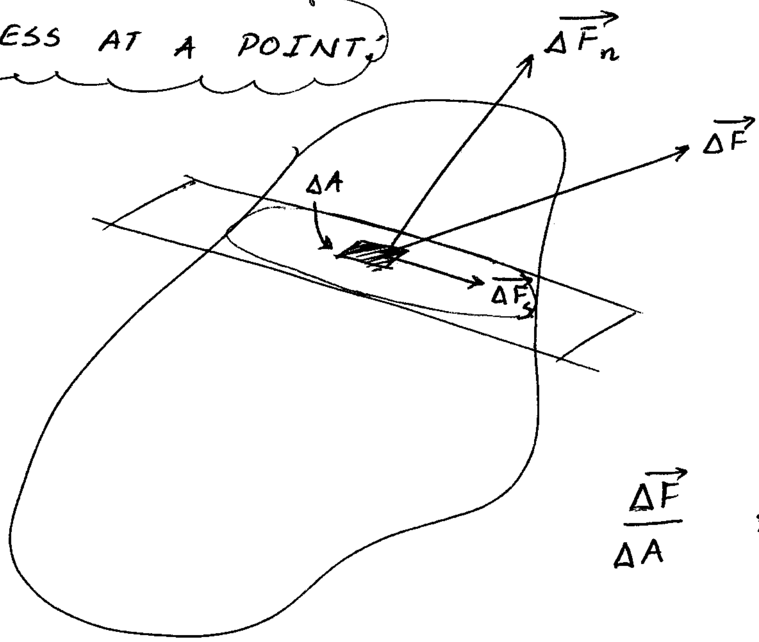
CE 592 : PLASTIC DESIGN

- STRESS AT A POINT



general body cut by plane passing through Q

$$\frac{\vec{\Delta F}}{\Delta A}, \quad \frac{\vec{\Delta F_n}}{\Delta A}, \quad \frac{\vec{\Delta F_s}}{\Delta A}$$

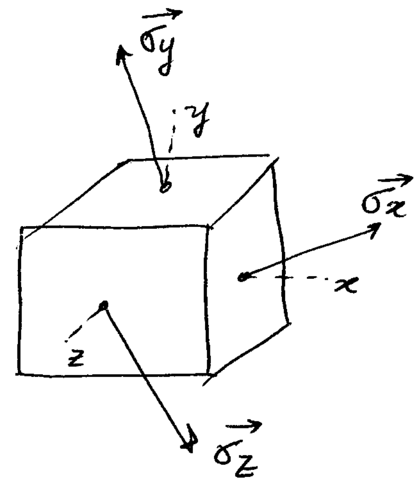
average stress, normal average stress, shear average stress

- STRESS NOTATION

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\sigma_{xy} = \sigma_{yx}$$
$$\sigma_{xz} = \sigma_{zx}$$
$$\sigma_{yz} = \sigma_{zy}$$

Stress vectors $\vec{\sigma_x}$, $\vec{\sigma_y}$, and $\vec{\sigma_z}$ are stress vectors acting on 3 perpendicular planes x, y, z in cartesian coordinate system.



$$\vec{\sigma_x} = \sigma_{xx}\,\hat{i} + \sigma_{xy}\,\hat{j} + \sigma_{xz}\,\hat{k}$$

$$\vec{\sigma_y} = \sigma_{yx}\,\hat{i} + \sigma_{yy}\,\hat{j} + \sigma_{yz}\,\hat{k}$$

$$\vec{\sigma_z} = \sigma_{zx}\,\hat{i} + \sigma_{zy}\,\hat{j} + \sigma_{zz}\,\hat{k}$$

STRESS VECTOR $\vec{\sigma_P}$ ON an arbitrary oblique plane P cuts the volume element into tetrahedron

unit normal to the plane P is

$$\vec{N} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$$

where $\sqrt{l^2 + m^2 + n^2} = 1$

$$\cancel{\vec{\sigma_P} = \vec{N}\cdot\begin{Bmatrix}\vec{\sigma_x}\\ \vec{\sigma_y}\\ \vec{\sigma_z}\end{Bmatrix}} \qquad \vec{\sigma_P} = l\,\vec{\sigma_x} + m\,\vec{\sigma_y} + n\,\vec{\sigma_z}$$

$$\vec{\sigma_P} = \sigma_{Px}\,\hat{i} + \sigma_{Py}\,\hat{j} + \sigma_{Pz}\,\hat{k}$$

$$\vec{\sigma_p} = \begin{Bmatrix} \vec{\sigma_{px}} \\ \vec{\sigma_{py}} \\ \vec{\sigma_{pz}} \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}_{3\times 3} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}_{3\times 1}$$

$$\vec{\sigma_{px}} = \sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n$$

$$\vec{\sigma_{py}} = \sigma_{yx} l + \sigma_{yy} m + \sigma_{yz} n$$

$$\vec{\sigma_{pz}} = \sigma_{zx} l + \sigma_{zy} m + \sigma_{zz} n$$

Normal Stress σ_{PN} on the plane P is the projection of the vector $\vec{\sigma_p}$ in the direction of $\vec{N}$

$$= \{l \quad m \quad n\} \begin{Bmatrix} \vec{\sigma_{px}} \\ \vec{\sigma_{py}} \\ \vec{\sigma_{pz}} \end{Bmatrix} = \vec{N} \cdot \vec{\sigma_p}$$

$$\sigma_{PN} = l^2 \sigma_{xx} + m^2 \sigma_{yy} + n^2 \sigma_{zz} + 2mn\, \sigma_{yz} + 2\, ln\, \sigma_{xz} + 2\, lm\, \sigma_{xy}$$

The max. value of σ_{PN} at a point is important of the infinite planes

$\sigma_{PN} \longrightarrow$ reaches max. value called max. principal stress

$$\sigma_{PS} = \sqrt{\sigma_p^2 - \sigma_{PN}^2} = \sqrt{\sigma_{px}^2 + \sigma_{py}^2 + \sigma_{pz}^2 - \sigma_{PN}^2}$$

STRESS TRANSFORMATION

Direction cosines

	x	y	z
X	l_1	m_1	n_1
Y	l_2	m_2	n_2
Z	l_3	m_3	n_3

Angles are measured from x, y, z to X, Y, Z

$$l_1 = \cos\theta_{xX}$$

$$l_2 = \cos\theta_{xY}$$

Since x, y, z & $X\,Y\,Z$ are orthogonal

$$l_1^2 + l_2^2 + l_3^2 = 1 \qquad l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$$

$$m_1^2 + m_2^2 + m_3^2 = 1 \qquad l_1 n_1 + l_2 n_2 + l_3 n_3 = 0$$

$$n_1^2 + n_2^2 + n_3^2 = 1 \qquad m_1 n_1 + m_2 n_2 + m_3 n_3 = 0$$

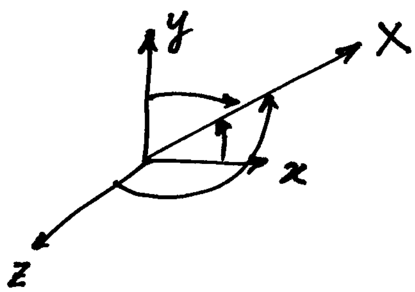
σ_{XX} $\quad$ σ_{xx}

σ_{YY} $\quad$ σ_{yy}

σ_{ZZ} $\quad$ σ_{zz}

Is this better?

Great!



Stress Transformation

Angles that X makes w.r.t. x-y-z

$$l_1 \qquad m_1 \qquad n_1$$

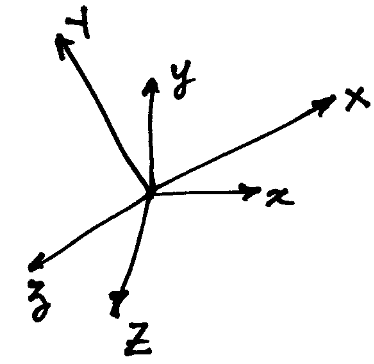
$$\therefore \quad \vec{\sigma_X} = \begin{Bmatrix} \sigma_{Xx} \\ \sigma_{Xy} \\ \sigma_{Xz} \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix}$$

Normal stress in X direction

$$\sigma_{XX} = \begin{bmatrix} l_1 & m_1 & n_1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \cdots & \\ & \ddots & \\ & & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix}$$

$$\sigma_{XY} = \begin{bmatrix} l_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \cdots & \\ & \ddots & \\ & & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix}$$

$$\sigma_{XZ} = \begin{bmatrix} l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \cdots & \\ & \ddots & \\ & & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix}$$



	x	y	z
X	l_1	m_1	n_1
Y	l_2	m_2	n_2
Z	l_3	m_3	n_3

---x---

Normal stress $\vec{\sigma}_{xx}$

takes a max. value $\longrightarrow$ principal stress

& when this happens $\longrightarrow$ the corresponding shear stress $\to 0$

Shear stress vanish on principal planes

---x---x---

$$\sigma_p = \sigma \times \vec{N}$$

$\sigma \to$ magnitude of principal stress

Assume $\vec{N} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$ relative to x-y-z

$$\therefore \sigma_{px} = l \cdot \sigma$$

$$\sigma_{py} = m \cdot \sigma$$

$$\sigma_{pz} = n \cdot \sigma$$

Great!

$$\vec{\sigma_p} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

$$\vec{\sigma}_p = \begin{bmatrix} \ell\,\sigma_{xx} + m\,\sigma_{xy} + n\,\sigma_{xz} \\ \ell\,\sigma_{yx} + m\,\sigma_{yy} + n\,\sigma_{yz} \\ \ell\,\sigma_{zx} + m\,\sigma_{zy} + n\,\sigma_{zz} \end{bmatrix} = \begin{Bmatrix} \ell\,\sigma \\ m\,\sigma \\ n\,\sigma \end{Bmatrix}$$

$$\therefore \quad \begin{aligned} \ell(\sigma_{xx} - \sigma) + m\,\sigma_{xy} + n\,\sigma_{xz} &= 0 \\ \ell\,\sigma_{yx} + m(\sigma_{yy} - \sigma) + n\,\sigma_{yz} &= 0 \\ \ell\,\sigma_{zx} + m\,\sigma_{zy} + n(\sigma_{zz} - \sigma) &= 0 \end{aligned}$$

$$\therefore \quad \begin{vmatrix} \cancel{\ell}(\sigma_{xx} - \sigma) & \cancel{m}\,\sigma_{xy} & \cancel{n}\,\sigma_{xz} \\ \sigma_{yx} & (\sigma_{yy} - \sigma) & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & (\sigma_{zz} - \sigma) \end{vmatrix} = 0$$

for non-zero ℓ, m, n

$$\therefore \quad \boxed{\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0}$$

cubic equation to solve for 3 roots
each will be principal stresses

$$\sigma_1 , \ \sigma_2 , \ \sigma_3$$

where, I_1, I_2, and I_3 are the stress invariants.

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix}$$

$$= \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix}$$

- Three roots $\sigma_1, \sigma_2, \sigma_3$

  Eigenvalue problem

  I_1, I_2, I_3

- In terms of principal stress

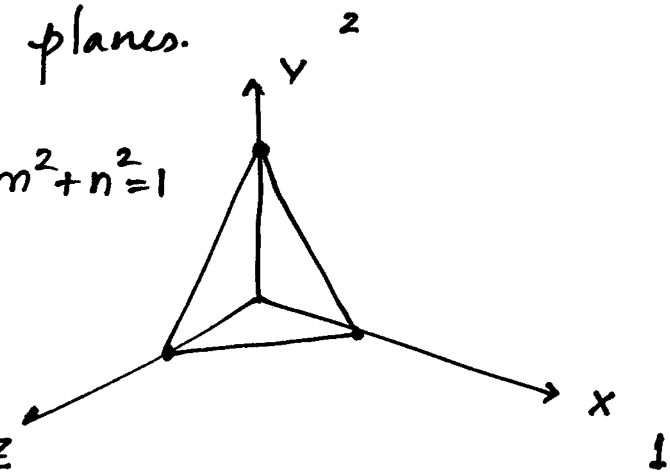
$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

OCTAHEDRAL STRESS

X, Y & Z are principal planes.



family of planes with $l^2+m^2+n^2=1$

& $l^2=m^2=n^2=1/3$

eight such planes

$\sigma_{oct} \rightarrow$ octahedral normal stress

$\tau_{oct} \rightarrow$ octahedral shear stress.

$$\sigma_{oct} = \frac{I_1}{3} = \frac{\sigma_1+\sigma_2+\sigma_3}{3}$$

$$\tau_{oct} = \sqrt{\frac{2}{9}I_1^2 - \frac{2}{3}I_2} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

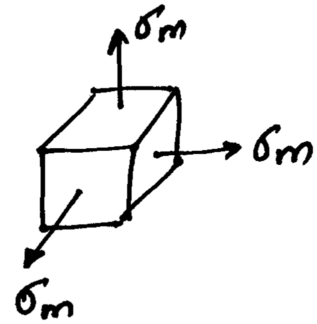
MEAN & DEVIATORIC STRESS

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_{xx}-\sigma_m & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}-\sigma_m & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}-\sigma_m \end{bmatrix}$$

where $\sigma_m = \dfrac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} = \dfrac{I_1}{3}$

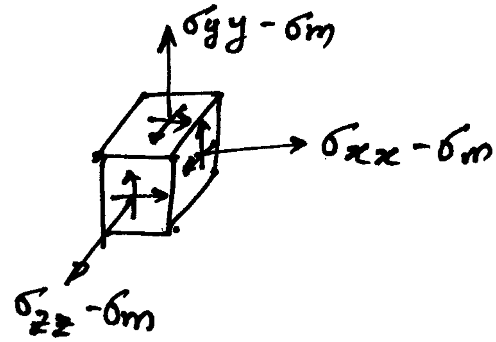
$\sigma_m \rightarrow$ hydrostatic pressure stress

$[\sigma]_D$ deviatoric stress tensor



hydrostatic stress

+



deviatoric stress

change in volume

no change in volume
only distortion of shape.

> Materials react differently to hydrostatic stress.

metals are typically hydrostatic stress independent
soils, concrete & other materials that can be compacted
are typically hydrostatic stress ~~in~~dependepent

> Plasticity & Fracture.

> Deviatoric Stress Tensor → central role in plasticity for metals.

$$J_1 = 0$$

$$J_2 = I_2 - \frac{1}{3} I_1^2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$J_3 = I_3 - \frac{1}{3} I_1 J_2 + \frac{2}{27} I_1^3$$

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CE592 - PLASTIC DESIGN

- PRINCIPAL STRESS
- STRESS INVARIANTS

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \;\cancel{/\,3}$$

$$I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1$$

$$I_3 = \sigma_1 \sigma_2 \sigma_3$$

} $\sigma_1, \sigma_2, \sigma_3$ are principal stresses.

$$[\sigma] = [\sigma]_m + [\sigma]_D$$

$$\sigma_m = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \rightarrow \text{hydrostatic stress}$$

$[\sigma]_D \rightarrow$ 3 stress invariants

$$J_1 = 0$$

$$J_2 = -\frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

$$J_3 = I_3 - \frac{1}{3} I_1 J_2 + \frac{2}{27} I_1^3$$

principal directions are the same

& Principal values

$$S_1 = \sigma_1 - \sigma_m = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3}$$

$$S_2 = \sigma_2 - \sigma_m \quad \vdots$$

$$S_3 = \sigma_3 - \sigma_m \quad \downarrow$$

2 D Plane Stress Transformation

$$\sigma_{XX} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\,\sigma_{xy}\sin\theta\cos\theta$$

$$\sigma_{YY} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\,\sigma_{xy}\sin\theta\cos\theta$$

$$\sigma_{XY} = -(\sigma_{xx}-\sigma_{yy})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$$

for principal directions $\tan 2\theta_p = \dfrac{2\,\tau_{xy}}{\sigma_x - \sigma_y} = \dfrac{2\,\sigma_{xy}}{\sigma_x - \sigma_y}$

— x —

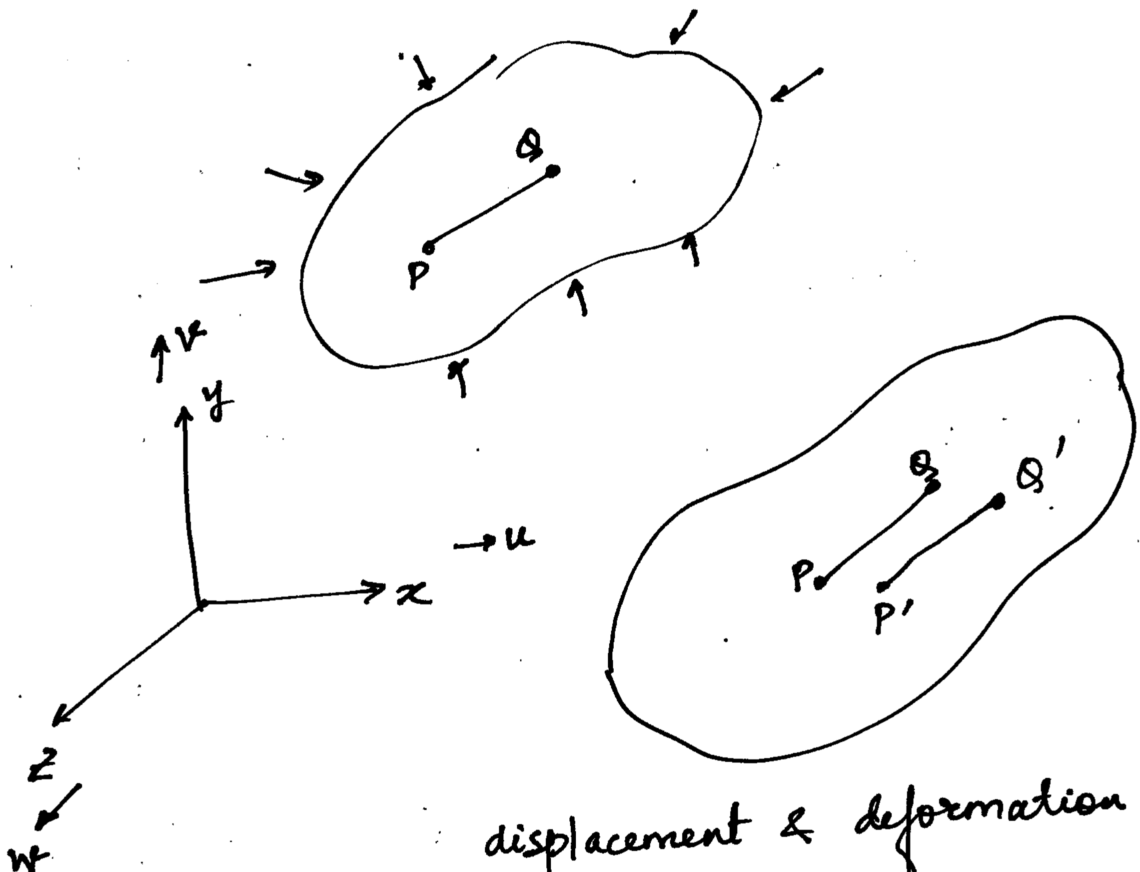
Differential Equations of Deformable Body

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z} + B_x = 0$$

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} + B_y = 0$$

$$\frac{\partial\sigma_{xz}}{\partial z} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + B_z = 0$$

Strains



displacement & deformation

$u(x, y, z)$

$v(x, y, z)$

$w(x, y, z)$.

