^2 x \text{ is self-adjoint, with } [\mathbf{M}] = E\frac{d^2}{dx^2} \text{ and } \mathbf{b} = \rho\omega^2 x.$$

$$\therefore \tilde{E} = \tfrac{1}{2}u * E\tfrac{d^2u}{dx^2} + u\rho\omega^2 x \Rightarrow J = \int_0^L \left\{\tfrac{1}{2}u * E\tfrac{d^2u}{dx^2} + u\rho\omega^2 x\right\} dx.$$

  - Problem: there is a nonzero boundary condition –

$$\tilde{B} = -\tfrac{1}{2}u(x=L) * \tfrac{F}{A} \text{ (Work done by applied force.)}$$

$$\therefore J = -\tfrac{1}{2}u(x=L) * \tfrac{F}{A} + \int_0^L \left\{\boxed{\tfrac{1}{2}u * E\tfrac{d^2u}{dx^2}} + u\rho\omega^2 x\right\} dx.$$

Needs to be integrated by parts!

2.2: Calculus of Variations (cont.)

- Solution:
  - Doing this gives:

$$J=-\tfrac{1}{2}u\left(x=L\right)*\tfrac{F}{A}+\left[\tfrac{1}{2}u*E\tfrac{du}{dx}\right]_{x=0}^{x=L}-\int_0^L\left\{\tfrac{1}{2}E\left(\tfrac{du}{dx}\right)^2\right\}dx+\int_0^L\left\{u\rho\omega^2x\right\}dx$$

$$=-u\left(x=L\right)*\tfrac{F}{A}-\int_0^L\left\{\tfrac{1}{2}E\left(\tfrac{du}{dx}\right)^2\right\}dx+\int_0^L\left\{u\rho\omega^2x\right\}dx.$$

  - Require the first variation to equal zero:

$$\delta J=-\delta u\left(x=L\right)*\tfrac{F}{A}+\int_0^L\left\{\delta u*\rho\omega^2x-E\tfrac{du}{dx}*\tfrac{d(\delta u)}{dx}\right\}dx=0.$$

2.2: Calculus of Variations (cont.)

- Solution:
  - Using the given approximate function:

$$u\left(x\right)=\mathrm{a}_1x+\mathrm{a}_2x^2\Rightarrow\delta u\left(x\right)=\delta\mathrm{a}_1x+\delta\mathrm{a}_2x^2.$$

$$\delta J=-\left\{\delta\mathrm{a}_1L+\delta\mathrm{a}_2L^2\right\}*\tfrac{F}{A}$$

$$+\int_0^L\left\{\left(\delta\mathrm{a}_1x+\delta\mathrm{a}_2x^2\right)*\rho\omega^2x-E\left(\mathrm{a}_1+2\mathrm{a}_2x\right)*\left(\delta\mathrm{a}_1+2\delta\mathrm{a}_2x\right)\right\}dx=0.$$

  - After some integrating, result is:

$$\delta J=\overset{=0}{\boxed{\left(\tfrac{1}{3}\rho\omega^2L^3-\tfrac{FL}{A}-EL\mathrm{a}_1-EL^2\mathrm{a}_2\right)}}\delta\mathrm{a}_1$$

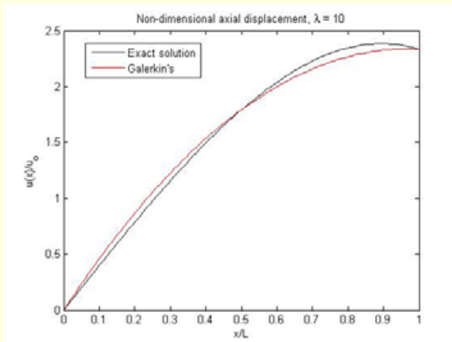
$$+\overset{=0}{\boxed{\left(\tfrac{1}{4}\rho\omega^2L^4-\tfrac{FL^2}{A}-EL^2\mathrm{a}_1-\tfrac{4}{3}EL^3\mathrm{a}_2\right)}}\delta\mathrm{a}_2=0.$$

2.2: Calculus of Variations (cont.)

- Solution:
  - Solve the two equations to get:

$$a_1 = -\frac{F}{EA} + \frac{7\rho\omega^2 L^2}{12E};\ a_2 = -\frac{\rho\omega^2 L}{4E} \Rightarrow u_{approx}(x) = u_o * \left\{\lambda\left[\tfrac{7}{12}\left(\tfrac{x}{L}\right) - \tfrac{1}{4}\left(\tfrac{x}{L}\right)^2\right] - \left(\tfrac{x}{L}\right)\right\}.$$

Same as Galerkin's method solution!



2.2: Calculus of Variations (cont.)

- What if we had forgotten about the BC?
  - Functional becomes:

$$J = \left[\tfrac{1}{2}u * E\tfrac{du}{dx}\right]_{x=0}^{x=L} - \int_0^L \left\{\tfrac{1}{2}E\left(\tfrac{du}{dx}\right)^2\right\}dx + \int_0^L \left\{u\rho\omega^2 x\right\}dx$$

$$= -\tfrac{1}{2}u(x=L) * \tfrac{F}{A} - \int_0^L \left\{\tfrac{1}{2}E\left(\tfrac{du}{dx}\right)^2\right\}dx + \int_0^L \left\{u\rho\omega^2 x\right\}dx.$$

  - So the first variation becomes:

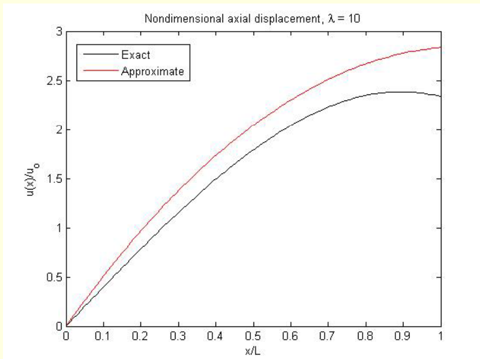
$$\delta J = -\delta u(x=L) * \boxed{\tfrac{F}{2A}} + \int_0^L \left\{\delta u * \rho\omega^2 x - E\tfrac{du}{dx} * \tfrac{d(\delta u)}{dx}\right\}dx = 0.$$

Force is cut in half!

2.2: Calculus of Variations (cont.)

- Solution:
  - Solution becomes:

$$a_1 = -\frac{F}{2EA} + \frac{7\rho\omega^2 L^2}{12E};\ a_2 = -\frac{\rho\omega^2 L}{4E} \Rightarrow u_{approx}(x) = u_o * \left\{ \lambda\left[\tfrac{7}{12}\left(\tfrac{x}{L}\right) - \tfrac{1}{4}\left(\tfrac{x}{L}\right)^2 \right] - \left(\tfrac{x}{2L}\right) \right\}.$$



Section 3: Implementation of Finite Element Analysis – the Constant Strain Triangle

1. Fundamental Concepts
2. Element Formulation
3. Assembly
4. Convergence and Other Issues
5. Examples

Will illustrate the details of these ideas in the context of a specific "simple" finite element called the Constant Strain Triangle (CST).

Section 3: Implementation of FEA – the CST

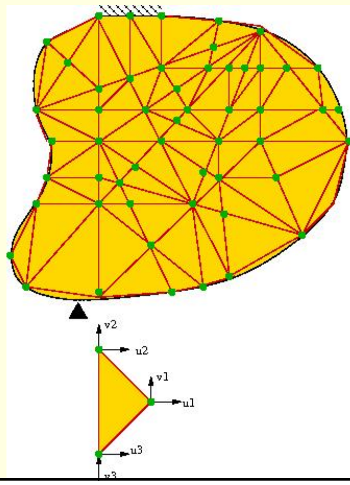
Overview: What is Finite Element Analysis?

- Assume a given problem is to be solved globally over some object. FEA proceeds as follows:
  1. Discretize the problem into a *finite* number of "local" approximate problems on regions called *elements*.
  2. Set up and solve each of the local approximate problems.
  3. Assemble the local solutions into a global solution.
  4. Check convergence. (If not converged, repeat steps #2 and #3.)
  5. Once converged, evaluate the solution.

Your responsibility! (Program does the rest.)

Section 3.1: Fundamental Concepts

What is an element?

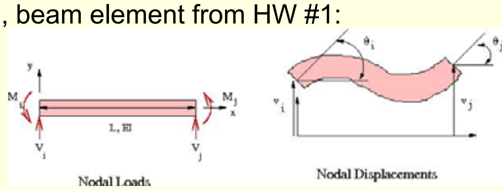


- An element is a small piece of an overall object characterized by:
  1. Its type (truss, beam, plate, solid, …)
  2. Its geometry (1D, 2D, or 3D; line, triangle, rectangle, tetrahedron, "brick", …)
  3. Its *nodes* (corner, interior, …) and *degrees of freedom* (displacement, rotation, …)
  4. Its shape functions

3.1: Fundamental Concepts (cont.)

Nodes and Degrees of Freedom –

- Elements are restricted in how they can behave.
  - E.g., beam element from HW #1:



  - No left/right motion allowed; only certain types of bending possible.
- It is required that the complete range of possible behaviors be determined by the locations of the nodes, the basic behaviors allowed at each node (*degrees of freedom*), and the shape functions used. $v(x)=v_i N_1(x)+\theta_i N_2(x)+v_j N_3(x)+\theta_j N_4(x).$

3.1: Fundamental Concepts (cont.)

Required Properties of the Approximate Solution:

- Well-posedness: the number of shape functions used in the approximate solution must equal the number of degrees of freedom.
  - Degrees of freedom are the unknowns in the local problem; need one equation per unknown to get unique solution.
  - E.g., Galerkin's method:

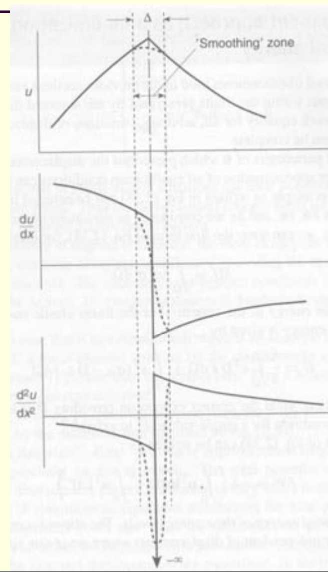
$$I_k(\mathbf{a}) = \int_V \mathbf{N}_k(\mathbf{x}) \cdot \mathbf{R}_E(\mathbf{x}, \mathbf{a})\, dV = 0 \text{ , } k = 1, 2, \ldots, n.$$

3.1: Fundamental Concepts (cont.)

Required Properties of the Approximate Solution:

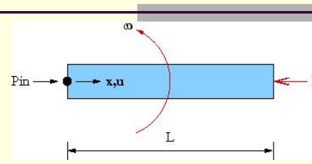
- Differentiability: The shape functions must be continuous and have continuous derivatives up to an order consistent with the variational principle used.
  - Defines behavior within element.
  - In general, if *n*th order derivative is in variational principle, then shape functions must have continuous derivatives of order *n-1.*
  - Prevents integrals from "blowing up" or becoming undefined.

3.1: Fundamental Concepts (cont.)



3.1: Fundamental Concepts (cont.)

- Example: 1D Axial Rod "dynamics"



- Variational principle <u>before</u> integrating by parts was

$$J = -\tfrac{1}{2} u\left(x = L\right) * \tfrac{F}{A} + \int_0^L \left\{ \tfrac{1}{2} u * E \tfrac{d^2u}{dx^2} + u \rho \omega^2 x \right\} dx.$$

Requires continuous 1st derivatives for *u(x)*.

- Variational principle <u>after</u> integrating by parts was

$$J = -u\left(x = L\right) * \tfrac{F}{A} - \int_0^L \left\{ \tfrac{1}{2} E \left(\tfrac{du}{dx}\right)^2 \right\} dx + \int_0^L \left\{ u \rho \omega^2 x \right\} dx.$$

Requires continuous *u(x)*.

3.1: Fundamental Concepts (cont.)

Required Properties of the Approximate Solution:

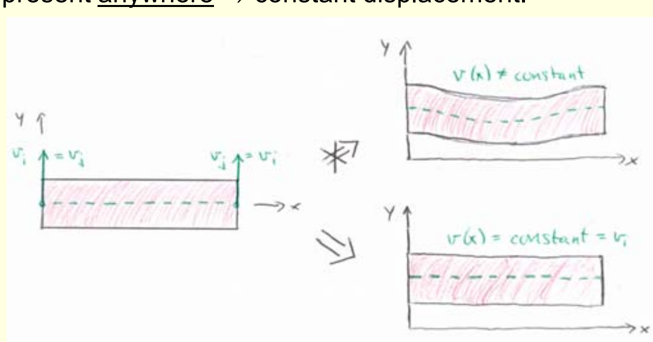
- Completeness: approximate solution must be able to represent two special displacement states exactly –

  1. Rigid body motion: all points in element have same values of displacement.
  2. Constant strain state: all normal strains and shearing strains have a fixed value everywhere in the element.

3.1: Fundamental Concepts (cont.)

Required Properties of the Approximate Solution:

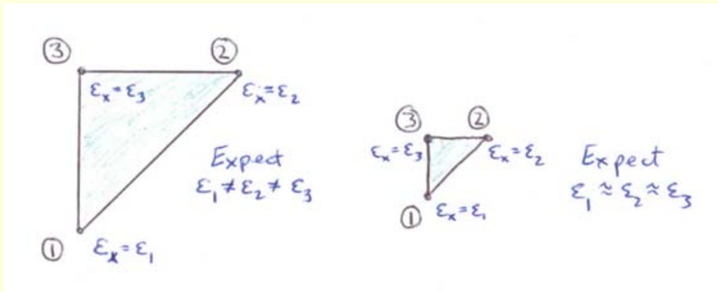
- Rigid body motions cannot create strain energy, so no strains present anywhere ⇒ constant displacement.



3.1: Fundamental Concepts (cont.)

Required Properties of the Approximate Solution:

- As element size becomes very small, expect internal strains to change very little at different points in element. If approximate solution cannot support this, expect problems in convergence.



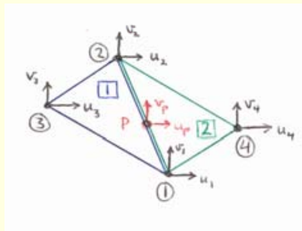
3.1: Fundamental Concepts (cont.)

Required Properties of the Approximate Solution:

- Compatibility: If two elements share a boundary, the shape functions must permit an appropriate level of continuity across the boundary.
  - Prevents “gaps” or “kinks” from developing in global solution.
  - Elements with this property are called *conforming.*
  - For more complicated elements, compatibility may be required only at selected points on the boundary. (Such elements are called *non-conforming*.)

3.1: Fundamental Concepts (cont.)

- Example: Two constant strain triangles



- For element #1, displacements at *P* depend on coordinates of point and the values *(u1, v1, u2, v2, u3, v3).*
  $\therefore \left(u_P, v_P\right) = \text{function}\,1\left(x_P, y_P, u_1, v_1, u_2, v_2, u_3, v_3\right).$
- For element #2, displacements at *P* depend on coordinates of point and the values *(u1, v1, u2, v2, u4, v4).*
  $\therefore \left(u_P, v_P\right) = \text{function}\,2\left(x_P, y_P, u_1, v_1, u_2, v_2, u_4, v_4\right).$

3.1: Fundamental Concepts (cont.)

- Displacement continuity then requires

$$\text{function}\,1\left(x_P, y_P, u_1, v_1, u_2, v_2, u_3, v_3\right)$$
$$= \text{function}\,2\left(x_P, y_P, u_1, v_1, u_2, v_2, u_4, v_4\right).$$

at <u>all points</u> *P* and for <u>all values</u> of *(u1, v1, u2, v2, u3, v3, u4, v4).*

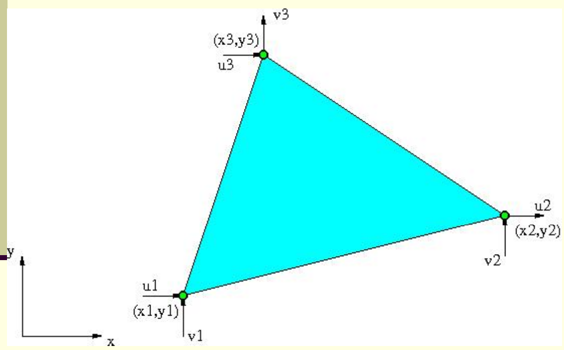
- Can show that this requirement will be satisfied if

$$\text{function}\,1 = \text{function}\,1\left(x_P, y_P, u_1, v_1, u_2, v_2\right) \text{ only.}$$
$$\text{function}\,2 = \text{function}\,2\left(x_P, y_P, u_1, v_1, u_2, v_2\right) \text{ only.}$$

(Displacements on a boundary depend <u>only</u> upon the nodes on that boundary!)

Section 3.2: Element Formulation

The Constant Strain Triangle element:



- 2D element used in plane stress and plane strain problems
- Nominal thickness = h (small; can be variable)
- Three corner nodes with coordinates (x_i, y_i)
- Two degrees of freedom per node: (u_i, v_i)

3.2: Element Formulation (cont.)

Two approaches for generating shape functions:

- "Interpolation approach":
  - Matrix-based method
  - Works best for small numbers of d.o.f.
- "Direct approach":
  - More geometric method
  - Works best for higher-order elements

(Other methods also exist; will not discuss these much.)

3.2: Element Formulation (cont.)

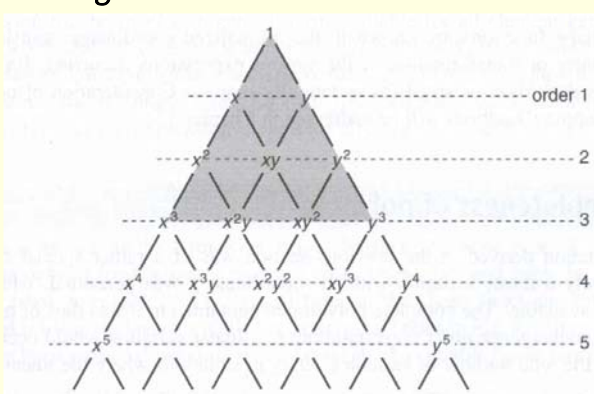
Some basic ideas (for both approaches):

- 6 d.o.f. total ⇒ 6 shape functions.
- Fundamental unknowns are horizontal displacement *u(x,y)* and vertical displacement *v(x,y)*.
- ∴ Each displacement expected to use 3 shape functions.

- Rule of thumb: simple shape functions = better shape functions. (easier to integrate, more widely applicable, …)
- For 2D elements, polynomials in *x* and *y* are most common choice for shape functions.
- If you have derivatives of order *n* in your variational principle, it is best to choose your shape functions so that they can form a *complete polynomial* of order *n.*
  (Gives control over errors, faster convergence, …)

3.2: Element Formulation (cont.)

Pascal's triangle



(Row *n+1* based upon expansion of $(x+y)^n$.)

3.2: Element Formulation (cont.)

"Interpolation approach":

- Approximate *u(x,y)* and *v(x,y)* by complete 1st order polynomials:

$$u(x,y) = \mathrm{a}_1 + \mathrm{a}_2 x + \mathrm{a}_3 y;\ v(x,y) = \mathrm{a}_4 + \mathrm{a}_5 x + \mathrm{a}_6 y.$$

- At each node, require $u(x_i,y_i) = u_i$ and $v(x_i,y_i) = v_i$:

$$u_1 = \mathrm{a}_1 + \mathrm{a}_2 x_1 + \mathrm{a}_3 y_1;\ v_1 = \mathrm{a}_4 + \mathrm{a}_5 x_1 + \mathrm{a}_6 y_1.$$
$$u_2 = \mathrm{a}_1 + \mathrm{a}_2 x_2 + \mathrm{a}_3 y_2;\ v_2 = \mathrm{a}_4 + \mathrm{a}_5 x_2 + \mathrm{a}_6 y_2.$$
$$u_3 = \mathrm{a}_1 + \mathrm{a}_2 x_3 + \mathrm{a}_3 y_3;\ v_3 = \mathrm{a}_4 + \mathrm{a}_5 x_3 + \mathrm{a}_6 y_3.$$

6 equations for the 6 unknowns!

3.2: Element Formulation (cont.)

"Interpolation approach":

- Write this in matrix form:

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{pmatrix} \mathrm{a}_1 \\ \mathrm{a}_2 \\ \mathrm{a}_3 \\ \mathrm{a}_4 \\ \mathrm{a}_5 \\ \mathrm{a}_6 \end{pmatrix} \Leftrightarrow (\mathbf{d}) = [\mathbf{C}](\mathbf{a}).$$

- Solution (in symbolic form) is $(\mathbf{a}) = [\mathbf{C}]^{-1}(\mathbf{d})$.

3.2: Element Formulation (cont.)

"Interpolation approach":

- Now, rewrite interpolation functions in matrix/vector form:

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{pmatrix} \mathrm{a}_1 \\ \mathrm{a}_2 \\ \mathrm{a}_3 \\ \mathrm{a}_4 \\ \mathrm{a}_5 \\ \mathrm{a}_6 \end{pmatrix} \Leftrightarrow (\mathbf{u}(\mathbf{x})) = [\mathbf{P}(\mathbf{x})](\mathbf{a}).$$

- Substitute previous result: $(\mathbf{u}(\mathbf{x})) = \underbrace{[\mathbf{P}(\mathbf{x})][\mathbf{C}]^{-1}}_{[\mathbf{N}(\mathbf{x})]}(\mathbf{d}).$

Matrix of shape functions!

3.2: Element Formulation (cont.)

"Interpolation approach":

- For CST, can show that

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{bmatrix} N_1(x,y) & 0 & N_3(x,y) & 0 & N_5(x,y) & 0 \\ 0 & N_2(x,y) & 0 & N_4(x,y) & 0 & N_6(x,y) \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix},$$

$$N_1(x,y) = N_2(x,y) = \left[(x_2y_3 - x_3y_2) + x(y_2 - y_3) + y(x_3 - x_2)\right]/2A;$$
$$N_3(x,y) = N_4(x,y) = \left[(x_3y_1 - x_1y_3) + x(y_3 - y_1) + y(x_1 - x_3)\right]/2A;$$
$$N_5(x,y) = N_6(x,y) = \left[(x_1y_2 - x_2y_1) + x(y_1 - y_2) + y(x_2 - x_1)\right]/2A;$$

$$A = \text{ area of triangle} = \tfrac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}.$$

3.2: Element Formulation (cont.)

Notes on "Interpolation approach":

- This approach generalizes to different shapes, different node locations, and different numbers of d.o.f. (See Prob. 3.1 and 3.2 in Schaum's Outline.)
- However, the matrix [**C**] is not always invertible for general choices of nodal locations.
- As number of d.o.f. increases, matrix inversion becomes more difficult, and thus exact functions become harder to determine.

3.2: Element Formulation (cont.)

"Direct approach": Need two "facts" about shape functions –

- *u(x,y)* and *v(x,y)* are complete 1st order polynomials:

$$u(x,y) = \mathrm{a}_1 + \mathrm{a}_2 x + \mathrm{a}_3 y = u_1 N_1(x,y) + u_2 N_2(x,y) + u_3 N_3(x,y);$$

$$v(x,y) = \mathrm{a}_4 + \mathrm{a}_5 x + \mathrm{a}_6 y = v_1 N_4(x,y) + v_2 N_5(x,y) + v_3 N_6(x,y).$$

$\therefore$ Shape functions must be linear in both *x* and *y*.

- Suppose I know the shape functions already:

$$u(x,y) = u_1 N_1(x,y) + u_2 N_2(x,y) + u_3 N_3(x,y).$$

$$\therefore u_i = u(x_i,y_i) = u_1 N_1(x_i,y_i) + u_2 N_2(x_i,y_i) + u_3 N_3(x_i,y_i)$$

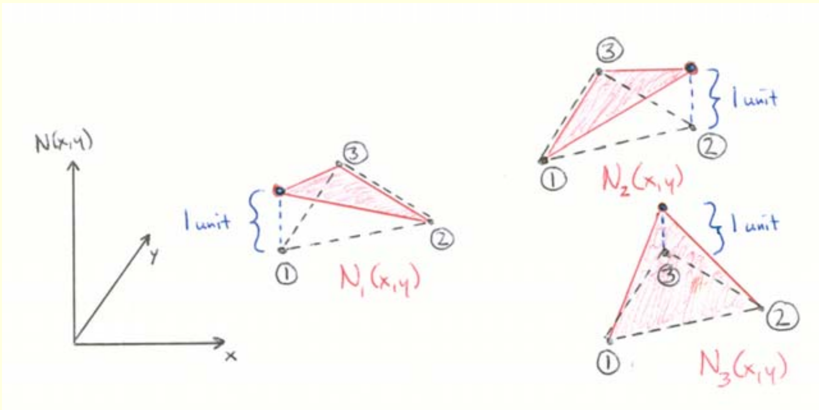
for any values of u_1, u_2, u_3.

$$\Rightarrow \text{Want to have } N_j(x_i,y_i) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

Kronecker delta property

3.2: Element Formulation (cont.)

- Visually, this looks like:



3.2: Element Formulation (cont.)

- Consider the shape function corresponding to u_1 :

Linear in x and $y \Rightarrow N_1(x,y) = \mathrm{b}_1 + \mathrm{b}_2 x + \mathrm{b}_3 y.$

Kronecker delta $\Rightarrow N_1(x_1,y_1) = 1; N_1(x_2,y_2) = 0; N_1(x_3,y_3) = 0.$

- Therefore, get a set of equations to solve:

$$1 = \mathrm{b}_1 + \mathrm{b}_2 x_1 + \mathrm{b}_3 y_1;$$
$$0 = \mathrm{b}_1 + \mathrm{b}_2 x_2 + \mathrm{b}_3 y_2;$$
$$0 = \mathrm{b}_1 + \mathrm{b}_2 x_3 + \mathrm{b}_3 y_3.$$

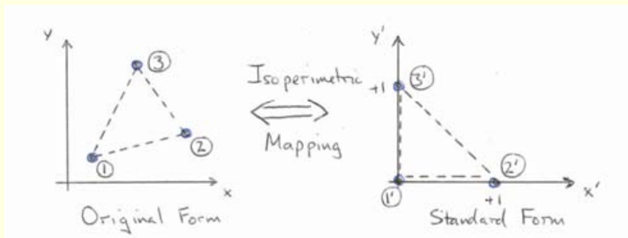
$$\Rightarrow N_1(x,y) = \left[(x_2 y_3 - x_3 y_2) + x(y_2 - y_3) + y(x_3 - x_2) \right] / 2A.$$

- Similar procedure to construct other shape functions.

3.2: Element Formulation (cont.)

Notes on "Direct approach":

- This approach is more commonly used as number of d.o.f. and/or order of polynomials used increases.
- Works best if shape functions are computed on "standard" geometries – leads to so-called "isoparametric formulation".



3.2: Element Formulation (cont.)

Check the required properties:

- Well-posedness: 6 d.o.f and 6 shape functions.✓
- Differentiability: Will show that only 1st derivatives show up in variational principle, so need continuous shape functions. ✓
- Completeness –
  - Rigid body motion: Suppose $u_1 = u_2 = u_3 = \hat{u}; v_1 = v_2 = v_3 = 0.$

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{bmatrix} N_1(x,y) & 0 & N_3(x,y) & 0 & N_5(x,y) & 0 \\ 0 & N_2(x,y) & 0 & N_4(x,y) & 0 & N_6(x,y) \end{bmatrix} \begin{pmatrix} \hat{u} \\ 0 \\ \hat{u} \\ 0 \\ \hat{u} \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{u} * (N_1 + N_3 + N_5) \\ 0 \end{pmatrix}.$$

$$N_1(x,y) + N_3(x,y) + N_5(x,y) = \left[(x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1) \right] / 2A = 1. \quad ✓$$

3.2: Element Formulation (cont.)

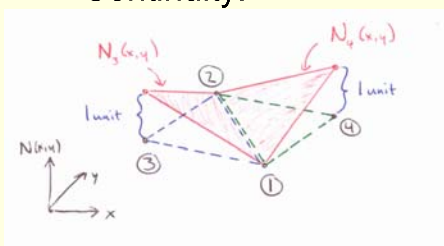
Check the required properties:

- Completeness –
  - Constant strain: Can $\varepsilon_x = \dfrac{\partial u}{\partial x} = \text{constant?}$

$$\varepsilon_x = \frac{\partial u}{\partial x} = u_1 \frac{\partial N_1}{\partial x} + u_2 \frac{\partial N_2}{\partial x} + u_3 \frac{\partial N_3}{\partial x} = \left[u_1 \left(y_2 - y_3 \right) + u_2 \left(y_3 - y_1 \right) + u_3 \left(y_1 - y_2 \right) \right] / 2A = \text{constant.} \checkmark$$

In fact, strain <u>must</u> be a constant !

- Continuity:



Neither $N_3(x,y)$ nor $N_4(x,y)$ can influence what happens on the line between pt. 1 and pt. 2.

$\therefore$ Only d.o.f. at pt. 1 and pt. 2 matter! $\checkmark$

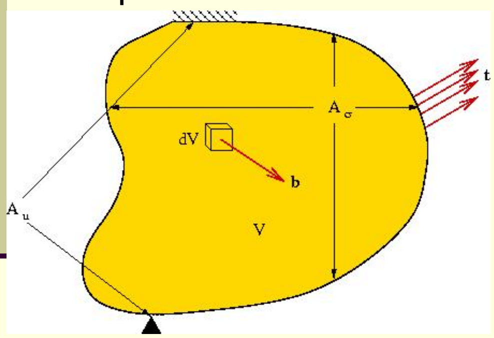
3.2: Element Formulation (cont.)

Next issue: deriving the approximate equations

- Determine *element (local) stiffness matrix*
  - Relates forces (stresses) to displacements (strains)
  - Term is used for <u>all</u> elements, not just elastic ones
- Determine *element (local) force vector*
  - Includes both body forces and surface tractions
  - Will change during the course of solving a problem

3.2: Element Formulation (cont.)

Goal: obtain approximate solution to 2D elasticity equations –



$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0,$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0,$$

$$\mathbf{u} - \mathbf{u}_0 = \mathbf{0} \text{ on } A_u,$$

$$[\boldsymbol{\sigma}](\hat{\mathbf{n}}) - \mathbf{t} = \mathbf{0} \text{ on } A_\sigma.$$

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}.$$

Galerkin, Calculus of Variations, Rayleigh-Ritz, …

3.2: Element Formulation (cont.)

Using Calculus of Variations (aka Principle of Virtual Displacements):

$$\int_V (\boldsymbol{\sigma}) \cdot (\boldsymbol{\delta\varepsilon}) dV - \int_V (\mathbf{b}) \cdot (\boldsymbol{\delta}\mathbf{u}) dV - \int_{A_\sigma} (\mathbf{t}) \cdot (\boldsymbol{\delta}\mathbf{u}) dA = 0.$$

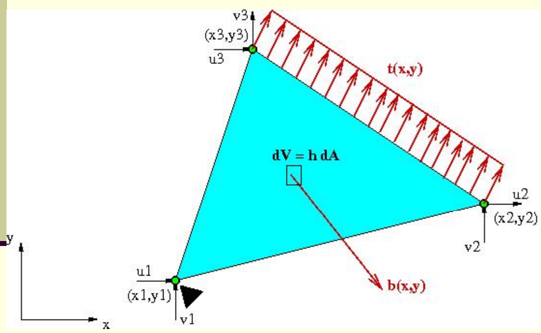
- Key Idea: solve <u>same</u> problem <u>locally</u> –

$$\int_{V_e} (\boldsymbol{\sigma}) \cdot (\boldsymbol{\delta\varepsilon}) dV - \int_{V_e} (\mathbf{b}) \cdot (\boldsymbol{\delta}\mathbf{u}) dV - \int_{A_{e,\sigma}} (\mathbf{t}) \cdot (\boldsymbol{\delta}\mathbf{u}) dA = 0.$$

$V_e =$ element volume; $A_{e,\sigma} =$ element surface.

3.2: Element Formulation (cont.)

New Goal: obtain approximate solution to 2D elasticity equations on each element –



$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0,$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0,$$

$$\mathbf{u} - \mathbf{u}_0 = \mathbf{0} \text{ on } A_u,$$

$$[\boldsymbol{\sigma}](\hat{\mathbf{n}}) - \mathbf{t} = \mathbf{0} \text{ on } A_\sigma.$$

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}.$$

3.2: Element Formulation (cont.)

Strain-Displacement Relations –

- Relate **u** to **ε** as follows: Derivative operator matrix, $[\partial]$

$$(\boldsymbol{\varepsilon}) = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \boxed{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}} \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}.$$

$$\therefore (\boldsymbol{\varepsilon}) = [\partial](\mathbf{u}) \Rightarrow (\delta\boldsymbol{\varepsilon}) = [\partial](\delta\mathbf{u}).$$

- Using shape functions:

$$(\mathbf{u}) = \underbrace{[\mathbf{N}(\mathbf{x})]}_{\text{shape functions}} \underbrace{(\mathbf{d})}_{\text{d.o.f.}} \Rightarrow (\boldsymbol{\varepsilon}) = \underbrace{[\partial][\mathbf{N}(\mathbf{x})]}_{[\mathbf{B}(\mathbf{x})]}(\mathbf{d}).$$

$$\therefore (\boldsymbol{\varepsilon}) = [\mathbf{B}(\mathbf{x})](\mathbf{d}) \Rightarrow (\delta\boldsymbol{\varepsilon}) = [\mathbf{B}(\mathbf{x})](\delta\mathbf{d}).$$

3.2: Element Formulation (cont.)

Stress-Strain Relations –

- Relate **σ** to **ε** as follows:

Not the same as used in interpolation approach!

$$(\boldsymbol{\sigma}) = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}}_{\text{elasticity matrix}} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = [\mathbf{C}](\boldsymbol{\varepsilon}).$$

- Using previous results:

$$\therefore (\boldsymbol{\sigma}) = [\mathbf{C}][\mathbf{B}(\mathbf{x})](\mathbf{d}).$$

3.2: Element Formulation (cont.)

Element (Local) Stiffness Matrix –

- Put these results into 1st term of PVD:

$$\int_{V_e} (\boldsymbol{\sigma}) \cdot (\boldsymbol{\delta\varepsilon})\, dV = h * \int_{triangle} \{[\mathbf{C}][\mathbf{B}(\mathbf{x})](\mathbf{d})\} \cdot \{[\mathbf{B}(\mathbf{x})](\delta\mathbf{d})\}\, dA$$

$$= h * \int_{triangle} (\delta\mathbf{d}) \cdot \left\{[\mathbf{B}(\mathbf{x})]^T [\mathbf{C}][\mathbf{B}(\mathbf{x})](\mathbf{d})\right\} dA$$

$$= (\delta\mathbf{d}) \cdot \boxed{\left\{ h * \int_{triangle} [\mathbf{B}(\mathbf{x})]^T [\mathbf{C}][\mathbf{B}(\mathbf{x})]\, dA \right\}} (\mathbf{d}).$$

Element stiffness matrix, [**k**].

3.2: Element Formulation (cont.)

Notes on Element (Local) Stiffness Matrix –

- The element stiffness matrix for <u>any</u> elastic element will follow the <u>exact same process</u>. (Details will change.)
- Element stiffness matrix is symmetric.

$$[\mathbf{k}]^T = \left\{ h * \int\limits_{triangle} [\mathbf{B}(\mathbf{x})]^T [\mathbf{C}][\mathbf{B}(\mathbf{x})] dA \right\}^T = h * \int\limits_{triangle} \left\{ [\mathbf{B}(\mathbf{x})]^T [\mathbf{C}][\mathbf{B}(\mathbf{x})] \right\}^T dA.$$

$$\text{But } \left\{ [\mathbf{M}_1][\mathbf{M}_2][\mathbf{M}_3] \right\}^T = [\mathbf{M}_3]^T [\mathbf{M}_2]^T [\mathbf{M}_1]^T .$$

$$\therefore [\mathbf{k}]^T = h * \int\limits_{triangle} \left\{ [\mathbf{B}(\mathbf{x})]^T [\mathbf{C}]^T [\mathbf{B}(\mathbf{x})] \right\} dA = h * \int\limits_{triangle} [\mathbf{B}(\mathbf{x})]^T [\mathbf{C}][\mathbf{B}(\mathbf{x})] dA = [\mathbf{k}].$$

3.2: Element Formulation (cont.)