3 NORMAL STRAINS

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

3 SHEAR STRAINS

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

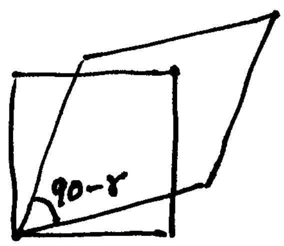
$$\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$ } need to satisfy Strain compatibility relations $\rightarrow$ 6 of them

engineering strain $\gamma_{xy} = 2\,\varepsilon_{xy}$

$\gamma_{xz} = 2\,\varepsilon_{xz}$

$\gamma_{yz} = 2\,\varepsilon_{yz}$



$\gamma \rightarrow$ engineering shear strain

All the discussions related to stresses now apply to strains with the modification above.

— x —

STRESS - STRAIN RELATIONSHIPS.

isotropic, homogenous elastic material

2 material constants $\rightarrow$ Elastic modulus E

$\rightarrow$ Poisson ratio ν

$$
\left.
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\,\sigma_{yy} - \nu\,\sigma_{zz}\right) \\
\varepsilon_{yy} &= \frac{1}{E}\left(\sigma_{yy} - \nu\,\sigma_{xx} - \nu\,\sigma_{zz}\right) \\
\varepsilon_{zz} &= \frac{1}{E}\left(\sigma_{zz} - \nu\,\sigma_{xx} - \nu\,\sigma_{yy}\right) \\
\varepsilon_{xy} &= \frac{\sigma_{xy}}{2G} = \frac{1+\nu}{E}\,\sigma_{xy} \\
\varepsilon_{yz} &= \frac{\sigma_{yz}}{2G} = \frac{1+\nu}{E}\,\sigma_{yz} \\
\varepsilon_{xz} &= \frac{\sigma_{xz}}{2G} = \frac{1+\nu}{E}\,\sigma_{xz}
\end{aligned}
\right\}
$$

6 stress-strain relationships

Inverse these relationship

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\varepsilon_{xx} + \nu\,\varepsilon_{yy} + \nu\,\varepsilon_{zz}\right]$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\varepsilon_{yy} + \nu\,\varepsilon_{xx} + \nu\,\varepsilon_{zz}\right]$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\,\varepsilon_{xx} + \nu\,\varepsilon_{yy} + (1-\nu)\,\varepsilon_{zz}\right]$$

$\sigma_{xy} = \frac{E}{1+\nu}\cdot\varepsilon_{xy}$ | $\sigma_{xz} = \frac{E}{1+\nu}\cdot\varepsilon_{xz}$ | $\sigma_{yz} = \frac{E}{1+\nu}\,\varepsilon_{yz}$

Example 1

drive shaft coupling has stress components relative to x, y, z axes as:

$\sigma_{xx} = 80$ MPa $\qquad \sigma_{xy} = 20$ MPa

$\sigma_{yy} = 60$ MPa $\qquad \sigma_{xz} = 40$ MPa

$\sigma_{zz} = 20$ MPa $\qquad \sigma_{yz} = 10$ MPa

$$\begin{bmatrix} 80 & 20 & 40 \\ 20 & 60 & 10 \\ 40 & 10 & 20 \end{bmatrix}$$

(a) determine the stress vector on a plane normal to the vector $\vec{R} = \hat{i} + 2\hat{j} + \hat{k}$

$$|\vec{R}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\therefore \vec{R} = \sqrt{6}\left[\frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}\right]$$

$\therefore$ direction cosines $l = 1/\sqrt{6}$

$m = 2/\sqrt{6}$

$n = 1/\sqrt{6}$

$$\vec{\sigma}_R = \begin{Bmatrix} \vec{\sigma}_{Rx} \\ \vec{\sigma}_{Ry} \\ \vec{\sigma}_{Rz} \end{Bmatrix} = \begin{bmatrix} 80 & 20 & 40 \\ 20 & 60 & 10 \\ 40 & 10 & 20 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{Bmatrix}$$

$$\vec{\sigma}_R = \begin{bmatrix} 160/\sqrt{6} \\ 150/\sqrt{6} \\ 80/\sqrt{6} \end{bmatrix} = \begin{Bmatrix} 65.32 \text{ MPa} \\ 61.23 \text{ MPa} \\ 32.66 \text{ MPa} \end{Bmatrix} = 65.32\,\hat{i} + 61.23\,\hat{j} + 32.66\,\hat{k}$$

(b) Determine Stress Invariants

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 160 \text{ MPa}$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix}$$

$$= 5500 \text{ MPa}^2$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} = 0$$

(c) Determine principal stresses

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = \sigma^3 - 160\sigma^2 + 5500\sigma = 0$$

$$\therefore \sigma_1 = 110 \text{ MPa}$$
$$\sigma_2 = 50 \text{ MPa}$$
$$\sigma_3 = 0$$

(d) max. shear stress $= \tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 55 \text{ MPa}$

(e) $\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2} = 44.97 \text{ MPa}$

$$\tau_{max} = 1.223\, \tau_{oct}$$

Strain Compatibility Relations

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = 2\,\frac{\partial^2 \varepsilon_{xy}}{\partial x\,\partial y}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\,\frac{\partial^2 \varepsilon_{xz}}{\partial x\,\partial z}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} = 2\,\frac{\partial^2 \varepsilon_{yz}}{\partial y\,dz}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x\,\partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial z^2} = \frac{\partial^2 \varepsilon_{yz}}{\partial z\,\partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial y\,\partial z}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x\,\partial z} + \frac{\partial^2 \varepsilon_{xz}}{\partial y^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial y\,\partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial x\,\partial y}$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y\,\partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xz}}{\partial x\,\partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial x\,\partial z}$$

STRESS- STRAIN RELATIONS

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}$$

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}\right)$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}\right)$$

$$\varepsilon_{xy} = \frac{1}{2G}\sigma_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$

$$\varepsilon_{yz} = \frac{1}{2G}\sigma_{yz} = \frac{1+\nu}{E}\sigma_{yz}$$

$$\varepsilon_{zx} = \frac{1}{2G}\sigma_{zx} = \frac{1+\nu}{E}\sigma_{zx}$$

Inverse the relations

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right]$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\right]$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right]$$

$$\sigma_{xy} = \frac{E}{1+\nu}\cdot\varepsilon_{xy} \qquad \sigma_{xz} = \frac{E}{1+\nu}\cdot\varepsilon_{xz}$$

$$\sigma_{yz} = \frac{E}{1+\nu}\cdot\varepsilon_{yz}$$

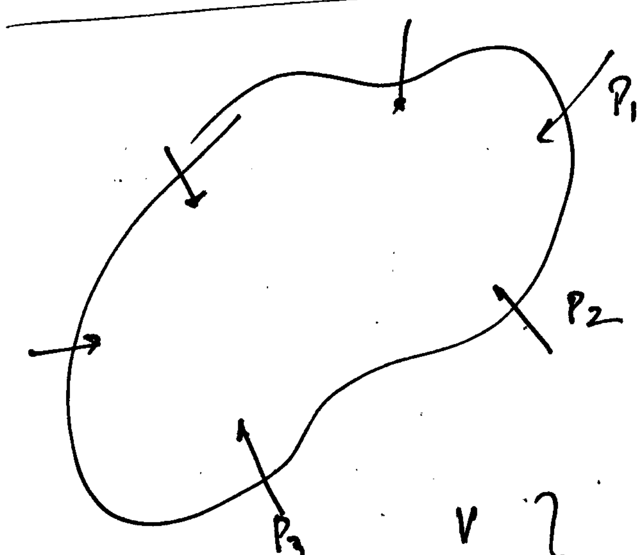
STRESS- STRAIN RELATIONSHIP

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}_{6\times 1} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{16} \\ C_{21} & C_{22} & \cdots & C_{26} \\ & \ddots & & \\ & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}_{6\times 1}$$

$C_{ij} \rightarrow f(E, \nu)$

$E \rightarrow$ elastic modulus

$\nu \rightarrow$ Poisson ratio



$\longrightarrow$ $u(x,y,z)$

$v(x,y,z)$

$w(x,y,z)$

diplacements of the body

$$\left.\begin{matrix} v \\ u \\ w \end{matrix}\right\} \longrightarrow \varepsilon_{ij} \xrightarrow[\substack{\text{elasticity} \\ E,\ \nu}]{} \sigma_{ij} \longrightarrow \left.\begin{matrix} \sigma_{p1} \\ \sigma_{p2} \\ \sigma_{p3} \end{matrix}\right\} \text{stress invariants}$$

limit to its elasticity

Elastic Behavior:

Plane Stress

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

& $\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$ but $\sigma_z = 0$

Plane Strain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1+2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

& $\varepsilon_z = 0$ but $\sigma_z = \nu(\sigma_x + \sigma_y)$

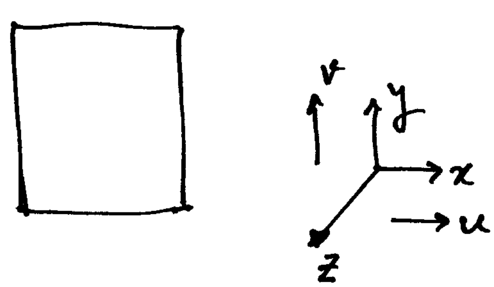
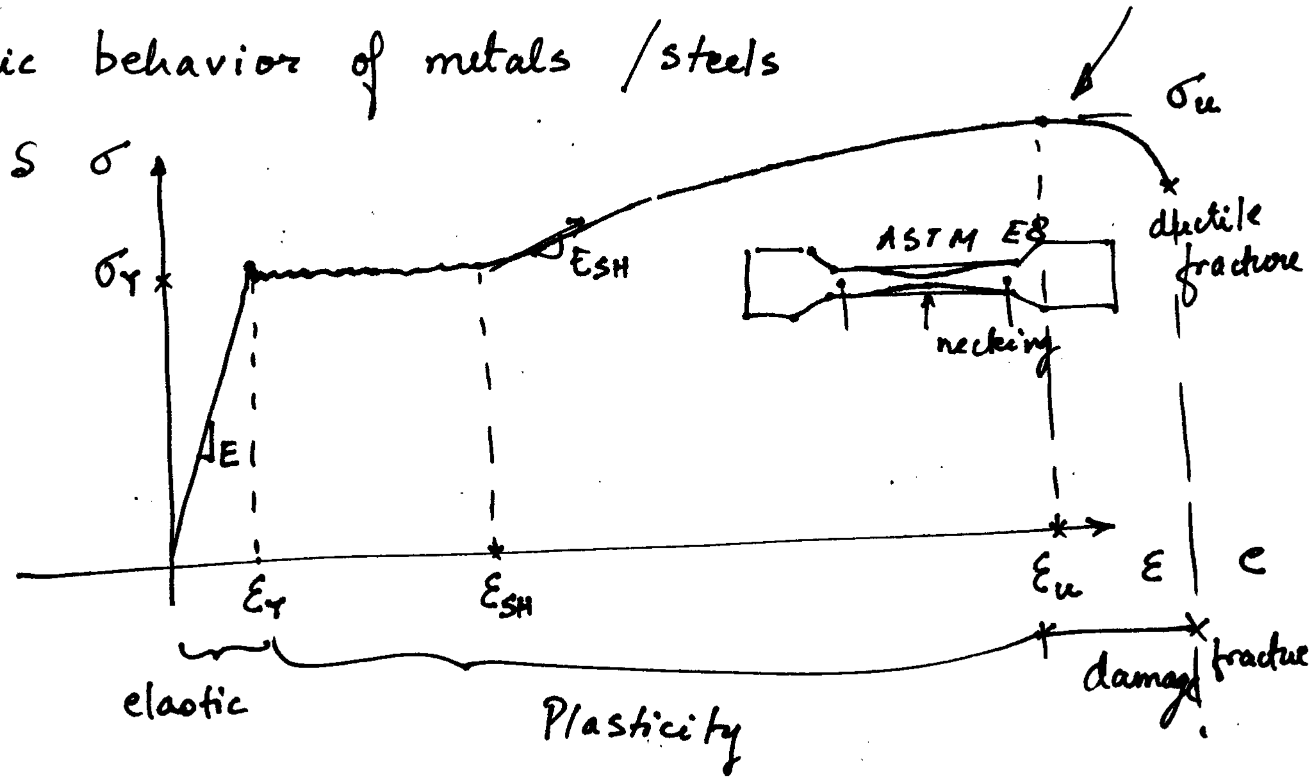


plate bending is mathematically a plane-stress problem.

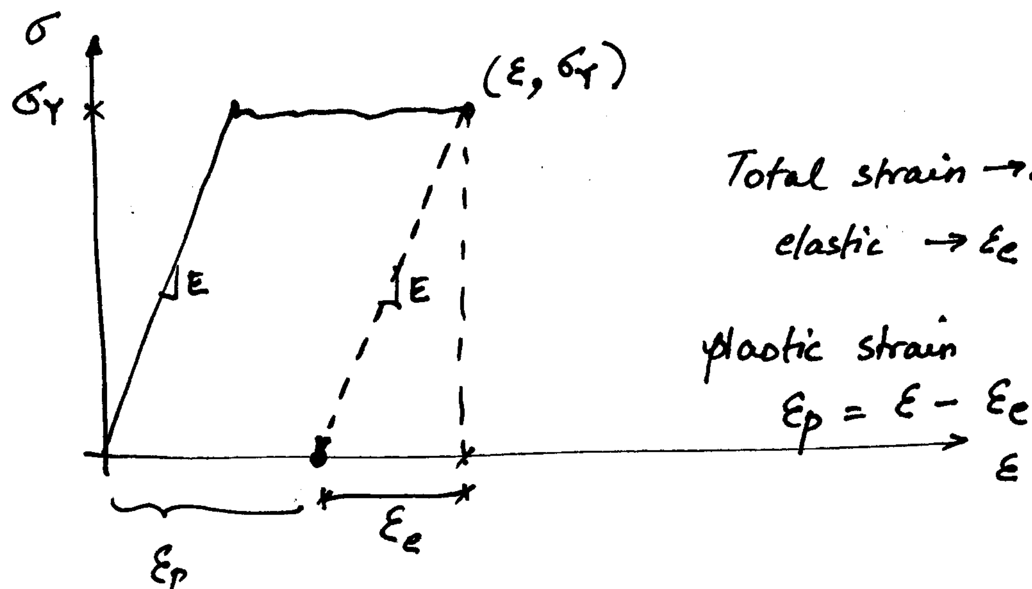
generalized plane stress

INELASTIC BEHAVIOR OF METALS.

Monotonic behavior of metals / steels



loading reversal.



Total strain → ε

elastic → $\varepsilon_e = \dfrac{\sigma_Y}{E}$

plastic strain

$\varepsilon_p = \varepsilon - \varepsilon_e$

plastic strain → irrecoverable strain

$\varepsilon_Y \rightarrow 0.002$

$\varepsilon_{SH} \rightarrow ? \approx 10-12\ \varepsilon_Y \quad 0.02$

$\varepsilon_u \rightarrow ?\ 60-100\ \varepsilon_Y \quad 0.2$

} plastic behavior range.

Plastic Design

$\varepsilon \rightarrow \dfrac{\text{change in length}}{\text{original length}} = \dfrac{\Delta L}{L_0}$

engineering strain $\rightarrow e$

True strain $= \dfrac{\text{change in length}}{\text{instantaneous length.}} = \dfrac{dl}{l} = \varepsilon$

$\varepsilon = \ln(1+e)$

true strain → engineering strain

stress engineering $= \dfrac{\text{Force}}{\text{Original area}} = \dfrac{F}{A_0} = S$

true stress $= \sigma = S(1+e)$

$\sigma - \varepsilon$ true stress-strain

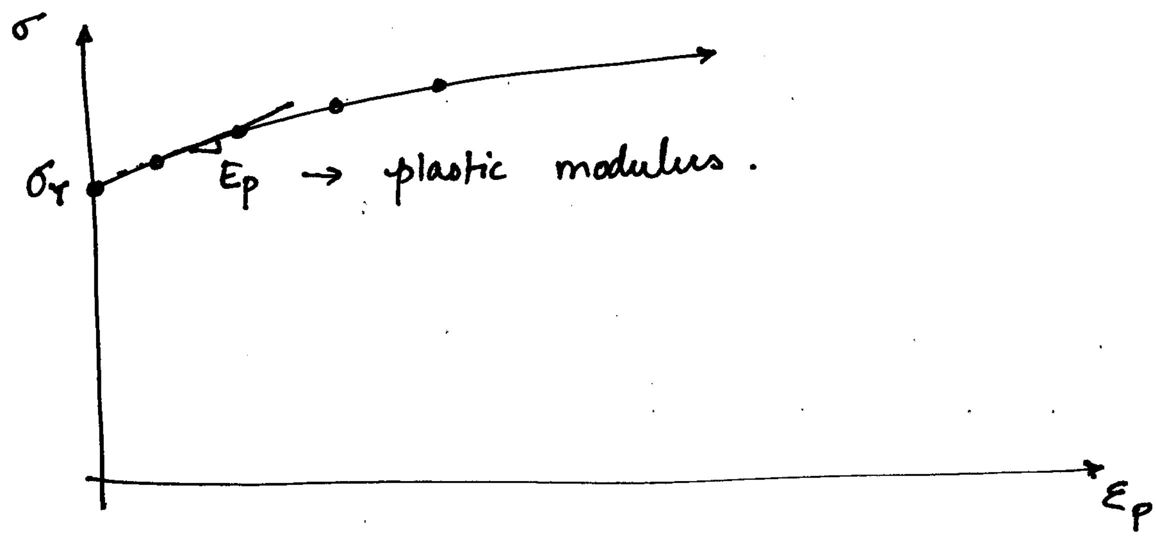
Engineering Stress-Strain Curve $(S-e)$ curve.

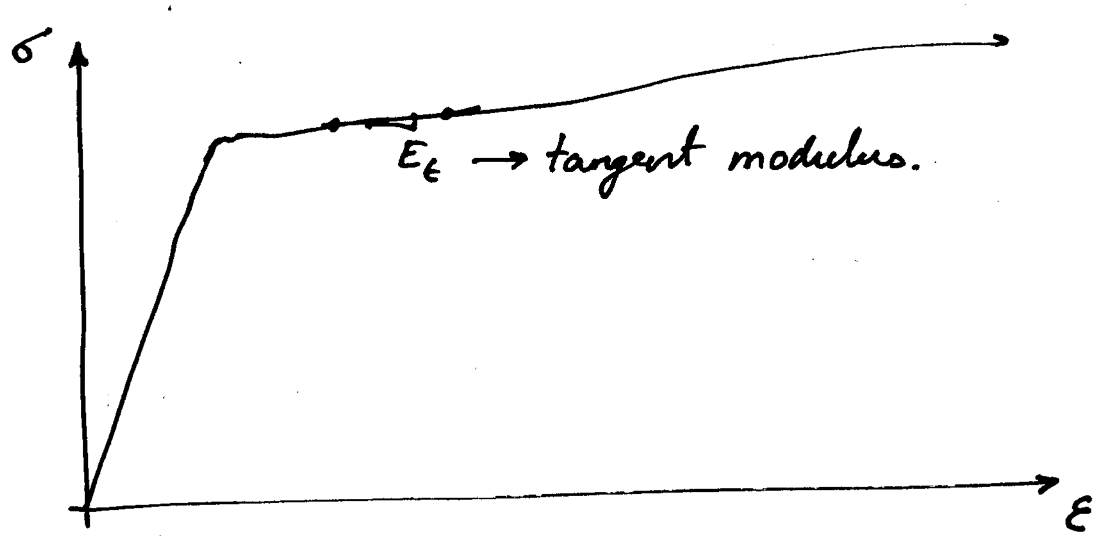
$$\left.\begin{aligned} \varepsilon &= \ln(1+e) \\ \sigma &= S(1+e) \end{aligned}\right\} \longrightarrow \text{True stress-strain curve.}$$

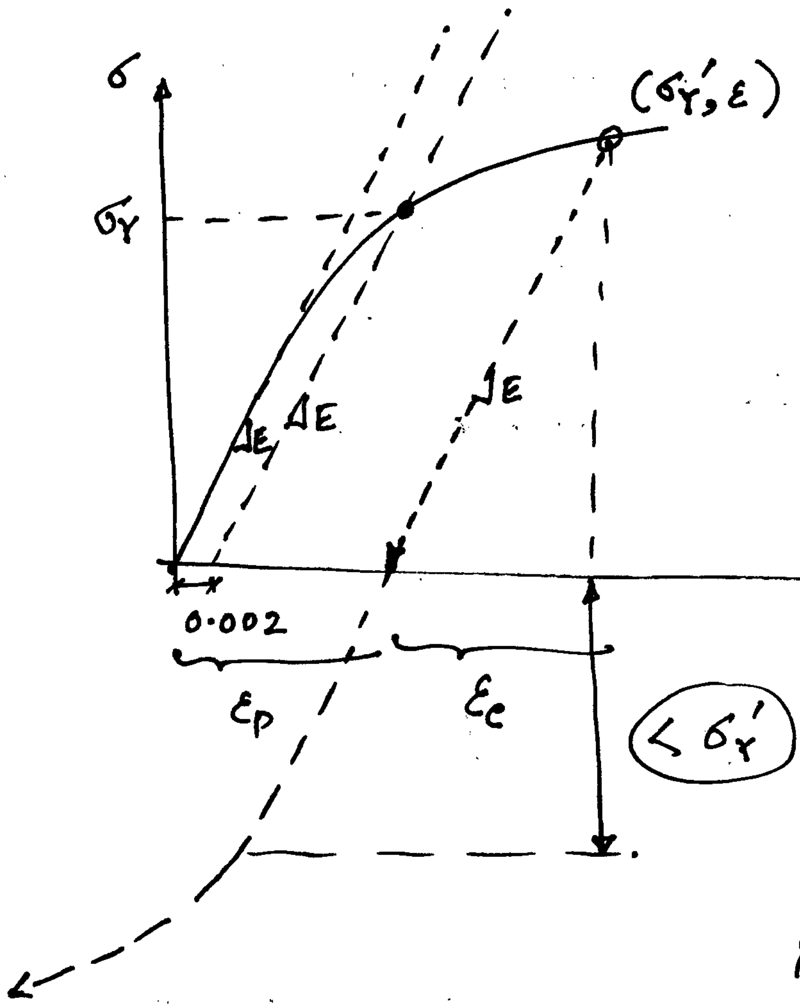
stop when necking begins.

$\sigma - \varepsilon_p$ curve

$$\varepsilon_p = \varepsilon - \frac{\sigma}{E} = \varepsilon - \varepsilon_e$$







reversal of loadi

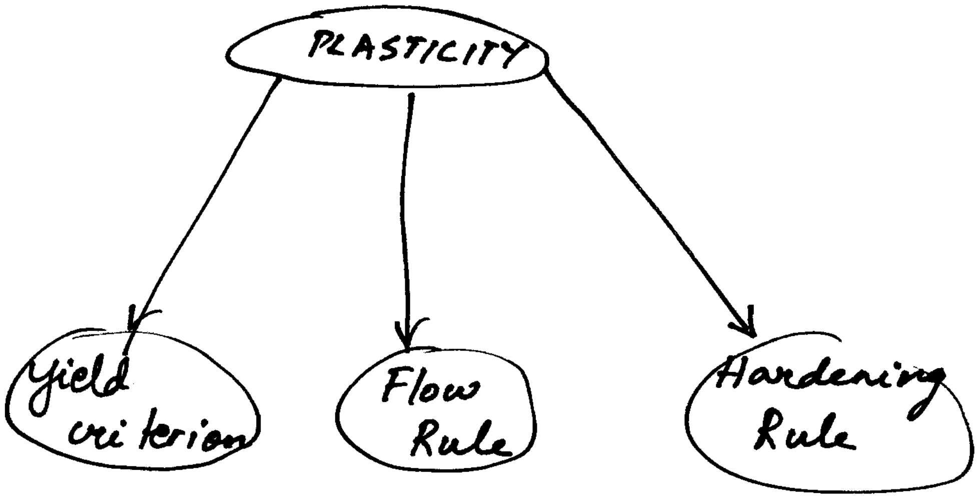
yields faster on stress reversal

Bauschinger's effects.

Monotonic → Cyclic

hardening is important but not complicated

hardening becomes complicated due to Bauschinger's effect uniaxial behavior $\sigma - \varepsilon$

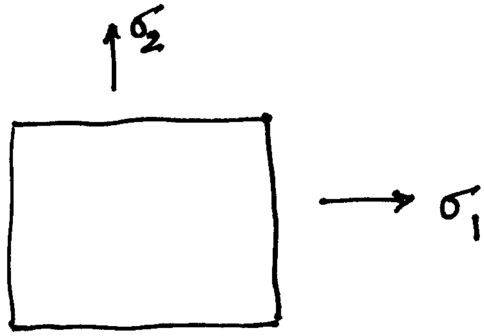


YIELD CRITERION:

for metals.

- $S = \dfrac{T}{A_o}$

$\sigma = \dfrac{\text{Applied force}}{\text{Instantaneous Area}}$



Yielding?

$\sigma_1 \rightarrow F_y$?

$\sigma_2 \rightarrow F_y$?

$f(\sigma_1, \sigma_2)$

TRESCA – 1864

yielding will occur when τ_{max} reaches a critical value.

k

σ_1, σ_2, σ_3 are the principal stresses

$$\tau_{max} = \mathrm{Max}\left(\frac{1}{2}|\sigma_1 - \sigma_2| \,,\ \frac{1}{2}|\sigma_2 - \sigma_3| \,,\ \frac{1}{2}|\sigma_3 - \sigma_1| \right) = k$$

If this is true, then it will be valid for uniaxial tension test

$$\sigma_1 = \sigma_Y \qquad \sigma_2 = 0 \qquad \sigma_3 = 0$$

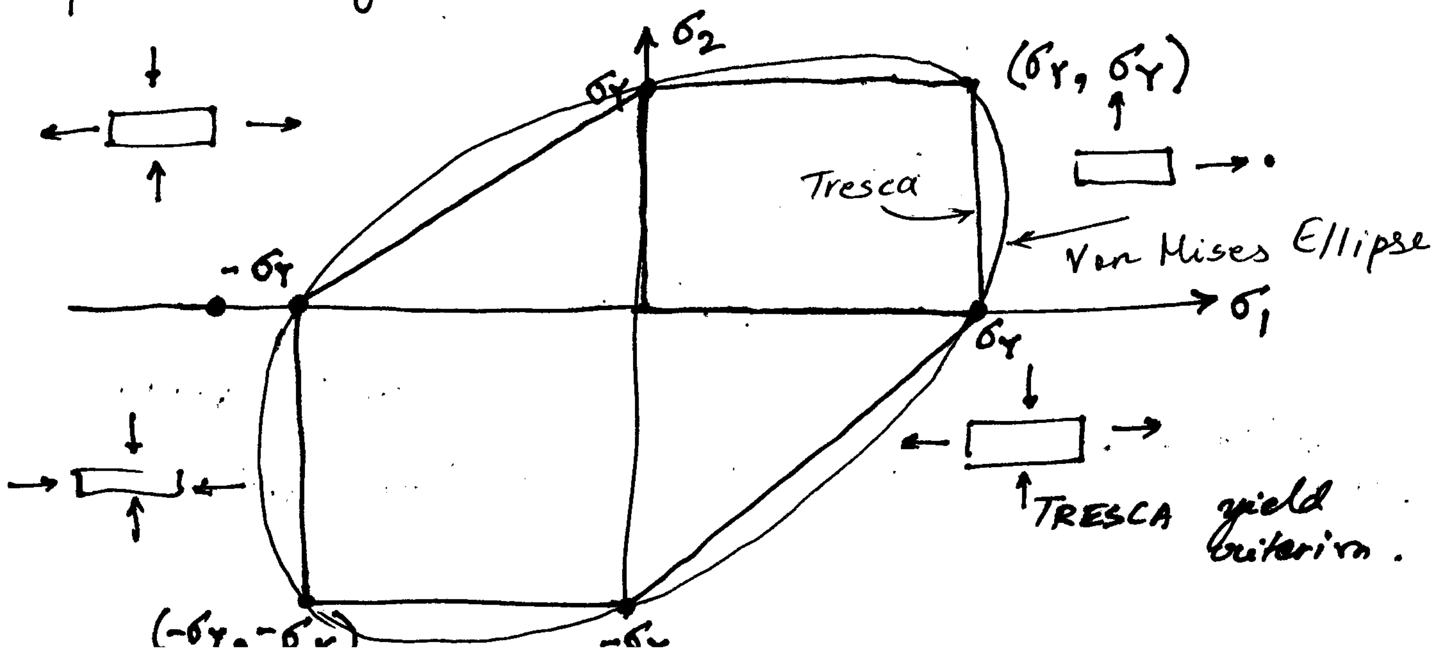
$$\therefore \tau_{max} = \frac{\sigma_Y}{2}$$

$$\therefore k \rightarrow \frac{\sigma_Y}{2}$$

Tresca criterion is $\tau_{max} = \dfrac{\sigma_Y}{2}$

or $\left(\frac{1}{2}|\sigma_1 - \sigma_2| \,,\ \frac{1}{2}|\sigma_2 - \sigma_3| \,,\ \frac{1}{2}|\sigma_3 - \sigma_1| \right)_{max} = \dfrac{\sigma_Y}{2}$

plot this yield criterion in 2D σ- space.



Assume $\sigma_1 > \sigma_2 > \sigma_3$

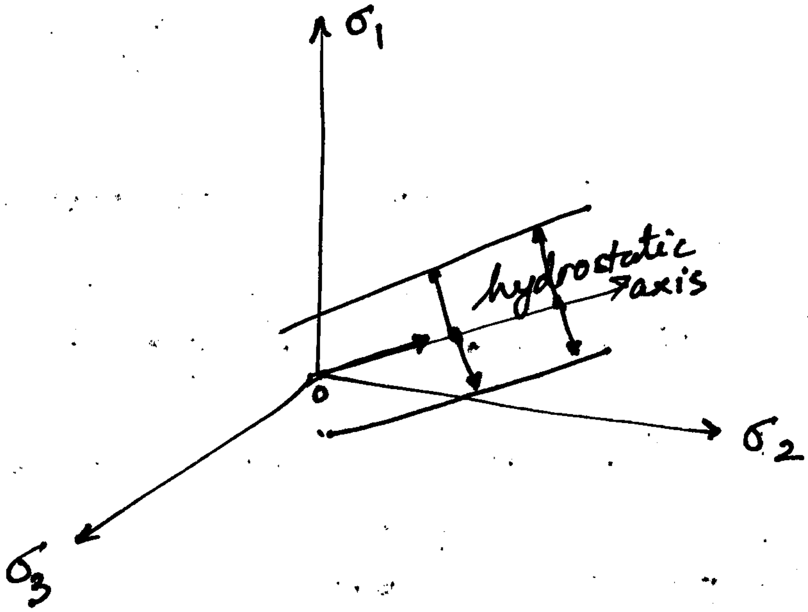
max. shear stress $= \frac{1}{2}(\sigma_1 - \sigma_3)$

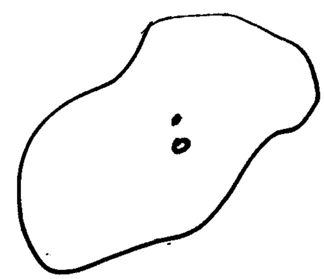
$$= \frac{1}{\sqrt{3}} J_2 \left[\cos\theta - \cos\left(\theta + \frac{2\pi}{3}\right)\right] = k$$

I_1 ? I_3 ? J_3 ? σ_2 $\qquad 0 \leqslant \theta \leqslant 60^\circ$

TRESCA YIELD SURFACE

$$f(J_2, \theta) = 2\sqrt{J_2}\,\sin\left(\theta + \frac{\pi}{3}\right) - \sigma_Y = 0 \qquad 0 \leqslant \theta \leqslant 60^\circ$$

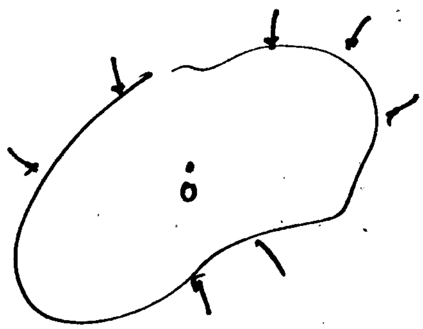




$$\sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}.$$

Von Mises (1913)

→ Maximum Strain Energy of Distortion Yield Criterion.