Notes on Element (Local) Stiffness Matrix –

- [**k**] for the CST element is:

$$\begin{bmatrix} b_1^2C_{11}+c_1^2C_{33} & b_1c_1C_{12}+b_1c_1C_{33} & b_1b_2C_{11}+c_1c_2C_{33} & b_1c_2C_{12}+b_2c_1C_{33} & b_1b_3C_{11}+c_1c_2C_{33} & b_1c_3C_{12}+b_3c_1C_{33} \\ & c_1^2C_{11}+b_1^2C_{33} & b_2c_1C_{12}+b_1c_2C_{33} & c_1c_2C_{11}+b_1b_2C_{33} & b_3c_1C_{12}+b_1c_3C_{33} & c_1c_3C_{11}+b_1b_3C_{33} \\ & & b_2^2C_{11}+c_2^2C_{33} & b_2c_2C_{12}+b_2c_2C_{33} & b_2b_3C_{11}+c_2c_3C_{33} & b_2c_3C_{12}+b_3c_2C_{33} \\ & \text{Symmetric} & & c_2^2C_{11}+b_2^2C_{33} & b_3c_2C_{12}+b_2c_3C_{33} & c_2c_3C_{11}+b_2b_3C_{33} \\ & & & & b_3^2C_{11}+c_3^2C_{33} & b_3c_3C_{12}+b_3c_3C_{33} \\ & & & & & c_3^2C_{11}+b_3^2C_{33} \end{bmatrix} t \div 2A$$

$$b_1 = y_2 - y_3; b_2 = y_3 - y_1; b_3 = y_1 - y_2; c_1 = x_3 - x_2; c_2 = x_1 - x_3; c_3 = x_2 - x_1; t = h; A = \text{area of triangle}.$$

(See Probs. 3.14, 3.19, and 3.40 in Schaum's.)

3.2: Element Formulation (cont.)

Element (Local) Force Vector –

- Evaluate 2nd and 3rd terms of PVD:

$$\int_{V_e} (\mathbf{b})\cdot(\boldsymbol{\delta}\mathbf{u})\,dV + \int_{A_{e,\sigma}} (\mathbf{t})\cdot(\boldsymbol{\delta}\mathbf{u})\,dA$$

$$= \int_{V_e} (\mathbf{b})\cdot[\mathbf{N}(\mathbf{x})](\boldsymbol{\delta}\mathbf{d})\,dV + \int_{A_{e,\sigma}} (\mathbf{t})\cdot[\mathbf{N}(\mathbf{x})](\boldsymbol{\delta}\mathbf{d})\,dA$$

$$= \int_{V_e} (\boldsymbol{\delta}\mathbf{d})\cdot[\mathbf{N}(\mathbf{x})]^T(\mathbf{b})\,dV + \int_{A_{e,\sigma}} (\boldsymbol{\delta}\mathbf{d})\cdot[\mathbf{N}(\mathbf{x})]^T(\mathbf{t})\,dA$$

$$= (\boldsymbol{\delta}\mathbf{d})\cdot\boxed{\left\{\int_{V_e}[\mathbf{N}(\mathbf{x})]^T(\mathbf{b})\,dV + \int_{A_{e,\sigma}}[\mathbf{N}(\mathbf{x})]^T(\mathbf{t})\,dA\right\}}.$$

Element force vector (**f**)

3.2: Element Formulation (cont.)

Notes on Element (Local) Force Vector –

- In general, **b** and **t** can depend upon position, so they are left inside of the integrals.
- A "physical" interpretation of **f**:

Let $(\mathbf{N}_i(\mathbf{x})) = i$th column of $[\mathbf{N}(\mathbf{x})]$.

($i.e.$, shape function associated with d.o.f. $\#i$.)

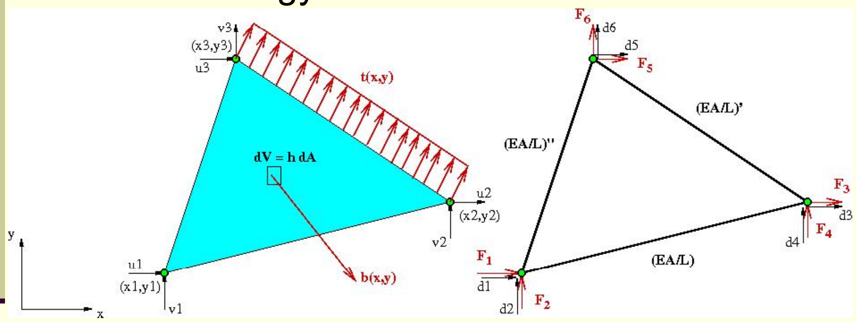
$$f_i = \int_{V_e} (\mathbf{N}_i(\mathbf{x}))^T(\mathbf{b})\,dV + \int_{A_{e,\sigma}} (\mathbf{N}_i(\mathbf{x}))^T(\mathbf{t})\,dA$$

= "average force" acting on d.o.f. $\#i$.

Called *equivalent nodal force*

3.2: Element Formulation (cont.)

A “truss analogy” for elements –



k_{ij} = "direct stiffness" between d.o.f. #i and #j.

(related to the various EA/L values).

f_i = "external force" acting parallel to d.o.f. #i.

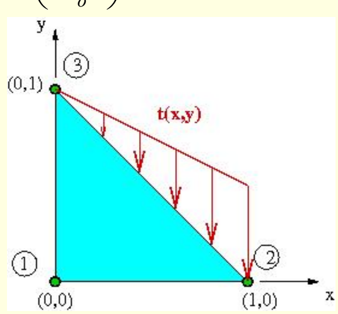
3.2: Element Formulation (cont.)

Example –

- Given: CST element shown has no body force and a surface traction applied to edge *23* expressed as

$$(\mathbf{t}) = \begin{pmatrix} 0 \\ -t_o x \end{pmatrix}.$$

- Required: Find (**f**).



3.2: Element Formulation (cont.)

Example –

- Solution: $(\mathbf{f}) = h * \int\limits_{\text{edge } 23} [\mathbf{N}(\mathbf{x})]^T (\mathbf{t}) \, d\ell$, where

$$\text{edge } 23 = \{(x,y) : y = 1-x, 0 \le x \le 1\},$$

$$d\ell = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{2}dx.$$

Can show that $N_1(x,y) = 1-x-y; N_2(x,y) = x; N_3(x,y) = y.$

$$\therefore [\mathbf{N}(\mathbf{x})] = \begin{bmatrix} 1-x-y & 0 & x & 0 & y & 0 \\ 0 & 1-x-y & 0 & x & 0 & y \end{bmatrix}.$$

3.2: Element Formulation (cont.)

Example –

- Solution:

$$\therefore [\mathbf{N}(\mathbf{x})]^T (\mathbf{t}) = \begin{bmatrix} 1-x-y & 0 \\ 0 & 1-x-y \\ x & 0 \\ 0 & x \\ y & 0 \\ 0 & y \end{bmatrix} \begin{pmatrix} 0 \\ -t_o x \end{pmatrix} = \begin{pmatrix} 0 \\ -t_o x(1-x-y) \\ 0 \\ -t_o x^2 \\ 0 \\ -t_o xy \end{pmatrix}.$$

$$\text{On edge } 23,\, y = 1-x \Rightarrow [\mathbf{N}(\mathbf{x})]^T (\mathbf{t}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -t_o x^2 \\ 0 \\ -t_o x(1-x) \end{pmatrix}.$$

3.2: Element Formulation (cont.)

Example –

- Solution:

$$\left(\mathbf{f}\right) = \sqrt{2}h * \int_0^1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -t_o x^2 \\ 0 \\ -t_o x(1-x) \end{pmatrix} dx = -\sqrt{2}ht_o \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/3 \\ 0 \\ 1/6 \end{pmatrix}.$$

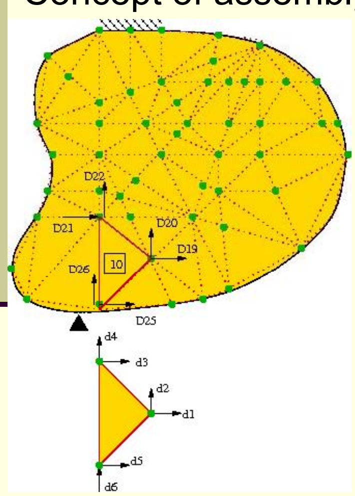
Section 3: Implementation of FEA – the CST

Overview: What is Finite Element Analysis?

- Assume a given problem is to be solved globally over some object. FEA proceeds as follows:
  1. Discretize the problem into a *finite* number of “local” approximate problems on regions called *elements*.
  2. Set up each of the local approximate problems.
  3. Assemble the local problems into a global problem.
  4. Solve the global problem and check convergence. (If not converged, repeat steps #2 and #3.)
  5. Once converged, evaluate the solution.

Section 3.3: Assembly

Concept of assembly:



- Each degree of freedom d_i in a given element corresponds to a unique degree of freedom D_k in the overall object.
- $\therefore$ Each local stiffness matrix $[\mathbf{k}]_e$ contributes to <u>part</u> of the global stiffness matrix of the object, **[K]**.
- Also, each local force vector $(\mathbf{f})_e$ contributes to <u>part</u> of the global force vector of the object, **(F)**.

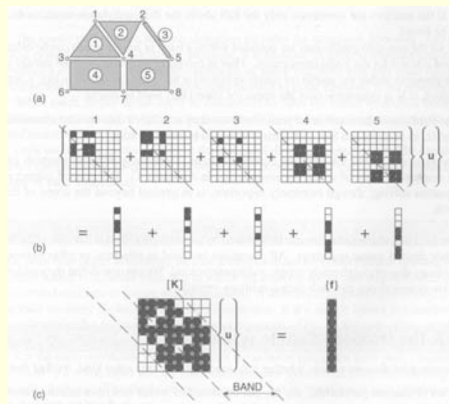
$$\boxed{[\mathbf{K}](\mathbf{D})=(\mathbf{F})} \Leftrightarrow \text{"}\sum_{e=1}^{n_e}\text{"}\left\{\boxed{[\mathbf{k}]_e(\mathbf{d})=(\mathbf{f})_e}\right\}.$$

Global Local

3.3: Assembly (cont.)

Concept of Assembly:

- Assembly is <u>not</u> straight addition –



3.3: Assembly (cont.)

How is assembly done in FEA programs?

- Each element has an associated "map" that contains connectivity information; *i.e.*, it links each local d.o.f. to corresponding global d.o.f. for the given element).
- Various names: "Connectivity vector", "destination array", "element-node array", …
- For picture on Slide 2:

$$\text{"Map"} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 19 & 20 & 21 & 22 & 25 & 26 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \leftarrow \text{Row 10}$$

3.3: Assembly (cont.)

Pseudocode for Assembly:

For *e* = 1, *numel* ← sum over all elements

For *i = 1, numdof(e)* ← sum over all local d.o.f.

For *j = i, numdof(e)* ← local d.o.f.

ii = map(e,i); jj = map(e,j); ← get global d.o.f.

K(ii,jj) = K(ii,jj) + k(e,i,j); ← assemble **[K]**

Continue

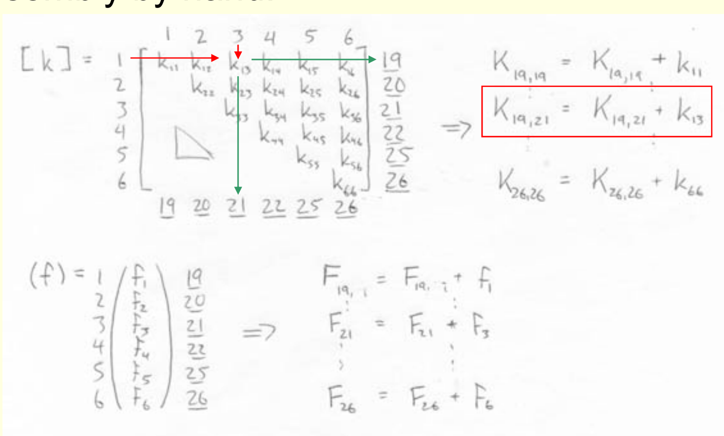
F(ii) = F(ii) + f(e,i); ← assemble **(F)**

Continue

Continue

3.3: Assembly (cont.)

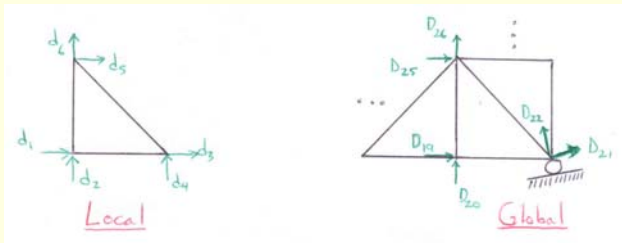
Assembly by hand:



3.3: Assembly (cont.)

Complication: what if local and global degrees of freedom are not parallel?

- Example: Roller support in global problem.



Easier to express boundary condition this way!

3.3: Assembly (cont.)

- Create a set of "new" local coordinates that <u>are</u> parallel –

$$\begin{pmatrix} d_3' \\ d_4' \end{pmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} d_3 \\ d_4 \end{pmatrix} \Rightarrow \begin{pmatrix} d_1 \\ d_2 \\ d_3' \\ d_4' \\ d_5 \\ d_6 \end{pmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cos\alpha & \sin\alpha & & \\ & & -\sin\alpha & \cos\alpha & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix},$$

$$\text{or } (\mathbf{d}') = [\mathbf{T}](\mathbf{d}).$$

- Can show that $(\mathbf{d}) = [\mathbf{T}]^T (\mathbf{d}')$.
- Will also need to transform element force vector:

$$(\mathbf{f}') = [\mathbf{T}](\mathbf{f}).$$

3.3: Assembly (cont.)

- Stiffness matrix then transforms –

$$[\mathbf{k}](\mathbf{d}) = (\mathbf{f}) \Rightarrow [\mathbf{T}][\mathbf{k}](\mathbf{d}) = [\mathbf{T}](\mathbf{f}) = (\mathbf{f}').$$

$$\text{But } (\mathbf{d}) = [\mathbf{T}]^T(\mathbf{d}') \Rightarrow \underbrace{[\mathbf{T}][\mathbf{k}][\mathbf{T}]^T}_{[\mathbf{k}']}(\mathbf{d}') = (\mathbf{f}').$$

These can now be assembled!

- Can show that same approach works for other types of "transformations" (*e.g.,* renumbering d.o.f., linking d.o.f, …)

Section 3.4: Boundary Conditions

- Traction boundary conditions used in PVD, but displacement boundary conditions aren't.

$$\int_V (\boldsymbol{\sigma})\cdot(\boldsymbol{\delta\varepsilon})\,dV - \int_V (\mathbf{b})\cdot(\boldsymbol{\delta}\mathbf{u})\,dV - \int_{A_\sigma} (\mathbf{t})\cdot(\boldsymbol{\delta}\mathbf{u})\,dA = 0.$$

$$\text{Must also have } (\mathbf{u}) = (\mathbf{u}_o) \text{ on } A_u!$$

- Shape functions must have Kronecker delta property $\therefore$ Cannot "choose" them to satisfy $(\mathbf{u}) = (\mathbf{u}_o)$!
- Must enforce displacement boundary conditions on global problem!

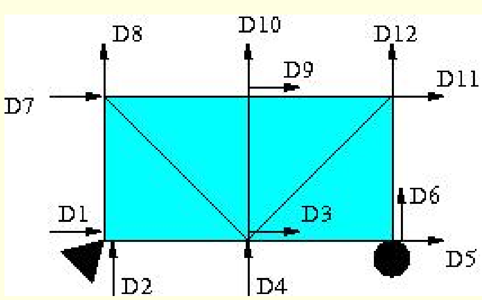
3.4: Boundary Conditions (cont.)

- Three general techniques for enforcing displacement boundary conditions:
  - Condensation
  - Penalty Method
  - Lagrange Multipliers
- In all methods, must discretize the boundary conditions to apply only at the nodes.

3.4: Boundary Conditions (cont.)

Condensation –

- Idea: formally remove constrained d.o.f. from the calculation, but keep their effects on other d.o.f.



$$D_1 = 0;$$
$$D_2 = 0;$$
$$D_6 = 0.$$

3.4: Boundary Conditions (cont.)

Condensation –

- For each constrained d.o.f., <u>remove</u> ("condense out") corresponding row and column from global equation.

$$
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} & K_{19} & K_{110} & K_{111} & K_{112} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} & K_{29} & K_{210} & K_{211} & K_{212} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} & K_{39} & K_{310} & K_{311} & K_{312} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} & K_{59} & K_{510} & K_{511} & K_{512} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} & K_{69} & K_{610} & K_{611} & K_{612} \\
K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} & K_{79} & K_{710} & K_{711} & K_{712} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
K_{121} & K_{122} & K_{123} & K_{124} & K_{125} & K_{126} & K_{127} & K_{128} & K_{129} & K_{1210} & K_{1211} & K_{1212}
\end{bmatrix}
\begin{pmatrix} 0 \\ 0 \\ D_3 \\ \vdots \\ D_5 \\ 0 \\ D_7 \\ \vdots \\ \vdots \\ D_{12} \end{pmatrix}
=
\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_5 \\ F_6 \\ F_7 \\ \vdots \\ \vdots \\ F_{12} \end{pmatrix}
$$

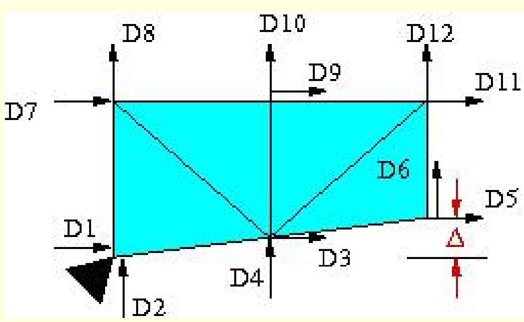
New (F) vector

New [K] matrix

What if $D_i \neq 0$?

3.4: Boundary Conditions (cont.)

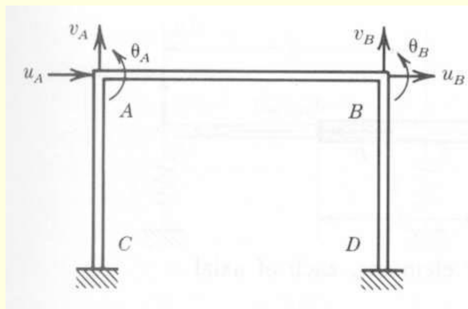
- Two possible situations where $D_i \neq 0$:
  - <u>*Single-point*</u> *boundary conditions*: b.c. involves only one d.o.f.; at least one must be nonzero.



$D_1 = 0;$
$D_2 = 0;$
$D_6 = \Delta.$

3.4: Boundary Conditions (cont.)

- *Multi-point boundary conditions*: b.c. involves more than one d.o.f.; may be zero or nonzero.



$$u_A - u_B = 0.$$

3.4: Boundary Conditions (cont.)

Procedure:

- Re-number the d.o.f. to put constrained d.o.f. together.
- Write the constraints in matrix/vector form (see Slide 14).

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{array}\right]}_{[\tilde{\mathbf{C}}]} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ \vdots \\ D_{12} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ \Delta \end{pmatrix}}_{(\mathbf{D}_o)} \Rightarrow \left[\begin{array}{cc} \tilde{\mathbf{C}}_{cc} & \tilde{\mathbf{C}}_{cr} \end{array} \right] \begin{pmatrix} (\mathbf{D}_c) \\ (\mathbf{D}_r) \end{pmatrix} = (\mathbf{D}_o).$$

- Solve for constrained d.of. in terms of regular d.o.f.:

$$\left[\tilde{\mathbf{C}}_{cc}\right](\mathbf{D}_c) + \left[\tilde{\mathbf{C}}_{cr}\right](\mathbf{D}_r) = (\mathbf{D}_o)$$

$$\Rightarrow (\mathbf{D}_c) = \left[\tilde{\mathbf{C}}_{cc}\right]^{-1}(\mathbf{D}_o) - \left[\tilde{\mathbf{C}}_{cc}\right]^{-1}\left[\tilde{\mathbf{C}}_{cr}\right](\mathbf{D}_r).$$

3.4: Boundary Conditions (cont.)

- This defines a transformation into "new" coordinates:

$$\begin{pmatrix} (\mathbf{D}_c) \\ (\mathbf{D}_r) \end{pmatrix} = \underbrace{\begin{bmatrix} -[\tilde{\mathbf{C}}_{cc}]^{-1}[\tilde{\mathbf{C}}_{cr}] \\ [\mathbf{I}] \end{bmatrix}}_{[\mathbf{T}]^T} (\mathbf{D}_r) + \underbrace{\begin{pmatrix} [\tilde{\mathbf{C}}_{cc}]^{-1}(\mathbf{D}_o) \\ (\mathbf{0}) \end{pmatrix}}_{(\tilde{\mathbf{D}}_o)} \Rightarrow (\mathbf{D}) = [\mathbf{T}]^T (\mathbf{D}_r) + (\tilde{\mathbf{D}}_o).$$

"old" coordinates

"new" coordinates

- Substitute "new" coordinates into global problem :

$$[\mathbf{K}](\mathbf{D}) = (\mathbf{F}) \Rightarrow [\mathbf{K}][\mathbf{T}]^T (\mathbf{D}_r) + [\mathbf{K}](\tilde{\mathbf{D}}_o) = (\mathbf{F}).$$

- Multiply by **[T]** and rearrange:

$$\boxed{[\mathbf{T}][\mathbf{K}][\mathbf{T}]^T}(\mathbf{D}_r) = \boxed{[\mathbf{T}](\mathbf{F}) - [\mathbf{T}][\mathbf{K}](\tilde{\mathbf{D}}_o)}.$$

New global stiffness matrix New global force vector

3.4: Boundary Conditions (cont.)

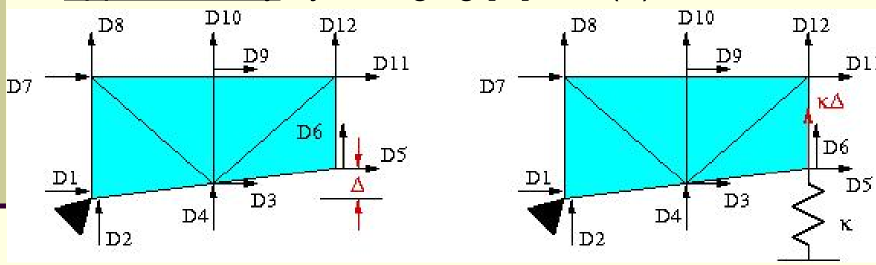
Notes on condensation –

- Very powerful method; can handle a variety of displacement boundary conditions exactly.
- Reduces bandwidth by eliminating d.o.f.
- If all boundary conditions are single-point, this method is equivalent to simply "condensing out" the constrained d.o.f. and adding in "extra" forces due to the imposed displacements.
- If there are many multi-point constraints, the process of re-numbering, transforming, and condensing can be very time-consuming.
- Condensation can sometimes be done at element level.

3.4: Boundary Conditions (cont.)

Penalty Method –

- Idea: enforce a displacement boundary condition <u>approximately</u> by changing [**K**] and (**F**).



3.4: Boundary Conditions (cont.)

Penalty Method –

- κ = stiffness of new spring; want $\kappa \gg \max\left|K_{ij}\right|$.
- Add new force $= \kappa\Delta$ to existing nodal force F_6.
- Add κ to existing stiffness K_{66}.
- Look at equation corresponding to d.o.f. D_6 :

  $$K_{16}D_1 + \ldots + \left(K_{66} + \kappa\right)D_6 + \ldots + K_{1212}D_{12} = F_6 + \kappa\Delta$$

  $$\Rightarrow \kappa D_6 \approx \kappa\Delta, \text{ or } D_6 \approx \Delta.$$
- Note: as κ gets bigger, approximation becomes better.

3.4: Boundary Conditions (cont.)

General Theory of the Penalty Method –

- Assume boundary conditions in "standard form"

$$\left[\tilde{\mathbf{C}}\right]\left(\mathbf{D}\right)=\left(\mathbf{D}_o\right) \qquad m \text{ equations}$$

- Define an *m* by *m* "penalty matrix"

$$\left[\boldsymbol{\kappa}\right]=\operatorname{diag}\left(\kappa_1,\kappa_2,\ldots,\kappa_m\right);\kappa_i \gg \max\left|K_{ij}\right|.$$

- Modify the global problem as follows:

$$\left\{\left[\mathbf{K}\right]+\boxed{\left[\tilde{\mathbf{C}}\right]^T\left[\boldsymbol{\kappa}\right]\left[\tilde{\mathbf{C}}\right]}\right\}\left(\mathbf{D}\right)=\left(\mathbf{F}\right)+\boxed{\left[\tilde{\mathbf{C}}\right]^T\left[\boldsymbol{\kappa}\right]\left(\mathbf{D}_o\right)}.$$

Added stiffness

Added force

3.4: Boundary Conditions (cont.)

Notes on the Penalty Method –

- Very easy to implement: no re-numbering, no transformations to apply, …
- Does not eliminate d.o.f., so no reduction in bandwidth.
- Assigning the penalty numbers κ_i can be tricky:
  - Too low ⇒ poor approximation to boundary condition
  - Too high ⇒ can create numerical problems *(e.g.*, locking, ill-conditioning, …)

3.4: Boundary Conditions (cont.)

- Why "Penalty Method"? Look at variational principle:

$$J\left(\mathbf{u}=\boxed{\mathbf{u}_{approx}}\right)=\tfrac{1}{2}(\mathbf{D})\square[\mathbf{K}](\mathbf{D})-(\mathbf{D})\square(\mathbf{F})$$

global approximate solution = "union" of all element solutions

- Set first variation to zero:

$$\delta J\left(\mathbf{u}=\mathbf{u}_{approx}\right)=2*\tfrac{1}{2}(\delta\mathbf{D})\square[\mathbf{K}](\mathbf{D})-(\delta\mathbf{D})\square(\mathbf{F})=0$$

$$\Rightarrow[\mathbf{K}](\mathbf{D})-(\mathbf{F})=(\mathbf{0}).$$

- Now, add in "penalty function": penalizes errors in satisfying b.c.'s

$$\tilde{J}\left(\mathbf{u}=\mathbf{u}_{approx}\right)=J\left(\mathbf{u}=\mathbf{u}_{approx}\right)+\boxed{\tfrac{1}{2}\left\{\left[\tilde{\mathbf{C}}\right](\mathbf{D})-(\mathbf{D}_o)\right\}\square[\boldsymbol{\kappa}]\left\{\left[\tilde{\mathbf{C}}\right](\mathbf{D})-(\mathbf{D}_o)\right\}}.$$

$$\delta\tilde{J}=0\Rightarrow\left\{[\mathbf{K}]+\left[\tilde{\mathbf{C}}\right]^T[\boldsymbol{\kappa}]\left[\tilde{\mathbf{C}}\right]\right\}(\mathbf{D})-(\mathbf{F})-\left[\tilde{\mathbf{C}}\right]^T[\boldsymbol{\kappa}](\mathbf{D}_o)=(\mathbf{0}).$$

3.4: Boundary Conditions (cont.)

Lagrange Multipliers –

- Idea: add extra d.o.f. into the problem, and use these d.o.f. to enforce the boundary conditions.

$$\tilde{J}\left(\mathbf{u}=\mathbf{u}_{approx}\right)=\tfrac{1}{2}(\mathbf{D})\square[\mathbf{K}](\mathbf{D})-(\mathbf{D})\square(\mathbf{F})+\boxed{(\boldsymbol{\lambda})}\square\left\{\left[\tilde{\mathbf{C}}\right](\mathbf{D})-(\mathbf{D}_o)\right\}.$$

Lagrange multipliers

- Note: Lagrange multipliers can be interpreted physically as constraint forces.
- Take first variation:

$$\delta\tilde{J}=(\delta\mathbf{D})\square\boxed{\left\{[\mathbf{K}](\mathbf{D})-(\mathbf{F})+\left[\tilde{\mathbf{C}}\right]^T(\boldsymbol{\lambda})\right\}}+(\delta\boldsymbol{\lambda})\square\boxed{\left\{\left[\tilde{\mathbf{C}}\right](\mathbf{D})-(\mathbf{D}_o)\right\}}.$$

must equal zero must equal zero

3.4: Boundary Conditions (cont.)

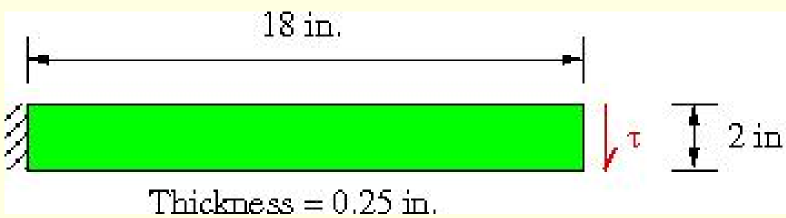
Lagrange Multipliers –

- Get an augmented global problem:

$$\begin{bmatrix} [\mathbf{K}] & [\tilde{\mathbf{C}}]^T \\ [\tilde{\mathbf{C}}] & [\mathbf{0}] \end{bmatrix} \begin{pmatrix} (\mathbf{D}) \\ (\boldsymbol{\lambda}) \end{pmatrix} = \begin{pmatrix} (\mathbf{F}) \\ (\mathbf{D}_o) \end{pmatrix}.$$

- Advantage: very effective at handling multi-point constraints.
- Disadvantage: more d.o.f. ⇒ longer solution time.

Section 3.5: Example Problem

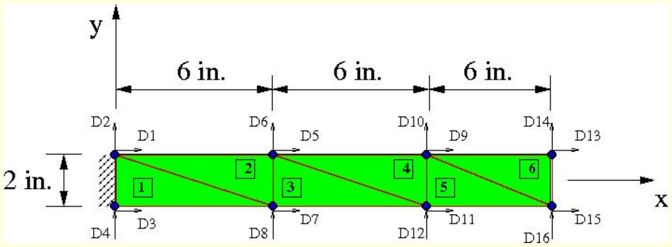


- Given: Cantilevered beam with dimensions shown; rigidly fixed at x = 0; applied traction $\mathbf{t} = -2.4\mathbf{j}$ ksi at x = 18. E = 27,000 ksi; ν = 0.25 .
- Required: Using CST plane stress elements, find the approximate deflection of the free end at y = 0.

3.5: Example (cont.)

Some preliminaries –

- Mesh the beam as shown:



6 elements;
16 total d.o.f.;
4 constraints.

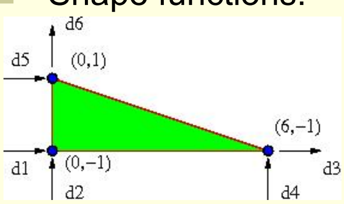
- Exact solution is known:

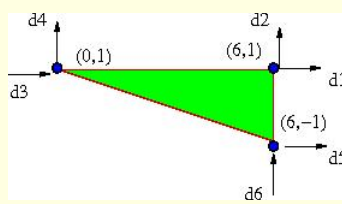
$$v\left(x=18, y=0\right)=-\frac{\tau AL^3}{3EI}=-\frac{\left(2.4\text{ ksi}\right)\left(0.5\text{ in}^2\right)\left(18\text{ in}\right)^3}{3\left(27000\text{ ksi}\right)\left(0.1667\text{ in}^4\right)}=-0.5184\text{ in.}$$

3.5: Example (cont.)

Formulate the elements –

- Shape functions:





Element #1:

$$N_1\left(x,y\right)=1-\tfrac{1}{6}x-\tfrac{1}{2}\left(y+1\right).$$
$$N_2\left(x,y\right)=\tfrac{1}{6}x.$$
$$N_3\left(x,y\right)=\tfrac{1}{2}\left(y+1\right).$$

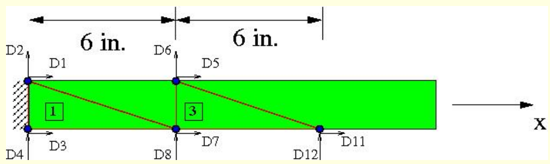
Element #2:

$$N_1\left(x,y\right)=1+\tfrac{1}{6}\left(x-6\right)+\tfrac{1}{2}\left(y-1\right).$$
$$N_2\left(x,y\right)=-\tfrac{1}{6}\left(x-6\right).$$
$$N_3\left(x,y\right)=-\tfrac{1}{2}\left(y-1\right).$$

3.5: Example (cont.)

Formulate the elements –

- Shape functions for other elements:



- Element #3 is simply Element #1 shifted from x = 0 to x = 6; thus, can "shift" each shape function:

$$N_1(x,y) = 1 - \tfrac{1}{6}(x-6) - \tfrac{1}{2}(y+1).$$
$$N_2(x,y) = \tfrac{1}{6}(x-6).$$
$$N_3(x,y) = \tfrac{1}{2}(y+1).$$

3.5: Example (cont.)

- Element #4 = Element #2 shifted from x = 6 to x = 12:

$$N_1(x,y) = 1 + \tfrac{1}{6}(x-12) + \tfrac{1}{2}(y-1),$$
$$N_2(x,y) = -\tfrac{1}{6}(x-12), N_3(x,y) = -\tfrac{1}{2}(y-1).$$

- Element #5 = Element #1 shifted from x = 0 to x = 12:

$$N_1(x,y) = 1 - \tfrac{1}{6}(x-12) - \tfrac{1}{2}(y+1),$$
$$N_2(x,y) = \tfrac{1}{6}(x-12), N_3(x,y) = \tfrac{1}{2}(y+1).$$

- Element #6 = Element #2 shifted from x = 6 to x = 18:

$$N_1(x,y) = 1 + \tfrac{1}{6}(x-18) + \tfrac{1}{2}(y-1),$$
$$N_2(x,y) = -\tfrac{1}{6}(x-18), N_3(x,y) = -\tfrac{1}{2}(y-1).$$

3.5: Example (cont.)

Formulate the elements –

- **[B]** matrix for Element #1:

$$
\left[\mathbf{B}(\mathbf{x})\right]_{\#1} = [\partial]\left[\mathbf{N}(\mathbf{x})\right]_{\#1} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{1}{2}-\frac{1}{6}x-\frac{1}{2}y & 0 & \frac{1}{6}x & 0 & \frac{1}{2}y+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}-\frac{1}{6}x-\frac{1}{2}y & 0 & \frac{1}{6}x & 0 & \frac{1}{2}y+\frac{1}{2} \end{bmatrix}
$$

$$
= \begin{bmatrix} -\frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 \end{bmatrix}.
$$

3.5: Example (cont.)

Formulate the elements –

- **[B]** matrix for Element #2:

$$
\left[\mathbf{B}(\mathbf{x})\right]_{\#2} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}+\frac{1}{6}x+\frac{1}{2}y & 0 & 1-\frac{1}{6}x & 0 & \frac{1}{2}-\frac{1}{2}y & 0 \\ 0 & -\frac{1}{2}+\frac{1}{6}x+\frac{1}{2}y & 0 & 1-\frac{1}{6}x & 0 & \frac{1}{2}-\frac{1}{2}y \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{2} & 0 \end{bmatrix} = -\left[\mathbf{B}(\mathbf{x})\right]_{\#1}.
$$

3.5: Example (cont.)

Other [**B**] matrices:

- Since Elements #3 and #5 are both simply shifts of Element #1, coefficients of *x* and *y* do not change.

$$\therefore \left[\mathbf{B}(\mathbf{x})\right]_{\#3} = \left[\mathbf{B}(\mathbf{x})\right]_{\#5} = \begin{bmatrix} -\frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 \end{bmatrix}.$$

- Likewise, Elements #4 and #6 are shifts of Element #2, so have same [**B**] matrices.

$$\therefore \left[\mathbf{B}(\mathbf{x})\right]_{\#4} = \left[\mathbf{B}(\mathbf{x})\right]_{\#6} = \begin{bmatrix} \frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{2} & 0 \end{bmatrix}.$$

3.5: Example (cont.)

Element stiffness matrices –

- Elastic matrix is:

$$\left[\mathbf{C}\right] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} = \begin{bmatrix} 28800 & 7200 & 0 \\ 7200 & 28800 & 0 \\ 0 & 0 & 10800 \end{bmatrix} \text{ksi}.$$

- Since [**B**] and [**C**] are constant matrices, the volume integral for [**k**] reduces to

$$[\mathbf{k}] = \int_{triangle} [\mathbf{B}]^T [\mathbf{C}][\mathbf{B}] * h\,dA \Rightarrow [\mathbf{k}] = [\mathbf{B}]^T [\mathbf{C}][\mathbf{B}] * hA_{triangle}.$$

3.5: Example (cont.)

- For Element #1:

$$[\mathbf{k}]_{\#1} = [\mathbf{B}]^T_{\#1}[\mathbf{C}][\mathbf{B}]_{\#1} * hA_{triangle} = (0.25*6)\begin{bmatrix} -\frac{1}{6} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{6} \\ \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 28800 & 7200 & 0 \\ 7200 & 28800 & 0 \\ 0 & 0 & 10800 \end{bmatrix}\begin{bmatrix} -\frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5250 & 2250 & -1200 & -1350 & -4050 & -900 \\ 2250 & 11250 & -900 & -450 & -1350 & -10800 \\ -1200 & -900 & 1200 & 0 & 0 & 900 \\ -1350 & -450 & 0 & 450 & 1350 & 0 \\ -4050 & -1350 & 0 & 1350 & 4050 & 0 \\ -900 & -10800 & 900 & 0 & 0 & 10800 \end{bmatrix} \text{kips/in.}$$

3.5: Example (cont.)