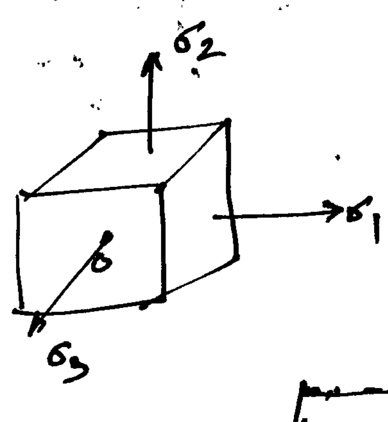
$O \rightarrow [\sigma] \quad [\varepsilon]$

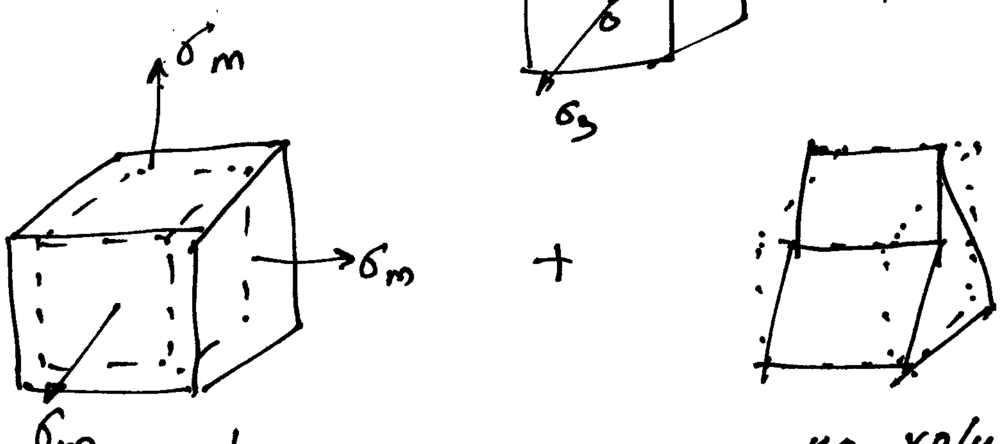
The strain energy per unit volume $= \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$

where $\left.\begin{matrix}\sigma_1, \sigma_2, \sigma_3 \\ \varepsilon_1, \varepsilon_2, \varepsilon_3\end{matrix}\right\}$ are principal values.

Strain energy associated with volume change
associated with distortion of volume.

$$\sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$





volume change

no volume change.

$$U_o = \frac{(\sigma_1+\sigma_2+\sigma_3)^2}{18K} + \frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{12G}$$

$$= \frac{1-2\nu}{6E} I_1^2 + \frac{1+\nu}{E} J_2$$

Strain energy due to volume dilation

strain energy due to distortion.

K = bulk modulus

$$= \frac{E}{3(1-2\nu)}$$

G = shear modulus

$$= \frac{E}{2(1+\nu)}$$

Yielding will occur when strain energy due to distortion achieves critical value.

$$U_{DO} = \frac{1}{12G} \times \left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

For simple tension test

$$\sigma_1 = \sigma_Y \;;\; \sigma_2 = 0 \;;\; \sigma_3 = 0$$

$$\therefore U_{DO} = \frac{\sigma_Y^2}{6G}$$

$$\therefore \frac{1}{\cancel{12G}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] = \frac{\sigma_Y^2}{\cancel{6G}} \times 2$$

Von Mises yield criterion

$$U_{DO} = \frac{1+\nu}{E} J_2 = \frac{\sigma_Y^2}{6G}$$

J_2 dependent yield criterion only

$$\frac{\cancel{1+\nu}}{\cancel{E}} \cdot J_2 = \frac{\sigma_Y^2 \times \cancel{2}(\cancel{1+\nu})}{3\cancel{6} \times \cancel{E}}.$$

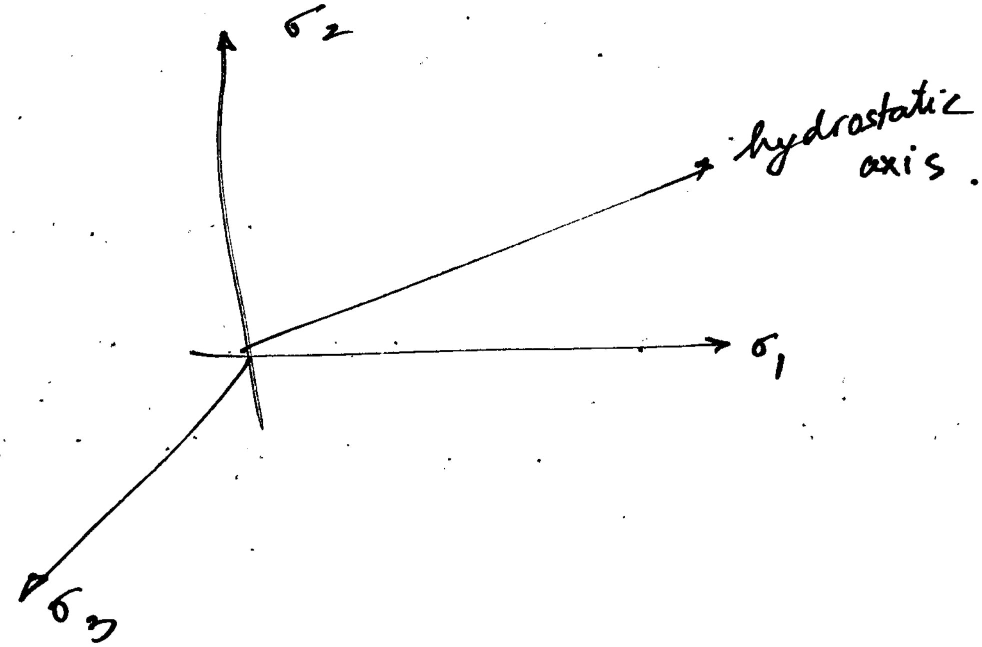
$$J_2 = \frac{\sigma_Y^2}{3}$$ yielding in material

Von Mises Yield Criterion.

hypothesis $U_{DO} \longrightarrow k$

calibrating the yield criterion.

I_1 independent



cylinder in $\sigma_1, \sigma_2, \sigma_3$ space.

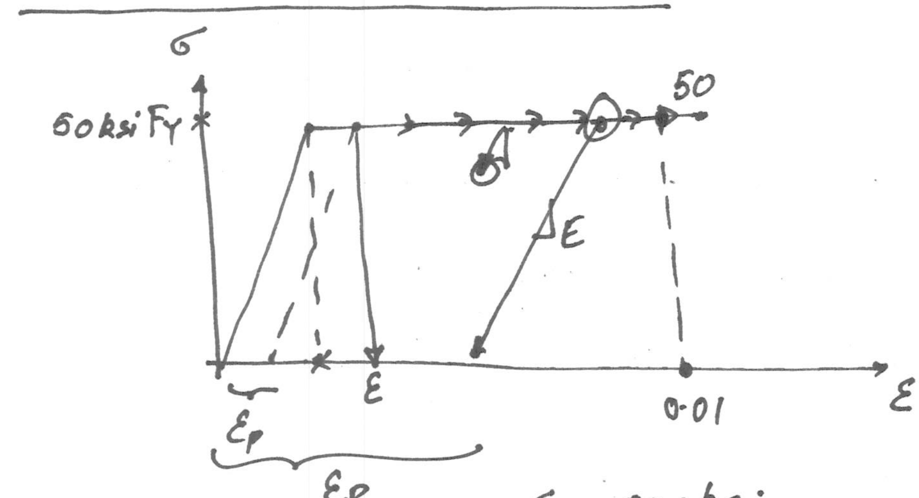
circle in π-plane (deviatoric plane)

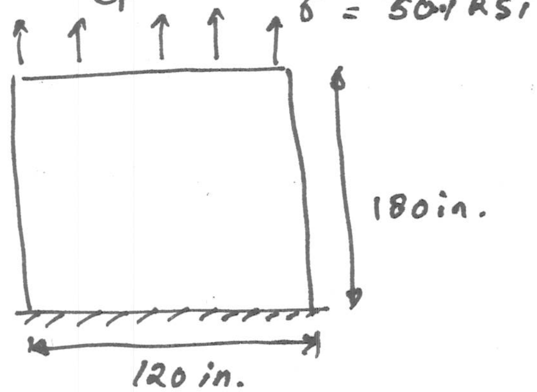
ellipse $\sigma_1 - \sigma_2$ space.

Equation of ellipse

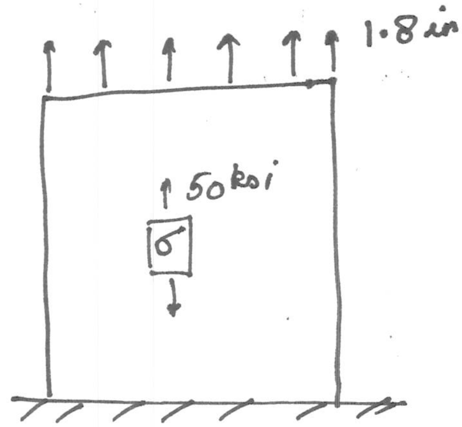
$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = \sigma_Y^2$$

PERFECTLY PLASTIC MATERIALS





The magnitude of plastic strain cannot be uniquely determined by the given state of stresses σ_{ij} and the increment $d\sigma_{ij}$



$$\varepsilon = \frac{1.8}{180} = 0.01$$

For given current stresses σ_{ij} and plastic strain increment $d\varepsilon_{ij}^{p}$, then the corresponding incremental stress state $d\sigma_{ij}$

Yield Criterion:

elastic behavior ε_{ij} σ_{ij} C_{ijkl}

stress invariants I_1, J_2 . . . .

deviatoric stresses → max. distortion strain energy.

Von Mises yield criteria

$$J_2 - \frac{\sigma_Y^2}{3} = 0 \rightarrow \text{for yielding to occur.}$$

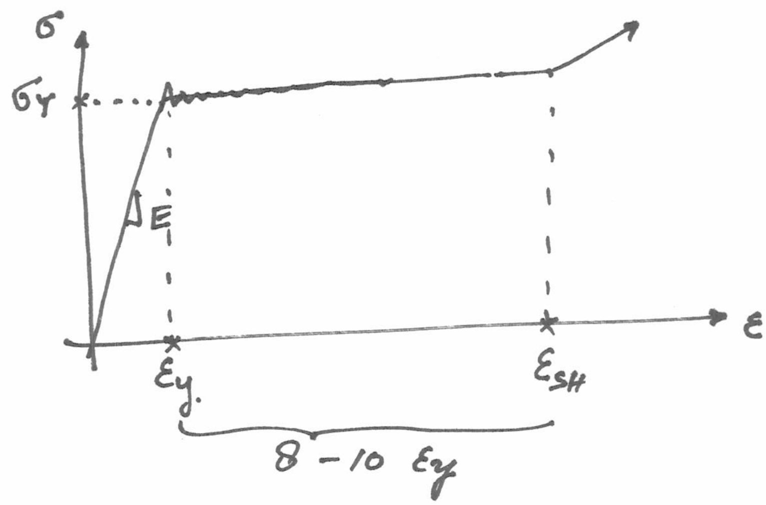
σ_Y → uniaxial yield stress

J_2 → 2nd invariant of deviatoric stress tensor.

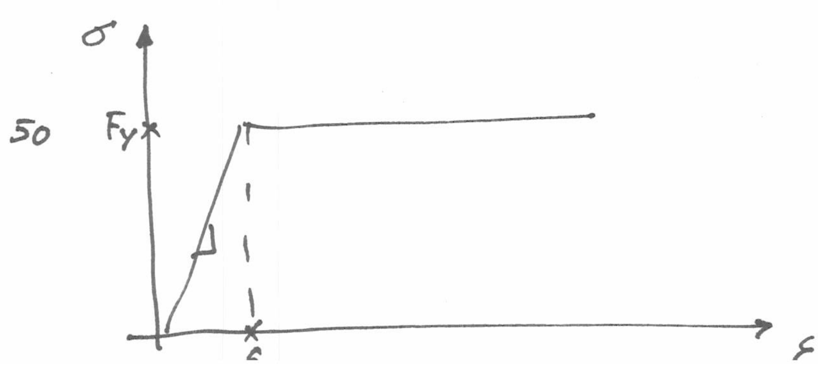
What happens after yielding occurs?

— × —

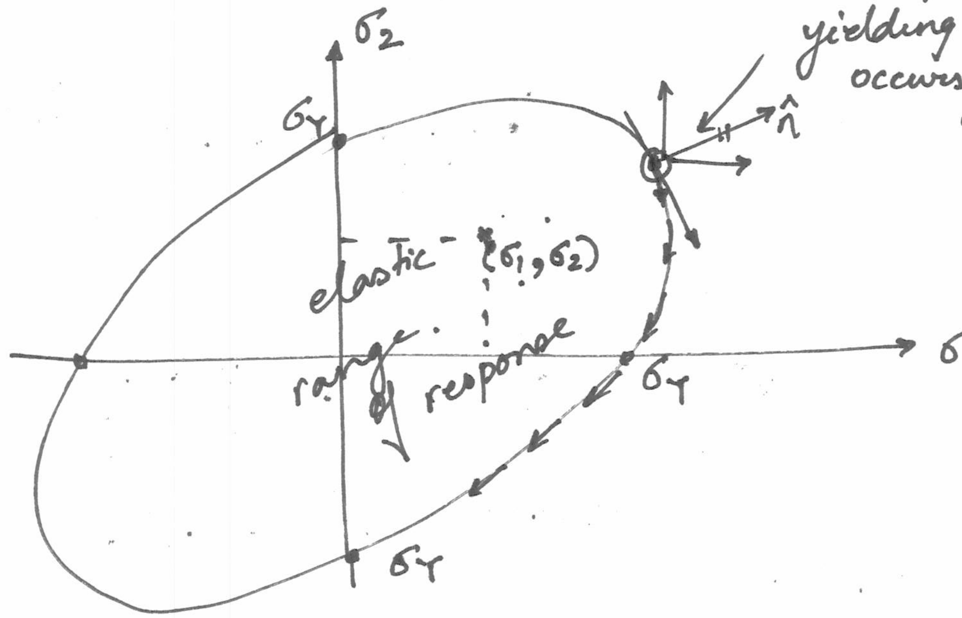
uniaxial case:



can be idealized as elastic - perfectly plastic



Consider a 2D problem:



$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = \sigma_Y^2$$

How will determine the corresponding $d\varepsilon_{ij}^p$ while plasticity is occurring

the stress-state σ_{ij} will lie on the yield criterion

$$f(\sigma_{ij}) = F(\sigma_{ij}) - K = 0$$

← yield surface in σ_{ij} space

$$= J_2 - \frac{\sigma_Y^2}{3} = 0$$

The new stress states $\rightarrow f(\sigma_{ij}) = 0$

consistency condition.

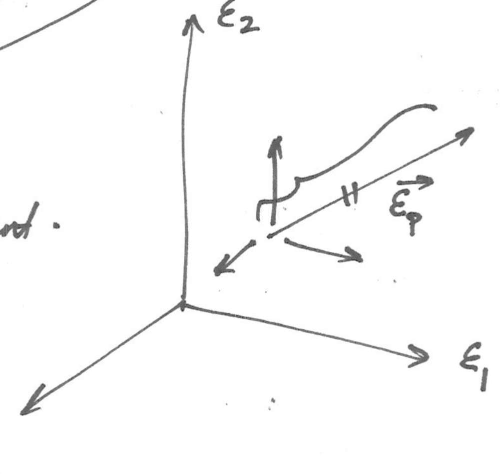
$$d\varepsilon_{ij}^p \rightarrow \begin{Bmatrix} d\varepsilon_{11}^p \\ d\varepsilon_{22}^p \\ d\varepsilon_{33}^p \\ d\varepsilon_{12}^p \\ d\varepsilon_{23}^p \\ d\varepsilon_{31}^p \end{Bmatrix} \longrightarrow \text{six components}$$

The incremental $d\varepsilon_{ij}^{p}$ may be represented geometrically by a vector with nine components in strain space or 3 components in principal strain space.

There are 2 things:

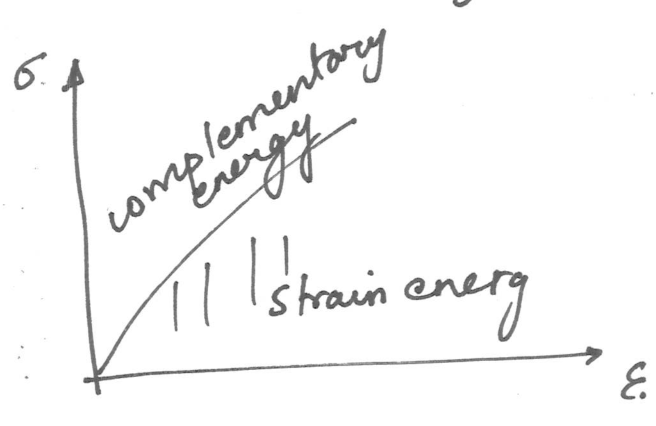
① magnitude of the plastic strain → lumped plastic strain

② direction of the plastic strain → relative ratios of the plastic strain increment.



What is elastic strain?

→ derived directly by differentiating the elastic potential function or complementary energy density function w.r.t. σ_{ij}



In 1928, von Mises proposed a similar concept.

Plastic potential function, which is a scalar function of σ_{ij}.

$$g(\sigma_{ij})$$

The plastic flow equation can be written as

$$d\varepsilon_{ij}^{p} = \underbrace{d\lambda}_{\text{scalar}} \underbrace{\frac{\partial g}{\partial \sigma_{ij}}}_{\text{vector}}$$

vector. what are its relative components.

- Assume plastic flow will occur ~~in~~ so as to cause maximum dissipation of plastic work.

$$dW_p = \sigma_{ij} \cdot d\varepsilon_{ij}^{p}$$

maximize the plastic work.

$$d\varepsilon_{ij}^{p} = \underbrace{d\lambda}_{\text{scalar}} \frac{\partial f}{\partial \sigma_{ij}}$$

f → yield criterion.

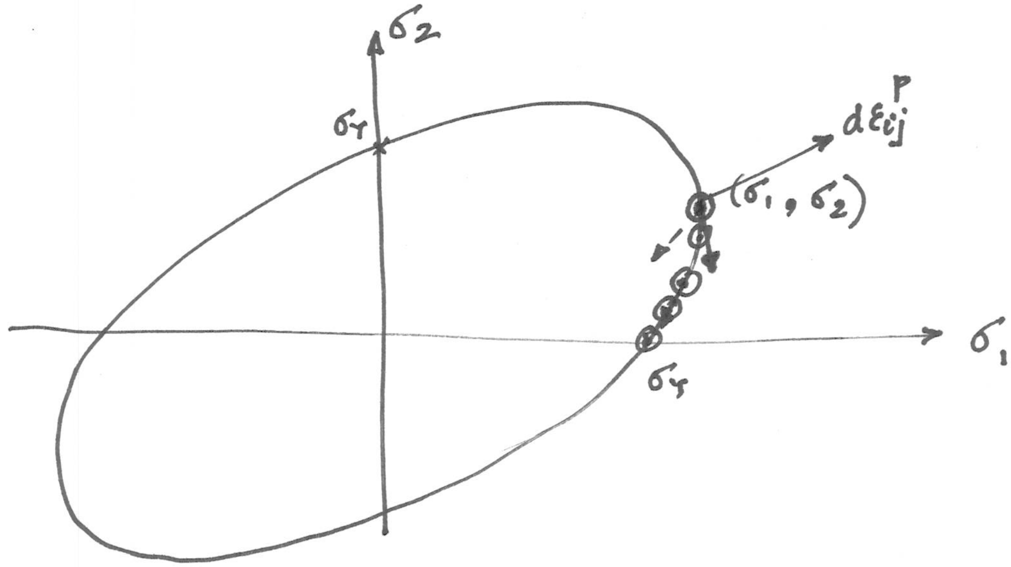
$$f(\sigma_{ij}) = J_2 - \frac{\sigma_Y^2}{3} = 0.$$

Associative Flow Rule

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

Assume plastic flow will occur so as to maximize dissipation of plastic work.

- For max. plastic dissipation, the incremental plastic strain vector $d\varepsilon_{ij}^{p}$ needs to $\perp$r to the incremental stress vector.



Since the incremental stress vector is tangential to the yield surface.

the incremental plastic strain vector will be normal to the yield surface.

Associative Flow Rule $\quad d\varepsilon_{ij}^{p} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}$

metals $\rightarrow$ J_2 governed

Associative Flow Rule $\rightarrow$ v. good assumption.

For hydrostatic pressure dependendent materials non-associative flow rule. $\partial \varepsilon_{ij}^{p} = d\lambda \dfrac{\partial g}{\partial \sigma_{ij}}$

where, $d\lambda \rightarrow$ positive scalar of ~~pta~~ proportionality [non-zero only when plastic deformations occur]

$g(\sigma_{ij}) \rightarrow$ surface in σ_{ij} space

$\dfrac{\partial g}{\partial \sigma_{ij}} \rightarrow$ gradient $\rightarrow$ normal to the surface.

If $g \rightarrow f$ yield function then $d\varepsilon_{ij}^{p} = d\lambda \dfrac{\partial f}{\partial \sigma_{ij}}$

or

$$\begin{Bmatrix} d\varepsilon_{11}^{p} \\ d\varepsilon_{22}^{p} \\ d\varepsilon_{33}^{p} \\ d\varepsilon_{12}^{p} \\ d\varepsilon_{13}^{p} \\ d\varepsilon_{23}^{p} \end{Bmatrix} = d\lambda \begin{Bmatrix} \dfrac{\partial f}{\partial \sigma_{11}} \\ \dfrac{\partial f}{\partial \sigma_{22}} \\ \vdots \\ \vdots \\ \dfrac{\partial f}{\partial \sigma_{23}} \end{Bmatrix}$$

Von Mises Yield function:

$$f(\sigma_{ij}) = J_2 - k^2 = 0$$

$$\text{or } J_2 - \frac{\sigma_Y^2}{3} = 0$$

But $J_2 = \frac{1}{2} s_{ij} s_{ji}$

what is $s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$

$$\begin{bmatrix} \sigma_{xx}-\sigma_m & \tau_{xy} & \tau_{xz} \\ & \sigma_{yy}-\sigma_m & \tau_{yz} \\ & & \sigma_{zz}-\sigma_m \end{bmatrix}$$

$$J_2 = \frac{1}{6}\left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2\right] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2$$

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial\left[\frac{1}{2} s_{ij} s_{ji} - \frac{\sigma_Y^2}{3}\right]}{\partial \sigma_{ij}}$$

$$= s_{ij}$$

Because $J_2 = \frac{1}{2}\left[s_{11}^2 + s_{22}^2 + s_{33}^2 + 2s_{12}^2 + 2s_{23}^2 + 2s_{31}^2\right]$

I $d\varepsilon_v^p = 0$ → volumetric plastic strain = 0

∴ no dilation

II the plastic strain increments depend only on the current deviatoric stress S_{ij}.

→ not on the stress increment $d\sigma_{ij}$ required to maintain plastic flow.

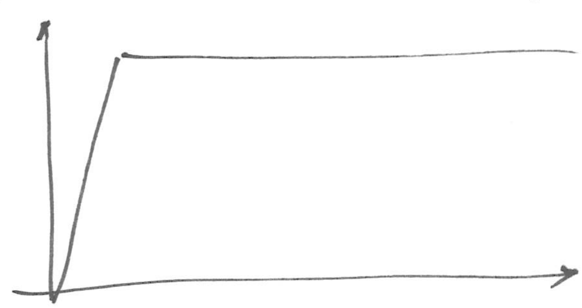
III The principal axes of stress σ_{ij} or deviatoric stress S_{ij} or the plastic strain increment $d\varepsilon_{ij}^p$ → COINCIDE

IV Equations are only about the ratio or relative magnitudes of the components of $d\varepsilon_{ij}^p$.

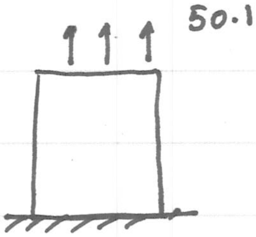
No direct information regarding absolute magnitude

V For perfectly plasticity, the magnitude of $d\varepsilon_{ij}^p$ cannot be determined uniquely using σ_{ij} or $d\sigma_{ij}$.

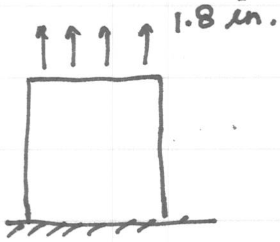
However, for given σ_{ij} and $d\varepsilon_{ij}^p$ → $d\sigma_{ij}$



Path to success



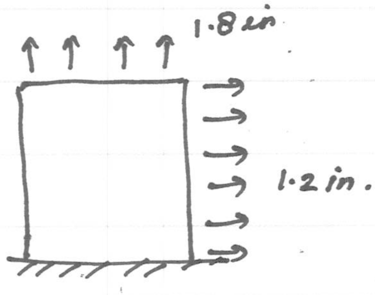
Step 1



Step 2

no. of increments

stress - strain

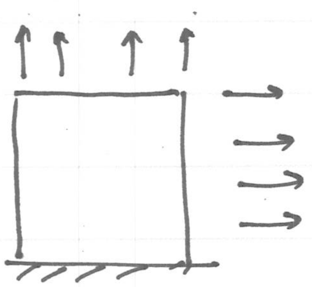


proportional loading

no. of increments

how to make sense

$\varepsilon_p^{11} = \varepsilon_p^{22}$ • $S_{11} = S_{22}$



Step 1 → 0.06 in.

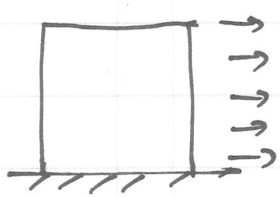
Step 2 → X = 0.12 in

Y = 0.18 in.

look @ ep11 ep22, S11 S22

ep33 PEEQ

$\frac{dep_{11}}{s_{11}}$ $\frac{dep_{22}}{s_{22}}$



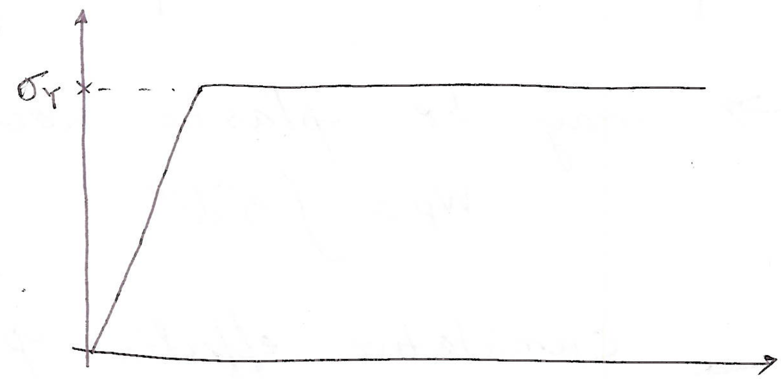
Step 1 → 0.06 in.

Step 2 → X = 0.12 in

Y = −0.18 in.

plot yield curve look @ $\sigma_e - \varepsilon_p$.

Elastic Perfectly Plastic ✓

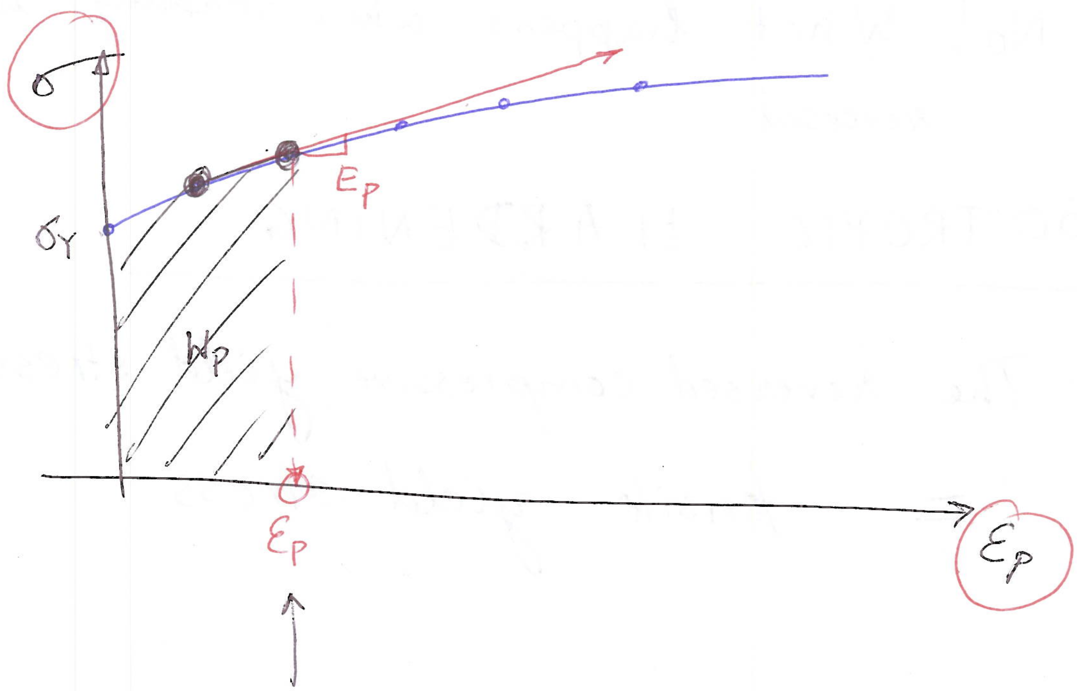


yield stress increases with further plastic straining known as work hardening or strain hardening.

HARDENING PARAMETER $\longrightarrow$ κ

to characteriz various stages of hardening

assume plastic modulus $(E_p) \rightarrow f(\kappa)$



K → represents how much plasticity has occurred!

→ may be plastic work W_p

$$W_p = \int \sigma \, d\varepsilon^p$$

→ cumulative effective plastic strain

$$\varepsilon^p = \int \left(d\varepsilon_{ij}^p \, d\varepsilon_{ij}^p \right)^{1/2}$$

Uniaxial test $\sigma - \varepsilon$ curve can be used to define the hardening parameter K

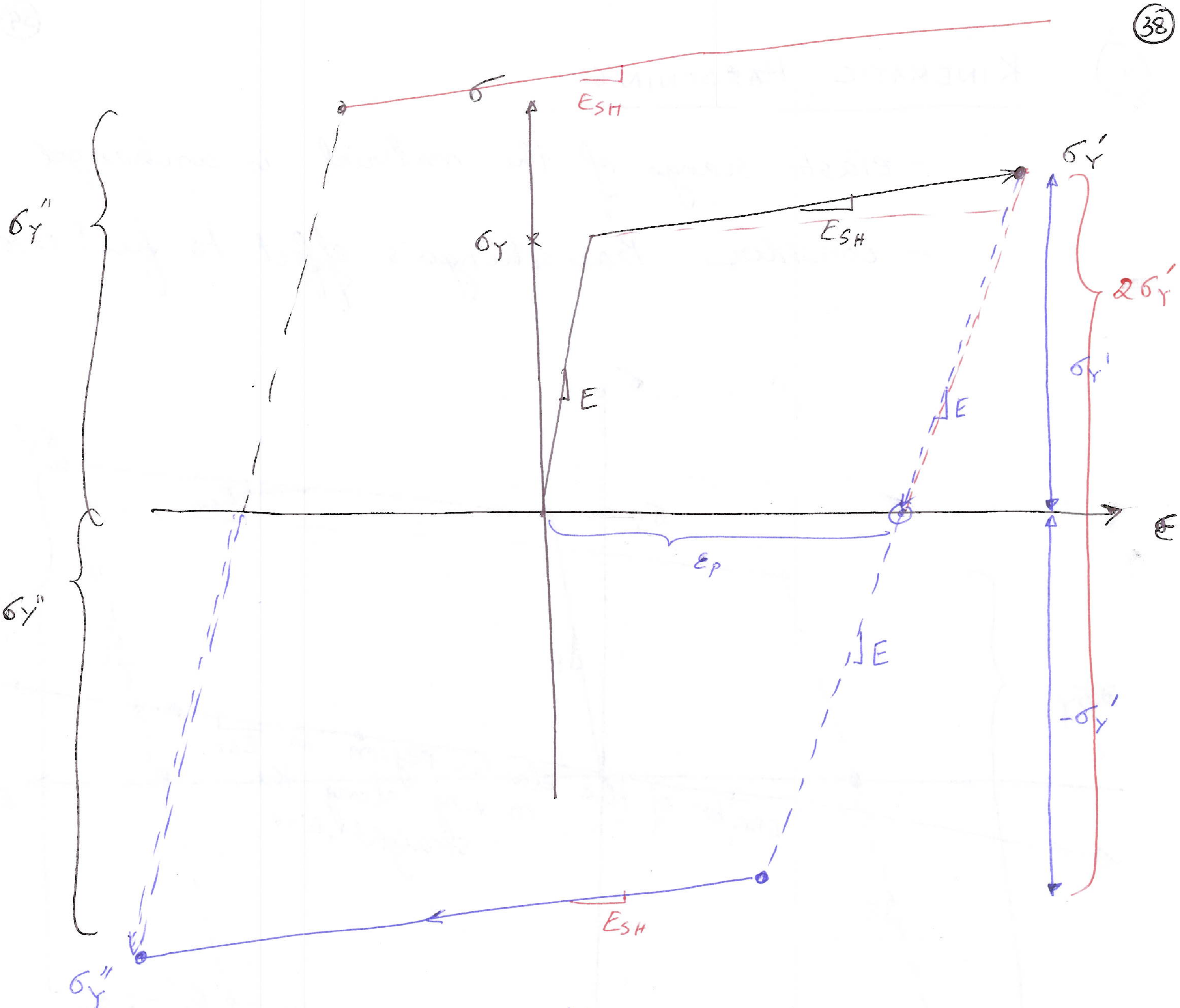
Is this enough?