- For the other elements, we notice the following:

$$[\mathbf{B}]_{\#n} = \pm[\mathbf{B}]_{\#1} \Rightarrow [\mathbf{k}]_{\#n} = \left\{\pm[\mathbf{B}]^T_{\#1}\right\}[\mathbf{C}]\left\{\pm[\mathbf{B}]_{\#1}\right\} * hA_{triangle} = [\mathbf{k}]_{\#1}.$$

All elements have the same stiffness matrix!

3.5: Example (cont.)

- Only element force vector for Element #6:

$$\left[\mathbf{N}(\mathbf{x})\right]_{\#6}^{T}(\mathbf{t})=\left\{\begin{bmatrix} \frac{1}{6}x+\frac{1}{2}y-\frac{5}{2} & 0 \\ 0 & \frac{1}{6}x+\frac{1}{2}y-\frac{5}{2} \\ 3-\frac{1}{6}x & 0 \\ 0 & 3-\frac{1}{6}x \\ \frac{1}{2}-\frac{1}{2}y & 0 \\ 0 & \frac{1}{2}-\frac{1}{2}y \end{bmatrix}\begin{pmatrix} 0 \\ -2.4 \end{pmatrix}\right\}_{x=18}=\left\{\begin{bmatrix} \frac{1}{2}y+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}y+\frac{1}{2} \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{2}-\frac{1}{2}y & 0 \\ 0 & \frac{1}{2}-\frac{1}{2}y \end{bmatrix}\begin{pmatrix} 0 \\ -2.4 \end{pmatrix}\right\}=\begin{pmatrix} 0 \\ -1.2y-1.2 \\ 0 \\ 0 \\ 0 \\ 1.2y-1.2 \end{pmatrix}.$$

$$(\mathbf{f})_{\#6}=\int_{A_{e,\sigma}}\left[\mathbf{N}(\mathbf{x})\right]^{T}(\mathbf{t})\,dA=h*\int_{y=-1}^{y=1}\begin{pmatrix} 0 \\ -1.2y-1.2 \\ 0 \\ 0 \\ 0 \\ 1.2y-1.2 \end{pmatrix}dy=\begin{pmatrix} 0 \\ -0.6 \\ 0 \\ 0 \\ 0 \\ -0.6 \end{pmatrix}\text{ kips}$$

3.5: Example (cont.)

Assembly –

- Element #1:

$$\begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 1 & 5250 & 2250 & -1200 & -1350 & -4050 & -900 & \mathbf{3} \\ 2 & 2250 & 11250 & -900 & -450 & -1350 & -10800 & \mathbf{4} \\ 3 & -1200 & -900 & 1200 & 0 & 0 & 900 & \mathbf{7} \\ 4 & -1350 & -450 & 0 & 450 & 1350 & 0 & \mathbf{8} \\ 5 & -4050 & -1350 & 0 & 1350 & 4050 & 0 & \mathbf{1} \\ 6 & -900 & -10800 & 900 & 0 & 0 & 10800 & \mathbf{2} \\ \hline & \mathbf{3} & \mathbf{4} & \mathbf{7} & \mathbf{8} & \mathbf{1} & \mathbf{2} & \end{array}$$

- Element #2:

$$\begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 1 & 5250 & 2250 & -1200 & -1350 & -4050 & -900 & \mathbf{5} \\ 2 & 2250 & 11250 & -900 & -450 & -1350 & -10800 & \mathbf{6} \\ 3 & -1200 & -900 & 1200 & 0 & 0 & 900 & \mathbf{1} \\ 4 & -1350 & -450 & 0 & 450 & 1350 & 0 & \mathbf{2} \\ 5 & -4050 & -1350 & 0 & 1350 & 4050 & 0 & \mathbf{7} \\ 6 & -900 & -10800 & 900 & 0 & 0 & 10800 & \mathbf{8} \\ \hline & \mathbf{5} & \mathbf{6} & \mathbf{1} & \mathbf{2} & \mathbf{7} & \mathbf{8} & \end{array}$$

3.5: Example (cont.)

Assembly –

- Stiffness matrix for #1 + #2 only:

$$\begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 5250 & 0 & -4050 & -1350 & -1200 & -900 & 0 & 2250 \\ 0 & 11250 & -900 & -10800 & -1350 & -450 & 2250 & 0 \\ -4050 & -900 & 5250 & 2250 & 0 & 0 & -1200 & -1350 \\ -1350 & -10800 & 2250 & 11250 & 0 & 0 & -900 & -450 \\ -1200 & -1350 & 0 & 0 & 5250 & 2250 & -4050 & -900 \\ -900 & -450 & 0 & 0 & 2250 & 11250 & -1350 & -10800 \\ 0 & 2250 & -1200 & -900 & -4050 & -1350 & 5250 & 0 \\ 2250 & 0 & -1350 & -450 & -900 & -10800 & 0 & 11250 \end{bmatrix} \end{array} \text{kips/in.}$$

- What about stiffness matrix for Element #3 + #4?
  - Only real difference between #3+#4 and #1+#2 is the numbering of the d.o.f. – all shifted by 4
  - $\therefore$ Have same matrix for different d.o.f.

3.5: Example (cont.)

- Stiffness matrix for #3 + #4 only:

$$\begin{array}{cc} & \begin{array}{cccccccc} 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} & \begin{bmatrix} 5250 & 0 & -4050 & -1350 & -1200 & -900 & 0 & 2250 \\ 0 & 11250 & -900 & -10800 & -1350 & -450 & 2250 & 0 \\ -4050 & -900 & 5250 & 2250 & 0 & 0 & -1200 & -1350 \\ -1350 & -10800 & 2250 & 11250 & 0 & 0 & -900 & -450 \\ -1200 & -1350 & 0 & 0 & 5250 & 2250 & -4050 & -900 \\ -900 & -450 & 0 & 0 & 2250 & 11250 & -1350 & -10800 \\ 0 & 2250 & -1200 & -900 & -4050 & -1350 & 5250 & 0 \\ 2250 & 0 & -1350 & -450 & -900 & -10800 & 0 & 11250 \end{bmatrix} \end{array} \text{kips/in.}$$

- Stiffness matrix for #5 + #6 only:

$$\begin{array}{cc} & \begin{array}{cccccccc} 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{array} \\ \begin{array}{c} 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{array} & \begin{bmatrix} 5250 & 0 & -4050 & -1350 & -1200 & -900 & 0 & 2250 \\ 0 & 11250 & -900 & -10800 & -1350 & -450 & 2250 & 0 \\ -4050 & -900 & 5250 & 2250 & 0 & 0 & -1200 & -1350 \\ -1350 & -10800 & 2250 & 11250 & 0 & 0 & -900 & -450 \\ -1200 & -1350 & 0 & 0 & 5250 & 2250 & -4050 & -900 \\ -900 & -450 & 0 & 0 & 2250 & 11250 & -1350 & -10800 \\ 0 & 2250 & -1200 & -900 & -4050 & -1350 & 5250 & 0 \\ 2250 & 0 & -1350 & -450 & -900 & -10800 & 0 & 11250 \end{bmatrix} \end{array} \text{kips/in.}$$

3.5: Example (cont.)

- Stiffness matrix for entire structure:

$$
[\mathbf{K}] = \begin{bmatrix}
5250 & 0 & -4050 & -1350 & -1200 & -900 & 0 & 2250 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 11250 & -900 & -10800 & -1350 & -450 & 2250 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4050 & -900 & 5250 & 2250 & 0 & 0 & -1200 & -1350 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1350 & -10800 & 2250 & 11250 & 0 & 0 & -900 & -450 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1200 & -1350 & 0 & 0 & 10500 & 2250 & -8100 & -2250 & -1200 & -900 & 0 & 2250 & 0 & 0 & 0 & 0 \\
-900 & -450 & 0 & 0 & 2250 & 22500 & -2250 & -21600 & -1350 & -450 & 2250 & 0 & 0 & 0 & 0 & 0 \\
0 & 2250 & -1200 & -900 & -8100 & -2250 & 10500 & 2250 & 0 & 0 & -1200 & -1350 & 0 & 0 & 0 & 0 \\
2250 & 0 & -1350 & -450 & -2250 & -21600 & 2250 & 22500 & 0 & 0 & -900 & -450 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1200 & -1350 & 0 & 0 & 10500 & 2250 & -8100 & -2250 & -1200 & -900 & 0 & 2250 \\
0 & 0 & 0 & 0 & -900 & -450 & 0 & 0 & 2250 & 22500 & -2250 & -21600 & -1350 & -450 & 2250 & 0 \\
0 & 0 & 0 & 0 & 0 & 2250 & -1200 & -900 & -8100 & -2250 & 10500 & 2250 & 0 & 0 & -1200 & -1350 \\
0 & 0 & 0 & 0 & 2250 & 0 & -1350 & -450 & -2250 & -21600 & 2250 & 22500 & 0 & 0 & -900 & -450 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1200 & -1350 & 0 & 0 & 5250 & 2250 & -4050 & -900 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -900 & -450 & 0 & 0 & 2250 & 11250 & -1350 & -10800 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2250 & -1200 & -900 & -4050 & -1350 & 5250 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2250 & 0 & -1350 & -450 & -900 & -10800 & 0 & 11250
\end{bmatrix} \text{kips/in.}
$$

3.5: Example (cont.)

- Force vector for entire structure:

$$
(\mathbf{F}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.6 \\ 0 \\ -0.6 \end{pmatrix} \text{kips.}
$$

3.5: Example (cont.)

Enforce constraints –

- Using condensation, you must simply eliminate the first 4 rows and columns of [**K**] and first 4 rows of (**F**):

$$[\mathbf{K}'] = \begin{bmatrix} 10500 & 2250 & -8100 & -2250 & -1200 & -900 & 0 & 2250 & 0 & 0 & 0 & 0 \\ 2250 & 22500 & -2250 & -21600 & -1350 & -450 & 2250 & 0 & 0 & 0 & 0 & 0 \\ -8100 & -2250 & 10500 & 2250 & 0 & 0 & -1200 & -1350 & 0 & 0 & 0 & 0 \\ -2250 & -21600 & 2250 & 22500 & 0 & 0 & -900 & -450 & 0 & 0 & 0 & 0 \\ -1200 & -1350 & 0 & 0 & 10500 & 2250 & -8100 & -2250 & -1200 & -900 & 0 & 2250 \\ -900 & -450 & 0 & 0 & 2250 & 22500 & -2250 & -21600 & -1350 & -450 & 2250 & 0 \\ 0 & 2250 & -1200 & -900 & -8100 & -2250 & 10500 & 2250 & 0 & 0 & -1200 & -1350 \\ 2250 & 0 & -1350 & -450 & -2250 & -21600 & 2250 & 22500 & 0 & 0 & -900 & -450 \\ 0 & 0 & 0 & 0 & -1200 & -1350 & 0 & 0 & 5250 & 2250 & -4050 & -900 \\ 0 & 0 & 0 & 0 & -900 & -450 & 0 & 0 & 2250 & 11250 & -1350 & -10800 \\ 0 & 0 & 0 & 0 & 0 & 2250 & -1200 & -900 & -4050 & -1350 & 5250 & 0 \\ 0 & 0 & 0 & 0 & 2250 & 0 & -1350 & -450 & -900 & -10800 & 0 & 11250 \end{bmatrix} \text{kips/in;} (\mathbf{F}') = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.6 \\ 0 \\ -0.6 \end{pmatrix} \text{kips.}$$

3.5: Example (cont.)

- This can now be solved for the remaining d.o.f:

$$(\mathbf{D}') = \begin{pmatrix} 1.759 \\ -6.563 \\ -1.702 \\ -6.487 \\ 2.845 \\ -21.37 \\ -2.674 \\ -21.29 \\ 3.245 \\ -40.28 \\ -2.959 \\ -40.21 \end{pmatrix} \times 10^{-3}\text{in.}$$

- Use this to interpolate the requested displacement:

$$v(x=18, y=0) = D_{14}N_1^{(6)}(x=18, y=0) + D_{10}N_2^{(6)}(x=18, y=0) + D_{16}N_3^{(6)}(x=18, y=0)$$
$$= (-40.28 \times 10^{-3}\text{in})(0.5) + (-21.37 \times 10^{-3}\text{in})(0) + (-40.21 \times 10^{-3}\text{in})(0.5) = -40.24 \times 10^{-3}\text{in.}$$

Very poor approximation!

3.5: Example (cont.)

What went wrong?

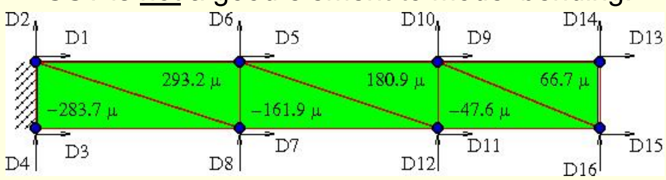
- Vertical d.o.f. are extremely stiff in this problem:

Equation #16: $2250D_9\text{-}1350D_{11}\text{-}450D_{12}\text{-}900D_{13}\boxed{-10800D_{14}+11250D_{16}=-0.6\text{ kips}}$.

(Changing aspect ratio will help.)

Dominant terms in equation.
$\therefore D_{14} \approx D_{16}$ = small number.

- CST is <u>not</u> a good element to model bending!



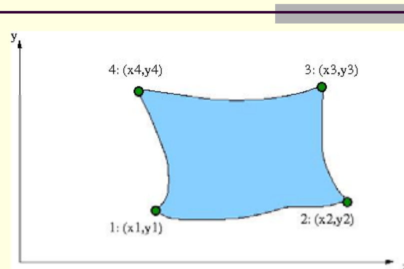
$$\varepsilon_x \neq -\frac{M_z y}{I_z}$$

Section 4: Implementation of Finite Element Analysis – Other Elements

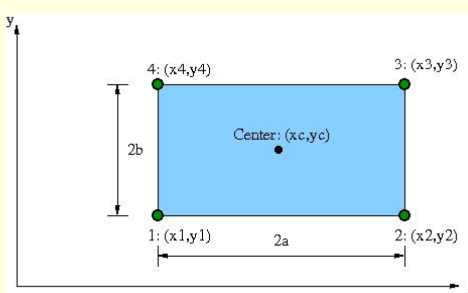
1. Quadrilateral Elements
2. Higher Order Triangular Elements
3. Isoparametric Elements

Section 4.1: Quadrilateral Elements

- Refers in general to <u>any</u> four-sided, 2D element.



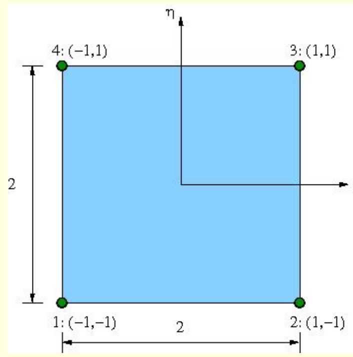
- We will start by considering rectangular elements with sides parallel to coordinate axes. (Thickness = *h*)



4.1: Quadrilateral Elements (cont.)

Normalized Element Geometry –

- "Standard" setting for calculations:



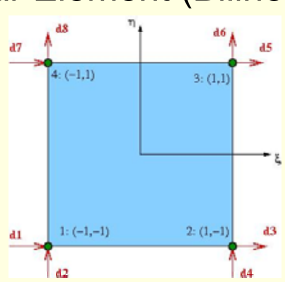
- Mapping between real and normalized coordinates:

$$\xi = \frac{x - x_c}{a}; \eta = \frac{y - y_c}{b} \Leftrightarrow x = a\xi + x_c; y = b\eta + y_c.$$

4.1: Quadrilateral Elements (cont.)

First Order Rectangular Element (Bilinear Quad):

- 4 nodes; 2 translational d.o.f. per node.



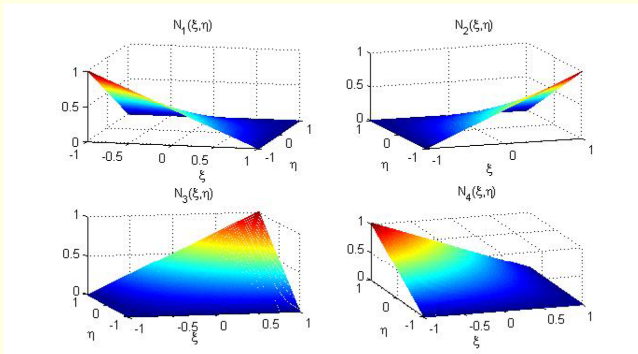
- Displacements interpolated as follows:

$$u(\xi(x),\eta(y)) = a_1 + a_2\xi + a_3\eta + \boxed{a_4\xi\eta}$$
$$v(\xi(x),\eta(y)) = a_5 + a_6\xi + a_7\eta + \boxed{a_8\xi\eta}$$

"Bilinear terms" – implies that all shape functions are products of linear functions of *x* and *y*.

4.1: Quadrilateral Elements (cont.)

Shape Functions:



$$N_1(\xi,\eta)=\tfrac{1}{4}(1-\xi)(1-\eta);N_2(\xi,\eta)=\tfrac{1}{4}(1+\xi)(1-\eta);$$
$$N_3(\xi,\eta)=\tfrac{1}{4}(1+\xi)(1+\eta);N_4(\xi,\eta)=\tfrac{1}{4}(1-\xi)(1+\eta).$$
$$\therefore N_k(\xi,\eta)=\tfrac{1}{4}(1+\xi_k\xi)(1+\eta_k\eta).$$

4.1: Quadrilateral Elements (cont.)

- Displacement interpolation becomes:

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \underbrace{\begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) \end{bmatrix}}_{\text{"}[\mathbf{N}(\mathbf{x})]\text{"}} \begin{pmatrix} d_1 \\ \vdots \\ d_8 \end{pmatrix}$$

- Need to compute [**B**] matrix:

$$[\mathbf{B}(\mathbf{x})]=[\partial][\mathbf{N}(\mathbf{x})]$$
$$=\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) \end{bmatrix} = [?].$$

4.1: Quadrilateral Elements (cont.)

- Chain rule:

$$\frac{\partial N_k}{\partial x}(\xi,\eta)=\frac{\partial N_k}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial N_k}{\partial \eta}\overset{0}{\cancel{\frac{\partial \eta}{\partial x}}}=\tfrac{1}{4}\xi_k\left(1+\eta_k\eta\right)*\tfrac{1}{a}=\tfrac{b\xi_k}{4ab}\left(1+\eta_k\eta\right);$$

$$\frac{\partial N_k}{\partial y}(\xi,\eta)=\frac{\partial N_k}{\partial \xi}\underset{0}{\cancel{\frac{\partial \xi}{\partial y}}}+\frac{\partial N_k}{\partial \eta}\frac{\partial \eta}{\partial y}=\tfrac{1}{4}\eta_k\left(1+\xi_k\xi\right)*\tfrac{1}{b}=\tfrac{a\eta_k}{4ab}\left(1+\xi_k\xi\right).$$

- Resulting **[B(x)]** matrix:

$$\left[\mathbf{B}(\mathbf{x})\right]=\tfrac{1}{4ab}\begin{bmatrix} -b(1-\eta) & 0 & b(1-\eta) & 0 & b(1+\eta) & 0 & -b(1+\eta) & 0 \\ 0 & -a(1-\xi) & 0 & -a(1+\xi) & 0 & a(1+\xi) & 0 & a(1-\xi) \\ -a(1-\xi) & -b(1-\eta) & -a(1+\xi) & b(1-\eta) & a(1+\xi) & b(1+\eta) & a(1-\xi) & -b(1+\eta) \end{bmatrix}$$

- Recall general expression for **[k]**:

$$\left[\mathbf{k}\right]=h*\int_{\boxed{area}}\left[\mathbf{B}(\mathbf{x})\right]^T\left[\mathbf{C}\right]\left[\mathbf{B}(\mathbf{x})\right]\boxed{dA}$$

Express in terms of ξ and η!

4.1: Quadrilateral Elements (cont.)

- Can show that

$$dA=dxdy=\left(\frac{dx}{d\xi}d\xi\right)\left(\frac{dy}{d\eta}d\eta\right)=ab*d\xi d\eta\Rightarrow\int_{area}dA=\int_{\eta=-1}^{\eta=1}\int_{\xi=-1}^{\xi=1}ab*d\xi d\eta.$$

$$\therefore\left[\mathbf{k}\right]=h*ab*\int_{\eta=-1}^{\eta=1}\int_{\xi=-1}^{\xi=1}\boxed{\left[\mathbf{B}\right]^T\left[\mathbf{C}\right]\left[\mathbf{B}\right]}d\xi d\eta.$$

Everything in terms of ξ and η!

- Can also show that

$$\begin{aligned}(\mathbf{f})&=\int_{V_e}\left[\mathbf{N}(\mathbf{x})\right]^T\left(\mathbf{b}(\mathbf{x})\right)dV+\int_{A_{e,\sigma}}\left[\mathbf{N}(\mathbf{x})\right]^T\left(\mathbf{t}(\mathbf{x})\right)dA\\ &=h*ab*\int_{\eta=-1}^{\eta=1}\int_{\xi=-1}^{\xi=1}\left[\mathbf{N}(\xi,\eta)\right]^T\left(\mathbf{b}(\xi,\eta)\right)d\xi d\eta\;+h*a\int_{\xi=-1}^{\xi=1}\left[\mathbf{N}(\xi,\eta=-1)\right]^T\left(\mathbf{t}(\xi,\eta=-1)\right)d\xi\\ &+h*b\int_{\eta=-1}^{\eta=1}\left[\mathbf{N}(\xi=1,\eta)\right]^T\left(\mathbf{t}(\xi=1,\eta)\right)d\eta\;+h*a\int_{\xi=-1}^{\xi=1}\left[\mathbf{N}(\xi,\eta=1)\right]^T\left(\mathbf{t}(\xi,\eta=1)\right)d\xi\\ &+h*b\int_{\eta=-1}^{\eta=1}\left[\mathbf{N}(\xi=-1,\eta)\right]^T\left(\mathbf{t}(\xi=-1,\eta)\right)d\eta.\end{aligned}$$

4.1: Quadrilateral Elements (cont.)

Gauss Quadrature:

- Let's take a closer look at one of the integrals for the element stiffness matrix (assume plane stress):

$$k_{22} = \frac{E}{1-\nu^2} * \frac{h}{16ab} \int_{\eta=-1}^{\eta=1} \int_{\xi=-1}^{\xi=1} \left\{ a^2 (1-\xi)^2 + \tfrac{1}{2} b^2 (1-\nu)(1-\eta)^2 \right\} d\xi \, d\eta.$$

- Can be solved exactly, but for various reasons FEA prefers to evaluate integrals like this approximately:
  - Historically, considered more efficient and reduced coding errors.
  - Only possible approach for isoparametric elements.
  - Can actually improve performance in certain cases!

4.1: Quadrilateral Elements (cont.)

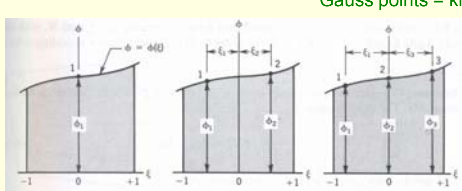
Gauss Quadrature:

- Idea: approximate integral by a sum of function values at predetermined points with optimal weights –

weights = known constants, depend on *n*

$$\text{1D case: } \int_{-1}^{1} \phi(\xi) \, d\xi \approx \sum_{i=1}^{n} \boxed{W_i} \phi\left(\boxed{\xi_i}\right).$$

Gauss points = known locations, depend on *n*



- *n* = order of quadrature; determines accuracy of integral.
  (Note: any polynomial of order *2n-1* can be integrated exactly using *n*th order Gauss quadrature.)

4.1: Quadrilateral Elements (cont.)

Gauss Quadrature:

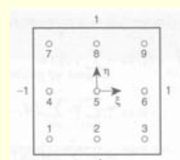
- Have tables for weights and Gauss points:

TABLE 6.4-1. SAMPLING POINTS AND WEIGHTS FOR GAUSS QUADRATURE OVER THE INTERVAL $\xi = -1$ TO $\xi = +1$.

Order n	Location ξ_i of Sampling Point	Weight Factor W_i
1	0.	2.
2	$\pm 0.57735\ 02691\ 89626 = \pm 1/\sqrt{3}$	1.
3	$\pm 0.77459\ 66692\ 41483 = \pm\sqrt{0.6}$	$0.55555\ 55555\ 55555 = \frac{5}{9}$
	0.	$0.88888\ 88888\ 88888 = \frac{8}{9}$
4	$\pm 0.86113\ 63115\ 94053 = \pm\left[\frac{3+2r}{7}\right]^{1/2}$	$0.34785\ 48451\ 37454 = \frac{1}{2} - \frac{1}{6r}$
	$\pm 0.33998\ 10435\ 84856 = \pm\left[\frac{3-2r}{7}\right]^{1/2}$ where $r = \sqrt{1.2}$	$0.65214\ 51548\ 62546 = \frac{1}{2} + \frac{1}{6r}$

- 2D case handled as two 1D cases:

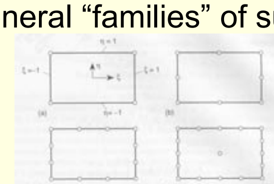
$$\int_{-1}^{1}\int_{-1}^{1}\phi(\xi,\eta)\,d\xi d\eta \approx \sum_{j=1}^{n}\sum_{i=1}^{n} W_i W_j \phi(\xi_i,\eta_j).$$



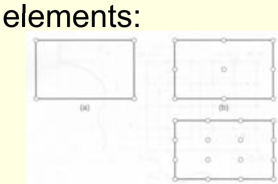
4.1: Quadrilateral Elements (cont.)

Higher Order Rectangular Elements

- More nodes; still 2 translational d.o.f. per node.
- "Higher order" ⇒ higher degree of complete polynomial contained in displacement approximations.
- Two general "families" of such elements:



Serendipity

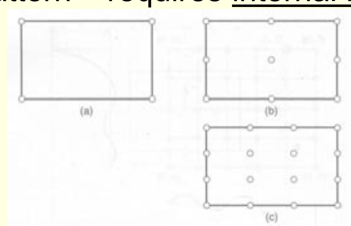


Lagrangian

4.1: Quadrilateral Elements (cont.)

Lagrangian Elements:

- Order *n* element has $(n+1)^2$ nodes arranged in square-symmetric pattern – requires <u>internal nodes</u>.

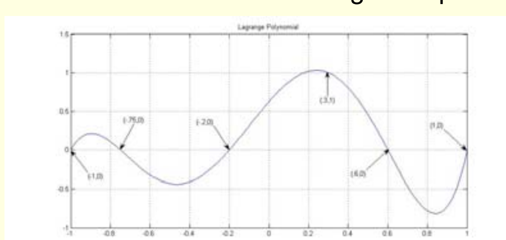


- Shape functions are products of *n*th order polynomials in each direction. ("biquadratic", "bicubic", …)
- Bilinear quad is a Lagrangian element of order *n* = 1.

4.1: Quadrilateral Elements (cont.)

Lagrangian Shape Functions:

- Uses a procedure that <u>automatically</u> satisfies the Kronecker delta property for shape functions.
  - Consider 1D example of 6 points; want function = 1 at $\xi_3 = 0.3$ and function = 0 at other designated points:



$$\xi_0 = -1;\quad \xi_1 = -.75;\quad \xi_2 = -.2;\quad \xi_3 = .3;\quad \xi_4 = .6;\quad \xi_5 = 1.$$

$$L_3^{(5)}(\xi) = \frac{(\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_4)(\xi-\xi_5)}{(\xi_3-\xi_0)(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)(\xi_3-\xi_5)}.$$

4.1: Quadrilateral Elements (cont.)

Lagrangian Shape Functions:

- Can perform this for <u>any</u> number of points at <u>any</u> designated locations.

$$L_k^{(m)}(\xi)=\frac{(\xi-\xi_0)(\xi-\xi_1)\cdots\boxed{(\xi-\xi_{k-1})(\xi-\xi_{k+1})}\cdots(\xi-\xi_m)}{(\xi_k-\xi_0)(\xi_k-\xi_1)\cdots\boxed{(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})}\cdots(\xi_k-\xi_m)}=\prod_{\substack{i=0\\ i\neq k}}^{m}\frac{(\xi-\xi_i)}{(\xi_k-\xi_i)}.$$

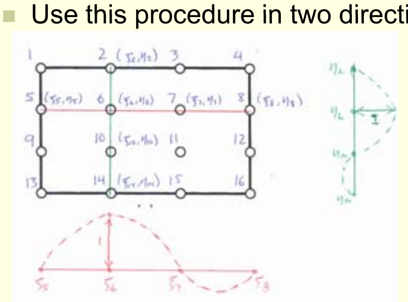
No ξ-ξ_k term!

Lagrange polynomial of order *m* at node *k*

4.1: Quadrilateral Elements (cont.)

Lagrangian Shape Functions:

- Use this procedure in two directions at each node:



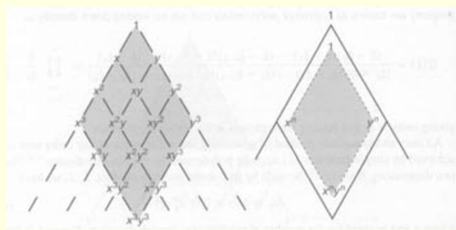
$$V_6^{(3)}(\eta)=\frac{(\eta-\eta_2)(\eta-\eta_{10})(\eta-\eta_{14})}{(\eta_6-\eta_2)(\eta_6-\eta_{10})(\eta_6-\eta_{14})}$$

$$H_6^{(3)}(\xi)=\frac{(\xi-\xi_5)(\xi-\xi_7)(\xi-\xi_8)}{(\xi_6-\xi_5)(\xi_6-\xi_7)(\xi_6-\xi_8)} \longrightarrow N_6(\xi,\eta)=H_6^{(3)}(\xi)V_6^{(3)}(\eta).$$

4.1: Quadrilateral Elements (cont.)

Notes on Lagrangian Elements:

- Once shape functions have been identified, there are no procedural differences in the formulation of higher order quadrilateral elements and the bilinear quad.
- Pascal's triangle for the Lagrangian quadrilateral elements:

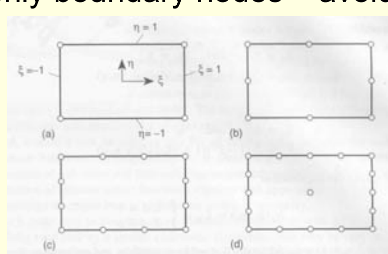


3 x 3 n x n

4.1: Quadrilateral Elements (cont.)

Serendipity Elements:

- In general, only boundary nodes – avoids internal ones.

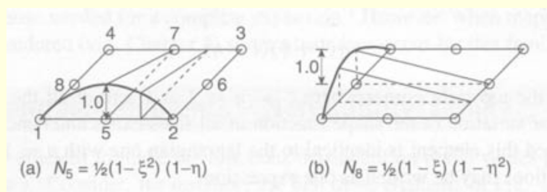


- Not as accurate as Lagrangian elements.
- However, more efficient than Lagrangian elements and avoids certain types of instabilities.

4.1: Quadrilateral Elements (cont.)

Serendipity Shape Functions:

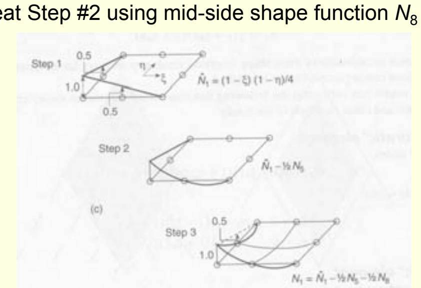
- Shape functions for **mid-side nodes** are products of an *n*th order polynomial parallel to side and a linear function perpendicular to the side.
  - E.g., quadratic serendipity element:



$$N_6 = \tfrac{1}{2}(1+\xi)(1-\eta^2); N_7 = \tfrac{1}{2}(1-\xi^2)(1+\eta).$$

4.1: Quadrilateral Elements (cont.)

- Shape functions for **corner nodes** are modifications of the shape functions of the bilinear quad.
  - Step #1: start with appropriate bilinear quad shape function, $\hat{N}_1$.
  - Step #2: subtract out mid-side shape function N_5 with appropriate weight $\hat{N}_1(\text{node }\#5) = \tfrac{1}{2}$
  - Step #3: repeat Step #2 using mid-side shape function N_8 and weight $\hat{N}_1(\text{node }\#8) = \tfrac{1}{2}$

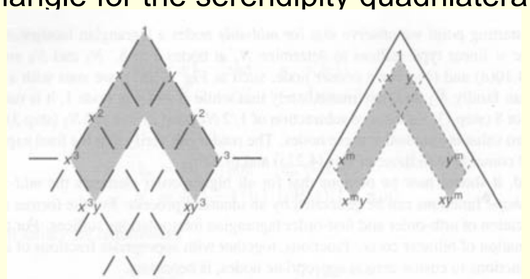


$$N_k = \tfrac{1}{4}(1+\xi_k\xi)(1+\eta_k\eta)(\xi_k\xi+\eta_k\eta-1); k = 1,2,3,4.$$

4.1: Quadrilateral Elements (cont.)

Notes on Serendipity Elements:

- Once shape functions have been identified, there are no procedural differences in the formulation of higher order quadrilateral elements and the bilinear quad.
- Pascal's triangle for the serendipity quadrilateral elements:



3 x 3 m x m

4.1: Quadrilateral Elements (cont.)

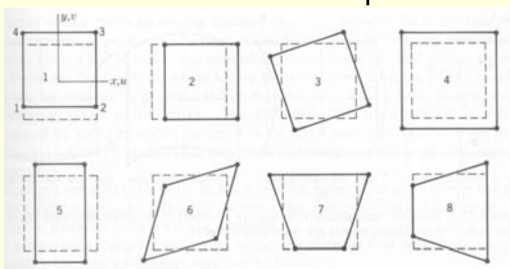
Zero-Energy Modes (Mechanisms; Kinematic Modes) –

- Instabilities for an element (or group of elements) that produce deformation without any strain energy.
- Typically caused by using an inappropriately low order of Gauss quadrature.
- If present, will dominate the deformation pattern.
- Can occur for all 2D elements except the CST.

4.1: Quadrilateral Elements (cont.)

Zero-Energy Modes –

- Deformation modes for a bilinear quad:

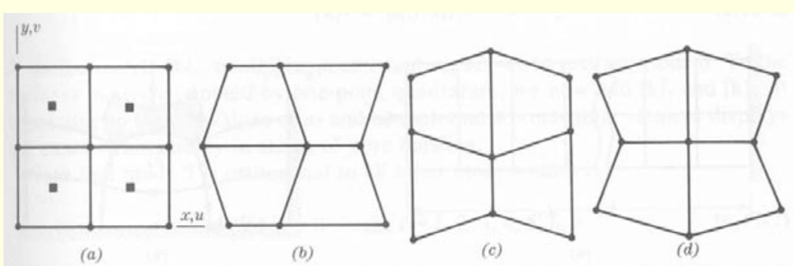


  - #1, #2, #3 = *rigid body modes*; can be eliminated by proper constraints.
  - #4, #5, #6 = *constant strain modes*; always have nonzero strain energy.
  - #7, #8 = *bending modes*; produce zero strain at origin.

4.1: Quadrilateral Elements (cont.)

Zero-Energy Modes –

- Mesh instability for bilinear quads using order 1 quadrature:

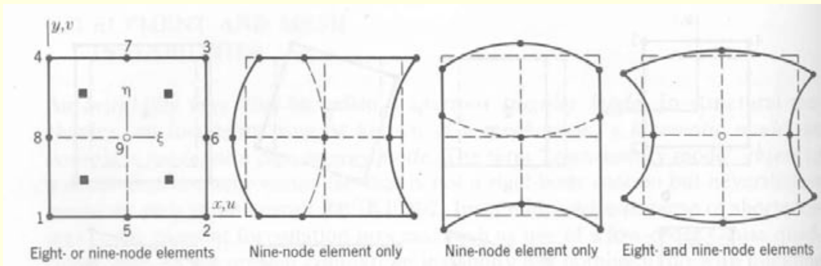


"Hourglass modes"

4.1: Quadrilateral Elements (cont.)

Zero-Energy Modes –

- Element instability for quadratic quadrilaterals using 2x2 Gauss quadrature:



"Hourglass modes"

4.1: Quadrilateral Elements (cont.)