No! What happens when loading is reversed

ISOTROPIC HARDENING.

The reversed compressive yield stress

= tensile yield stress



ISOTROPIC HARDENING.

Completely neglects Bauschinger's effect

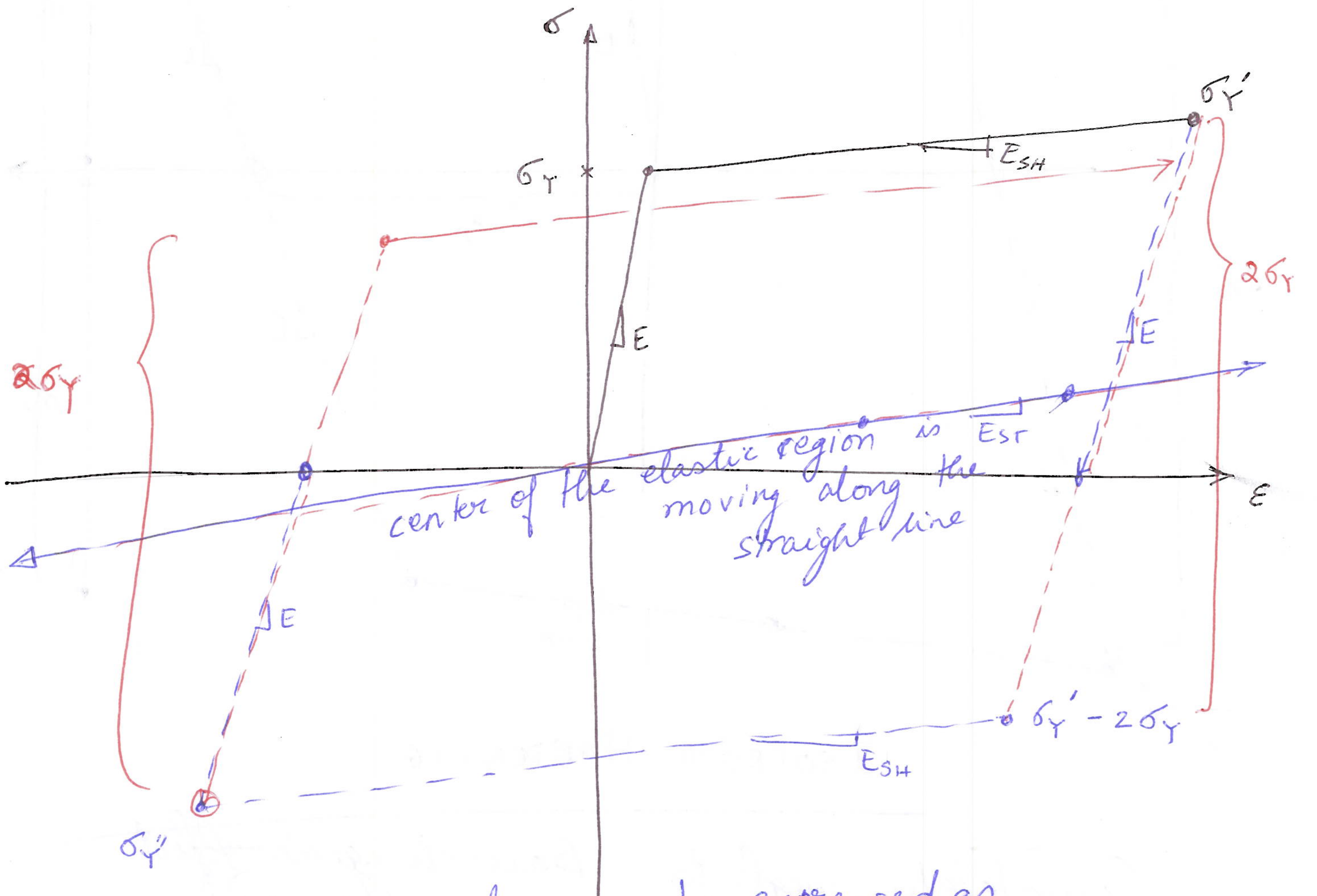
$$|\sigma| = |\sigma(K)|$$

where $\sigma(K)$ is a function of the hardening parameter K

& $K \longrightarrow$ +ve scalar

2 KINEMATIC HARDENING

- elastic range of the matirial is unchanged
- considers Bauschinger's effect to full extent



Hardening rule may be expressed as

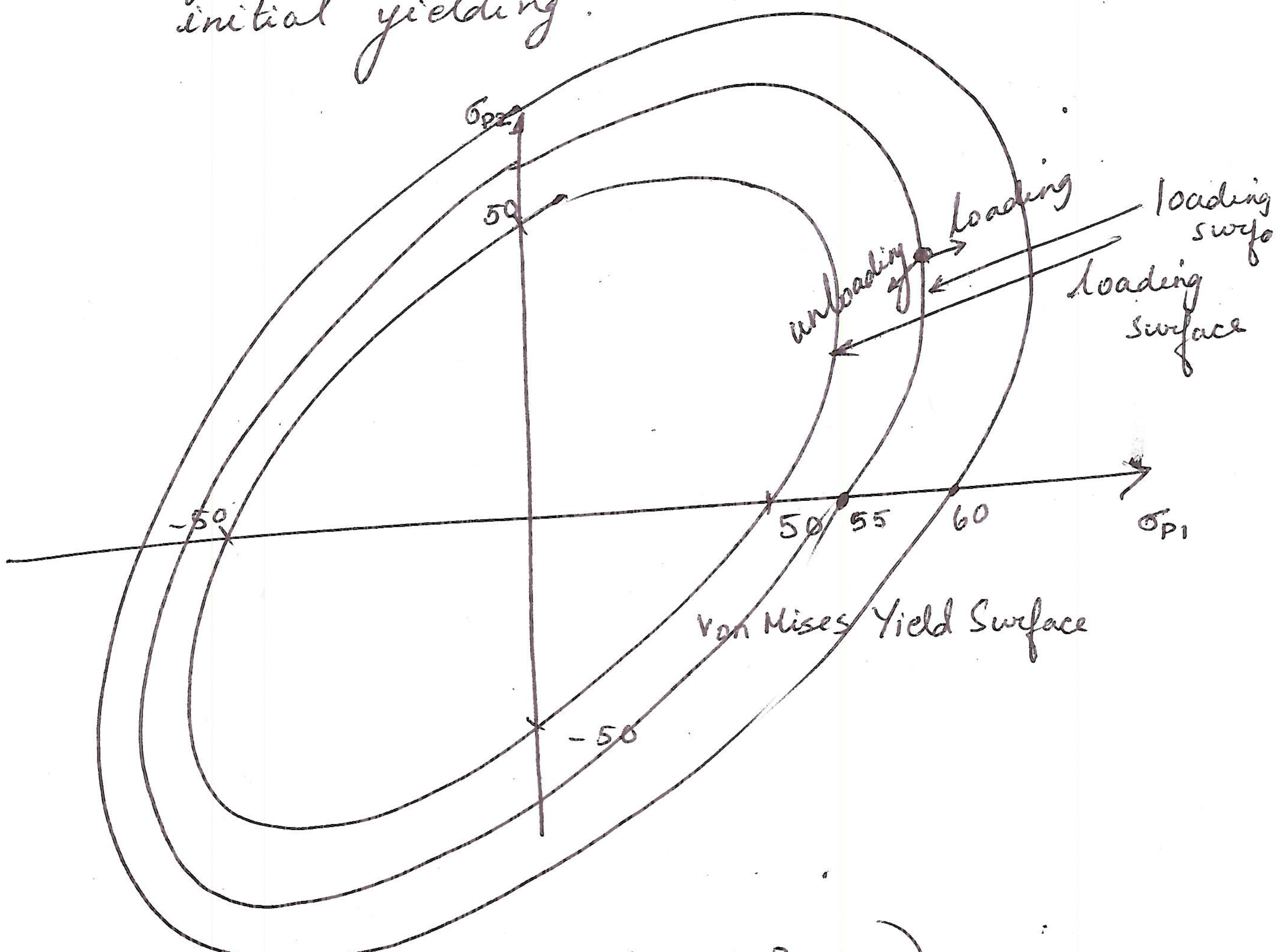
$$|\sigma - c(k)| = \sigma_Y$$

ISOTROPIC HARDENING

Multiaxial world of plasticity.

$K \rightarrow$ hardening parameter

LOADING SURFACE : subsequent yield surface for plastically deformed material after initial yielding.



Yield criterion. $f(\sigma_{ij}, \varepsilon_{ij}^{P}, K) = 0$

stresses — plastic strains — hardening parameter

$$F(\sigma_{ij}) = K^2(\varepsilon^p_{eff})$$

$$\underbrace{J_2 = \frac{1}{2} s_{ij} s_{ij} = K^2(\varepsilon^p_{eff})}$$

Define Effective Stress

$$\sigma_e = \sqrt{3 J_2}$$

yield criterion

$$\frac{3}{2} s_{ij} s_{ij} - \underbrace{\sigma_e^2(\varepsilon_p^{eff})} = 0 .$$

hardening parameter related though uniaxial stress-strain curve.

hardening rule $\longrightarrow$ rule for evolution of the subsequent yield surfaces or "loading surface"

$$f(\sigma_{ij}, \varepsilon_{ij}^{p}, K) = F(\sigma_{ij}, \varepsilon_{ij}^{p}) - K^2(\varepsilon_p) = 0$$

$K^2(\varepsilon_p) \longrightarrow$ represents size of loading surface

$F(\sigma_{ij}, \varepsilon_{ij}^{p}) \longrightarrow$ defines shape.

$K^2 \longrightarrow$ fun. (ε_{eff}^{p}) $\longrightarrow$ <u>effective plastic strain</u>

integrated or cumulative ~~incr~~ of incremental plastic strains

$\varepsilon_{eff}^{p} \longrightarrow$ depends on loading path

---

ISOTROPIC HARDENING

Simplest Assumption $\longrightarrow$ initial yield surface expands without any distortion

Size depends on $K^2 \longrightarrow$ function (ε_{eff}^{p})

What is σ_e?

At any point in the material,

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\downarrow$$

$$\sigma_m \delta_{ij} + S_{ij}$$

$$= \begin{bmatrix} \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \end{bmatrix} + \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$$\sigma_e = \frac{1}{2} S_{ij} S_{ij}$$

Scalar quantity

$$= S_{11}^2 + S_{22}^2 + S_{33}^2 + S_{12}^2 + S_{23}^2 + S_{31}^2$$

$\sigma_e \longrightarrow$ Scalar $\longrightarrow$ representing the multiaxial stressstate in a lumped manner.

$\varepsilon_p^{eff} \longrightarrow$ effective plastic strain

Scalar function representing the amount of plastic work $= \int \sigma_{ij} \, d\varepsilon_{ij}^p$

or

cumulative effective plastic strain

$$= \int d\varepsilon_p^{eff}$$

$$d\varepsilon_p^{eff} = \sqrt{\frac{2}{3} \, d\varepsilon_{ij}^p \cdot d\varepsilon_{ij}^p}$$

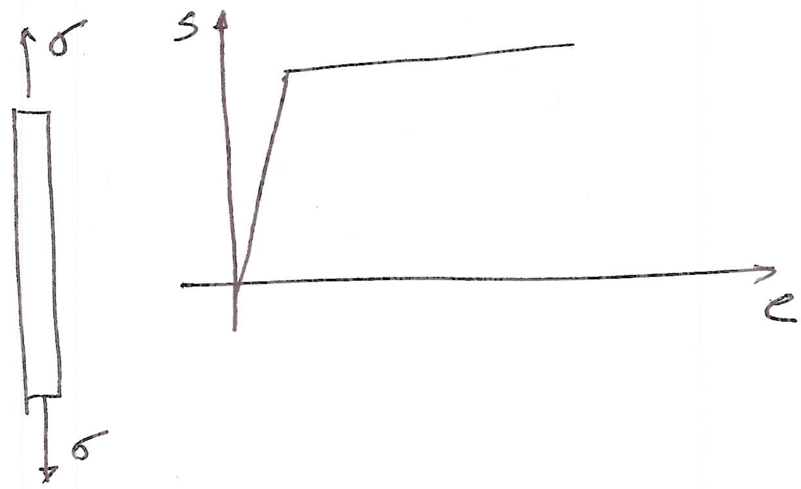
$$= \sqrt{\frac{2}{3} \left((d\varepsilon_{11}^p)^2 + (d\varepsilon_{22}^p)^2 + (d\varepsilon_{33}^p)^2 + (d\varepsilon_{12}^p)^2 + (d\varepsilon_{23}^p)^2 + (d\varepsilon_{13}^p)^2 \right)}$$

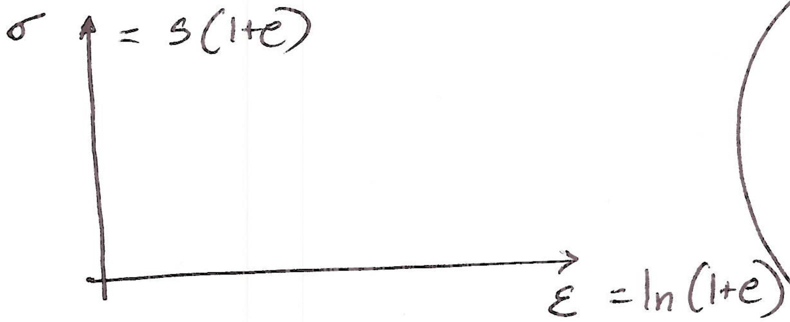
$\varepsilon_p^{eff} \longrightarrow$ Scalar $\longrightarrow$ representing the multiaxial plastic strain state in a lumped manner.

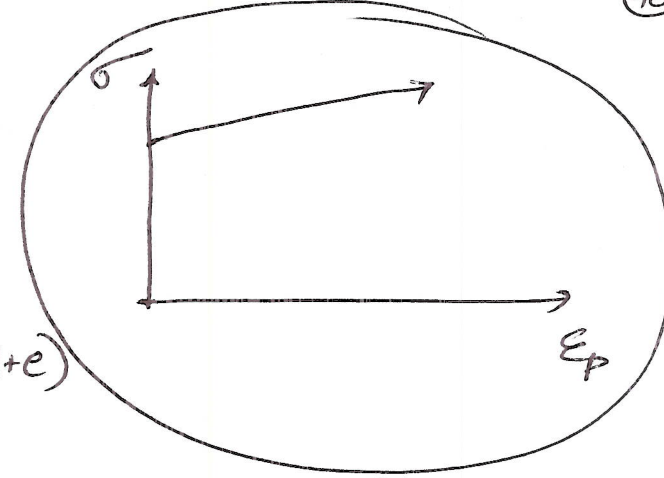
Simple combination of the incremenal plastic strains that is alway positive and increasing.

Where did the 3 & 2/3 appear in the definitions of σ_e & ε_{eff}^{p}

because it was calibrated to work for uniaxial case.







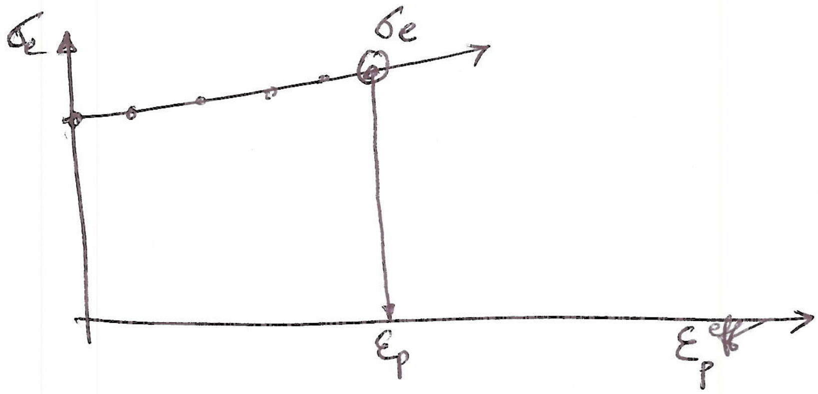
$$\sigma_e = \sigma$$

$$\varepsilon_p^{eff} = \varepsilon_p$$

when you get the uniaxial data

↓

$\sigma_e - \varepsilon_p^{eff}$ curve.