Zero-Energy Modes –

- How can you prevent this?
  - Use higher order Gauss quadrature in formulation.
  - Can artificially "stiffen" zero-energy modes via penalty functions.
  - Avoid elements with known instabilities!

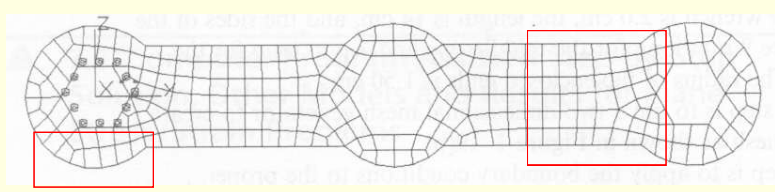
Section 4: Implementation of Finite Element Analysis – Other Elements

1. Quadrilateral Elements
2. Isoparametric Elements
3. Higher Order Triangular Elements

Note: <u>any</u> type of geometry can be used for isoparametric elements; we will only look at quadrilateral elements.

Section 4.2: Isoparametric Elements

- For various reasons, need elements that do not "fit" the standard geometry.

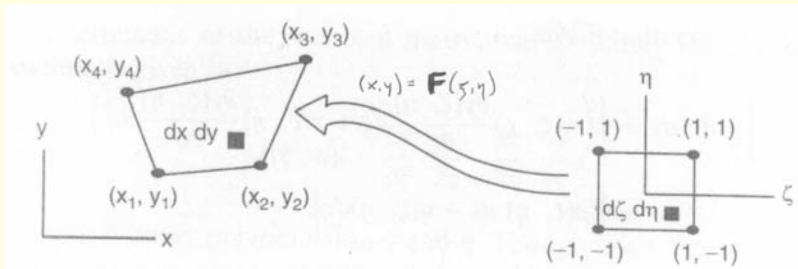


Curved boundaries

Transition regions

4.2: Isoparametric Elements (cont.)

- Problem: How do you map a general quadrilateral onto the normalized geometry?



$$(x,y)=\mathbf{F}(\xi,\eta)\Leftrightarrow(\xi,\eta)=\mathbf{F}^{-1}(x,y);\ \mathbf{F}=?$$

4.2: Isoparametric Elements (cont.)

- Idea: Approximate the mapping using "shape functions".

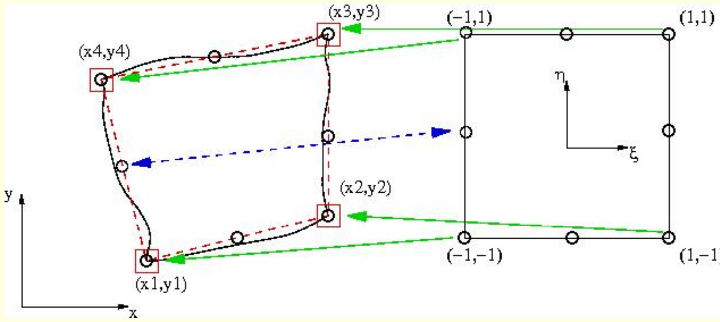
$$x\approx\mathbf{N}_1^*(\xi,\eta)x_1+\mathbf{N}_2^*(\xi,\eta)x_2+\mathbf{N}_3^*(\xi,\eta)x_3+\cdots+\mathbf{N}_n^*(\xi,\eta)x_n;$$
$$y\approx\mathbf{N}_1^*(\xi,\eta)y_1+\mathbf{N}_2^*(\xi,\eta)y_2+\mathbf{N}_3^*(\xi,\eta)y_3+\cdots+\mathbf{N}_n^*(\xi,\eta)y_n.$$

- Require $\mathbf{N}_k^*(\xi,\eta)$ to have Kronecker delta property.

- $\mathbf{N}_k^*(\xi,\eta)$ not required to be the actual shape functions of the element; *n* can be as large or as small as you want.

4.2: Isoparametric Elements (cont.)

- Approximate "serendipity element" shown using bilinear quad shape functions and approximation points at corners



$$x = \tfrac{1}{4}(1-\xi)(1-\eta)x_1 + \tfrac{1}{4}(1+\xi)(1-\eta)x_2 + \tfrac{1}{4}(1+\xi)(1+\eta)x_3 + \tfrac{1}{4}(1-\xi)(1+\eta)x_4.$$

4.2: Isoparametric Elements (cont.)

- For an <u>isoparametric element</u>, the number of approximation points equals the actual number of nodes for the element; also, the approximation functions are the actual shape functions for the element:

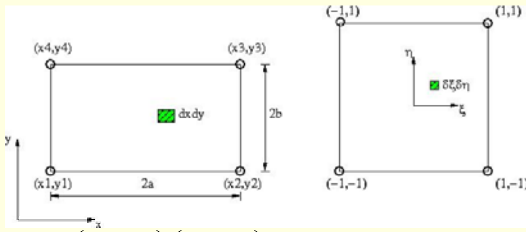
$$x = \mathbf{N}_1(\xi,\eta)x_1 + \mathbf{N}_2(\xi,\eta)x_2 + \mathbf{N}_3(\xi,\eta)x_3 + \cdots + \mathbf{N}_n(\xi,\eta)x_n;$$
$$y = \mathbf{N}_1(\xi,\eta)y_1 + \mathbf{N}_2(\xi,\eta)y_2 + \mathbf{N}_3(\xi,\eta)y_3 + \cdots + \mathbf{N}_n(\xi,\eta)y_n;$$
$$u(x,y) = \mathbf{N}_1(\xi,\eta)u_1 + \mathbf{N}_2(\xi,\eta)u_2 + \mathbf{N}_3(\xi,\eta)u_3 + \cdots + \mathbf{N}_n(\xi,\eta)u_n;$$
$$v(x,y) = \mathbf{N}_1(\xi,\eta)v_1 + \mathbf{N}_2(\xi,\eta)v_2 + \mathbf{N}_3(\xi,\eta)v_3 + \cdots + \mathbf{N}_n(\xi,\eta)v_n;$$

  - If # of approx. pts. > # of nodes, element is called <u>superparametric</u>; if # of approx. pts. < # of nodes, element is called <u>subparametric</u>.

4.2: Isoparametric Elements (cont.)

Formulating an Isoparametric Element:

- Recall the formulation of "standard" bilinear quad:



$$dA = dxdy = \left(\frac{dx}{d\xi}d\xi\right)\left(\frac{dy}{d\eta}d\eta\right) = ab * d\xi d\eta \Rightarrow \int_{area} dA = \int_{\eta=-1}^{\eta=1}\int_{\xi=-1}^{\xi=1} ab * d\xi d\eta.$$

$$\therefore [\mathbf{k}] = h * ab * \int_{\eta=-1}^{\eta=1}\int_{\xi=-1}^{\xi=1} [\mathbf{B}]^T [\mathbf{C}][\mathbf{B}] d\xi d\eta.$$

How does this work for an isoparametric element?

4.2: Isoparametric Elements (cont.)

Formulating an Isoparametric Element:

- Calculating the **[B]** matrix (assume isoparametric bilinear quad element):

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \underbrace{\begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) \end{bmatrix}}_{\text{"}[\mathbf{N}(\mathbf{x})]\text{"}} \begin{pmatrix} d_1 \\ \vdots \\ d_8 \end{pmatrix}.$$

$$[\mathbf{B}(\mathbf{x})] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) \end{bmatrix} = [?].$$

Need to apply the chain rule!

4.2: Isoparametric Elements (cont.)

Formulating an Isoparametric Element:

- Chain rule: compute inverse rule first –

$$\frac{\partial N_k}{\partial \xi} = \frac{\partial N_k}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_k}{\partial y}\frac{\partial y}{\partial \xi}; \frac{\partial N_k}{\partial \eta} = \frac{\partial N_k}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_k}{\partial y}\frac{\partial y}{\partial \eta}.$$

- Using the approximate mapping:

$$x = \sum_{i=1}^{n} N_i(\xi,\eta)x_i \Rightarrow \frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi}(\xi,\eta)x_i; \frac{\partial x}{\partial \eta} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta}(\xi,\eta)x_i.$$

$$\text{Similarly, } \frac{\partial y}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi}(\xi,\eta)y_i; \frac{\partial y}{\partial \eta} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta}(\xi,\eta)y_i.$$

4.2: Isoparametric Elements (cont.)

Formulating an Isoparametric Element:

- Put all of this together –

$$\begin{pmatrix} \frac{\partial N_k}{\partial \xi} \\ \frac{\partial N_k}{\partial \eta} \end{pmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{pmatrix} \frac{\partial N_k}{\partial x} \\ \frac{\partial N_k}{\partial y} \end{pmatrix}$$

$$= \boxed{\begin{bmatrix} \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi}(\xi,\eta)x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi}(\xi,\eta)y_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta}(\xi,\eta)x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta}(\xi,\eta)y_i \end{bmatrix}} \begin{pmatrix} \frac{\partial N_k}{\partial x} \\ \frac{\partial N_k}{\partial y} \end{pmatrix}.$$

The *Jacobian matrix* **[J]** of the mapping.

4.2: Isoparametric Elements (cont.)

Formulating an Isoparametric Element:

- Can now compute the regular chain rule –

$$\begin{pmatrix} \dfrac{\partial N_k}{\partial \xi} \\ \dfrac{\partial N_k}{\partial \eta} \end{pmatrix} = [\mathbf{J}] \begin{pmatrix} \dfrac{\partial N_k}{\partial x} \\ \dfrac{\partial N_k}{\partial y} \end{pmatrix} \Rightarrow \begin{pmatrix} \dfrac{\partial N_k}{\partial x} \\ \dfrac{\partial N_k}{\partial y} \end{pmatrix} = [\mathbf{J}]^{-1} \begin{pmatrix} \dfrac{\partial N_k}{\partial \xi} \\ \dfrac{\partial N_k}{\partial \eta} \end{pmatrix}.$$

$$[\mathbf{J}]^{-1} = \frac{1}{J} \begin{bmatrix} \sum_{i=1}^{n} \dfrac{\partial N_i}{\partial \eta}(\xi,\eta) y_i & -\sum_{i=1}^{n} \dfrac{\partial N_i}{\partial \xi}(\xi,\eta) y_i \\ -\sum_{i=1}^{n} \dfrac{\partial N_i}{\partial \eta}(\xi,\eta) x_i & \sum_{i=1}^{n} \dfrac{\partial N_i}{\partial \xi}(\xi,\eta) x_i \end{bmatrix}; \boxed{J = \det[\mathbf{J}]}.$$

"Jacobian" of the mapping

4.2: Isoparametric Elements (cont.)

Formulating an Isoparametric Element:

- *J* is a (nonconstant) scaling factor that relates area in original geometry to area in normalized geometry; can show that $dxdy = J * d\xi d\eta$.

- For a well-defined mapping, *J* must have same sign at <u>all points</u> in normalized geometry.

- Large variations in *J* imply highly distorted mappings – leads to badly formed elements.

4.2: Isoparametric Elements (cont.)

Formulating an Isoparametric Element:

- Calculating the **[B]** matrix: $(\mathbf{u})=\left[\mathbf{N}(\mathbf{x})\right](\mathbf{d}) \Rightarrow (\boldsymbol{\varepsilon})=\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{pmatrix}=\left[\mathbf{B}(\mathbf{x})\right](\mathbf{d});$

$$\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{pmatrix}=\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&1&0\end{bmatrix}\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\end{pmatrix};\ \begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\end{pmatrix}=\begin{bmatrix}\left[\mathbf{J}\right]^{-1}&\left[\mathbf{0}\right]\\ \left[\mathbf{0}\right]&\left[\mathbf{J}\right]^{-1}\end{bmatrix}\begin{pmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\\ \frac{\partial v}{\partial \xi}\\ \frac{\partial v}{\partial \eta}\end{pmatrix};\ \begin{pmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\\ \frac{\partial v}{\partial \xi}\\ \frac{\partial v}{\partial \eta}\end{pmatrix}=\begin{bmatrix}\frac{\partial N_1}{\partial \xi}&0&\frac{\partial N_2}{\partial \xi}&0&\frac{\partial N_3}{\partial \xi}&0&\frac{\partial N_4}{\partial \xi}&0\\ \frac{\partial N_1}{\partial \eta}&0&\frac{\partial N_2}{\partial \eta}&0&\frac{\partial N_3}{\partial \eta}&0&\frac{\partial N_4}{\partial \eta}&0\\ 0&\frac{\partial N_1}{\partial \xi}&0&\frac{\partial N_2}{\partial \xi}&0&\frac{\partial N_3}{\partial \xi}&0&\frac{\partial N_4}{\partial \xi}\\ 0&\frac{\partial N_1}{\partial \eta}&0&\frac{\partial N_2}{\partial \eta}&0&\frac{\partial N_3}{\partial \eta}&0&\frac{\partial N_4}{\partial \eta}\end{bmatrix}\begin{pmatrix}d_1\\ d_2\\ \vdots\\ d_8\end{pmatrix}.$$

$$\therefore\left[\mathbf{B}\right]=\frac{1}{J}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&1&0\end{bmatrix}\begin{bmatrix}\sum_{i=1}^{n}\frac{\partial N_i}{\partial \eta}y_i & -\sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}y_i & 0 & 0\\ -\sum_{i=1}^{n}\frac{\partial N_i}{\partial \eta}x_i & \sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}x_i & 0 & 0\\ 0 & 0 & \sum_{i=1}^{n}\frac{\partial N_i}{\partial \eta}y_i & -\sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}y_i\\ 0 & 0 & -\sum_{i=1}^{n}\frac{\partial N_i}{\partial \eta}x_i & \sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}x_i\end{bmatrix}\begin{bmatrix}\frac{\partial N_1}{\partial \xi}&0&\frac{\partial N_2}{\partial \xi}&0&\frac{\partial N_3}{\partial \xi}&0&\frac{\partial N_4}{\partial \xi}&0\\ \frac{\partial N_1}{\partial \eta}&0&\frac{\partial N_2}{\partial \eta}&0&\frac{\partial N_3}{\partial \eta}&0&\frac{\partial N_4}{\partial \eta}&0\\ 0&\frac{\partial N_1}{\partial \xi}&0&\frac{\partial N_2}{\partial \xi}&0&\frac{\partial N_3}{\partial \xi}&0&\frac{\partial N_4}{\partial \xi}\\ 0&\frac{\partial N_1}{\partial \eta}&0&\frac{\partial N_2}{\partial \eta}&0&\frac{\partial N_3}{\partial \eta}&0&\frac{\partial N_4}{\partial \eta}\end{bmatrix}.$$

4.2: Isoparametric Elements (cont.)

Calculating the element stiffness matrix:

$$\left[\mathbf{k}\right]=h*\int_{\eta=-1}^{\eta=1}\int_{\xi=-1}^{\xi=1}\left[\mathbf{B}\right]^T\left[\mathbf{C}\right]\left[\mathbf{B}\right]*\boxed{J(\xi,\eta)}\,d\xi\, d\eta.$$

area scaling factor – polynomial function of (ξ,η)

- Note: **[B]** is proportional to J^{-1}:

$$\left[\mathbf{B}\right]=\frac{\left[\textbf{polynomials functions of }(\xi,\eta)\right]}{J(\xi,\eta)}.$$

$$\therefore\left[\mathbf{k}\right]=h*\int_{\eta=-1}^{\eta=1}\int_{\xi=-1}^{\xi=1}\frac{\left[\textbf{new polynomials functions of }(\xi,\eta)\right]}{J(\xi,\eta)}d\xi\, d\eta.$$

In general, you are integrating ratios of polynomial functions, which typically don't have exact integrals ⇒ use Gauss quadrature to evaluate!

4.2: Isoparametric Elements (cont.)

Calculating the element nodal forces:

$$(\mathbf{f}) = \int_{V_e} [\mathbf{N}(\mathbf{x})]^T (\mathbf{b}(\mathbf{x}))dV + \int_{A_{e,\sigma}} [\mathbf{N}(\mathbf{x})]^T (\mathbf{t}(\mathbf{x}))dA = ?$$

- Body force contribution: What do you do with this?

$$\int_{V_e} [\mathbf{N}(\mathbf{x})]^T (\mathbf{b}(\mathbf{x}))dV = h * \int_{\eta=-1}^{\eta=1} \int_{\xi=-1}^{\xi=1} [\mathbf{N}(\xi,\eta)]^T \boxed{(\mathbf{b}(x,y))} * J(\xi,\eta)\,d\xi\,d\eta.$$

- Surface traction contribution:

$$\int_{A_{e,\sigma}} [\mathbf{N}(\mathbf{x})]^T (\mathbf{t}(\mathbf{x}))dA = \sum_{\text{all edges}} h * \int_{edge\#k} \left[\mathbf{N}(\xi,\eta)_{edge\#k}\right]^T \boxed{(\mathbf{t}(x,y)_{edge\#k})} \boxed{d\ell_{edge\#k}}$$

What do you do with these?

4.2: Isoparametric Elements (cont.)

Converting body force and surface tractions:

- Idea #0: If body force = constant and/or surface traction on edge #*k* = constant, do nothing!
- Idea #1: Use the isoparametric mapping to modify force functions:

$$x \approx \sum_{i=1}^{n} N_i(\xi,\eta)x_i,\ y \approx \sum_{i=1}^{n} N_i(\xi,\eta)y_i \Rightarrow (\mathbf{b}(x,y)) \approx \underbrace{\left(\mathbf{b}\left(\sum_{i=1}^{n} N_i(\xi,\eta)x_i, \sum_{i=1}^{n} N_i(\xi,\eta)y_i\right)\right)}_{\text{"}(\hat{\mathbf{b}}(\xi,\eta))\text{"}}.$$

$$\therefore \int_{V_e} [\mathbf{N}(\mathbf{x})]^T (\mathbf{b}(\mathbf{x}))dV \approx h * \int_{\eta=-1}^{\eta=1} \int_{\xi=-1}^{\xi=1} [\mathbf{N}(\xi,\eta)]^T (\hat{\mathbf{b}}(\xi,\eta)) * J(\xi,\eta)\,d\xi\,d\eta.$$

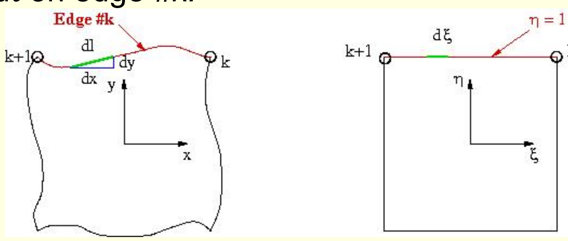
- Idea #2: Make an isoparametric approximation for the forces:

$$(\mathbf{b}(x,y)) \approx \sum_{i=1}^{n} N_i(\xi,\eta) * (\mathbf{b}(x_i,y_i))$$

$$\therefore \int_{V_e} [\mathbf{N}(\mathbf{x})]^T (\mathbf{b}(\mathbf{x}))dV \approx h * \int_{\eta=-1}^{\eta=1} \int_{\xi=-1}^{\xi=1} \left\{\sum_{i=1}^{n} N_i(\xi,\eta) * [\mathbf{N}(\xi,\eta)]^T (\mathbf{b}(x_i,y_i))\right\} * J(\xi,\eta)\,d\xi\,d\eta.$$

4.2: Isoparametric Elements (cont.)

Converting dℓ on edge #*k:*



- In general:

$$d\ell = \sqrt{(dx)^2 + (dy)^2};\ dx = \frac{\partial x}{\partial \xi}d\xi + \frac{\partial x}{\partial \eta}d\eta \text{ and } dy = \frac{\partial y}{\partial \xi}d\xi + \frac{\partial y}{\partial \eta}d\eta.$$

- On the given edge #*k*, $d\eta = 0$:

$$d\ell_{edge\#k} = d\xi\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2_{edge\#k} + \left(\frac{\partial y}{\partial \xi}\right)^2_{edge\#k}} = d\xi \underbrace{\sqrt{\left(\sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}(\xi,\eta=1)x_i\right)^2 + \left(\sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}(\xi,\eta=1)y_i\right)^2}}_{L(\xi)}.$$

4.2: Isoparametric Elements (cont.)

- Thus, the contribution from surface tractions on edge #*k* is:

$$h * \int_{edge\#k}\left[\mathbf{N}(\xi,\eta)_{edge\#k}\right]^T\left(\mathbf{t}(x,y)_{edge\#k}\right)d\ell_{edge\#k}$$

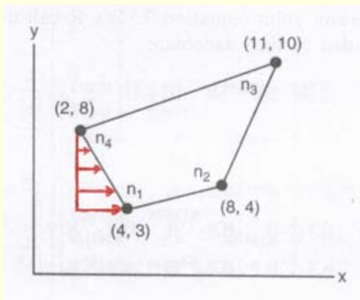
Idea #1!

$$= h * \int_{\xi=-1}^{\xi=+1}\left[\mathbf{N}(\xi,\eta=1)\right]^T \boxed{\left(\mathbf{t}\left(\sum_{i=1}^{n}N_i(\xi,\eta=1)x_i, \sum_{i=1}^{n}N_i(\xi,\eta=1)y_i\right)\right)} * L(\xi)d\xi.$$

- Note: $N_i(\xi,\eta=1) = 0$ unless *i* = *k* or *i* = *k+1* !

4.2: Isoparametric Elements (cont.)

Example: Formulating an Isoparametric Bilinear Quad –



$$\mathbf{t}(x,y)=\begin{pmatrix}0.4*(8-y)\\0\end{pmatrix}\text{ ksi}$$

- Given: 4-node plane stress element has E = 30,000 ksi, ν = 0.25, h = 0.50 in, no body force, and surface traction shown.
- Required: Find **[k]** and **(f)**. Use 2 x 2 Gauss quadrature for **[k]**.

4.2: Isoparametric Elements (cont.)

Solution:

- Isoparametric mapping:

$$\begin{aligned}x&=\tfrac{1}{4}(1-\xi)(1-\eta)x_1+\tfrac{1}{4}(1+\xi)(1-\eta)x_2+\tfrac{1}{4}(1+\xi)(1+\eta)x_3+\tfrac{1}{4}(1-\xi)(1+\eta)x_4\\&=\tfrac{1}{4}(1-\xi)(1-\eta)*4+\tfrac{1}{4}(1+\xi)(1-\eta)*8+\tfrac{1}{4}(1+\xi)(1+\eta)*11+\tfrac{1}{4}(1-\xi)(1+\eta)*2\\&=\tfrac{25}{4}+\tfrac{13}{4}\xi+\tfrac{1}{4}\eta+\tfrac{5}{4}\xi\eta;\\y&=\tfrac{1}{4}(1-\xi)(1-\eta)*3+\tfrac{1}{4}(1+\xi)(1-\eta)*4+\tfrac{1}{4}(1+\xi)(1+\eta)*10+\tfrac{1}{4}(1-\xi)(1+\eta)*8\\&=\tfrac{25}{4}+\tfrac{3}{4}\xi+\tfrac{11}{4}\eta+\tfrac{1}{4}\xi\eta;\end{aligned}$$

- Jacobian matrix and Jacobian:

$$[\mathbf{J}]=\begin{bmatrix}\dfrac{\partial x}{\partial\xi}&\dfrac{\partial y}{\partial\xi}\\[2mm]\dfrac{\partial x}{\partial\eta}&\dfrac{\partial y}{\partial\eta}\end{bmatrix}=\begin{bmatrix}\frac{13}{4}+\frac{5}{4}\eta&\frac{3}{4}+\frac{1}{4}\eta\\\frac{1}{4}+\frac{5}{4}\xi&\frac{11}{4}+\frac{1}{4}\xi\end{bmatrix};J=\det[\mathbf{J}]=\tfrac{35}{4}-\tfrac{1}{8}\xi+\tfrac{27}{8}\eta.$$

4.2: Isoparametric Elements (cont.)

Solution:

- **[B]** matrix:

$$[\mathbf{B}]=\frac{1}{J}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&1&0\end{bmatrix}\begin{bmatrix}\sum\limits_{i=1}^{n}\frac{\partial N_i}{\partial\eta}y_i & -\sum\limits_{i=1}^{n}\frac{\partial N_i}{\partial\xi}y_i & 0 & 0\\ -\sum\limits_{i=1}^{n}\frac{\partial N_i}{\partial\eta}x_i & \sum\limits_{i=1}^{n}\frac{\partial N_i}{\partial\xi}x_i & 0 & 0\\ 0 & 0 & \sum\limits_{i=1}^{n}\frac{\partial N_i}{\partial\eta}y_i & -\sum\limits_{i=1}^{n}\frac{\partial N_i}{\partial\xi}y_i\\ 0 & 0 & -\sum\limits_{i=1}^{n}\frac{\partial N_i}{\partial\eta}x_i & \sum\limits_{i=1}^{n}\frac{\partial N_i}{\partial\xi}x_i\end{bmatrix}\begin{bmatrix}\frac{\partial N_1}{\partial\xi}&0&\frac{\partial N_2}{\partial\xi}&0&\frac{\partial N_3}{\partial\xi}&0&\frac{\partial N_4}{\partial\xi}&0\\ \frac{\partial N_1}{\partial\eta}&0&\frac{\partial N_2}{\partial\eta}&0&\frac{\partial N_3}{\partial\eta}&0&\frac{\partial N_4}{\partial\eta}&0\\ 0&\frac{\partial N_1}{\partial\xi}&0&\frac{\partial N_2}{\partial\xi}&0&\frac{\partial N_3}{\partial\xi}&0&\frac{\partial N_4}{\partial\xi}\\ 0&\frac{\partial N_1}{\partial\eta}&0&\frac{\partial N_2}{\partial\eta}&0&\frac{\partial N_3}{\partial\eta}&0&\frac{\partial N_4}{\partial\eta}\end{bmatrix}$$

$$=\frac{8}{70-\xi+27\eta}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&1&0\end{bmatrix}\begin{bmatrix}\frac{11}{4}+\frac{1}{4}\xi & -\frac{3}{4}-\frac{1}{4}\eta & 0 & 0\\ -\frac{1}{4}-\frac{5}{4}\xi & \frac{13}{4}+\frac{5}{4}\eta & 0 & 0\\ 0 & 0 & \frac{11}{4}+\frac{1}{4}\xi & -\frac{3}{4}-\frac{1}{4}\eta\\ 0 & 0 & -\frac{1}{4}-\frac{5}{4}\xi & \frac{13}{4}+\frac{5}{4}\eta\end{bmatrix}\begin{bmatrix}-\frac{1}{4}+\frac{1}{4}\eta&0&\frac{1}{4}-\frac{1}{4}\eta&0&\frac{1}{4}+\frac{1}{4}\eta&0&-\frac{1}{4}-\frac{1}{4}\eta&0\\ -\frac{1}{4}+\frac{1}{4}\xi&0&-\frac{1}{4}-\frac{1}{4}\xi&0&\frac{1}{4}+\frac{1}{4}\xi&0&\frac{1}{4}-\frac{1}{4}\xi&0\\ 0&-\frac{1}{4}+\frac{1}{4}\eta&0&\frac{1}{4}-\frac{1}{4}\eta&0&\frac{1}{4}+\frac{1}{4}\eta&0&-\frac{1}{4}-\frac{1}{4}\eta\\ 0&-\frac{1}{4}+\frac{1}{4}\xi&0&-\frac{1}{4}-\frac{1}{4}\xi&0&\frac{1}{4}+\frac{1}{4}\xi&0&\frac{1}{4}-\frac{1}{4}\xi\end{bmatrix}$$

$$=\frac{1}{70-\xi+27\eta}\begin{bmatrix}-4+6\eta-2\xi&0&7-5\eta+2\xi&0&4+5\eta-\xi&0&-7-6\eta+\xi&0\\ 0&-6-3\eta+9\xi&0&-7-2\eta-9\xi&0&6+2\eta+4\xi&0&7+3\eta-4\xi\\ -6-3\eta+9\xi&-4+6\eta-2\xi&-7-2\eta-9\xi&7-5\eta+2\xi&6+2\eta+4\xi&4+5\eta-\xi&7+3\eta-4\xi&-7-6\eta+\xi\end{bmatrix}$$

4.2: Isoparametric Elements (cont.)

Solution:

- **[k]** matrix:

$$[\mathbf{C}]=\begin{bmatrix}32000&8000&0\\8000&32000&0\\0&0&12000\end{bmatrix}\text{ksi;}$$

$$[\mathbf{k}]=(0.5\text{ in})\int_{-1}^{1}\int_{-1}^{1}[\mathbf{B}]^T[\mathbf{C}][\mathbf{B}]*Jd\xi d\eta$$

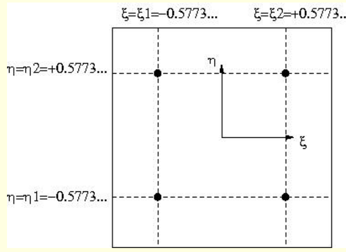
$$=\int_{-1}^{1}\int_{-1}^{1}\underbrace{\frac{8}{70-\xi+27\eta}*\begin{bmatrix}31.25*\left(236-276\eta-196\xi+315\eta^2-354\eta\xi+275\xi^2\right)&\cdots&31.25*\left(70+231\eta-203\xi+90\eta^2-231\eta\xi+43\xi^2\right)\\ \vdots&\ddots&\vdots\\ \text{sym}&\cdots&31.25*\left(539+588\eta-490\xi+180\eta^2-228\eta\xi+131\xi^2\right)\end{bmatrix}}_{[\mathbf{k}'(\xi,\eta)]}d\xi d\eta$$

4.2: Isoparametric Elements (cont.)

Solution:

- 2 x 2 Gauss quadrature:

$$W_i = W_j = 1;\ i, j = 1, 2.$$



$$[\mathbf{k}] \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j * \left[\mathbf{k}'\left(\xi = \xi_i, \eta = \eta_j\right)\right] = \left[\mathbf{k}'\left(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right)\right] + \left[\mathbf{k}'\left(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)\right] + \left[\mathbf{k}'\left(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right)\right] + \left[\mathbf{k}'\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)\right]$$

$$\therefore [\mathbf{k}] \approx \begin{bmatrix} 7028.9 & \cdots & 1260.6 \\ \vdots & \ddots & \vdots \\ 1260.6 & \cdots & 8489.9 \end{bmatrix} \text{kips/in.}$$

$$\text{Note: } [\mathbf{k}]_{exact} = \begin{bmatrix} 7136.6 & \cdots & 1263.9 \\ \vdots & \ddots & \vdots \\ 1263.9 & \cdots & 8499.0 \end{bmatrix} \text{kips/in.}$$

4.2: Isoparametric Elements (cont.)

Solution:

- Element nodal forces:

Finite Element Modeling and Analysis

CE 595: Course Part 2
Amit H. Varma

Discussion of planar elements

- Constant Strain Triangle (CST) - easiest and simplest finite element
  - Displacement field in terms of generalized coordinates

$$u = \beta_1 + \beta_2 x + \beta_3 y \qquad (3.2\text{-}1)$$
$$v = \beta_4 + \beta_5 x + \beta_6 y$$

  - Resulting strain field is

$$\varepsilon_x = \beta_2 \qquad \varepsilon_y = \beta_6 \qquad \gamma_{xy} = \beta_3 + \beta_5 \qquad (3.2\text{-}2)$$

  - Strains do not vary within the element. Hence, the name constant strain triangle (CST)
    - Other elements are not so lucky.
    - Can also be called linear triangle because displacement field is linear in x and y - sides remain straight.

Constant Strain Triangle

- The strain field from the shape functions looks like:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \qquad (3.2\text{-}3)$$

  - Where, x_i and y_i are nodal coordinates (i=1, 2, 3)
  - $x_{ij} = x_i - x_j$ and $y_{ij}=y_i - y_j$
  - 2A is twice the area of the triangle, $2A = x_{21}y_{31}-x_{31}y_{21}$
- Node numbering is arbitrary except that the sequence 123 must go clockwise around the element if A is to be positive.

Constant Strain Triangle

- Stiffness matrix for element k $=B^TEB\, tA$
- The CST gives good results in regions of the FE model where there is little strain gradient
  - Otherwise it does not work well.

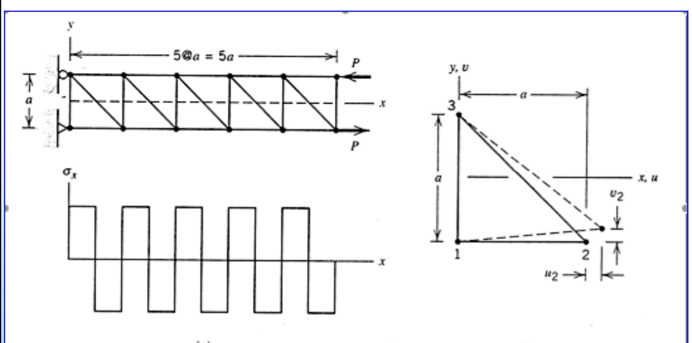


Fig. 3.2-2. (a) Stress σ_x along the x axis in a beam modeled by CSTs and loaded in pure bending. (b) Deformation of the lower-left CST in the model.

If you use CST to model bending.

See the stress along the x-axis - it should be zero.

The predictions of deflection and stress are poor

Spurious shear stress when bent

Mesh refinement will help.

Linear Strain Triangle

- Changes the shape functions and results in quadratic displacement distributions and linear strain distributions within the element.

$$
\begin{aligned}
u &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 \\
v &= \beta_7 + \beta_8 x + \beta_9 y + \beta_{10} x^2 + \beta_{11} xy + \beta_{12} y^2
\end{aligned}
\qquad (3.3\text{-}1)
$$

$$
\begin{aligned}
\varepsilon_x &= \beta_2 + 2\beta_4 x + \beta_5 y \\
\varepsilon_y &= \beta_9 + \beta_{11} x + 2\beta_{12} y \\
\gamma_{xy} &= (\beta_3 + \beta_8) + (\beta_5 + 2\beta_{10})x + (2\beta_6 + \beta_{11})y
\end{aligned}
\qquad (3.3\text{-}2)
$$

Linear Strain Triangle

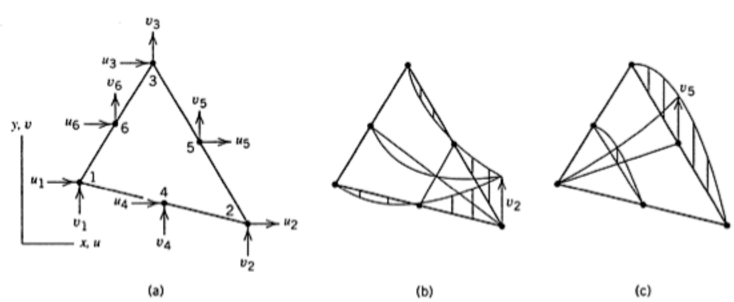
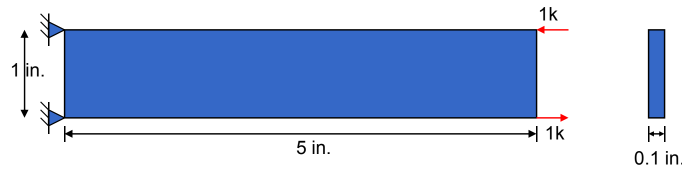


Fig. 3.3-1. (a) A linear strain triangle and its six nodal d.o.f. (b) Displacement mode associated with nodal d.o.f. v_2. (c) Displacement mode associated with nodal d.o.f. v_5. (For visualization only, imagine that displacement occurs normal to the plane of the element.) (*b* and *c* reprinted from [2.2] by permission of John Wiley & Sons, Inc.)

- Will this element work better for the problem?

Example Problem

- Consider the problem we were looking at:



$$I = 0.1 \times 1^3 / 12 = 0.008333\ in^4$$

$$\sigma = \frac{M \times c}{I} = \frac{1 \times 0.5}{0.008333} = 60\ ksi$$

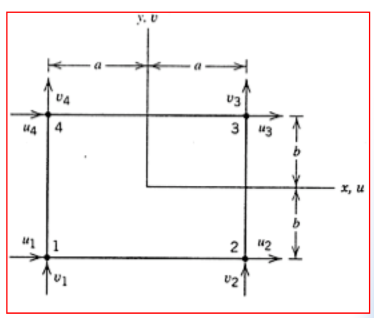
$$\varepsilon = \frac{\sigma}{E} = 0.00207$$

$$\delta = \frac{ML^2}{2EI} = \frac{25}{2 \times 29000 \times 0.008333} = 0.0517\ in.$$

Bilinear Quadratic

- The Q4 element is a quadrilateral element that has four nodes. In terms of generalized coordinates, its displacement field is:

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy \qquad (3.4\text{-}1)$$
$$v = \beta_5 + \beta_6 x + \beta_7 y + \beta_8 xy$$



$$\varepsilon_x = \beta_2 + \beta_4 y$$
$$\varepsilon_y = \beta_7 + \beta_8 x \qquad (3.4\text{-}2)$$
$$\gamma_{xy} = (\beta_3 + \beta_6) + \beta_4 x + \beta_8 y$$

Bilinear Quadratic

- Shape functions and strain-displacement matrix

$$N_1 = \frac{(a-x)(b-y)}{4ab} \qquad N_2 = \frac{(a+x)(b-y)}{4ab}$$
$$N_3 = \frac{(a+x)(b+y)}{4ab} \qquad N_4 = \frac{(a-x)(b+y)}{4ab} \qquad (3.4\text{-}3)$$

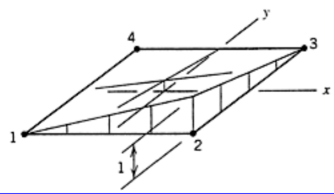
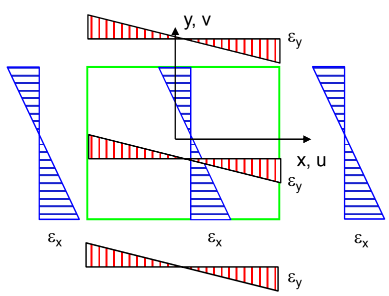


Fig. 3.4-3. Shape function N_2 of the bilinear quadrilateral. (For visualization only, imagine that displacement occurs normal to the xy plane.)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{4ab} \begin{bmatrix} -(b-y) & 0 & (b-y) & 0 & \cdots \\ 0 & -(a-x) & 0 & -(a+x) & \cdots \\ -(a-x) & -(b-y) & -(a+x) & (b-y) & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_4 \end{Bmatrix} \qquad (3.4\text{-}4)$$

Bilinear Quadratic

- The element stiffness matrix is obtained the same way
- A big challenge with this element is that the displacement field has a bilinear approximation, *which means that the* strains vary linearly in the two directions. But, the linear variation does not change along the length of the element.



$$\varepsilon_x = \beta_2 + \beta_4 y$$
$$\varepsilon_y = \beta_7 + \beta_8 x$$
$$\gamma_{xy} = (\beta_3 + \beta_6) + \beta_4 x + \beta_8 y$$

ε_x varies with y but not with x
ε_y varies with x but not with y

Bilinear Quadratic

- So, this element will struggle to model the behavior of a beam with moment varying along the length.
  - Inspite of the fact that it has linearly varying strains - it will struggle to model when M varies along the length.
- Another big challenge with this element is that the displacement functions force the edges to remain straight - no curving during deformation.

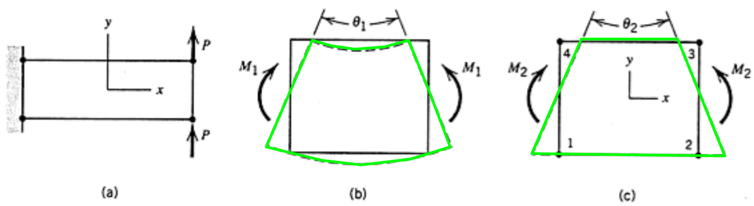


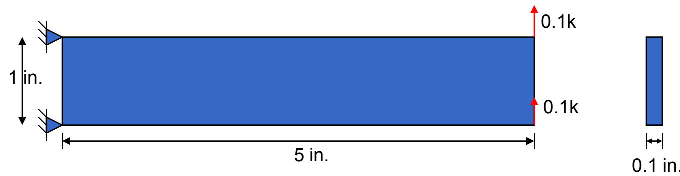
Fig. 3.4-2. (a) A one-element cantilever beam under transverse tip loading. (b) Correct deformation mode of a rectangular block in pure bending. (c) Deformation mode of the bilinear quadrilateral under bending load.

Bilinear Quadratic

- The sides of the element remain straight - as a result the angle between the sides changes.
  - Even for the case of pure bending, the element will develop a change in angle between the sides - which corresponds to the development of a spurious shear stress.
  - The Q4 element will resist even pure bending by developing both normal and shear stresses. This makes it too stiff in bending.
- The element converges properly with mesh refinement and in most problems works better than the CST element.

Example Problem

- Consider the problem we were looking at:



$$I = 0.1 \times 1^3 / 12 = 0.008333in^4$$

$$\sigma = \frac{M \times c}{I} = \frac{1 \times 0.5}{0.008333} = 60ksi$$

$$\varepsilon = \frac{\sigma}{E} = 0.00207$$

$$\delta = \frac{PL^3}{3EI} = \frac{0.2 \times 125}{3 \times 29000 \times 0.008333} = 0.0345in.$$

Quadratic Quadrilateral Element

- The 8 noded quadratic quadrilateral element uses quadratic functions for the displacements

$$\begin{aligned} u &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^2 y + \beta_8 xy^2 \\ v &= \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} x^2 + \beta_{13} xy + \beta_{14} y^2 + \beta_{15} x^2 y + \beta_{16} xy^2 \end{aligned} \quad (3.5\text{-}1)$$

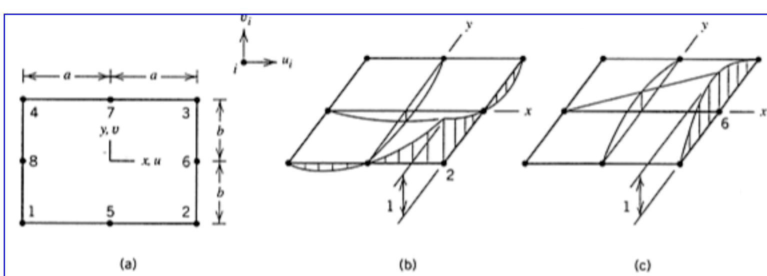


Fig. 3.5-1. (a) A quadratic quadrilateral. (b,c) Shape functions N_2 and N_6. (For visualization only, imagine that displacement occurs normal to the xy plane.)

Quadratic Quadrilateral Element

- Shape function examples:

$$u = \sum N_i \, u_i \qquad v = \sum N_i \, v_i \tag{3.5-2}$$

where index i runs from 1 to 8, which explains the "8" in the name Q8. As examples, two of the eight shape functions are

$$\begin{aligned} N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta) - \tfrac{1}{4}(1+\xi)(1-\eta^2) \\ N_6 &= \tfrac{1}{2}(1+\xi)(1-\eta^2) \end{aligned} \tag{3.5-3}$$

- Strain distribution within the element

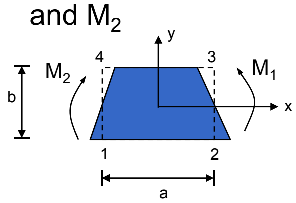
$$\begin{aligned} \varepsilon_x &= \beta_2 + 2\beta_4 x + \beta_5 y + 2\beta_7 xy + \beta_8 y^2 \\ \varepsilon_y &= \beta_{11} + \beta_{13} x + 2\beta_{14} y + \beta_{15} x^2 + 2\beta_{16} xy \\ \gamma_{xy} &= (\beta_3 + \beta_{10}) + (\beta_5 + 2\beta_{12})x + (2\beta_6 + \beta_{13})y \\ &\quad + \beta_7 x^2 + 2(\beta_8 + \beta_{15})xy + \beta_{16} y^2 \end{aligned} \tag{3.5-4}$$

Quadratic Quadrilateral Element

- Should we try to use this element to solve our problem?
- Or try fixing the Q4 element for our purposes.
  - Hmm… tough choice.

Improved Bilinear Quadratic (Q6)

- The principal defect of the Q4 element is its overstiffness in bending.
  - For the situation shown below, you can use the strain displacement relations, stress-strain relations, and stress resultant equation to determine the relationship between M_1 and M_2



$$M_2 = \frac{1}{1+\upsilon}\left[\frac{1}{1-\upsilon} + \frac{1}{2}\left(\frac{a}{b}\right)^2\right] M_1$$

  - M_2 increases infinitely as the element aspect ratio (a/b) becomes larger. This phenomenon is known as locking.
  - It is recommended to not use the Q4 element with too large aspect ratios - as it will have infinite stiffness

Improved bilinear quadratic (Q6)

- One approach is to fix the problem by making a simple modification, which results in an element referred sometimes as a Q6 element
  - Its displacement functions for *u* and *v* contain six shape functions instead of four.

$$\begin{aligned} u &= \sum_{i=1}^{4} N_i u_i \ + \ (1-\xi^2)g_1 \ + \ (1-\eta^2)g_2 \\ v &= \sum_{i=1}^{4} N_i v_i \ + \ (1-\xi^2)g_3 \ + \ (1-\eta^2)g_4 \end{aligned} \qquad (3.6\text{-}2)$$

  - The displacement field is augmented by modes that describe the state of constant curvature.
  - Consider the modes associated with degrees of freedom g_2 and g_3.

Improved Bilinear Quadratic

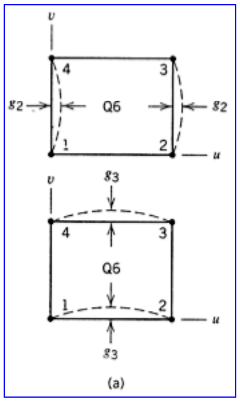


Fig. 3.6-2. (a) Displacement modes $u = (1 - \eta^2)g_2$ and $v = (1 - \xi^2)g_3$ in the Q6 element.

- These corrections allow the elements to curve between the nodes and model bending with x or y axis as the neutral axis.
- In pure bending the shear stress in the element will be

$$\gamma_{xy} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial y} u_i + \sum_{i=1}^{4} \frac{\partial N_i}{\partial x} v_i - \frac{2y}{b^2} g_2 - \frac{2x}{a^2} g_3$$

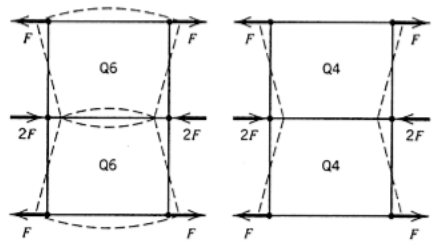
- The negative terms balance out the positive terms.
  - The error in the shear strain is minimized.

Improved Bilinear Quadratic

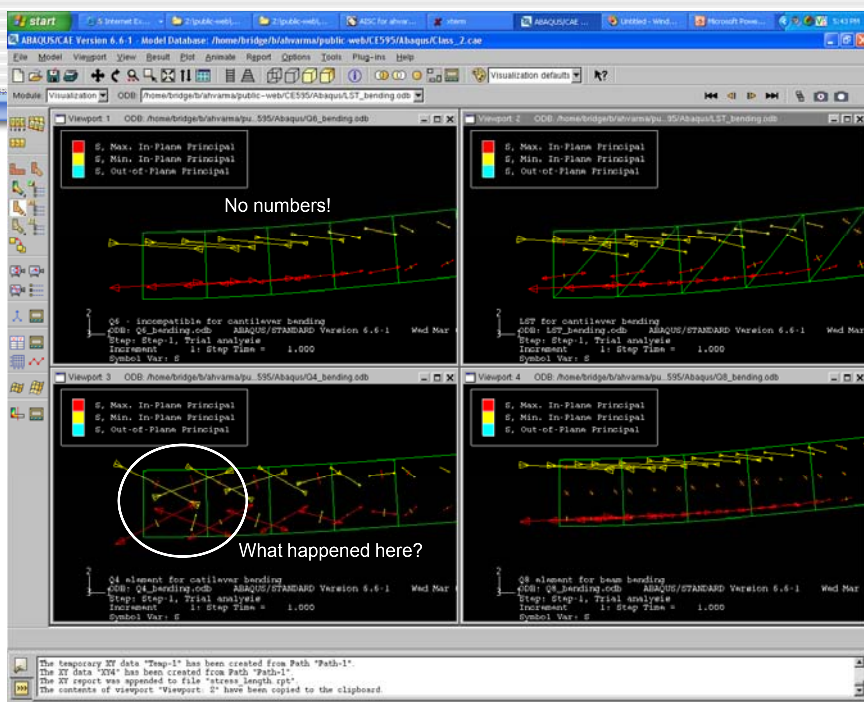
- The additional degrees of freedom g_1 - g_4 are condensed out before the element stiffness matrix is developed. Static condensation is one of the ways.
  - The element can model pure bending exactly, if it is rectangular in shape.
  - This element has become very popular and in many softwares, they don't even tell you that the Q4 element is actually a modified (or tweaked) Q4 element that will work better.
  - Important to note that g_1-g_4 are internal degrees of freedom and unlike nodal d.o.f. they are not connected to to other elements.
  - Modes associated with d.o.f. g_i are incompatible or non-conforming.

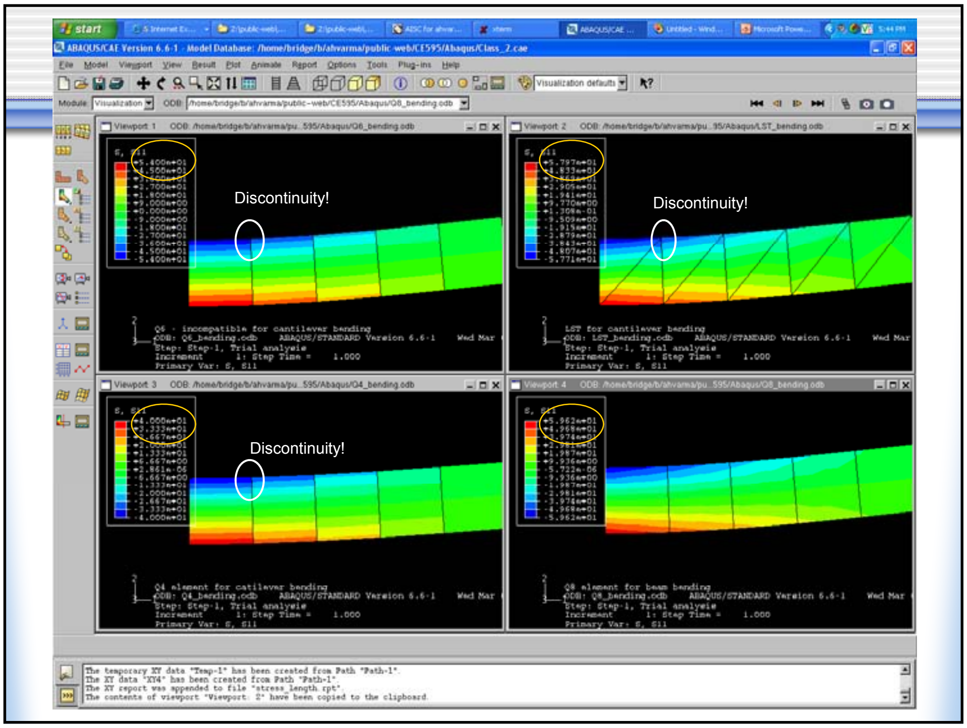
Improved bilinear quadratic

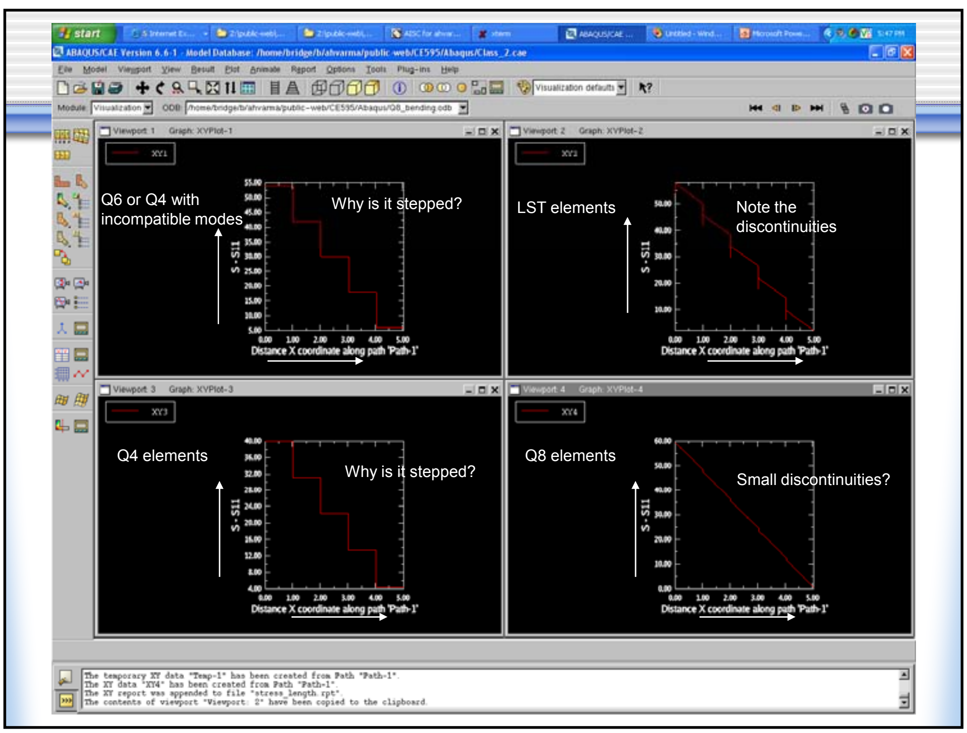
- Under some loading, on overlap or gap may be present between elements
  - Not all but some loading conditions this will happen.
  - This is different from the original Q4 element and is a violation of physical continuum laws.
  - Then why is it acceptable?

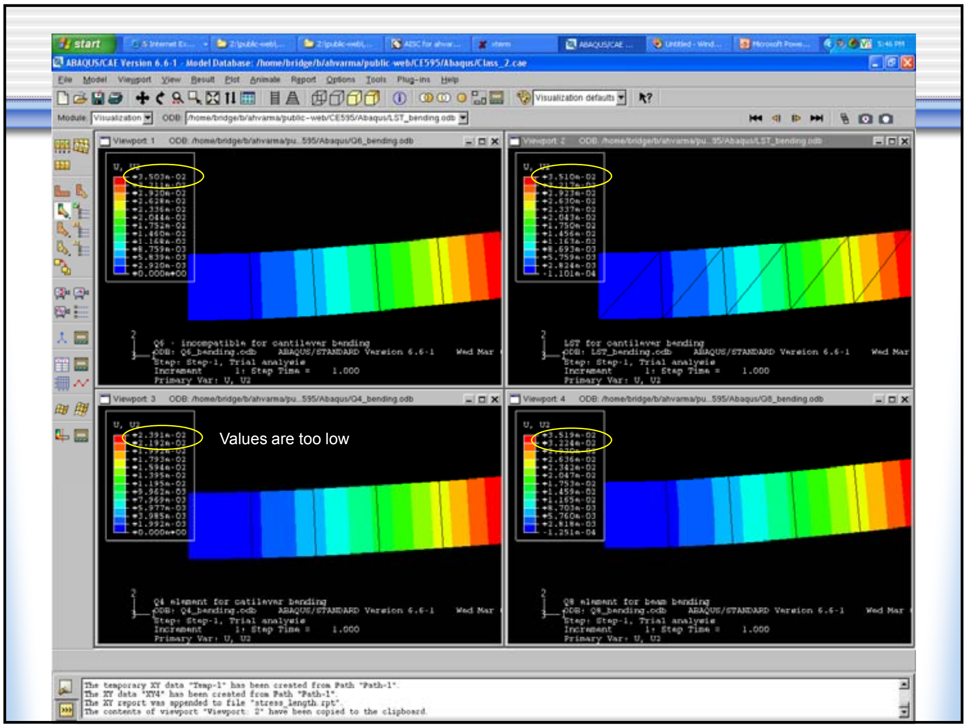


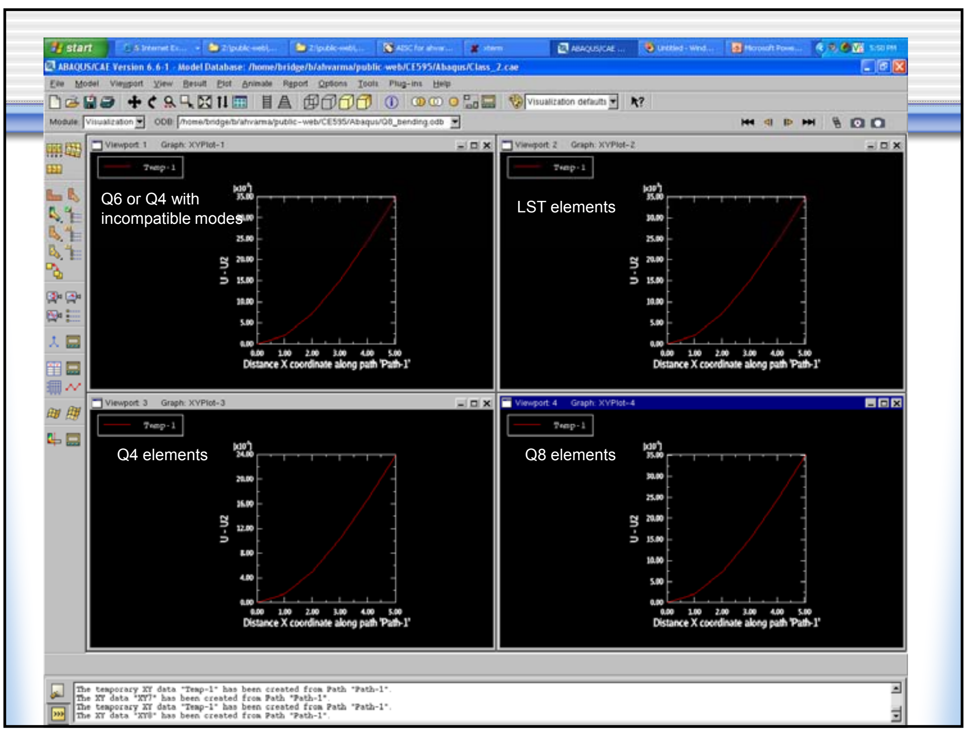
Elements approach a state
Of cons

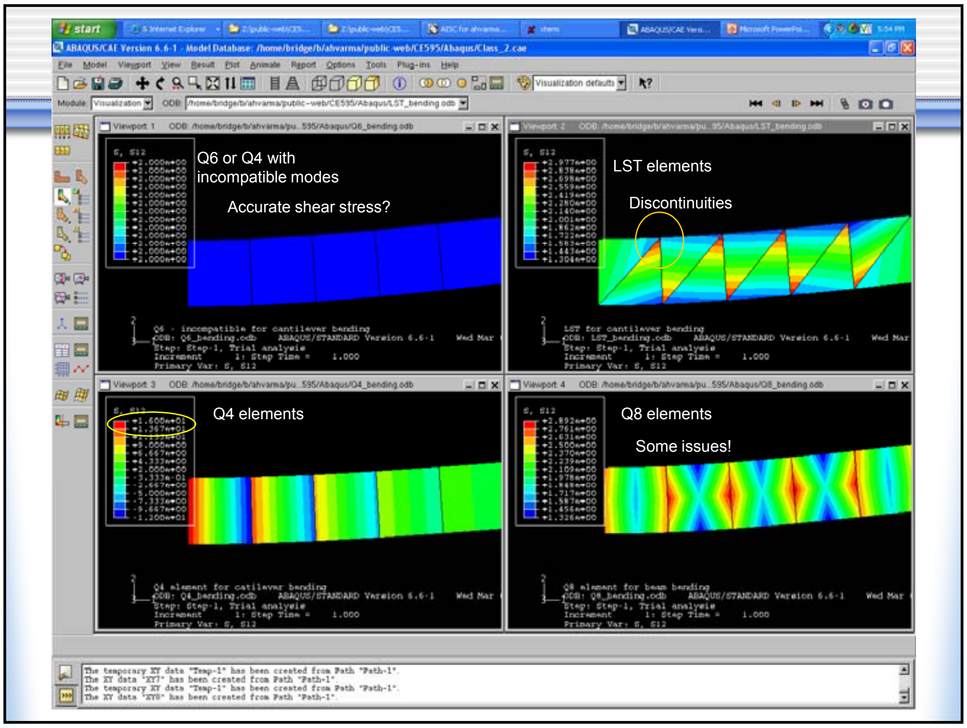


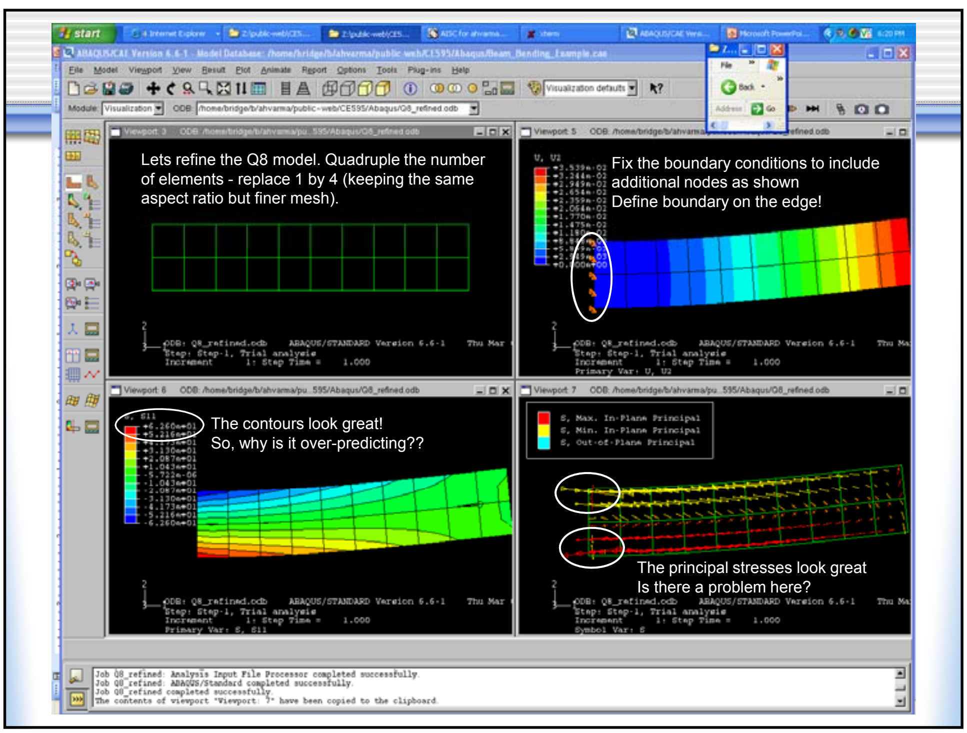

Discontinuity!
Discontinuity!
Discontinuity!

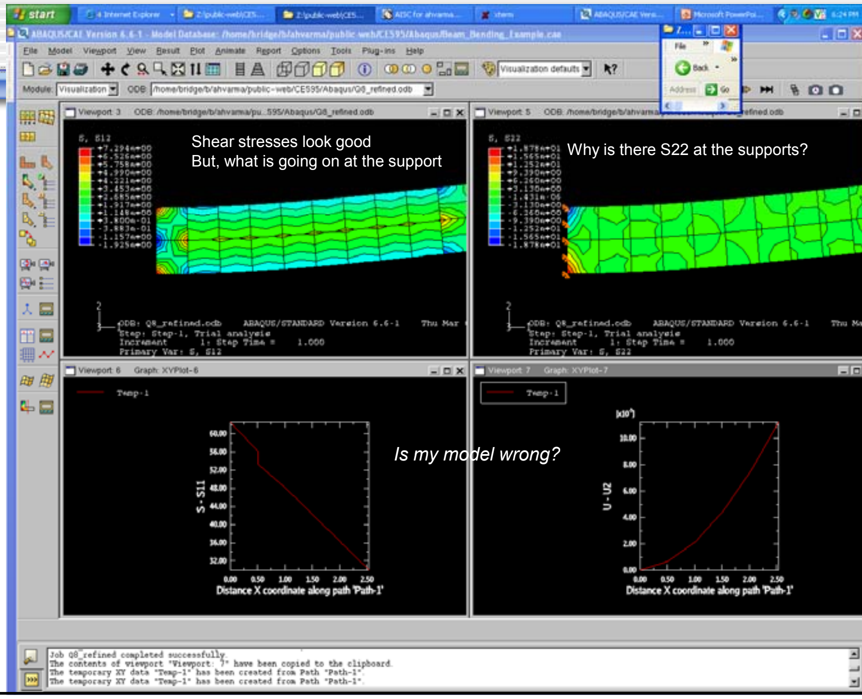

Q6 or Q4 with incompatible modes
Why is it stepped?
LST elements
Note the discontinuities
Q4 elements
Why is it stepped?
Q8 elements
Small discontinuities?


Values are too low


Q6 or Q4 with
incompatible modes
LST elements
Q4 elements
Q8 elements
Distance X coordinate along path 'Path-1'


Q6 or Q4 with
incompatible modes
Accurate shear stress?
LST elements
Discontinuities
Q4 elements
Q8 elements
Some issues!


Lets refine the Q8 model. Quadruple the number of elements - replace 1 by 4 (keeping the same aspect ratio but finer mesh).
Fix the boundary conditions to include additional nodes as shown
Define boundary on the edge!
The contours look great!
So, why is it over-predicting??
The principal stresses look great
Is there a problem here?



Reading assignment

- Section 3.8
- Figure 3.10-2 and associated text
- Mechanical loads consist of concentrated loads at nodes, surface tractions, and body forces.
  - Traction and body forces cannot be applied directly to the FE model. Nodal loads can be applied.
  - They must be converted to equivalent nodal loads. Consider the case of plane stress with translational d.o.f at the nodes.
  - A surface traction can act on boundaries of the FE mesh. Of course, it can also be applied to the interior.

Equivalent Nodal Loads

- Traction has arbitrary orientation with respect to the boundary but is usually expressed in terms of the components normal and tangent to the boundary.

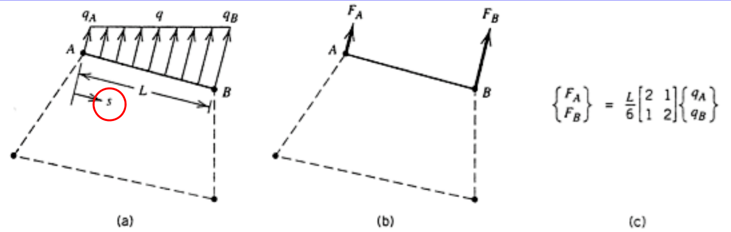


Fig. 3.9-1. (a) Linearly varying distributed load on a linear-displacement edge. (b,c) Work-equivalent nodal loads.

$$F_A v'_A + F_B v'_B = \int_0^L v'(q\, ds) \qquad (3.9\text{-}1)$$

where $v' = (L - s)v'_A/L + sv'_B/L$, for all values of v'_A and v'_B.

Principal of equivalent work

- The boundary tractions (and body forces) acting on the element sides are converted into equivalent nodal loads.
  - The work done by the nodal loads going through the nodal displacements is equal to the work done by the the tractions (or body forces) undergoing the side displacements

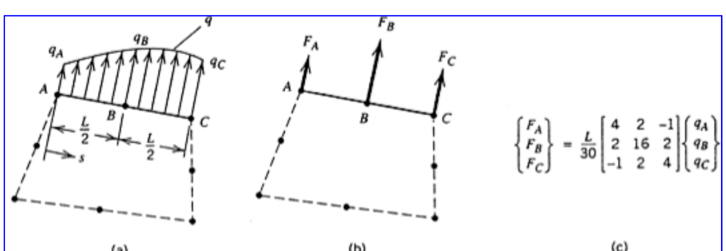


Fig. 3.9-3. (a) Quadratically varying distributed load on a quadratic-displacement edge. (b,c) Work-equivalent nodal forces.

Body Forces

- Body force (weight) converted to equivalent nodal loads. Interesting results for LST and Q8

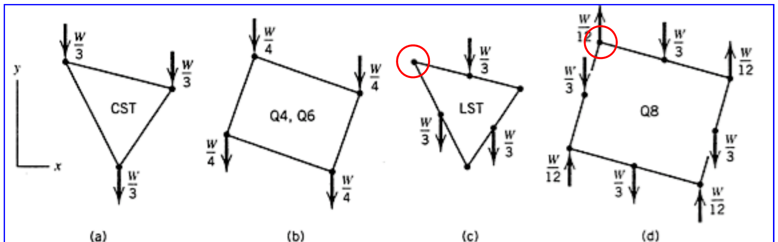
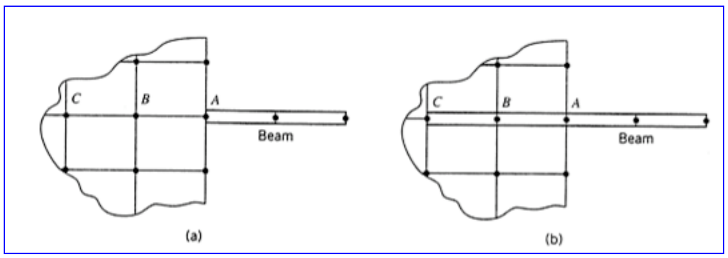


Fig. 3.9-5. Work-equivalent nodal forces associated with element weight W, for triangular and rectangular quadrilateral elements.

Important Limitation

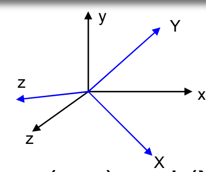
- These elements have displacement degrees of freedom only. So what is wrong with the picture below?



Is this the way to fix it?

Stress Analysis

- Stress tensor $\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$



- If you consider two coordinate systems (xyz) and (XYZ) with the same origin
  - The cosines of the angles between the coordinate axes (x,y,z) and the axes (X, Y, Z) are as follows
  - Each entry is the cosine of the angle between the coordinate axes designated at the top of the column and to the left of the row. (Example, l_1=cos θ_{xX}, l_2=cos θ_{xY})

	x	y	z
X	l_1	m_1	n_1
Y	l_2	m_2	n_2
Z	l_3	m_3	n_3

Stress Analysis

- The direction cosines follow the equations:
  - For the row elements: $l_i^2+m_i^2+n_i^2=1$ for I=1..3

    $l_1l_2+m_1m_2+n_1n_2=0$

    $l_1l_3+m_1m_3+n_1n_3=0$

    $l_3l_2+m_3m_2+n_3n_2=0$
  - For the column elements: $l_1^2+l_2^2+l_3^2=1$

    Similarly, sum $(m_i^2)=1$ and sum$(n_i^2)=1$

    $l_1m_1+l_2m_2+l_3m_3=0$

    $l_1n_1+l_2n_2+l_3n_3=0$

    $n_1m_1+n_2m_2+n_3m_3=0$
  - The stresses in the coordinates XYZ will be:

Stress Analysis

$$\sigma_{XX} = l_1^2\sigma_{xx} + m_1^2\sigma_{yy} + n_1^2\sigma_{zz} + 2m_1n_1\tau_{yz} + 2n_1l_1\tau_{zx} + 2l_1m_1\tau_{xy}$$
$$\sigma_{YY} = l_2^2\sigma_{xx} + m_2^2\sigma_{yy} + n_2^2\sigma_{zz} + 2m_2n_2\tau_{yz} + 2n_2l_2\tau_{zx} + 2l_2m_2\tau_{xy}$$
$$\sigma_{ZZ} = l_3^2\sigma_{xx} + m_3^2\sigma_{yy} + n_3^2\sigma_{zz} + 2m_3n_3\tau_{yz} + 2n_3l_3\tau_{zx} + 2l_3m_3\tau_{xy}$$
$$\tau_{XY} = l_1l_2\sigma_{xx} + m_1m_2\sigma_{yy} + n_1n_2\sigma_{zz} + (m_1n_2 + m_2n_1)\tau_{yz} + (l_1n_2 + l_2n_1)\tau_{xz} + (l_1m_2 + l_2m_1)\tau_{xy}$$
$$\tau_{Xz} = l_1l_3\sigma_{xx} + m_1m_3\sigma_{yy} + n_1n_3\sigma_{zz} + (m_1n_3 + m_3n_1)\tau_{yz} + (l_1n_3 + l_3n_1)\tau_{xz} + (l_1m_3 + l_3m_1)\tau_{xy}$$
$$\tau_{YZ} = l_3l_2\sigma_{xx} + m_3m_2\sigma_{yy} + n_3n_2\sigma_{zz} + (m_2n_3 + m_3n_2)\tau_{yz} + (l_2n_3 + l_3n_2)\tau_{xz} + (l_3m_2 + l_2m_3)\tau_{xy}$$

Equations A

- Principal stresses are the normal stresses on the principal planes where the shear stresses become zero
  - $\sigma_P=\sigma$ **N** where σ is the magnitude and **N** is unit normal to the principal plane
  - Let N = l **i** + m **j** +n **k** (direction cosines)
  - Projections of σ_P along x, y, z axes are $\sigma_{Px}=\sigma$ l, $\sigma_{Py}=\sigma$ m, $\sigma_{Pz}=\sigma$ n

Stress Analysis

- Force equilibrium requires that:

$$l\,(\sigma_{xx}-\sigma) + m\,\tau_{xy} + n\,\tau_{xz}=0$$
$$l\,\tau_{xy} + m\,(\sigma_{yy}-\sigma) + n\,\sigma_{yz} = 0$$
$$l\,\sigma_{xz} + m\,\sigma_{yz} + n\,(\sigma_{zz}-\sigma) = 0$$

Equations B

- Therefore,

$$\begin{vmatrix} \sigma_{xx}-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy}-\sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}-\sigma \end{vmatrix} = 0$$

$$\boxed{\therefore \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0}$$

Equation C

where,

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \tau_{xz} \\ \tau_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \tau_{yz} \\ \tau_{yz} & \sigma_{zz} \end{vmatrix} = \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \tau_{xy}{}^2 - \tau_{xz}{}^2 - \tau_{yz}{}^2$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix}$$

Stress Analysis

- The three roots of the equation are the principal stresses (3). The three terms I_1, I_2, and I_3 are stress invariants.
  - That means, any xyz direction, the stress components will be different but I_1, I_2, and I_3 will be the same.
  - Why? --- Hmm….
  - In terms of principal stresses, the stress invariants are:

$$I_1 = \sigma_{p1}+\sigma_{p2}+\sigma_{p3}\ ;$$

$$I_2=\sigma_{p1}\sigma_{p2}+\sigma_{p2}\sigma_{p3}+\sigma_{p1}\sigma_{p3}\ ;$$

$$I_3 = \sigma_{p1}\sigma_{p2}\sigma_{p3}$$

- In case you were wondering, the directions of the principal stresses are calculated by substituting $\sigma=\sigma_{p1}$ and calculating the corresponding l, m, n using Equations (B).