46A

$$F(\sigma_{ij}) = K^2(\varepsilon^p_{eff})$$

$$J_2 = \frac{1}{2} S_{ij} S_{ij} = K^2(\varepsilon_p^{eff})$$

Define $\longrightarrow$ Effective Stress $\sigma_e = \sqrt{3J_2}$

$$\frac{3}{2} S_{ij} S_{ij} - \underbrace{\sigma_e^2(\varepsilon_p)}_{\downarrow} = 0$$

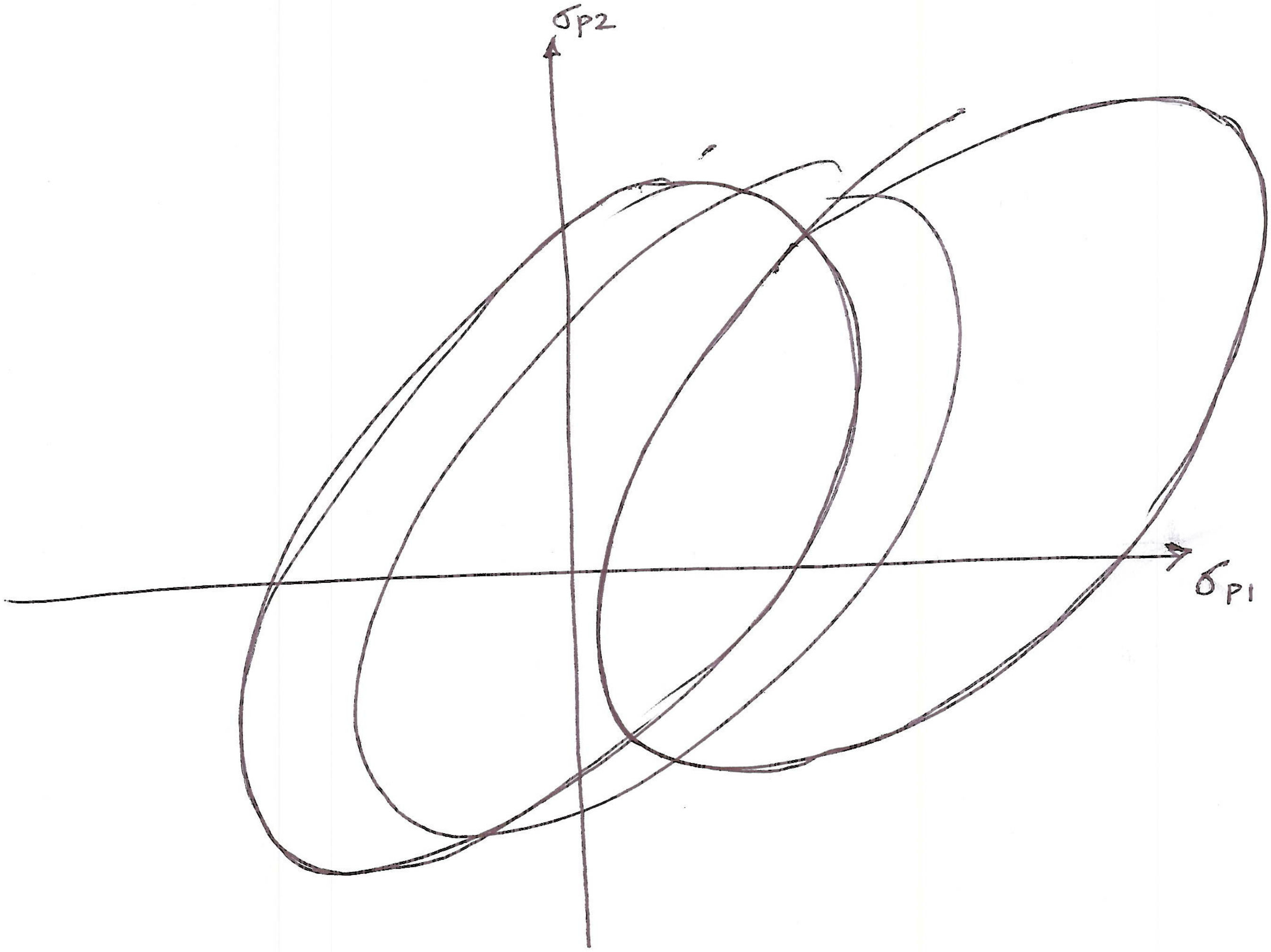
hardening parameter related to ε_p^{eff} through uniaxial tess.

$\varepsilon_p \longrightarrow$ scalar function of ~~the~~ plastic work $\int \sigma_{ij}\, d\varepsilon_{ij}^p$ or cumulative effective plastic strain

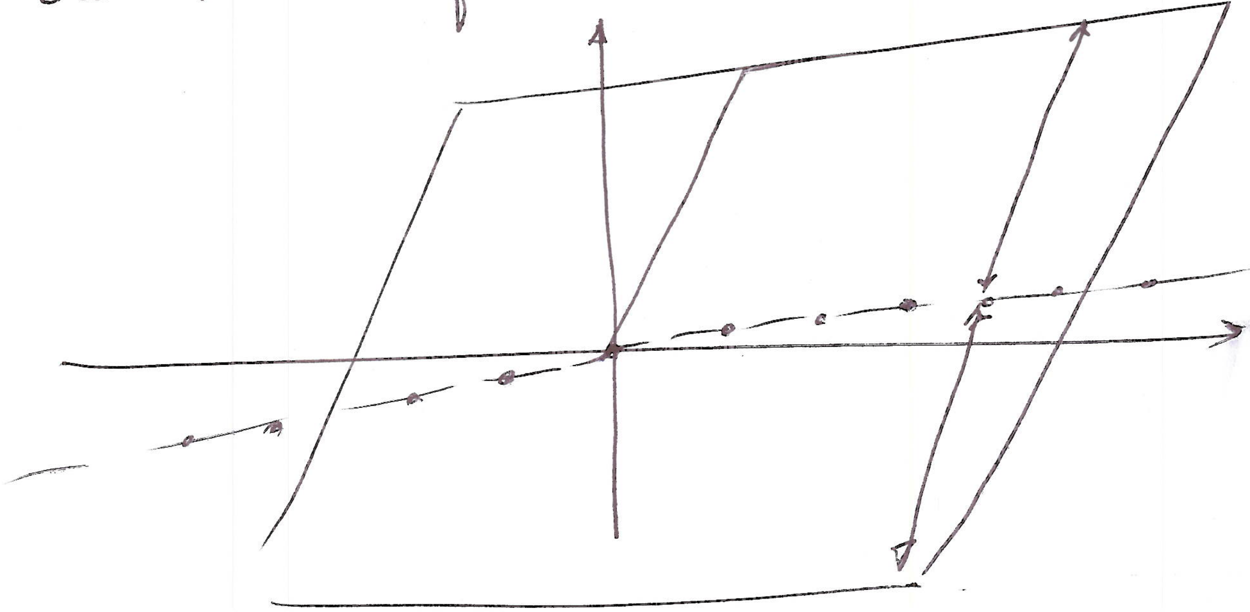
$$\varepsilon^p_{eff} = \int d\varepsilon_p^{eff}$$

$$= \int \sqrt{\frac{2}{3} d\varepsilon_{ij}^p\, d\varepsilon_{ij}^p}$$

KINEMATIC HARDENING



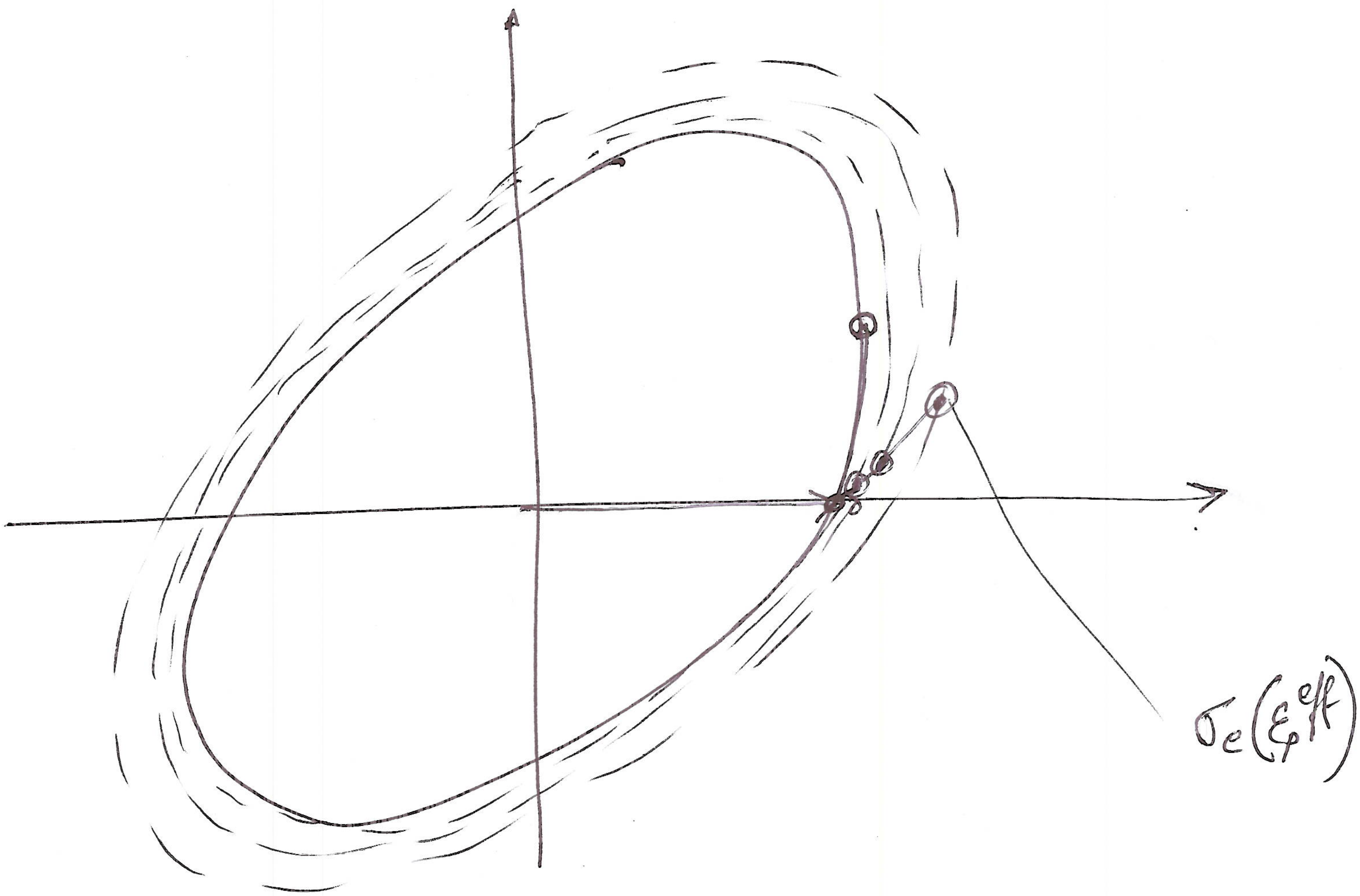
size of ellipse does not change but the surface is moving in stress-space

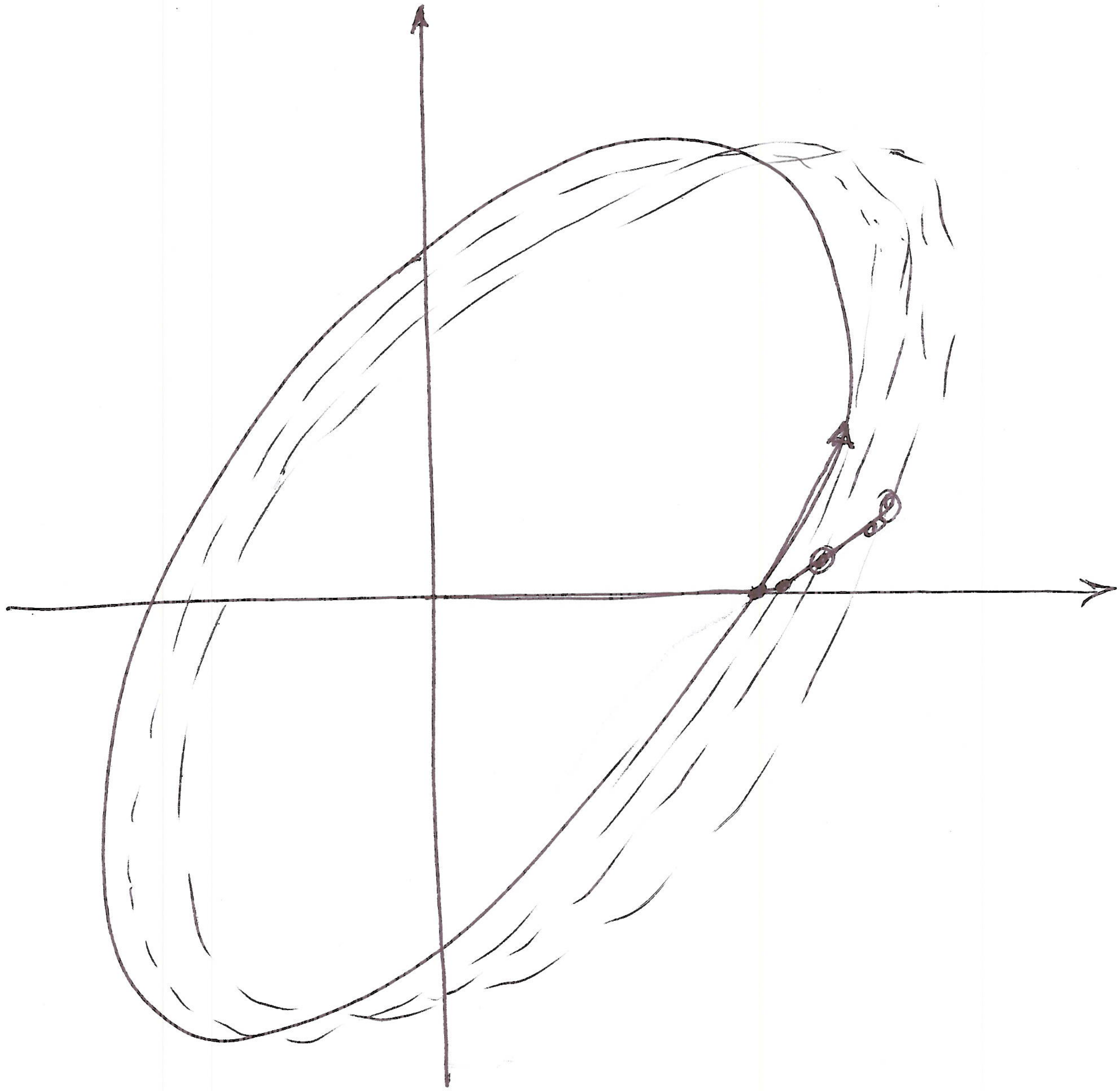


What does that mean?

different behavior for isotropic & kinematic.

definitely if you apply reversed cyclic loading

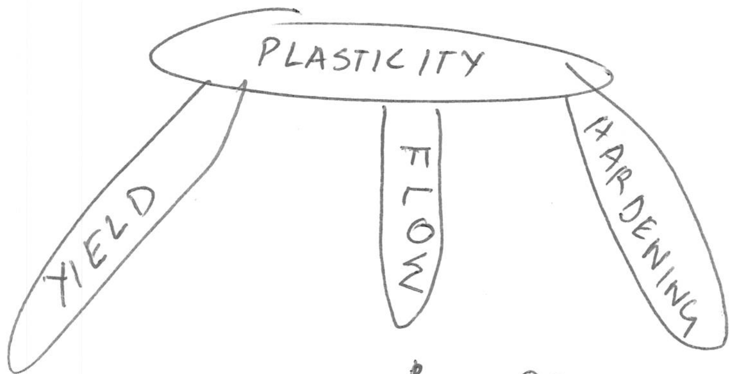




CE 592 → Plastic Design.

What do I not know?

This I know?



$f = 0$ $\quad d\varepsilon_{ij}^{p} = d\lambda \dfrac{\partial f}{\partial \sigma_{ij}}$ $\quad \kappa(\varepsilon_p^{eff})$

$\cancel{\sigma_{ij}^{e} \overset{C_{ijkl}}{\Rightarrow} \varepsilon_{ij}^{e}}$ $\qquad \sigma_{ij}^{e} = C_{ijkl}\,\varepsilon_{kl}^{e}$

σ_{ij}^{e} → elastic

C_{ijkl} → elasticity matrix

ε_{kl}^{e} → elastic strains

E , ν

E → modulus of elasticity

ν → Poisson Ratio.

$$\sigma_{ij} = \underline{\quad ? \quad}\left(\varepsilon_{kl}^{e} + \varepsilon_{kl}^{p}\right)$$

$$d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p} = d\varepsilon_{ij}$$

$$d\sigma_{ij}^{e} \qquad d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

$$\longrightarrow d\varepsilon_{ij}^{tot} \xrightarrow{D_{ijkl}} d\sigma_{ij}$$

> Dont have the relationship between stresses and strains in the inelastic range.

> Mathematical manipulations of

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}$$

elasticity matrix — 1 week — stresses to strains

> $$K_e = \iiint [B]^T [D] [B] \, dx\, dy\, dz$$

$$u_i(x, y, z) = N(x, y, z) \begin{Bmatrix} u_i \\ v_i \\ \vdots \end{Bmatrix}$$

N → Shape functions.

1 week of mathematical manipulation

$[B] \longrightarrow$ derivatives of shape functions.

$[D] \longrightarrow$ inelastic matrix not available

so I dont have the finite element formulation

finite element formulation or $[K]_e$

3 How to perform nonlinear ~~in~~ inelastic analysis.

$$\underbrace{\{F\} = [K]_s \{u\}}_{\text{nonlinear equations of equilibrium.}}$$

Newton - Raphson Iteration Method to solve nonlinear equations.