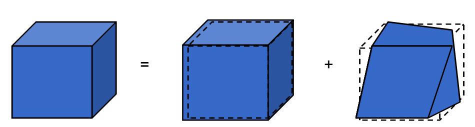
Stress Analysis

- The stress tensor can be discretized into two parts:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_{xx}-\sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy}-\sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}-\sigma_m \end{bmatrix}$$

$$where,\quad \sigma_m = \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} = \frac{I_1}{3}$$

Stress Tensor = Mean Stress Tensor + Deviatoric Stress Tensor



Original element Volume change Distortion only - no volume change

σ_m is referred as the mean stress, or hydostatic pressure, or just pressure (PRESS)

Stress Analysis

- In terms of principal stresses

$$\begin{bmatrix} \sigma_{p1} & 0 & 0 \\ 0 & \sigma_{p2} & 0 \\ 0 & 0 & \sigma_{p3} \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_{p1}-\sigma_m & 0 & 0 \\ 0 & \sigma_{p2}-\sigma_m & 0 \\ 0 & 0 & \sigma_{p3}-\sigma_m \end{bmatrix}$$

$$where, \quad \sigma_m = \frac{\sigma_{p1}+\sigma_{p2}+\sigma_{p3}}{3} = \frac{I_1}{3}$$

$$\therefore \text{Deviatoric Stress Tensor} = \begin{bmatrix} \frac{2\sigma_{p1}-\sigma_{p2}-\sigma_{p3}}{3} & 0 & 0 \\ 0 & \frac{2\sigma_{p2}-\sigma_{p1}-\sigma_{p3}}{3} & 0 \\ 0 & 0 & \frac{2\sigma_{p3}-\sigma_{p1}-\sigma_{p2}}{3} \end{bmatrix}$$

$$\therefore \text{The stress invariants of deviatoric stress tensor}$$

$$J_1 = 0$$

$$J_2 = -\frac{1}{6}\left[(\sigma_{p1}-\sigma_{p2})^2 + (\sigma_{p2}-\sigma_{p3})^2 + (\sigma_{p3}-\sigma_{p1})^2\right] = I_2 - \frac{I_1^2}{3}$$

$$J_3 = \left(\frac{2\sigma_{p1}-\sigma_{p2}-\sigma_{p3}}{3}\right) \times \left(\frac{2\sigma_{p2}-\sigma_{p1}-\sigma_{p3}}{3}\right) \times \left(\frac{2\sigma_{p3}-\sigma_{p1}-\sigma_{p2}}{3}\right) = I_3 + \frac{I_1 I_2}{3} + \frac{2I_1^3}{27}$$

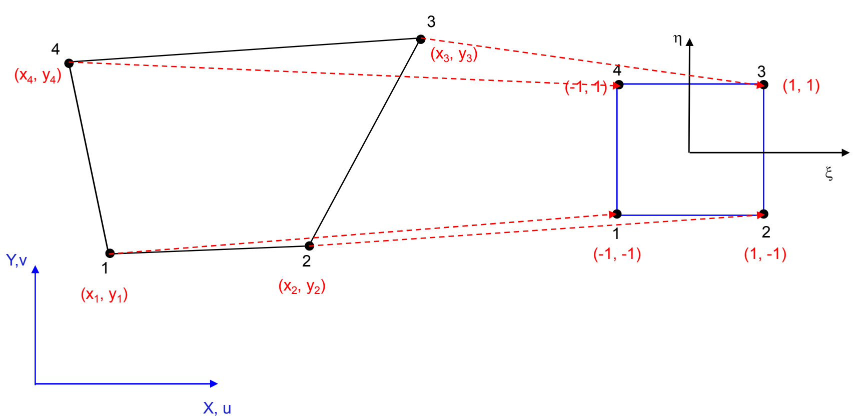
Stress Analysis

- The Von-mises stress is $\sqrt{3 \bullet J_2}$
- The Tresca stress is max {(σ_{p1}-σ_{p2}), (σ_{p1}-σ_{p3}), (σ_{p2}-σ_{p3})}
- Why did we obtain this? Why is this important? And what does it mean?
  - Hmmm….

Isoparametric Elements and Solution

- Biggest breakthrough in the implementation of the finite element method is the development of an isoparametric element with capabilities to model structure (problem) geometries of any shape and size.
- The whole idea works on mapping.
  - The element in the real structure is mapped to an ‘imaginary’ element in an ideal coordinate system
  - The solution to the stress analysis problem is easy and known for the ‘imaginary’ element
  - These solutions are mapped back to the element in the real structure.
  - All the loads and boundary conditions are also mapped from the real to the ‘imaginary’ element in this approach

Isoparametric Element



Isoparametric element

- The mapping functions are quite simple:

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Basically, the x and y coordinates of any point in the element are interpolations of the nodal (corner) coordinates.

From the Q4 element, the bilinear shape functions are borrowed to be used as the interpolation functions. They readily satisfy the boundary values too.

Isoparametric element

- Nodal shape functions for displacements

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

- The displacement strain relationships:

$$\varepsilon_x = \frac{\partial u}{\partial X} = \frac{\partial u}{\partial \xi} \bullet \frac{\partial \xi}{\partial X} + \frac{\partial u}{\partial \eta} \bullet \frac{\partial \eta}{\partial X}$$

$$\varepsilon_y = \frac{\partial v}{\partial Y} = \frac{\partial v}{\partial \xi} \bullet \frac{\partial \xi}{\partial Y} + \frac{\partial v}{\partial \eta} \bullet \frac{\partial \eta}{\partial Y}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial v}{\partial Y} \\ \frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial X} & \frac{\partial \eta}{\partial X} & 0 & 0 \\ 0 & 0 & \frac{\partial \xi}{\partial Y} & \frac{\partial \eta}{\partial Y} \\ \frac{\partial \xi}{\partial Y} & \frac{\partial \eta}{\partial Y} & \frac{\partial \xi}{\partial X} & \frac{\partial \eta}{\partial X} \end{bmatrix} \bullet \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

But, it is too difficult to obtain $\frac{\partial \xi}{\partial X}$ *and* $\frac{\partial \eta}{\partial X}$

Isoparametric Element

Hence we will do it another way

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial X} \bullet \frac{\partial X}{\partial \xi} + \frac{\partial u}{\partial Y} \bullet \frac{\partial Y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial X} \bullet \frac{\partial X}{\partial \eta} + \frac{\partial u}{\partial Y} \bullet \frac{\partial Y}{\partial \eta}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial \xi} & \frac{\partial Y}{\partial \xi} \\ \frac{\partial X}{\partial \eta} & \frac{\partial Y}{\partial \eta} \end{bmatrix} \bullet \begin{Bmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial u}{\partial Y} \end{Bmatrix}$$

It is easier to obtain $\frac{\partial X}{\partial \xi}$ *and* $\frac{\partial Y}{\partial \xi}$

$$J = \begin{bmatrix} \frac{\partial X}{\partial \xi} & \frac{\partial Y}{\partial \xi} \\ \frac{\partial X}{\partial \eta} & \frac{\partial Y}{\partial \eta} \end{bmatrix} = Jacobian$$

defines coordinate transformation

$$\frac{\partial X}{\partial \xi} = \Sigma \frac{\partial N_i}{\partial \xi} X_i \qquad \frac{\partial Y}{\partial \xi} = \Sigma \frac{\partial N_i}{\partial \xi} Y_i$$

$$\frac{\partial X}{\partial \eta} = \Sigma \frac{\partial N_i}{\partial \eta} X_i \qquad \frac{\partial Y}{\partial \eta} = \Sigma \frac{\partial N_i}{\partial \eta} Y_i$$

$$\therefore \begin{Bmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial u}{\partial Y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

Isoparametric Element

$$\varepsilon_x = \frac{\partial u}{\partial X} = J_{11}^* \frac{\partial u}{\partial \xi} + J_{12}^* \frac{\partial u}{\partial \eta}$$

where J_{11}^* *and* J_{12}^* *are coefficients in the first row of* $[J]^{-1}$

and $\frac{\partial u}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} u_i$ *and* $\frac{\partial u}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} u_i$

The remaining strains ε_y and ε_{xy} are computed similarly

The element stiffness matrix

$$[k] = \int [B]^T [E][B] dV = \int_{-1}^{1}\int_{-1}^{1} [B]^T [E][B]\, t\, |J|\, d\xi\, d\eta$$

dX dY=|J| dξdη

Gauss Quadrature

- The mapping approach requires us to be able to evaluate the integrations within the domain (-1…1) of the functions shown.
- Integration can be done analytically by using closed-form formulas from a table of integrals (Nah..)
  - Or numerical integration can be performed
- Gauss quadrature is the more common form of numerical integration - better suited for numerical analysis and finite element method.
- It evaluated the integral of a function as a sum of a finite number of terms

$$I = \int_{-1}^{1} \phi \, d\xi \qquad becomes \qquad I \approx \sum_{i=1}^{n} W_i \phi_i$$

Gauss Quadrature

- W_i is the 'weight' and ϕ_i is the value of $f(\xi=i)$

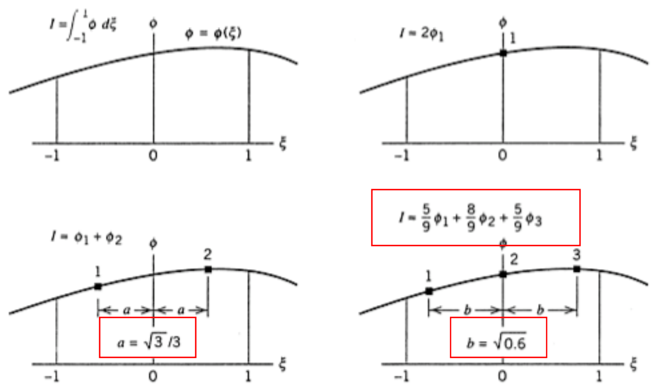


Fig. 4.5-1. Integration of a function $\phi = \phi(\xi)$ in one dimension by Gauss quadrature of orders 1, 2, and 3. Gauss points are numbered.

Gauss Quadrature

- If $\phi=\phi(\xi)$ is a polynomial function, then n-point Gauss quadrature yields the exact integral if ϕ is of degree 2n-1 or less.
  - The form $\phi=c_1+c_2\xi$ is integrated exactly by the one point rule
  - The form $\phi=c_1+c_2\xi+c_2\xi^2$ is integrated exactly by the two point rule
  - And so on…
  - Use of an excessive number of points (more than that required) still yields the exact result
- If ϕ is not a polynomial, Gauss quadrature yields *an* approximate result.
  - Accuracy improves as more Gauss points are used.
  - Convergence toward the exact result may not be monotonic

Gauss Quadrature

- In two dimensions, integration is over a quadrilateral and a Gauss rule of order n uses n^2 points

$$I = \int_{-1}^{1}\int_{-1}^{1} \phi(\xi, \eta)\, d\xi\, d\eta \approx \sum_{i=1}^{n}\sum_{j=1}^{m} W_i W_j\, \phi(\xi_i, \eta_j) \qquad (4.5\text{-}2)$$

- Where, W_iW_j is the product of one-dimensional weights. Usually m=n.
  - If m = n = 1, ϕ is evaluated at ξ and η=0 and $I=4\phi_1$
  - For Gauss rule of order 2 - need 2^2=4 points
  - For Gauss rule of order 3 - need 3^2=9 points

Gauss Quadrature

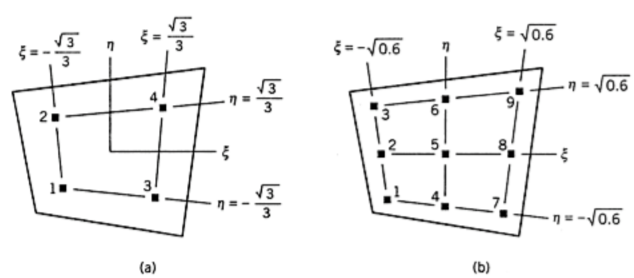


Fig. 4.5-2. Gauss point locations for integration of a function $\phi = \phi(\xi, \eta)$ in two dimensions, using orders 2 and 3. (Reprinted from [2.2] by permission of John Wiley & Sons, Inc.)

$$I \approx \phi_1 + \phi_2 + \phi_3 + \phi_4 \qquad \textit{for rule of order} = 2$$

$$I \approx \frac{25}{81}(\phi_1 + \phi_3 + \phi_7 + \phi_9) + \frac{40}{81}(\phi_2 + \phi_4 + \phi_6 + \phi_8) + \frac{64}{81}\phi_5$$

Number of Integration Points

- All the isoparametric solid elements are integrated numerically. Two schemes are offered: "full" integration and "reduced" integration.
  - For the second-order elements Gauss integration is always used because it is efficient and it is especially suited to the polynomial product interpolations used in these elements.
  - For the first-order elements the single-point reduced-integration scheme is based on the "uniform strain formulation": the strains are not obtained at the first-order Gauss point but are obtained as the (analytically calculated) average strain over the element volume.
  - The uniform strain method, first published by Flanagan and Belytschko (1981), ensures that the first-order reduced-integration elements pass the patch test and attain the accuracy when elements are skewed.
  - Alternatively, the "centroidal strain formulation," which uses 1-point Gauss integration to obtain the strains at the element center, is also available for the 8-node brick elements in ABAQUS/Explicit for improved computational efficiency.

Number of Integration Points

- The differences between the uniform strain formulation and the centroidal strain formulation can be shown as follows:

For the 8-node brick elements the interpolation function given above can be rewritten as

$$\mathbf{u} = N^I(g,h,r)\mathbf{u}^I \text{ sum on } I.$$

The isoparametric shape functions N^I can be written as

$$N^I(g,h,r) = \frac{1}{8}\Sigma^I + \frac{1}{4}g\Lambda_1^I + \frac{1}{4}h\Lambda_2^I + \frac{1}{4}r\Lambda_3^I + \frac{1}{2}hr\Gamma_1^I + \frac{1}{2}gr\Gamma_2^I + \frac{1}{2}gh\Gamma_3^I + \frac{1}{2}ghr\Gamma_4^I,$$

where

$$\Sigma^I = [+1,+1,+1,+1,+1,+1,+1,+1],$$

$$\Lambda_1^I = [-1,+1,+1,-1,-1,+1,+1,-1],$$

$$\Lambda_2^I = [-1,-1,+1,+1,-1,-1,+1,+1],$$

$$\Lambda_3^I = [-1,-1,-1,-1,+1,+1,+1,+1],$$

$$\Gamma_1^I = [+1,+1,-1,-1,-1,-1,+1,+1],$$

$$\Gamma_2^I = [+1,-1,-1,+1,-1,+1,+1,-1],$$

$$\Gamma_3^I = [+1,-1,+1,-1,+1,-1,+1,-1],$$

$$\Gamma_4^I = [-1,+1,-1,+1,+1,-1,+1,-1],$$

and the superscript *I* denotes the node of the element. The last four vectors, Γ_α^I (α has a range of four), are the hourglass base vectors, which are the deformation modes associated with no energy in the 1-point integration element but resulting in a nonconstant strain field in the element.

Number of Integration Points

In the uniform strain formulation the gradient matrix B^I is defined by integrating over the element as

$$B_i^I = \frac{1}{V_{el}} \int_{V_{el}} N_i^I(g,h,r)\, dV_{el},$$

$$N_i^I(g,h,r) = \frac{\partial N^I}{\partial x_i},$$

where V_{el} is the element volume and i has a range of three.

In the centroidal strain formulation the gradient matrix B^I is simply given as

$$B_i^I = N_i^I(0,0,0),$$

which has the following antisymmetric property:

$$B_i^1 = -B_i^7,$$

$$B_i^3 = -B_i^5,$$

$$B_i^2 = -B_i^8,$$

$$B_i^4 = -B_i^6.$$

It can be seen from the above that the centroidal strain formulation reduces the amount of effort required to compu

Number of integration points

- Numerical integration is simpler than analytical, but it is not exact. [k] is only approximately integrated regardless of the number of integration points
  - Should we use fewer integration points for quick computation
  - Or more integration points to improve the accuracy of calculations.
  - Hmm….

Reduced Integration

- A FE model is usually inexact, and usually it errs by being too stiff. Overstiffness is usually made worse by using more Gauss points to integrate element stiffness matrices because additional points capture more higher order terms in [k]
- These terms resist some deformation modes that lower order tems do not and therefore act to stiffen an element.
- On the other hand, use of too few Gauss points produces an even worse situation known as: instability, spurious singular mode, mechanics, zero-energy, or hourglass mode.
  - Instability occurs if one of more deformation modes happen to display zero strain at all Gauss points.
  - If Gauss points sense no strain under a certain deformation mode, the resulting [k] will have no resistance to that deformation mode.

Reduced Integration

- Reduced integration usually means that an integration scheme one order less than the full scheme is used to integrate the element's internal forces and stiffness.
  - Superficially this appears to be a poor approximation, but it has proved to offer significant advantages.
  - For second-order elements in which the isoparametric coordinate lines remain orthogonal in the physical space, the reduced-integration points have the Barlow point property (Barlow, 1976): the strains are calculated from the interpolation functions with higher accuracy at these points than anywhere else in the element.
  - For first-order elements the uniform strain method yields the exact average strain over the element volume. Not only is this important with respect to the values available for output, it is also significant when the constitutive model is **nonlinear**, since the strains passed into the constitutive routines are a better representation of the actual strains.

Reduced Integration

- Reduced integration decreases the number of constraints introduced by an element when there are internal constraints in the continuum theory being modeled, such as incompressibility, or the Kirchhoff transverse shear constraints if solid elements are used to analyze bending problems.
- In such applications fully integrated elements will “lock”—they will exhibit response that is orders of magnitude too stiff, so the results they provide are quite unusable. The reduced-integration version of the same element will often work well in such cases.
- Reduced integration lowers the cost of forming an element. The deficiency of reduced integration is that the element stiffness matrix will be rank deficient.
- This most commonly exhibits itself in the appearance of singular modes (“hourglass modes”) in the response. These are nonphysical response modes that can grow in an unbounded way unless they are controlled.

Reduced Integration

- The reduced-integration second-order serendipity interpolation elements in two dimensions—the 8-node quadrilaterals—have one such mode, but it is benign because it cannot propagate in a mesh with more than one element.
- The second-order three-dimensional elements with reduced integration have modes that can propagate in a single stack of elements. Because these modes rarely cause trouble in the second-order elements, no special techniques are used in ABAQUS to control them.
- In contrast, when reduced integration is used in the first-order elements (the 4-node quadrilateral and the 8-node brick), hourglassing can often make the elements unusable unless it is controlled.
- In ABAQUS the artificial stiffness method given in Flanagan and Belytschko (1981) is used to control the hourglass modes in these elements.

Reduced Integration

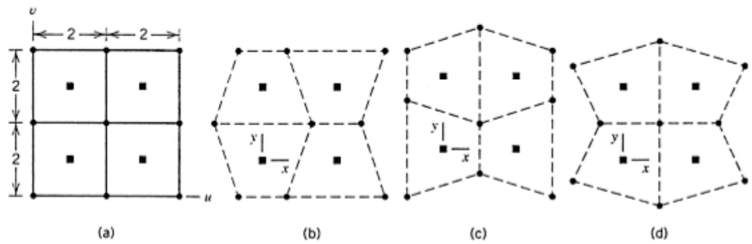


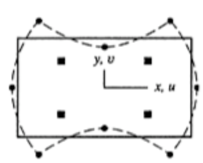
Fig. 4.6-1. (a) Undeformed plane 2 by 2 four-node square elements. Gauss points are shown by solid squares. (*b,c,d*) "Instability" displacement modes. (Reprinted from [2.2] by permission of John Wiley & Sons, Inc.)

three instabilities shown have the respective forms (b) $u = cxy$, $v = 0$; (c) $u = 0$, $v = -cxy$; and (d) $u = cy(1 - x)$, $v = cx(y - 1)$. We easily check that each of these displacement fields produces strains $\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$ at the Gauss point, $x = y = 0$. *Non*rectangular ele-

The FE model will have no resistance to loads that activate these modes. The stiffness matrix will be singular.

Reduced Integration

- Hourglass mode for 8-node element with reduced integration to four points



(b)

- This mode is typically non-communicable and will not occur in a set of elements.

Reduced Integration

- The hourglass control methods of Flanagan and Belytschko (1981) are generally successful for linear and mildly nonlinear problems but may break down in strongly nonlinear problems and, therefore, may not yield reasonable results.
- Success in controlling hourglassing also depends on the loads applied to the structure. For example, a point load is much more likely to trigger hourglassing than a distributed load.
- Hourglassing can be particularly troublesome in eigenvalue extraction problems: the low stiffness of the hourglass modes may create many unrealistic modes with low eigenfrequencies.
- Experience suggests that the reduced-integration, second-order isoparametric elements are the most cost-effective elements in ABAQUS for problems in which the solution can be expected to be smooth.

Solving Linear Equations

- Time independent FE analysis requires that the global equations [K]{D}={R} be solved for {D}
- This can be done by direct or iterative methods
- The direct method is usually some form of Gauss elimination.
- The number of operations required is dictated by the number of d.o.f. and the topology of [K]
- An iterative method requires an uncertain number of operations; calculations are halted when convergence criteria are satisfied or an iteration limit is reached.

Solving Linear Equations

- If a Gauss elimination is driven by node numbering, forward reduction proceeds in node number order and back substitution in reverse order, so that numerical values of d.o.f at first numbered node are determined last.
- If Gauss elimination is driven by element numbering, assembly of element matrices may alternate with steps of forward reduction.
  - Some eliminations are carried out as soon as enough information has been assembled, then more assembly is carried out, then more eliminations, and so on…
  - The assembly-reduction process is like a ‘wave’ that moves over the structure.
  - A solver that works this way is called a wavefront or ‘frontal’ equation solver.

Solving Linear Equations

- The computation time of a direct solution is roughly proportional to nb^2, where n is the order of [K] and b is the bandwidth.
  - For 3D structures, the computation time becomes large because *b* becomes large.
  - Large *b* indicates higher connectivity between the degrees of freedom.
  - For such a case, an iterative solver may be better because connectivity speeds convergence.

Solving Linear Equations

- In most cases, the structure must be analyzed to determine the effects of several different load vectors {R}.
  - This is done more effectively by direct solvers because most of the effort is expended to reduce the [K] matrix.
  - As long as the structure [K] does not change, the displacements for the new load vectors can be estimated easily.
  - This will be more difficult for iterative solvers, because the complete set of equations need to be re-solved for the new load vector.
  - Iterative solvers may be best for parallel processing computers and nonlinear problems where the [K] matrix changes from step i to i+1. Particularly because the solution at step i will be a good initial estimate.

Symmetry conditions

- Types of symmetry include reflective, skew, axial and cyclic. If symmetry can be recognized and used, then the models can be made smaller.
  - The problem is that not only the structure, but the boundary conditions and the loading needs to be symmetric too.
  - The problem can be anti-symmetric
  - If the problem is symmetric
  - Translations have no component normal to a plane of symmetry
  - Rotation vectors have no component parallel to a plane of symmetry.

Symmetry conditions

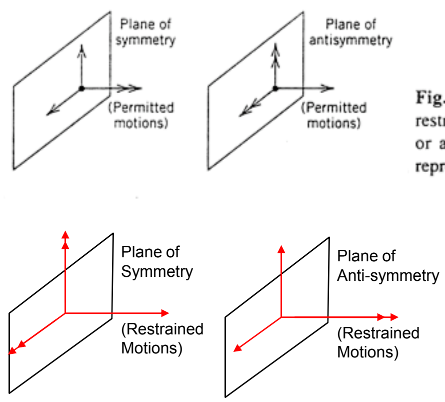


Fig. 4.12-2. The d.o.f. permitted (i.e., not restrained) at a node in a plane of symmetry or antisymmetry. A double-headed arrow represents a rotational d.o.f.

Symmetry Conditions

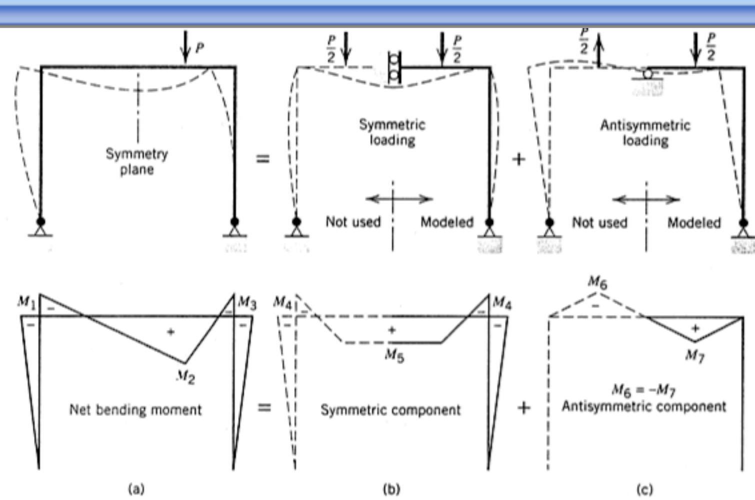


Fig. 4.12-3. Modeling a plane frame problem as the sum of symmetric and antisymmetric cases. (Reproduced from [5.4] by permission of the publisher.)

Constraints

- Special conditions for the finite element model.
  - A constraint equation has the general form [C]{D}-{Q}=0
  - Where [C] is an mxn matrix; m is the number of constraint equation, and n is the number of d.o.f. in the global vector {D}
  - {Q} is a vector of constants and it is usually zero.
  - There are two ways to impose the constraint equations on the global equation [K]{D}={R}
- Lagrange Multiplier Method
  - Introduce additional variables known as Lagrange multipliers $\lambda=\{\lambda_1\ \lambda_2\ \lambda_3 \dots \lambda_m\}^T$
  - Each constraint equation is written in homogenous form and multiplied by the corresponding λ_I which yields the equation λ
  - $\lambda^T\{[C]\{D\} - \{Q\}\}=0$
  - Final Form

$$\begin{bmatrix} K & C^T \\ C & 0 \end{bmatrix} \begin{Bmatrix} D \\ \lambda \end{Bmatrix} = \begin{Bmatrix} R \\ Q \end{Bmatrix}$$

Solved by Gaussian Elimination

Constraints

- Penalty Method
  - t=[C]{D}-{Q}
  - t=0 implies that the constraints have been satisfied
  - $\alpha=[\alpha_1\ \alpha_2\ \alpha_1 \dots \alpha_m]$ is the diagonal matrix of "penalty numbers."
  - Final form $\{[K]+[C]^T[\alpha][C]\}\{D\}=\{R\}+[C]^T[\alpha]\{Q\}$
    - $[C]^T[\alpha][C]$ is called the penalty matrix
    - If a is zero, the constraints are ignored
    - As a becomes large, the constraints are very nearly satisfied
    - Penalty numbers that are too large produce numerical ill-conditioning, which may make the computed results unreliable and may "lock" the mesh.
    - The penalty numbers must be large enough to be effective but not so large as to cause numerical difficulties

3D Solids and Solids of Revolution

- 3D solid - three-dimensional solid that is unrestricted as to the shape, loading, material properties, and boundary conditions.
- All six possible stresses (three normal and three shear) must be taken into account.
  - The displacement field involves all three components (u, v, and w)
  - Typical finite elements for 3D solids are tetrahedra and hexahedra, with three translational d.o.f. per node.

3D Solids

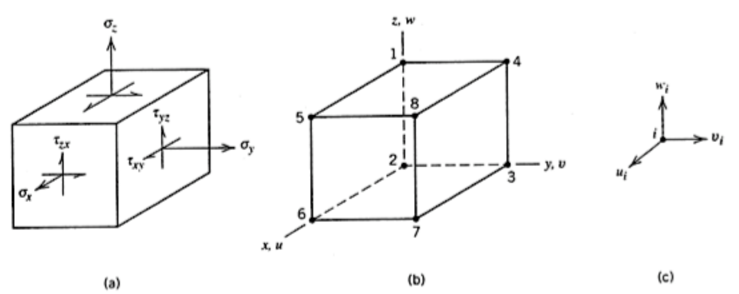


Fig. 6.1-1. (a) 3D state of stress. (b) An eight-node hexahedron FE. (c) The d.o.f. at a typical node ($i = 1, 2, \ldots, 8$).

3D Solids

- Problems of beam bending, plane stress, plates and so on can all be regarded as special cases of 3D solids.
  - Does this mean we can model everything using 3D finite element models?
  - Can we just generalize everything as 3D and model using 3D finite elements.
- Not true! 3D models are very demanding in terms of computational time, and difficult to converge.
  - They can be very stiff for several cases.
  - More importantly, the 3D finite elements do not have rotational degrees of freedom, which are very important for situations like plates, shells, beams etc.

3D Solids

- Strain-displacement relationships

Strain–Displacement Relations. Let $u = u(x, y, z)$, $v = v(x, y, z)$, and $w = w(x, y, z)$ be displacement components of an arbitrary material point in the x, y, and z directions, respectively. If strains and rotations are small, strains and displacement gradients in Cartesian coordinates are related by the equations

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} & \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \varepsilon_z &= \frac{\partial w}{\partial z} & \gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{aligned} \tag{6.1-5}$$

3D Solids

- Stress-strain-temperature relations

Stress–Strain–Temperature Relations. As usual, the constitutive relation of a linearly elastic material is written as

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} + \boldsymbol{\sigma}_0 \qquad (6.1\text{-}1)$$

For an isotropic material in three dimensions, with initial stress $\boldsymbol{\sigma}_0$ produced by temperature change, Eq. 6.1-1 symbolizes the relation

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} (1-\nu)c & \nu c & \nu c & 0 & 0 & 0 \\ & (1-\nu)c & \nu c & 0 & 0 & 0 \\ & & (1-\nu)c & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & \text{symmetric} & & & G & 0 \\ & & & & & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} - \frac{E\alpha\,\Delta T}{1-2\nu} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (6.1\text{-}2)$$

3D Solids

- The process for assembling the element stiffness matrix is the same as before.
  - {u}=[N] {d}
  - Where, [N] is the matrix of shape functions
  - The nodes have three translational degrees of freedom.
  - If n is the number of nodes, then [N] has 3n columns

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \end{Bmatrix} \qquad (6.1\text{-}7)$$

3D Solids

- Substitution of {u}=[N]{d} into the strain-displacement relation yields the strain-displacement matrix [B]
- The element stiffness matrix takes the form:

General solids:

$$\underset{6\times1}{\boldsymbol{\varepsilon}} = \underset{6\times3n}{\mathbf{B}}\ \underset{3n\times1}{\mathbf{d}}$$

$$\underset{3n\times3n}{\mathbf{k}} = \int\int\int \mathbf{B}^T \underset{6\times6}{\mathbf{E}}\ \mathbf{B}\, dx\, dy\, dz$$

3D Solid Elements

- Solid elements are direct extensions of plane elements discussed earlier. The extensions consist of adding another coordinate and displacement component.
  - The behavior and limitations of specific 3D elements largely parallel those of their 2D counterparts.
- For example:
  - Constant strain tetrahedron
  - Linear strain tetrahedron
  - Trilinear hexahedron
  - Quadratic hexahedron
- Hmm…
  - Can you follow the names and relate them back to the planar elements

3D Solids

- Pictures of solid elements

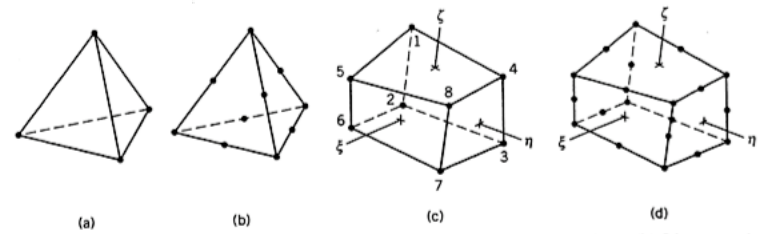
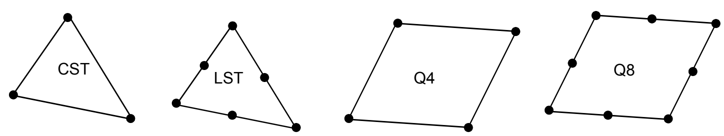


Fig. 6.2-1. Common 3D elements. (a) Constant strain (four-node) tetrahedron. (b) Linear strain (ten-node) tetrahedron. (c) Trilinear (eight-node) hexahedron. (d) Quadratic (20-node) hexahedron.



3D Solids

- Constant Strain Tetrahedron. The element has three translational d.o.f. at each of its four nodes.
  - A total of 12 d.o.f.
  - In terms of generalized coordinates β_i its displacement field is given by.

$$
\begin{aligned}
u &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 z \\
v &= \beta_5 + \beta_6 x + \beta_7 y + \beta_8 z \qquad (6.2\text{-}1) \\
w &= \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} z
\end{aligned}
$$

  - Like the constant strain triangle, the constant strain tetrahedron is accurate only when strains are almost constant over the span of the element.
  - The element is poor for bending and twisting specially if the axis passes through the element of close to it.

3D Solids

- Linear strain tetrahedron - This element has 10 nodes, each with 3 d.o.f., which is a total of 30 d.o.f.
  - Its displacement field includes quadratic terms.
  - Like the 6-node LST element, the 10-node tetrahedron element has linear strain distributions
- Trilinear tetrahedron - The element is also called an eight-node brick or continuum element.

$$\boxed{\begin{aligned} u &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 z + \beta_5 xy + \beta_6 yz + \beta_7 zx + \beta_8 xyz \\ v &= \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} z + \beta_{13} xy + \beta_{14} yz + \beta_{15} zx + \beta_{16} xyz \qquad (6.2\text{-}2) \\ w &= \beta_{17} + \beta_{18} x + \beta_{19} y + \beta_{20} z + \beta_{21} xy + \beta_{22} yz + \beta_{23} zx + \beta_{24} xyz \end{aligned}}$$

  - Each of three displacement expressions contains all modes in the expression $(c_1+c_2x)(c_3+c_4y)(c_5+c_6z)$, which is the product of three linear polynomials

3D Solids

- The hexahedral element can be of arbitrary shape if it is formulated as an isoparametric element.

$$u = \sum N_i u_i \qquad v = \sum N_i v_i \qquad w = \sum N_i w_i \qquad (6.2\text{-}3)$$

Index i runs from 1 to 8 in each summation. Shorthand for the shape functions is

$$N_i = \tfrac{1}{8}(1 \pm \xi)(1 \pm \eta)(1 \pm \zeta) \qquad (6.2\text{-}4)$$

in which all signs are negative for N_2, all signs are positive for N_8, and so on.

$$\mathbf{k} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T \mathbf{E}\, \mathbf{B}\, |\mathbf{J}|\, d\xi\, d\eta\, d\zeta \qquad (6.2\text{-}5)$$

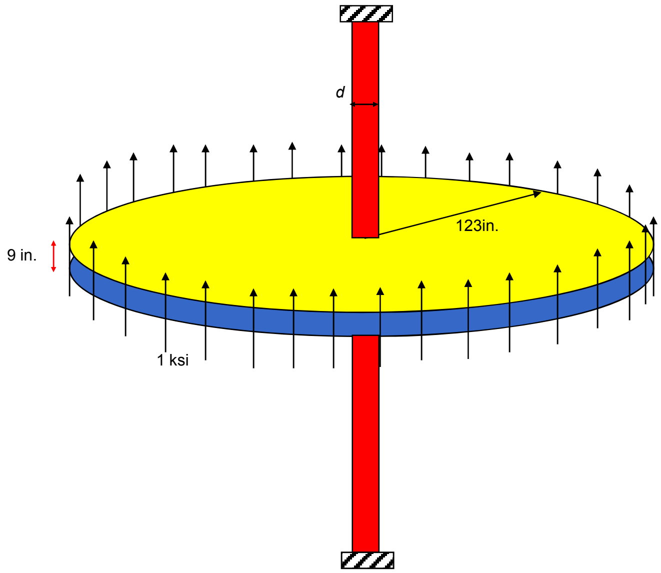
3D Solids

- The determinant |J| can be regarded as a scale factor. Here it expresses the volume ratio of the differential element *dX dY dZ* to the $d\xi\, d\eta\, d\zeta$
- The integration is performed numerically, usually by 2 x 2 x 2 Gauss quadrature rule.
- Like the bilinear quadrilateral (Q4) element, the trilinear tetrahedron does not model beam action well because the sides remain straight as the element deforms.
- If elongated it suffers from shear locking when bent.
- Remedy from locking - use incompatible modes - additional degress of freedom for the sides that allow them to curve

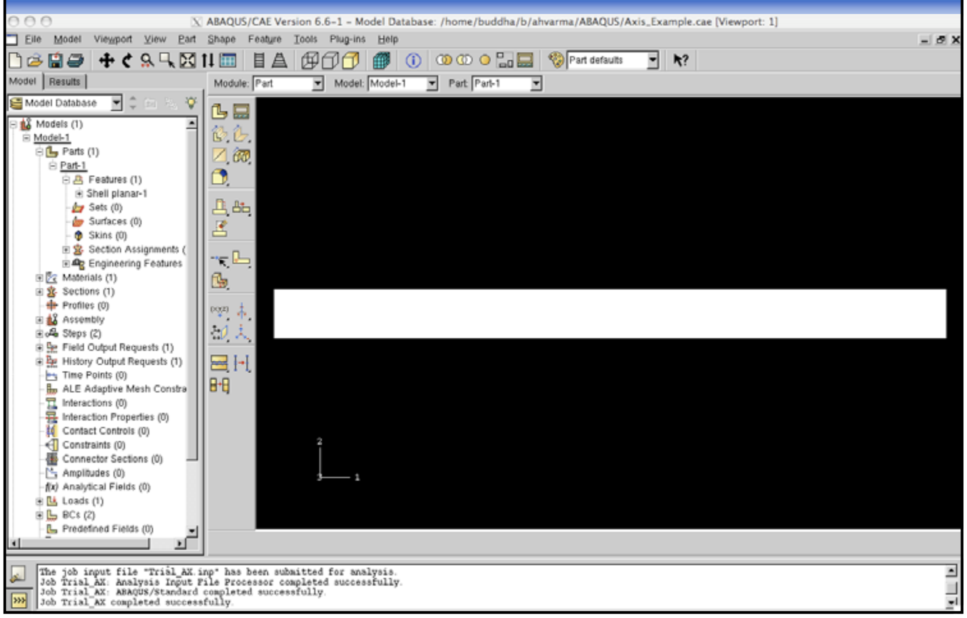
3D Solids

- Quadratic Hexahedron
  - Direct extension of the quadratic quadrilateral Q8 element presented earlier.
  - [B] is now a 6 x 60 rectangular matrix.
  - If [k] is integrated by a 2 x 2 Gauss Quadrature rule, three “hourglass” instabilities will be possible.
  - These hourglass instabilities can be communicated in 3D element models.
  - Stabilization techniques are used in commercial FE packages. Their discussion is beyond the scope.

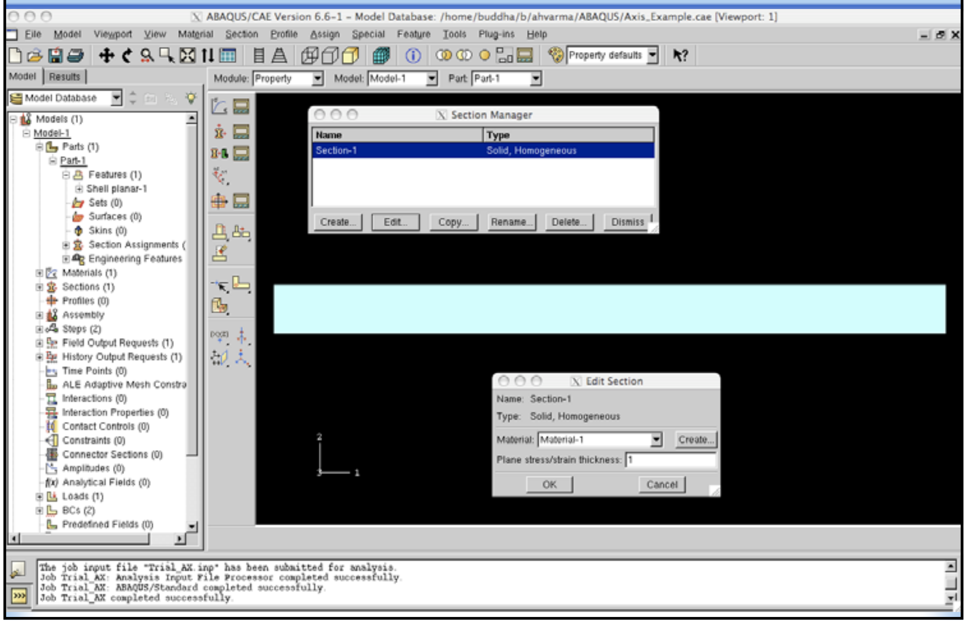
Example - Axisymmetric elements



Example



Example



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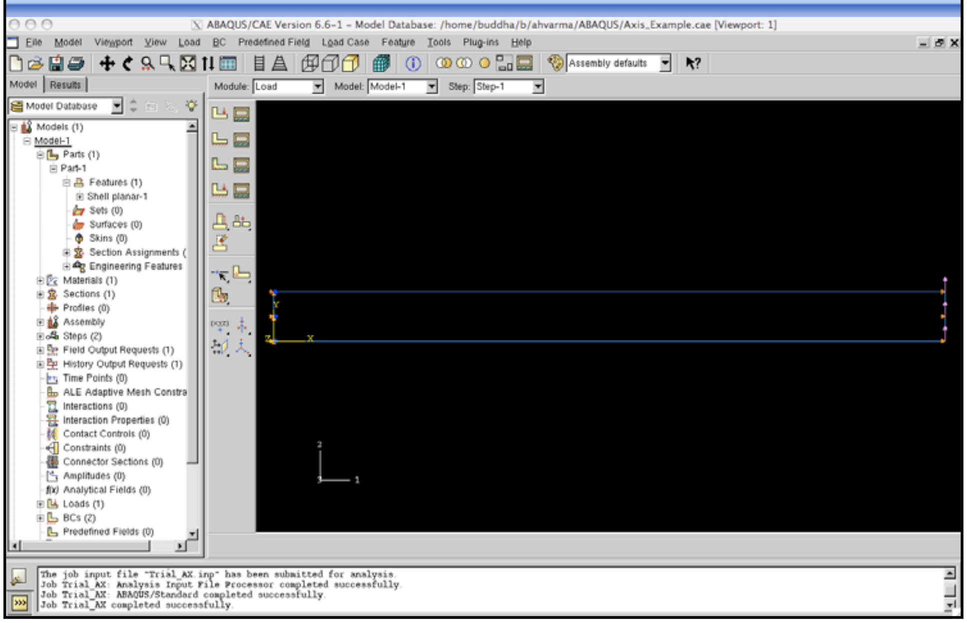

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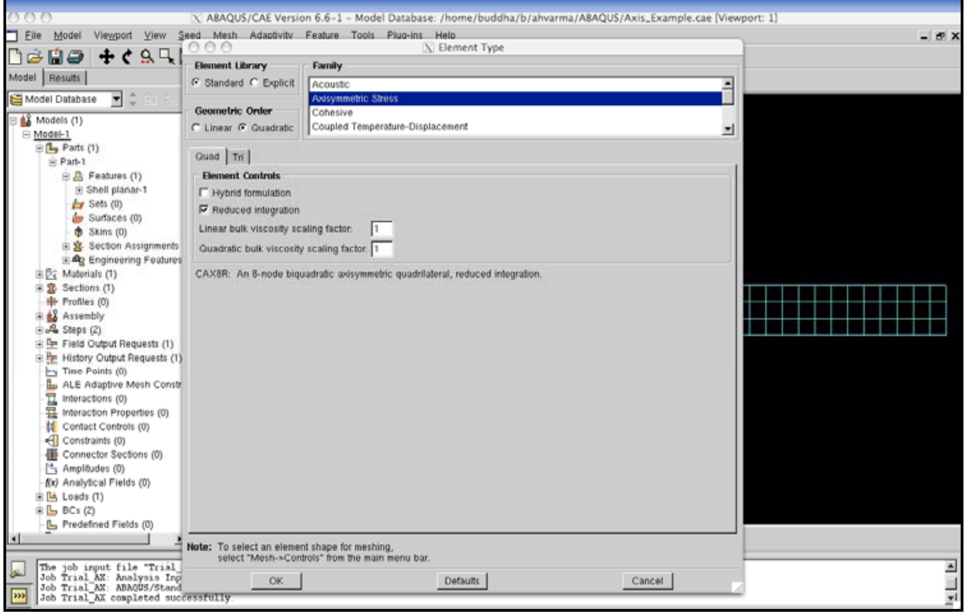

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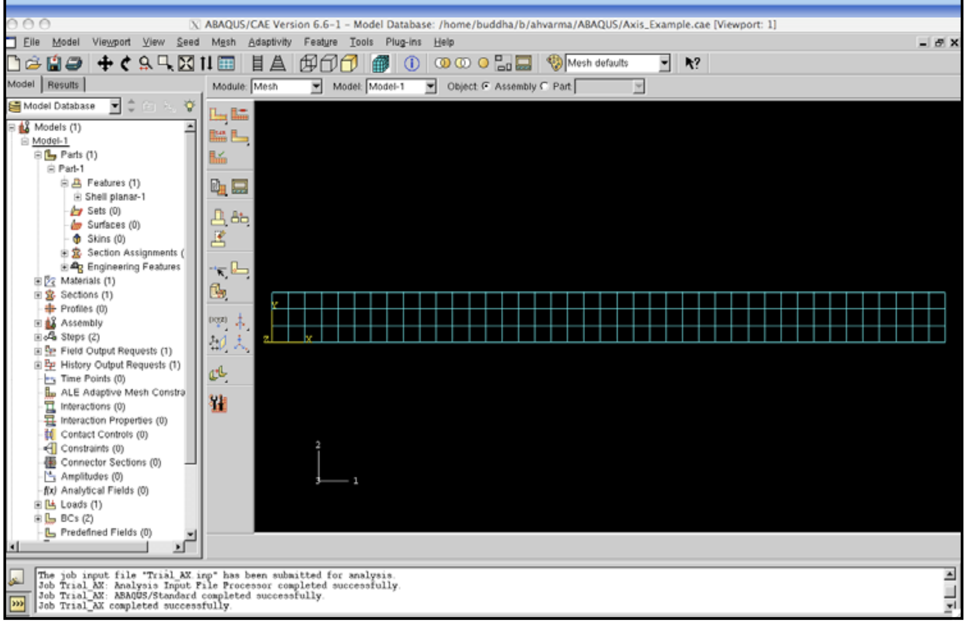

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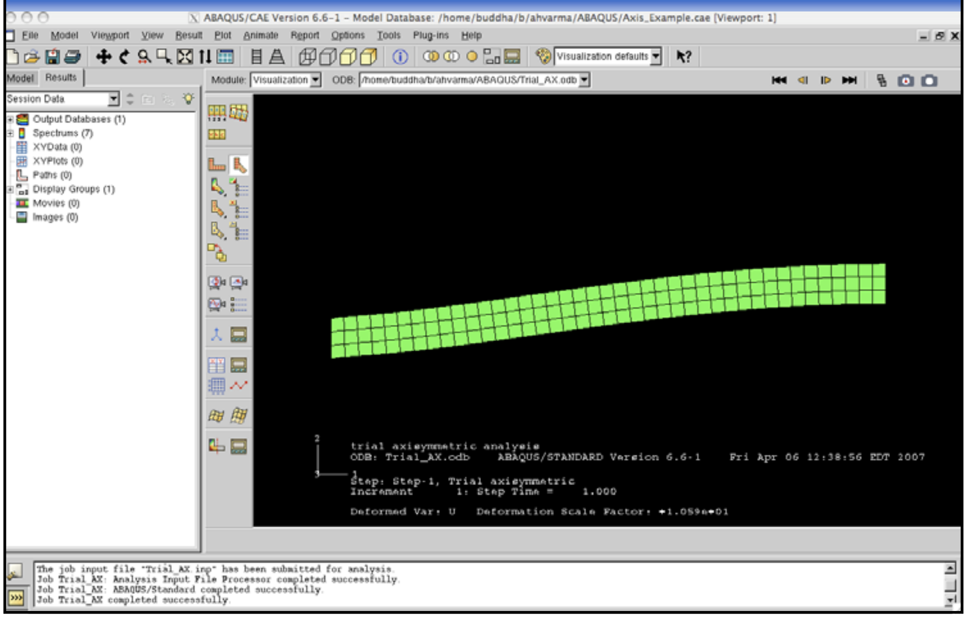

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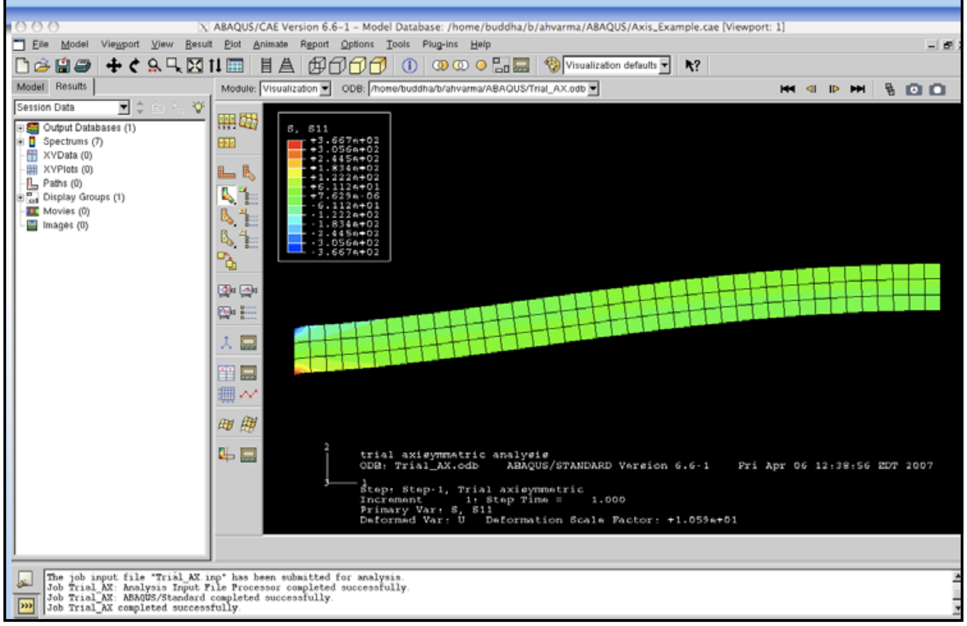

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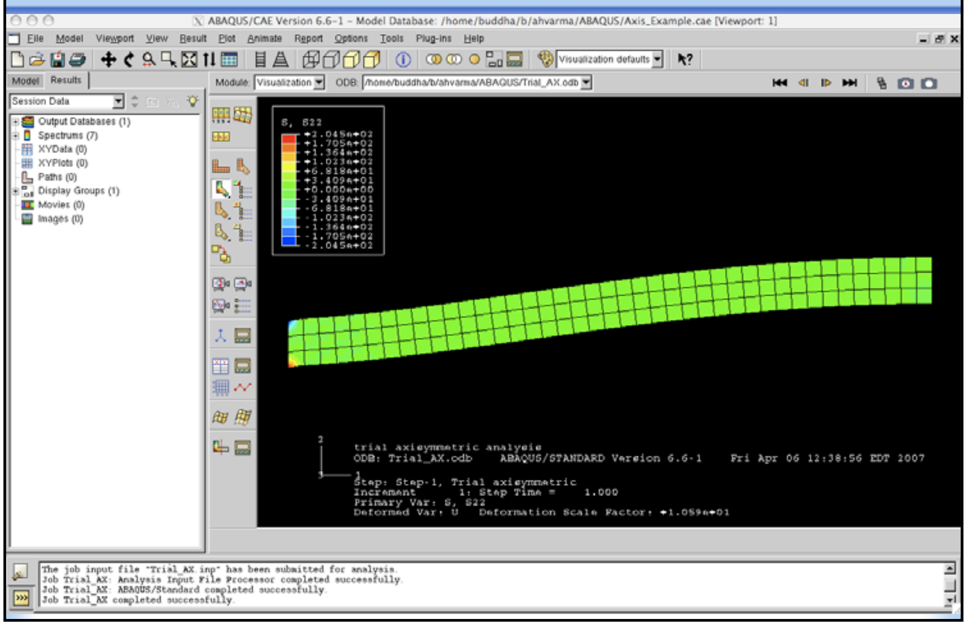

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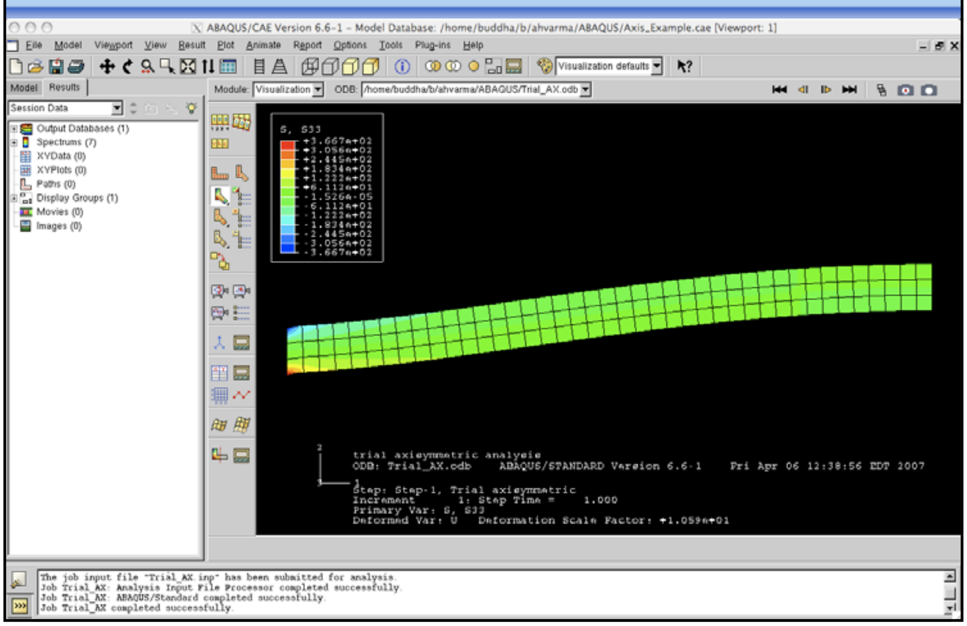

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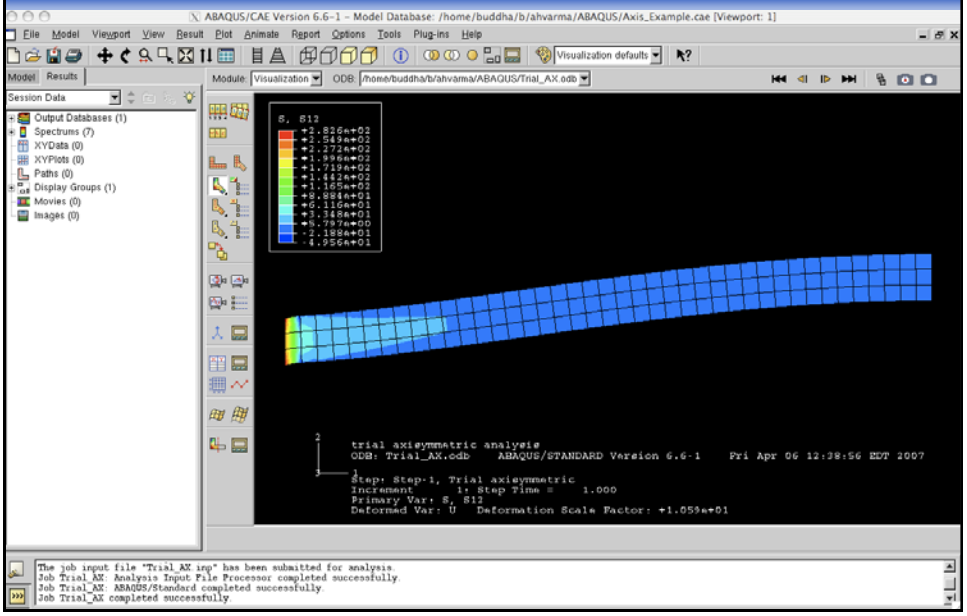

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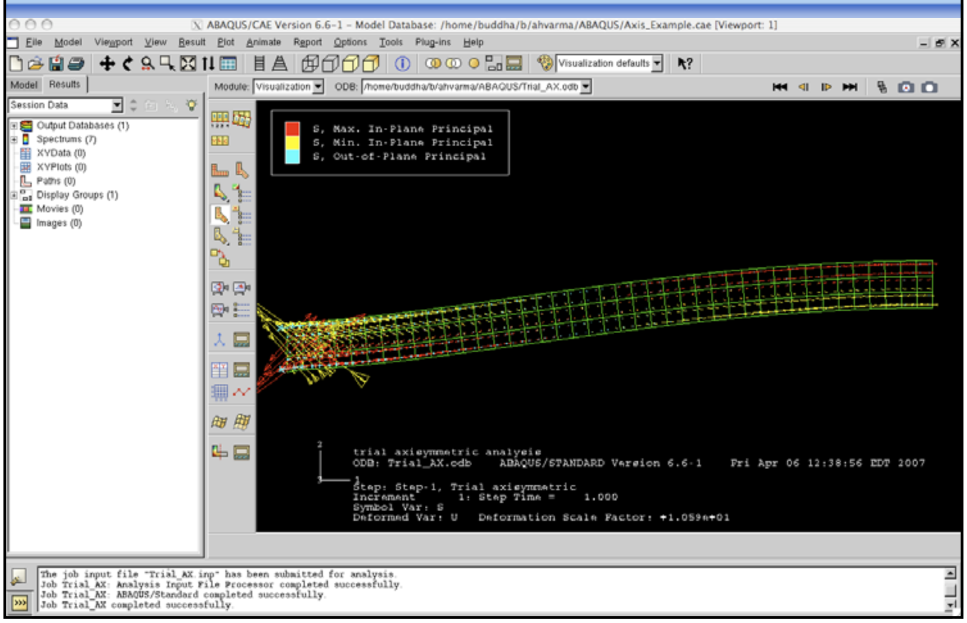

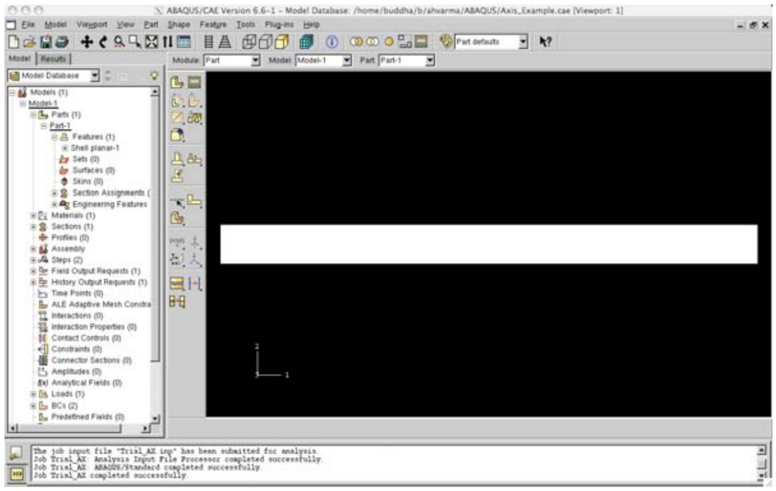
ABAQUS/CAE Version 6.6-1 - Model Database: /home/buddha/b/ahvarma/ABAQUS/Axis_Example.cae [Viewport: 1]
File Model Viewport View Part Shape Feature Tools Plug-ins Help
Part defaults
Model Results
Module: Part
Model: Model-1
Part: Part-1
Model Database
Models (1)
Model-1
Parts (1)
Part-1
Features (1)
Shell planar-1
Sets (0)
Surfaces (0)
Skins (0)
Section Assignments (
Engineering Features
Materials (1)
Sections (1)
Profiles (0)
Assembly
Steps (2)
Field Output Requests (1)
History Output Requests (1)
Time Points (0)
ALE Adaptive Mesh Constra
Interactions (0)
Interaction Properties (0)
Contact Controls (0)
Constraints (0)
Connector Sections (0)
Amplitudes (0)
Analytical Fields (0)
Loads (1)
BCs (2)
Predefined Fields (0)
The job input file "Trial_AX.inp" has been submitted for analysis.
Job Trial_AX: Analysis Input File Processor completed successfully.
Job Trial_AX: ABAQUS/Standard completed successfully.
Job Trial_AX completed successfully.

Linear Algebraic Equations

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1 \\
&a_{21}x_1 + a_{22}x_1 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2 \\
&a_{31}x_1 + a_{32}x_1 + a_{33}x_3 + \ldots + a_{3n}x_n = c_3 \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n
\end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

Linear Algebraic Equations

- In Matrix Format

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{c}}$$

- Solution:

$$\mathbf{A}^{-1}\mathbf{A}\vec{\mathbf{x}} = \mathbf{A}^{-\mathbf{1}}\vec{\mathbf{c}} \Rightarrow \vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{c}}$$

Gaussian Elimination

- Divide each row by the leading element.
- Subtract row 1 from all other rows.
- Move to the second row an continue the process.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} \frac{a_{11}}{a_{11}} & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{11}}{a_{11}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{a_{31}}{a_{31}} & \frac{a_{32}}{a_{31}} & \frac{a_{33}}{a_{31}} & \cdots & \frac{a_{3n}}{a_{31}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{a_{n1}} & \frac{a_{n2}}{a_{n1}} & \frac{a_{n3}}{a_{n1}} & \cdots & \frac{a_{nn}}{a_{n1}} \end{bmatrix} \begin{bmatrix} \frac{c_1}{a_{11}} \\ \vdots \\ \frac{c_3}{a_{31}} \\ \vdots \\ \frac{c_n}{a_{n1}} \end{bmatrix} =$$

Gaussian Elimination

$$\begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{11}}{a_{11}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{a_{32}}{a_{31}} - \frac{a_{12}}{a_{11}} & \frac{a_{33}}{a_{31}} - \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{3n}}{a_{31}} - \frac{a_{11}}{a_{11}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{a_{n2}}{a_{n1}} - \frac{a_{12}}{a_{11}} & \frac{a_{n3}}{a_{n1}} - \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{nn}}{a_{n1}} - \frac{a_{11}}{a_{11}} \end{bmatrix} \begin{bmatrix} \frac{c_1}{a_{11}} \\ \vdots \\ \frac{c_3}{a_{31}} - \frac{c_1}{a_{11}} \\ \vdots \\ \frac{c_n}{a_{n1}} - \frac{c_1}{a_{11}} \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 1 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & 1 & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & 1 & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} c'_1 \\ c'_2 \\ c'_3 \\ \vdots \\ c'_n \end{bmatrix}$$

- Back substitution

$$\mathbf{x_n} = c'_n$$

$$\mathbf{x_i} = c'_i - \sum_{j=i+1}^{n} a'_{ij} \mathbf{x_j}$$

Gaussian Elimination

- Division by zero: May occur in the forward elimination steps.
- Round-off error: Prone to round-off errors.

Gaussian Elimination

Consider the system of equations:

Use five significant figures with chopping

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

At the end of Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

Gaussian Elimination

Back Substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

$$x_3 = \frac{15004}{15005} = 0.99993$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001} = -1.5$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = -0.3500$$

Gaussian Elimination

Compare the calculated values with the exact solution

$$[X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

Improvements

Increase the number of significant digits

- Decreases round off error
- Does not avoid division by zero

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

Partial Pivoting

Gaussian Elimination with partial pivoting applies row switching to normal Gaussian Elimination.

How?

At the beginning of the k^th^ step of forward elimination, find the maximum of

$$|a_{kk}|,|a_{k+1,k}|,\ldots\ldots\ldots\ldots\ldots,|a_{nk}|$$

If the maximum of the values is $|a_{pk}|$ In the p^th^ row, $k \le p \le n,$

then switch rows p and k.

Partial Pivoting

What does it Mean?

Gaussian Elimination with Partial Pivoting ensures that each step of Forward Elimination is performed with the pivoting element $|a_{kk}|$ having the largest absolute value.

Partial Pivoting: Example

Consider the system of equations

$$
\begin{aligned}
&10x_1 - 7x_2 = 7 \\
&-3x_1 + 2.099x_2 + 3x_3 = 3.901 \\
&5x_1 - x_2 + 5x_3 = 6
\end{aligned}
$$

In matrix form

$$
\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}
$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

|10|, |-3|, and |5| or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

|-0.001| and |2.5| or 0.0001 and 2.5

The largest absolute value is 2.5, so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_2}{2.5} = 1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad [X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Gauss Method

An *iterative* method.

Basic Procedure:

-Algebraically solve each linear equation for x_i

-Assume an initial guess solution array

-Solve for each x_i and repeat

-Use absolute relative approximate error after each iteration to check if error is within a prespecified tolerance.

Gauss Method

Why?

The Gauss Method allows the user to control round-off error.

Elimination methods such as Gaussian Elimination and VU Factorization are prone to round-off error.

Also: If the physics of the problem is understood, a close initial guess can be made, decreasing the number of iterations needed.

Gauss Method

Algorithm

A set of *n* equations and *n* unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

If: the diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for x_1

Second equation, solve for x_2

Gauss Method

Algorithm

Rewriting each equation

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$ ← From Equation 1

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$ ← From equation 2

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$ ← From equation n-1

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$ ← From equation n

Gauss Method

Algorithm

General Form of each equation

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j \neq 1}}^{n} a_{1j}x_j}{a_{11}} \qquad x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j \neq n-1}}^{n} a_{n-1,j}x_j}{a_{n-1,n-1}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j \neq 2}}^{n} a_{2j}x_j}{a_{22}} \qquad x_n = \frac{c_n - \sum_{\substack{j=1 \\ j \neq n}}^{n} a_{nj}x_j}{a_{nn}}$$

Gauss Method

Gauss Algorithm

General Form for any row 'i'

$$x^{(k)}{}_i = \frac{c_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x^{(k-1)}{}_j}{a_{ii}}, i = 1,2,\ldots,n.$$

How or where can this equation be used?

Gauss Method

Solve for the unknowns

Assume an initial guess for [X]

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n\text{-}1} \\ x_n \end{bmatrix}$$

Use rewritten equations to solve for each value of xi.

Important: Remember to use the most recent value of x_i. Which means to apply values calculated to the calculations remaining in the **current** iteration.

Gauss Method

Calculate the Absolute Relative Approximate Error

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

So when has the answer been found?

The iterations are stopped when the absolute relative approximate error is less than a pre-specified tolerance for all unknowns.

Gauss-Seidel Method

- Improved method

$$x_i^{(k)} = \frac{c_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}}{a_{ii}}$$

Gauss-Seidel Method: Pitfall

Diagonally dominant: The coefficient on the diagonal must be at least equal to the sum of the other coefficients in that row and at least one row with a diagonal coefficient greater than the sum of the other coefficients in that row.

Which coefficient matrix is diagonally dominant?

$$[A] = \begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix} \qquad [B] = \begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

Most physical systems do result in simultaneous linear equations that have diagonally dominant coefficient matrices.

Gauss-Seidel Method: Example

Given the system of equations

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The coefficient matrix is:

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

Will the solution converge using the Gauss-Seidel method?

Gauss-Seidel Method: Example

Checking if the coefficient matrix is diagonally dominant

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

The inequalities are all true and at least one row is *strictly* greater than:

Therefore: The solution should converge using the Gauss-Seidel Method

Gauss-Siedel Method: Example

Rewriting each equation

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}$$

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

Gauss-Siedel Method: Example

The absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.50000-1.0000}{0.50000}\right|\times 100 = 67.662\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.9000-0}{4.9000}\right|\times 100 = 100.00\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.0923-1.0000}{3.0923}\right|\times 100 = 67.662\%$$

The maximum absolute relative error after the first iteration is 100%

Gauss-Siedel Method: Example

After Iteration #1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

Substituting the x values into the equations

$$x_1 = \frac{1-3(4.9000)+5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28-(0.14679)-3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76-3(0.14679)-7(4.900)}{13} = 3.8118$$

After Iteration #2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

Gauss-Siedel Method: Example

Iteration #2 absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 = 240.62\%$$

$$\left|\in_a\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 = 31.887\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 = 18.876\%$$

The maximum absolute relative error after the first iteration is 240.62%

This is much larger than the maximum absolute relative error obtained in iteration #1. Is this a problem?

Gauss-Siedel Method: Example

Repeating more iterations, the following values are obtained

Iteration	a_1	$\left\|\varepsilon_a\right\|_1$	a_2	$\left\|\varepsilon_a\right\|_2$	a_3	$\left\|\varepsilon_a\right\|_3$
1	0.50000	67.662	4.900	100.00	3.0923	67.662
2	0.14679	240.62	3.7153	31.887	3.8118	18.876
3	0.74275	80.23	3.1644	17.409	3.9708	4.0042
4	0.94675	21.547	3.0281	4.5012	3.9971	0.65798
5	0.99177	4.5394	3.0034	0.82240	4.0001	0.07499
6	0.99919	0.74260	3.0001	0.11000	4.0001	0.00000

The solution obtained $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$

is close to the exact solution of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

Comparison of different methods

- Gauss is slow convergent; however, more stable.
- Gauss-Seidel might not be stable; however, when stable, converges fast.
- Hotelling and bodewig uses an improved iterative method for matrix inversion.
- In Relaxation techniques small corrections will be applied to each element at each iteration.

CE595: Finite Elements in Elasticity

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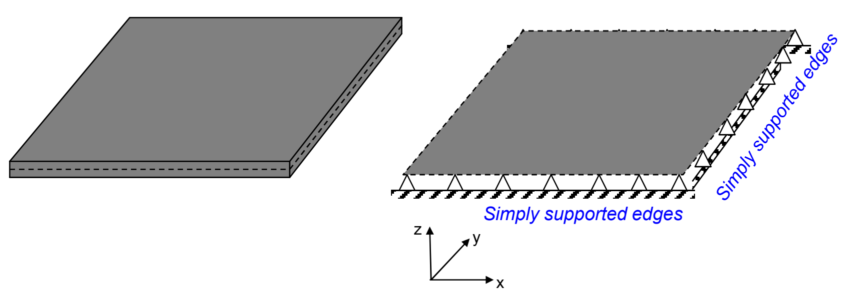
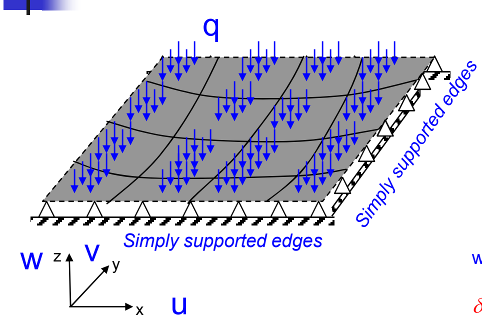
Behavior of Plates

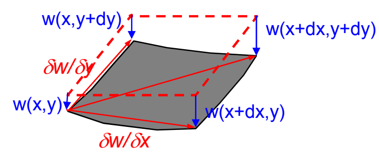
- The behavior of plates is similar to that of beams. They both carry transverse loads by bending action.
  - Plates carry transverse loads by bending and shear just like beams, but they have some peculiarities
  - We will focus on isotropic homogenous plates.



Behavior of Plates



- Plates undergo bending which can be represented by the deflection (*w*) of the middle plane of the plate



- The middle plane of the plate undergoes deflections w(x,y). The top and bottom surfaces of the plate undergo deformations almost like a rigid body along with the middle surface.

Behavior of Plates

Thin plate theory - does not include transverse shear deformations

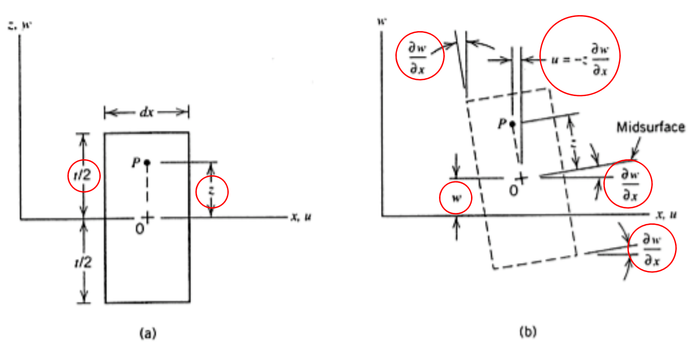


Fig. 7.1-1. Differential slice of a plate of thickness *t*. (a) Before loading. (b) Displacements after loading, according to Kirchhoff theory. Transverse shear deformation is neglected, so right angles in the cross section are preserved. Displacements in a *yz*-parallel plane are similar. (Reprinted from [2.2] by permission of John Wiley & Sons, Inc.)

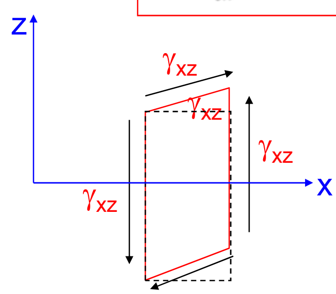
Behavior of Plates

$$u = -z\frac{\partial w}{\partial x} \qquad v = -z\frac{\partial w}{\partial y}$$

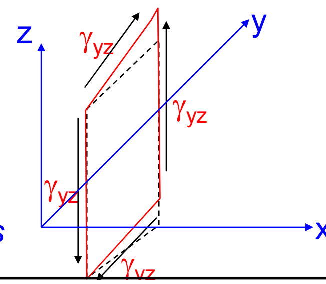
$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\frac{\partial^2 w}{\partial x \partial y} - z\frac{\partial^2 w}{\partial x \partial y} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\, \partial y} \qquad (7.1\text{-}1)$$

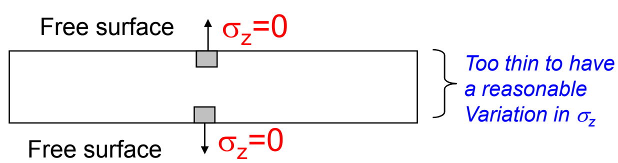


ZERO - transverse shear deformations



Behavior of Plates

- The normal stress in the direction of the plate thickness (z) is assumed to be negligible.

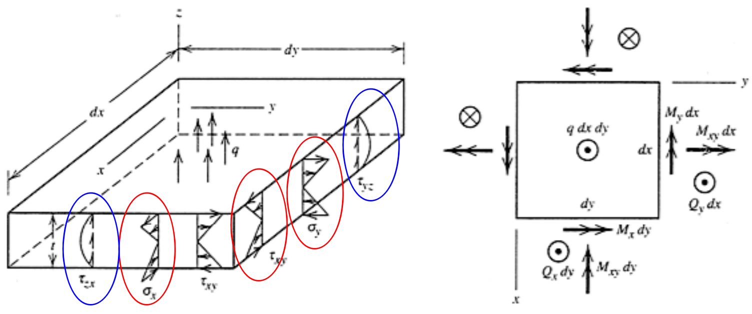


- Note that $\sigma_z=0$, not necessarily ε_z (normal strain in thickness direction).
- Plane stress equations relating 3D stresses to strains will work - not a plane stress situation - just mathematically!

$$\mathbf{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \quad \text{for plane stress} \qquad (3.1\text{-}3)$$

Behavior of Plates

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix} = -z\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \end{Bmatrix} \qquad \tau_{xy} = -2zG\frac{\partial^2 w}{\partial x \, \partial y} \qquad (7.1\text{-}2)$$



(a) (b)

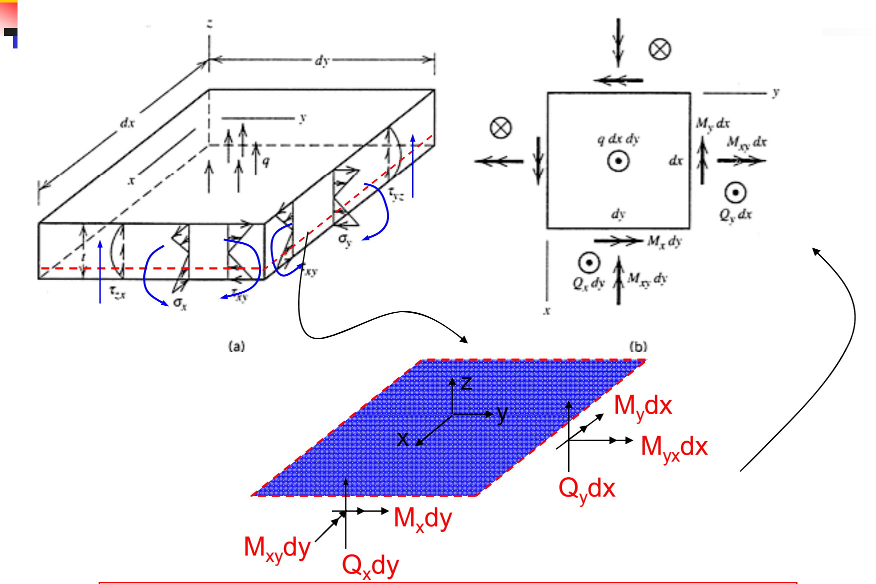
Fig. 7.1-2. Differential element of a plate. (a) Stresses on cross sections and distributed lateral load $q = q(x, y)$. (b) Differential forces and moments. Arrows that represent forces normal to the plate midsurface are viewed end-on. (Reprinted from [2.2] by permission of John Wiley & Sons, Inc.)

Behavior of Plates

- Note that the stresses vary linearly from the middle surface. Just like bending stresses in beams.
- Also note that the shear stresses (τ_{xy}) produced by bending also vary linearly from the middle surface.
- The shear stresses τ_{yz} and τ_{zx} are present and required for equilibrium, although the corresponding strains are assumed negligible. Parabolic variations of the stresses are assumed.
- The bending stresses can be simplified to resultant moments (M_{xx}, M_{yy}, M_{xy}). These moments are resultants of the linear stress variations through the thickness

$$M_x = \int_{-t/2}^{t/2} \sigma_x z\,dz \qquad M_y = \int_{-t/2}^{t/2} \sigma_y z\,dz \qquad M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z\,dz \qquad (7.1\text{-}3)$$

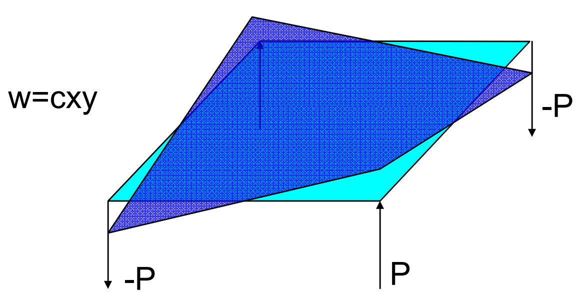
Behavior of Plates



$$M_x = \int_{-t/2}^{t/2} \sigma_x z\,dz \qquad M_y = \int_{-t/2}^{t/2} \sigma_y z\,dz \qquad M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z\,dz \qquad (7.1\text{-}3)$$

Behavior of Plates

- The stress states in plates differ from those in beams. The twisting moment is a major difference.
  - The twisting moment is the only only component if the deflection w=c*x*y. This is called the state of pure twist
  - It can be realized by applying two equal upward forces at diagonally opposite corners, and two equal downward forces at the other two diagonally opposite corners.



Behavior of Plates

- Another difference between beams and plates is that if we apply a moment M_x along a beam length
  - The beam deforms in the x-z plane
  - The beam has a narrow cross-section, so the normal stress σ_y is zero on its sides, and almost zero in between
  - Due to Poisson's effect, the top and bottom edges of the cross-section become curved in the y-z plane
- In contrast, the top and bottom edges of plates are long and do not become curved due to bending moment M_x
  - So, there is no curvature ($\delta^2 w/\delta y^2$), when the plate is subjected to cylindrical bending producing $\delta^2 w/\delta x^2$ due to M_x
  - The equations show that the stress σ_x is accompanied by a stress σ_y

Behavior of Plates

- The stress σ_y (and the resultant M_y) constrain the curvature of the plate $\delta^2 w/\delta y^2$
  - *This results in stiffening of the plate. The amount of stiffening is proportional to $1/(1-\nu^2)$*
  - *A unit weight of the plate has rigidity $Et^3/12(1-\nu^2)$*
  - *The corresponding beam would have rigidity $Et^3/12$*
- *This theory is called 'thin plate bending' or Kirchhoff plate bending theory.*
  - *It ignores the effects of transverse shear deformations.*
  - *If the plate thickness is less than smallest width/10, then this is a reasonable assumption*
- *Alternative is the Mindlin Plate Theory.*

Mindlin Plate Theory

- The transverse shear deformation effects are included by relaxing the assumption that plane sections remain perpendicular to middle surface, i.e., the right angles in the BPS element are no longer preserved.
  - Planes initially normal to the middle surface may experience different rotations than the middle surface itself
  - Analogy is the Timoshenko beam theory.

Mindlin Plate Theory

- θ_x and θ_y are rotations of lines perpendicular to the middle surface

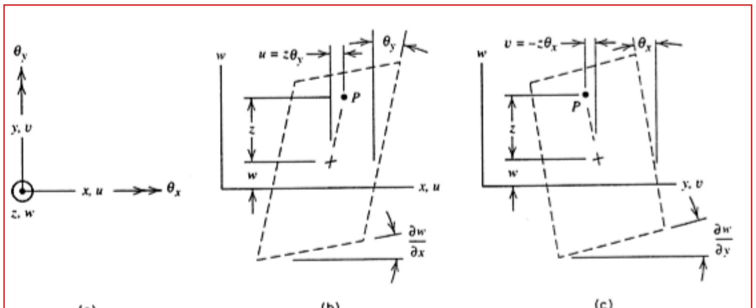


Fig. 7.1-3. (a) Displacements and positive directions for rotations θ_x and θ_y, viewed normal to the xy plane. (b) Mindlin theory displacements in an xz-parallel cross section. (c) Mindlin theory displacements in a yz-parallel cross section.

Mindlin Plate Theory

- Strain displacement relationships
  - Interesting

$$u = z\theta_y \qquad \varepsilon_x = z\frac{\partial \theta_y}{\partial x} \qquad \gamma_{xy} = z\left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right)$$

$$v = -z\theta_x \qquad \varepsilon_y = -z\frac{\partial \theta_x}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} - \theta_x \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \theta_y \qquad (7.1\text{-}4)$$

  - What is the real difference?
  - Consider Timoshenko beam theory.
    - There are two differential equations instead of one
    - One for bending and the second for shear force equilibrium.

Behavior of Plates

- Loads
  - Distributed or concentrated loads can be applied to plates.
  - At any point where a concentrated force is applied, Kirchoff theory predicts infinite bending moments. Mindlin theory predicts infinite bending moments and displacements.
  - In reality no force can be concentrated, and in plate theory the infinite values disappear if the load is applied over a small area.
  - Of course, the FEM will not compute infinite values.
- Supports
  - You can have pin supports, roller supports, fixed supports and free edges.
  - You can have the plates supported along edges or at discrete locations.

Behavior of Plates

- Large displacements and membrane forces
  - The simply supported plate subjected to distributed loads will have vertical deflections. If the horizontal loads are restrained by the supports, then membrane forces can develop for large deformations.
  - These membrane forces add to the stiffness of the plate, and reduce deflections.
  - For example, consider a beam -

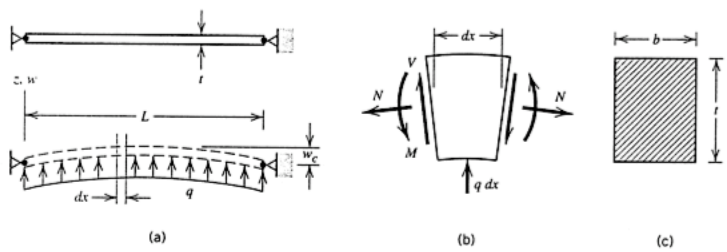


Fig. 7.1-4. (a) Beam of length L with immovable hinge supports. (b) Differential element, showing "string action" axial force N. (c) Cross section.

FINITE ELEMENTS FOR PLATES

- How many degrees of freedom are we talking about?

- Kirchhoff plate element - The stiffness matrix can be calculated from the standard equation.
  - E is replaced by a matrix of flexural rigidities
  - B is contrived to produce curvatures when it operates on nodal d.o.f. that describe the lateral displacement field w(x,y)
  - The behavior of a Kirchhoff element depends on the assumed w field, which is a polynomial in x and y, and the nodal values of w, dw/dx, and dw/dy

Finite elements for plates

- A 12 d.o.f. rectangular Kirchhoff element.
  - It is incompatible, i.e., the normal direction (n) to the element edge is not continuous between elements for some loading conditions.
  - The element cannot guarantee a lower bound on computed displacements
  - A compatible rectangular element with corner nodes only requires that the twist ($d^2w/dxdy$) also be used as a nodal d.o.f.
  - It is quite difficult to obtain a *triangular* Kirchhoff element that can represent states of constant curvature and twist, and has no preferred directions, and gives good results.
- It is a lot easier to formulate plate elements that allow for the shear deformations - Mindlin plate theory.

Finite Elements for Plates

- A Mindlin theory based plate element has three fields; $w(x,y)$, $\theta_x(x,y)$, and $\theta_y(x,y)$.
  - Each of these is interpolated from nodal values.
  - If all interpolations use the same polynomical

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} = \mathbf{Nd} \qquad (7.2\text{-}1)$$

  - Using the strain-displacement relations, the [B] matrix can be derived.
  - The $[E]_{5x5}$ matrix includes the 3x3 of the plane stress and the 2x2 shear moduli associated with the two transverse shear strains
  - Integration in the plane of the element is done numerically if the element is isoparametric.

Finite Elements for Plates

- Four node quadrilateral. Eight node quadrilateral also possible.
  - In any z=constant layer, strains vary in the same way as in the corresponding plane element. So, the behavior of the Mindlin plate element can be understood.
  - However, the integration rules are modified.
- Selective integration is used for the plate elements
  - One-point quadrature for the transverse shear strains (to reduce the effects of spurious shear stresses similar to the Q4 elements)
  - Four-point quadrature for the bending strains
  - Selective integration is common for the plate elements

- Reduced integration for plate elements

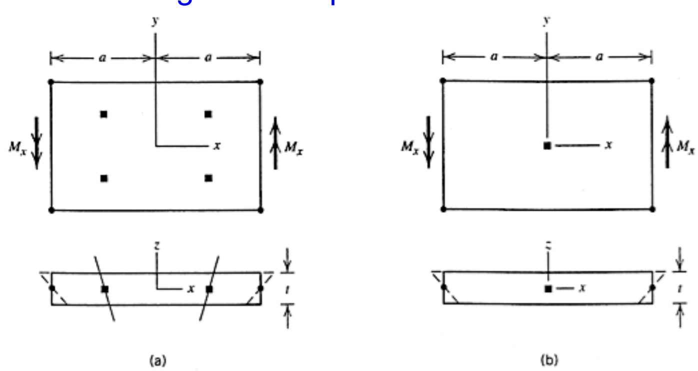


Fig. 7.2-3. A bending deformation in a four-node Mindlin plate element. (a) Four-point integration rule. (b) One-point integration rule.

Finite Elements for Plates

- Tricky to select and use. In many cases, user will not be sure that they understand or follow the formulation or the tweaks to make it better.
- The best way is to explore the elements provided by the software for simple test cases problems with known solutions.
- Discrete Kirchoff elements
  - Essential feature is that the transverse shear strain is set to zero at a finite number of points in the element, rather than at every point as in classical theory.
  - Thin plate elements - triangular in shape - incompatible
  - The elements are built after many manipulations. It is not apparent how a discrete Kirchoff plate element behaviors.
  - As with Mindlin plate element, the analyst should use numerical experiments to learn about behavior.

Plate modeling in ABAQUS

- Shell elements are used to model structures in which one dimension, the thickness, is significantly smaller than the other dimensions.
  - Conventional shell elements use this condition to discretize a body by defining the geometry at a reference surface.
  - In this case the thickness is defined through the section property definition.
  - Conventional shell elements have displacement and rotational degrees of freedom.
  - The “top” surface of a conventional shell element is the surface in the positive normal direction and is referred to as the positive (SPOS) face for contact definition.
  - The “bottom” surface is in the negative direction along the normal and is referred to as the negative (SNEG) face for contact definition.

Plate Modeling in ABAQUS

- Positive and negative are also used to designate top and bottom surfaces when specifying offsets of the reference surface from the shell's midsurface.
- The positive normal direction defines the convention for pressure load application and output of quantities that vary through the thickness of the shell.

Figure 23.6.1–2 Positive normals for three-dimensional conventional shells.

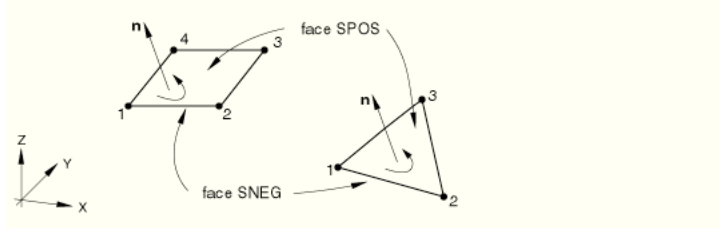


Plate Bending in ABAQUS

- Numbering of section points through the shell thickness
  - For a homogeneous section the total number of section points is defined by the number of integration points through the thickness
  - For general shell sections, output can be obtained at three section points. Section point 1 is always on the bottom surface of the shell.
  - For shell sections integrated during the analysis, you can define the number of integration points through the thickness. The default is five for Simpson's rule and three for Gauss quadrature.
  - For shell sections integrated during the analysis, section point 1 is exactly on the bottom surface of the shell if Simpson's rule is used, and it is the point that is closest to the bottom surface if Gauss quadrature is used.

Plate Bending in ABAQUS

- Default output points
  - The default output points through the thickness are on the bottom and top surfaces of the shell section.
  - For example, if five integration points are used through a single layer shell, output will be provided for section points 1 (bottom) and 5 (top).

Plate Bending in ABAQUS

- The ABAQUS/Standard shell element library includes:

  * elements for three-dimensional shell geometries

  * elements for axisymmetric geometries with axisymmetric deformation

  * elements for axisymmetric geometries with general deformation that is symmetric about one plane

  * elements for stress/displacement, heat transfer, and fully coupled temperature-displacement analysis

  * general-purpose elements, as well as elements specifically suitable for the analysis of “thick” or “thin” shells

  * general-purpose, three-dimensional, first-order elements that use reduced or full integration

  * elements that account for finite membrane strain

  * elements that use five degrees of freedom per node where possible, as well as elements that always use six degrees of freedom per node and

  * continuum shell elements.

Plate Bending in ABAQUS

- Naming convention.The naming convention for shell elements depends on the element dimensionality.
- Three-dimensional shell elements. Three-dimensional shell elements in ABAQUS are named as follows:

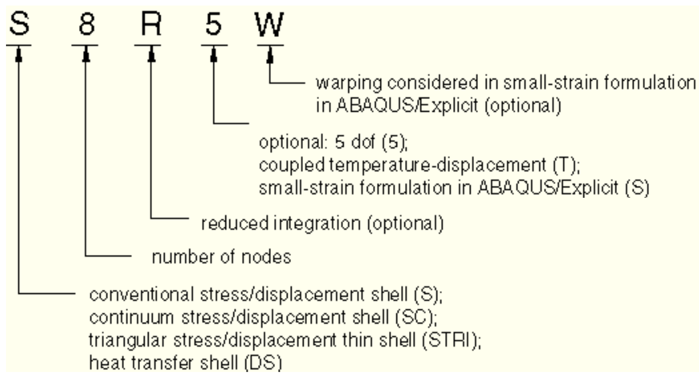


Plate Bending in ABAQUS

- Conventional stress/displacement shell elements
  - Can be used in 3D or axisymmetric analysis. They use linear or quadratic interpolation and allow mechanical and/or thermal (uncoupled) loading.
  - These elements can be used in static or dynamic procedures.
  - Some elements include the effect of transverse shear deformation and thickness change, while others do not.
  - Some elements allow large rotations and finite membrane deformation, while others allow large rotations but small strains.

Plate Bending in ABAQUS

- "Thick" versus "thin" conventional shell elements
  - ABAQUS includes general-purpose, conventional shell elements
  - As well as conventional shell elements that are valid for thick and thin shell problems.
- The general-purpose, conventional shell elements provide robust solutions for most applications
- In certain cases, for specific applications, enhanced performance may be obtained with the thin or thick conventional shell elements.
  - For example, if only small strains occur and five degrees of freedom per node are desired.

Plate Bending in ABAQUS

- General-purpose conventional shell elements
  - These elements allow transverse shear deformation.
  - They use thick shell theory as the shell thickness increases and become discrete Kirchhoff thin shell elements as the thickness decreases
  - The transverse shear deformation becomes very small as the shell thickness decreases.
- Element types S3/S3R, S3RS, S4, S4R, S4RS, S4RSW, SAX1, SAX2, SAX2T, SC6R, and SC8R are general-purpose shells.

Plate Bending in ABAQUS

- Thick conventional shell elements
  - Thick shells are needed where transverse shear flexibility is important and second-order interpolation is desired.
  - This occurs when the thickness is more than about 1/15 of a characteristic length on the surface of the shell, such as the distance between supports for a static case
- ABAQUS/Standard provides element types S8R and S8RT for use only in thick shell problems.

Plate Bending in ABAQUS

- Thin conventional shell elements
  - Thin shells are needed in cases where transverse shear flexibility is negligible and the Kirchhoff constraint must be satisfied accurately (i.e., the shell normal remains orthogonal to the reference surface).
  - For homogeneous shells this occurs when the thickness is less than about 1/15 of a characteristic length on the shell surface.
- ABAQUS has two types of thin shell elements: those that solve thin shell theory (the Kirchhoff constraint is satisfied analytically) and those that converge to thin shell theory as the thickness decreases (the Kirchhoff constraint is satisfied numerically).
  - The element that solves thin shell theory is STRI3. STRI3 has six degrees of freedom at the nodes and is a flat, faceted element (initial curvature is ignored). If STRI3 is used to model a thick shell problem, the element will always predict a thin shell solution.
  - The elements that impose the Kirchhoff constraint numerically are S4R5, STRI65, S8R5, S9R5, SAXA1n, and SAXA2n. These elements should not be used for applications in which transverse shear deformation is important. If these elements are used to model a thick shell problem, the elements may predict inaccurate results.

Plate Bending in ABAQUS

- Finite-strain versus small-strain shell elements
  - ABAQUS has both finite-strain and small-strain shell elements.
  - Finite-strain shell elements. Element types S3/S3R, S4, S4R, SAX1, SAX2, SAX2T, SAXA1n, and SAXA2n account for finite membrane strains and arbitrarily large rotations; therefore, they are suitable for large-strain analysis.
- Small-strain shell elements
  - In ABAQUS the three-dimensional "thick" and "thin" element types STRI3, S4R5, STRI65, S8R, S8RT, S8R5, and S9R5 provide for arbitrarily large rotations but only *small strains*.
  - The change in thickness with deformation is ignored in these elements.

Plate Bending in ABAQUS

- Five degree of freedom shells versus six degree of freedom shells
  - Two types of 3D conventional shell elements are provided
  - Ones that use five degrees of freedom (three displacement components and two in-surface rotation components)
  - And ones that use six degrees of freedom (three displacement components and three rotation components) at all nodes.
- The elements that use five degrees of freedom (S4R5, STRI65, S8R5, S9R5) can be more economical. However, they are available only as "thin" shells (they cannot be used as "thick" shells) and cannot be used for finite-strain applications (although they model large rotations with small strains accurately).

My recommendation

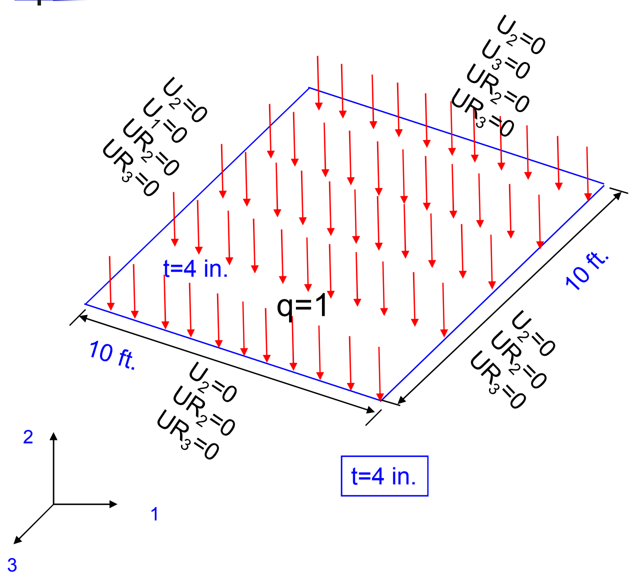
- Using S4 elements
  - Element type S4 is a fully integrated, general-purpose, finite-membrane-strain shell element available in ABAQUS/Standard.
  - The element's membrane response is treated with an assumed strain formulation that gives accurate solutions to in-plane bending problems, is not sensitive to element distortion, and avoids parasitic locking.
  - Element type S4 does not have hourglass modes in either the membrane or bending response of the element; hence, the element does not require hourglass control.
  - The element has four integration locations per element compared with one integration location for S4R, which makes the element computationally more expensive.
  - S4 is compatible with both S4R and S3R.
  - S4 can be used for problems prone to membrane- or bending-mode hourglassing, in areas where greater solution accuracy is required, or for problems where in-plane bending is expected. In all of these situations S4 will outperform element type S4R. S4 cannot be used with the hyperelastic or hyperfoam material definitions.

Summary

- STRI3 - triangular 3-node element for Kirchhoff thin plate bending
- S4R5 - quadrilateral 4-node element for Kirchoff thin plate bending with 5 d.o.f. per node.
- S8R - quadrilateral 8-node element for Mindlin thick plate bending with 6 d.o.f per node.
- S4 - quadrilateral general purpose finite element with finite strains.

- If you see a ‘5’ in the element name - it had 5 d.o.f. per node and will be a thin shell element.

Example Problem

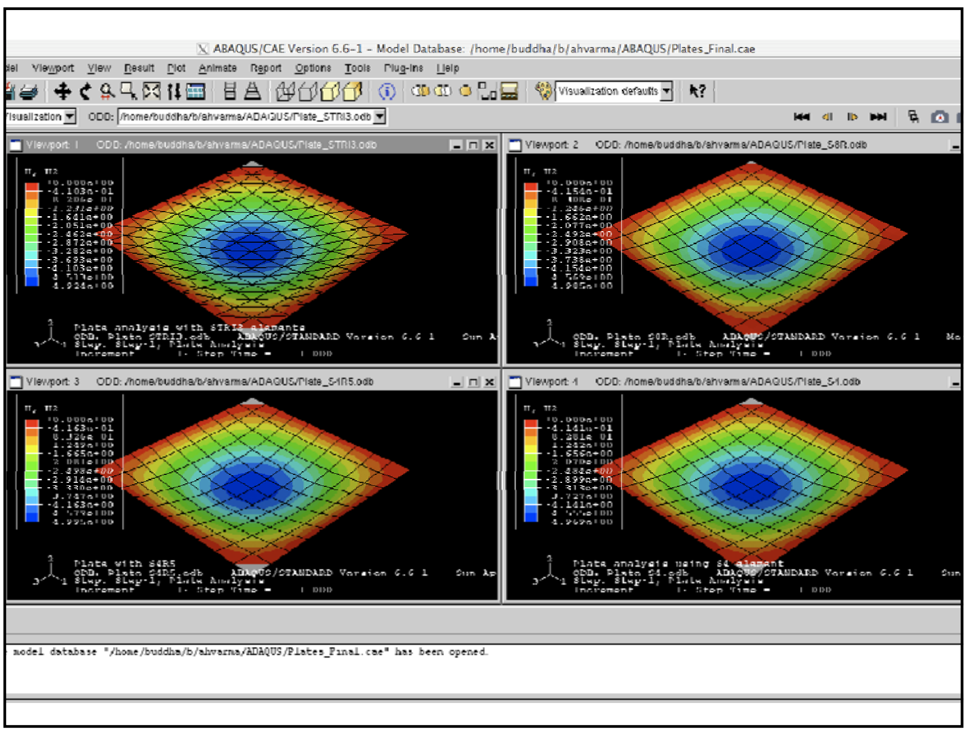


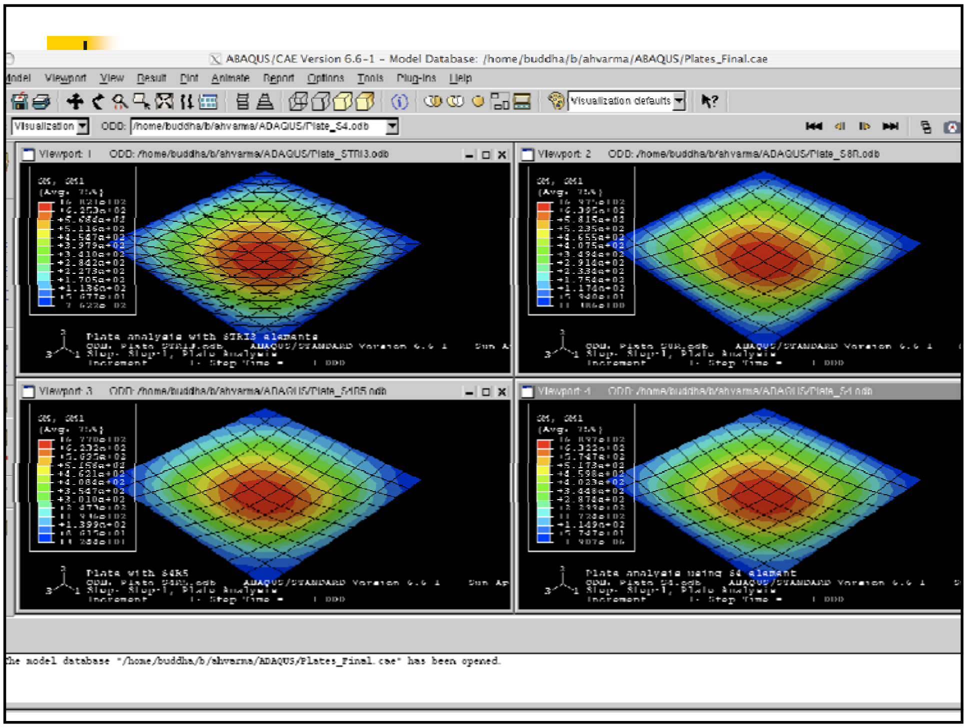
Example Problem

- Solved using Kirchhoff's plate bending theory and assuming small strains etc.
- $w_{max} = \alpha\ q\ a^4 / D$
  - Where α =0.00406
  - q = 1 kip/in. and a = 120 in.
  - $D = Et^3/12(1-\nu^2) = 29000 \times 4^3 / (12 \times 0.91) = 169963.37$ k-in
  - There w_{max} = 4.95 in.
- $M_{x\text{-}max} = M_{y\text{-}max} = \beta\ q\ a^2$
  - $= 0.0479 \times 1 \times 120^2 = 689.76$ k-in/in
- $Q_{x\text{-}max} = \gamma\ q\ a$
  - $= 0.338 \times 1 \times 120 = 40.56$ k / in.

Finite Element Analysis

- Models were developed and analyzed using ABAQUS
  - Element STRI3
  - Element S4R5
  - Element S8R
  - Element S4
- Compare the results in the next few slides
  - Note that the transverse shear stresses are not provided as output for thin shell theory elements.
  - The section forces and moments can be obtained from the analysis
  - The stresses can be looked at the various section points 1, 2, 3, 4, and 5
  - SM1, SM2, SM3 are the resulting M_x, M_y, and M_{xy} per unit length. The corresponding stresses are s11, s22, and s12

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Visualization defaults
Plate analysis with STRI3 elements
Plate with S4R5
Plate analysis using S4 element
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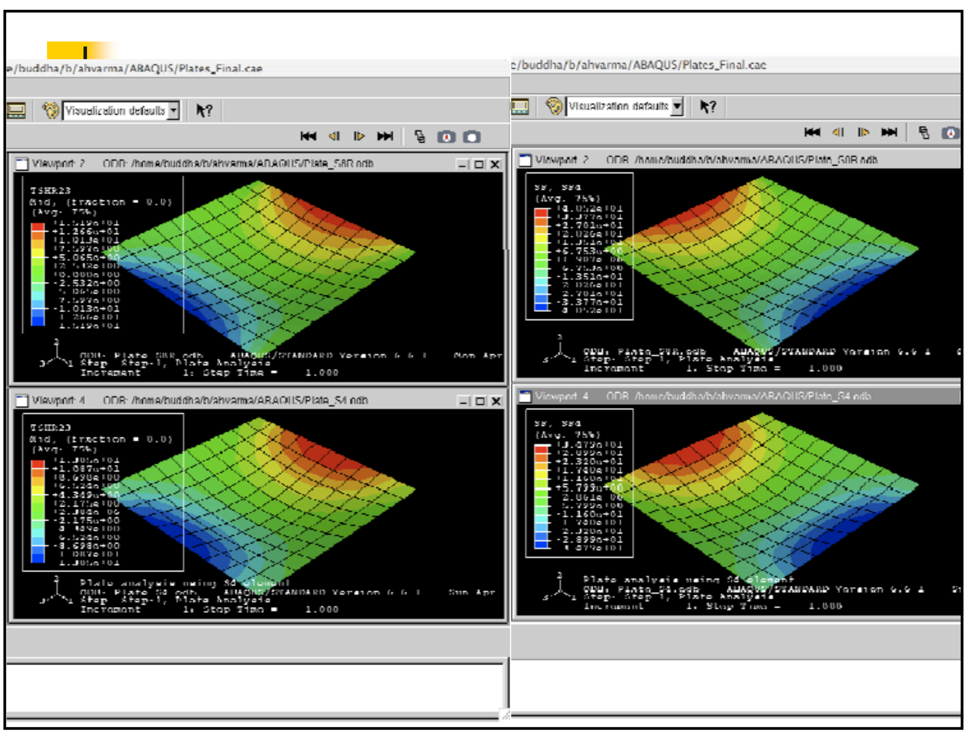
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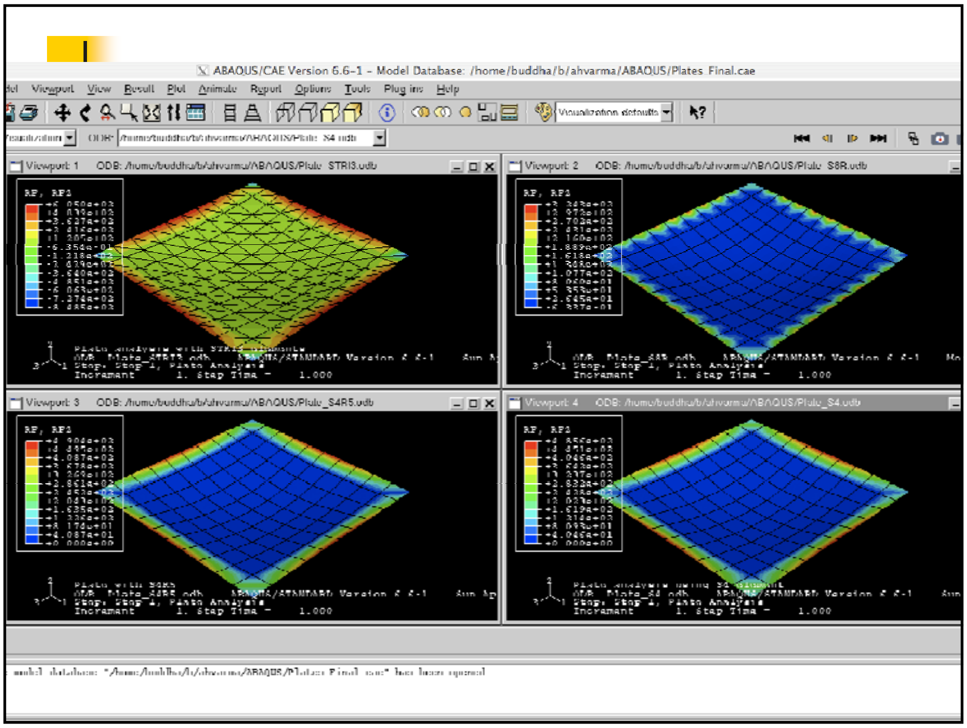
Visualization defaults
Plate analysis using S4 element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