NONLINEAR SOLUTION STRATEGIES. → 1-2weeks of Math & Programming!

1 $[D_{ij}]_{ep}$

2 $[K]_e$

3 N-R iteration for solving the nonlinear problem.

→ CE 675

5-6weeks

YIELD CRITERION, LOADING SURFACES

FLOW RULE, PLASTIC WORK HARDENING

EFFECTIVE STRESS, EFFECTIVE PLASTIC STRAIN,

Why Not?

Ans. Because I dont want to!

DESIGN

- Section Plasticity 3
- Member Plasticity 2
- Frame / Structure Plasticity 3

} DESIGN.

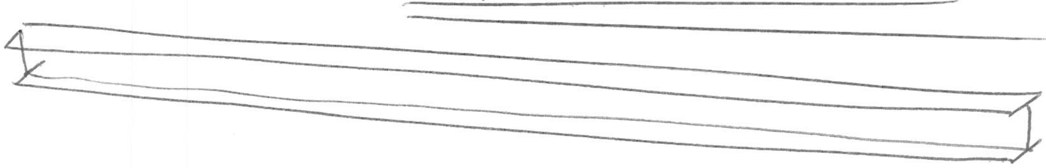
Pushover Analysis (Static) ⟶ Member, Section & Structure } Plasticity.

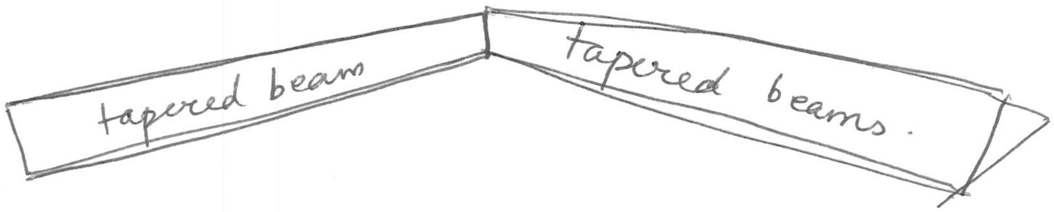
Section Plasticity:

Beam, Column, Frame } Member ⟶ Cross-Sections

Prismatic members.

Tapered members





MBMA structures → Metal Building Manufacturer Association.

AISC → hot rolled, design-built

1 - 3 stories

cold-formed, designed structured thin gage steel ← AISI

MBMA → prefabricated predesigned building

Section Behavior.

CURVATURE ϕ holds the key!

What exactly is curvature

ϕ → mystical quantity

hold the kinematics & equilibrium.

deform

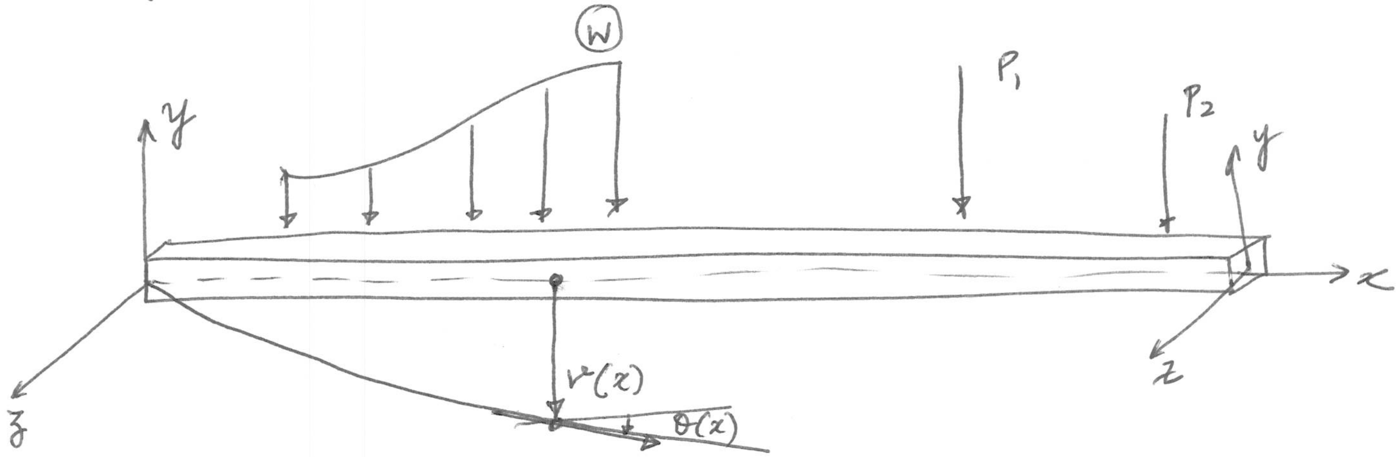
force / equilibrium.

$$\phi = \frac{d\theta}{dx} = \frac{d^2 v}{dx^2} = -\frac{\varepsilon}{y} = -\frac{\sigma}{yE} = +\frac{M}{EI} = \frac{1}{R}$$

v, θ, ε, y, σ, M, R

v → displacement in y direction

y → upwards as shown

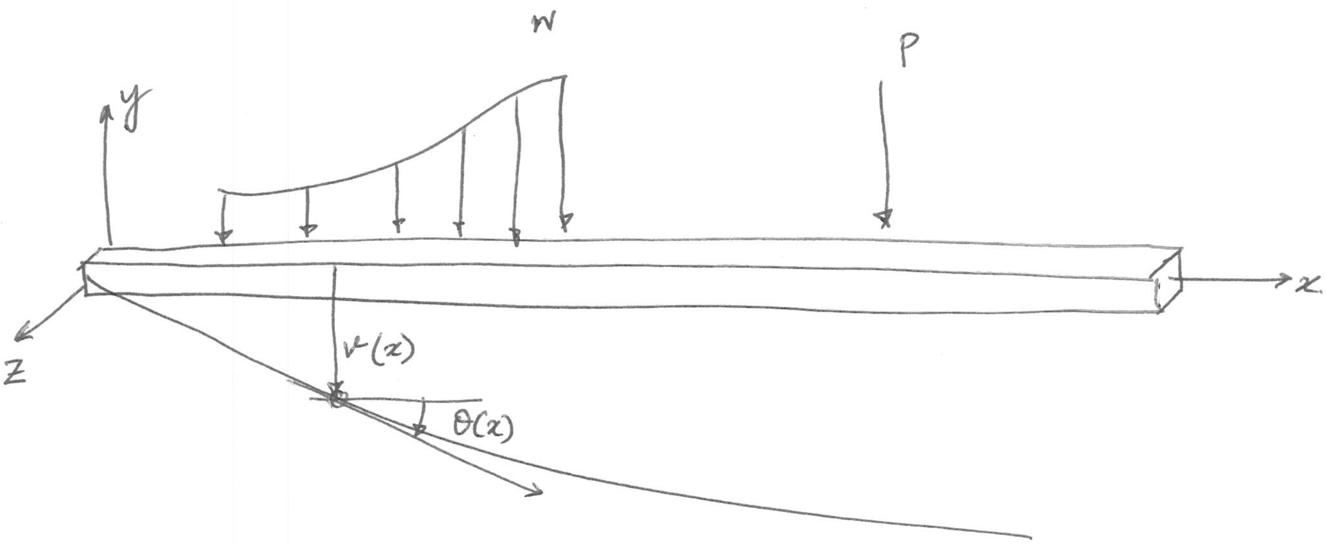


$\theta(x)$ ⟶ angle that tangent @ x makes with respect to horizontal

What ~~is~~ are the kinematics of this behavior?

plane sections remain plane before and after bending

CURVATURE HOLDS THE KEY!



$\phi(x) \rightarrow$ What is curvature

plane sections remain plane before & after bending (and perpendicular to the neutral axis.) Navier-Bernoulli beam

shear deformations are not negligible. ← Timoshenko beam

$\frac{l}{d} \geq 10$

Summary:

1. When plane sections remain plane & $\perp r$ to the neutral axis

   $\longrightarrow$ Strain diagram will be linear over the cross-section

2. If the ~~applied~~ ~~as~~ internal axial force $P(x)$ is zero $\rightarrow$ then the neutral axis will coincide with centroidal axis

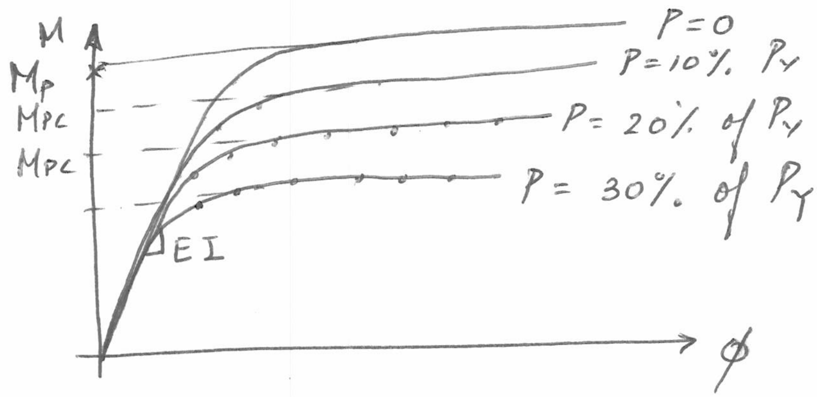
3. If $P(x) \neq 0$, then you need to determine the neutral axis location.

4. The internal M and the curvature ϕ have a fundamental relationship. In the elastic range, this is linear defined by

   EI $\longrightarrow$ ~~fs~~ section flexural stiffness

   referred sometimes as generalized stress-strain relationship.

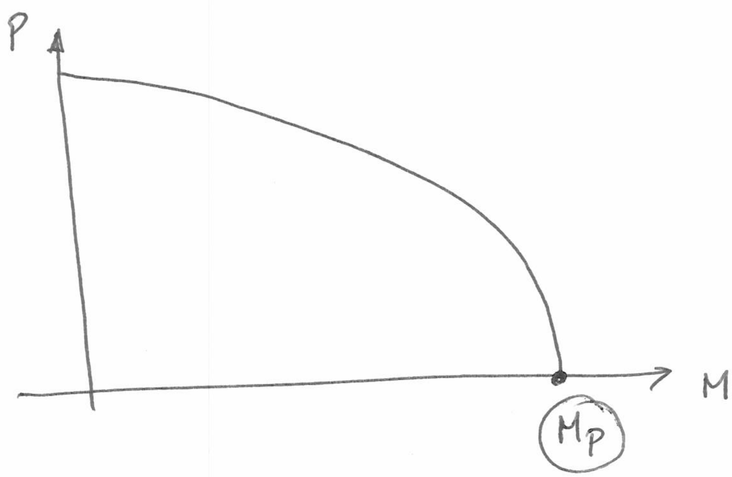
5. The internal forces $P(x)$, $V(x)$ & $M(x)$ are stress-resultants obtained by integrating stresses over cross-section.



$P_y = A_s F_y$

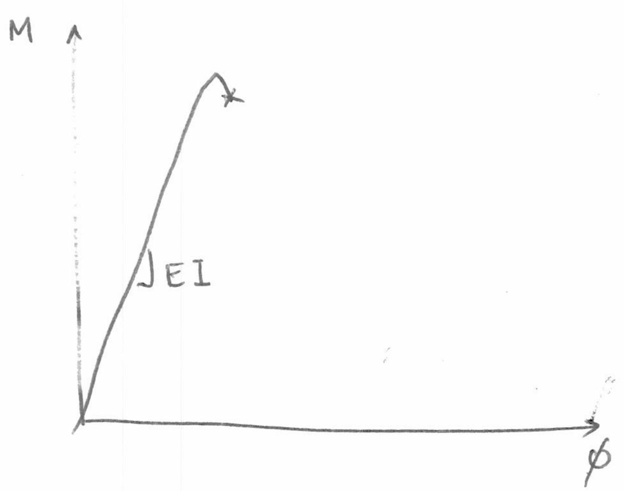
Doubly symmetric

Singly symmetric ⟶ complex for Axial Forces

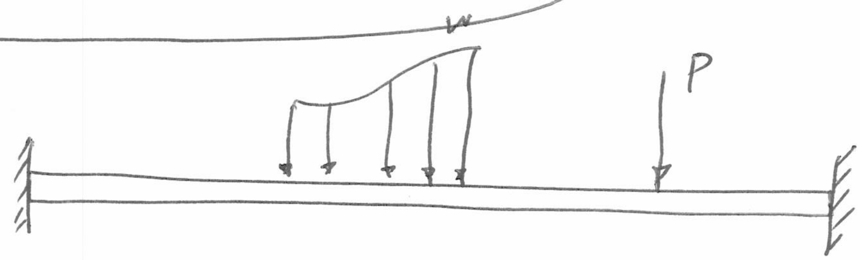


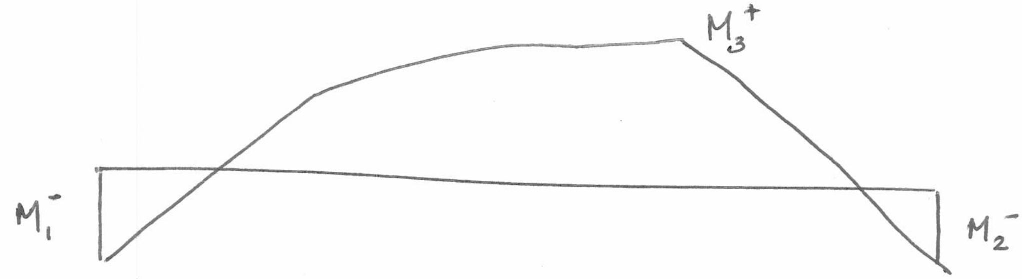
ε_0

$P - \varepsilon - M - \phi - T$

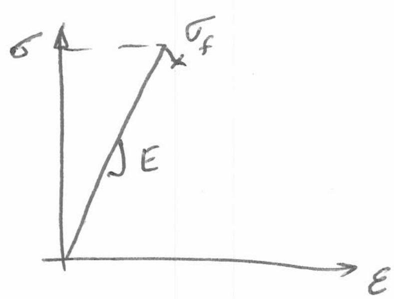


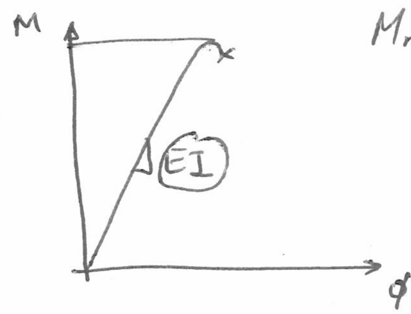
Why is this relevant?





What would be the load capacity of this structural member:





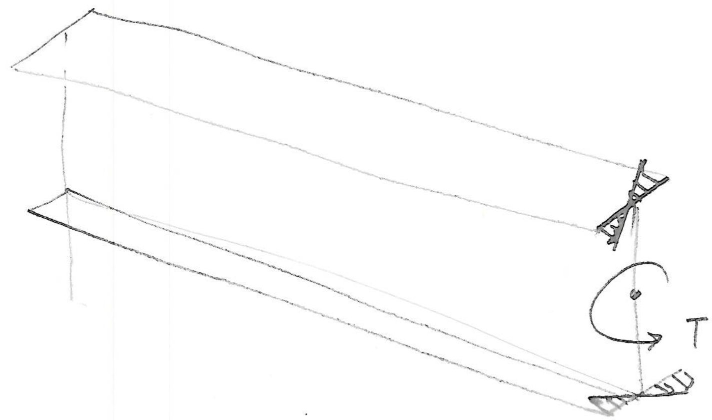
elastic modulus

$$M_{max} = \sigma_f \times \underbrace{\frac{I}{y}}_{S}$$

$$= \sigma_f \cdot S.$$

$$M_{max} \longrightarrow \underbrace{M_3^+ \longrightarrow M_3^+ = M_{max}}_{\text{load capacity}}.$$

$$\sigma = \frac{My}{I}$$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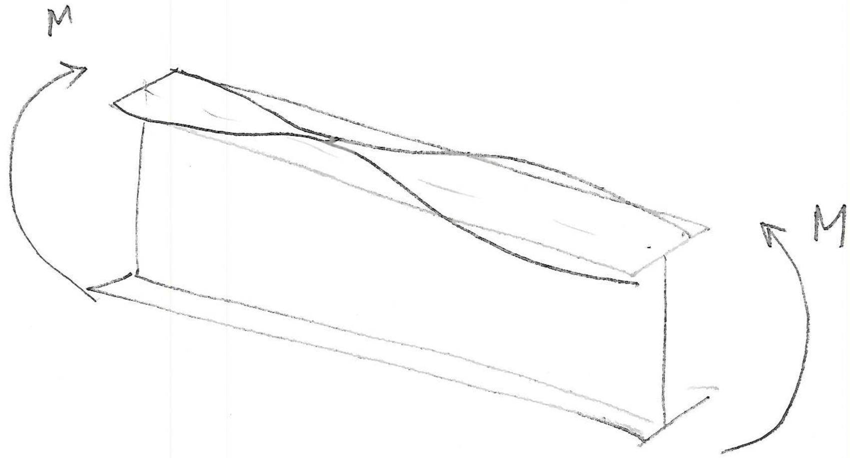
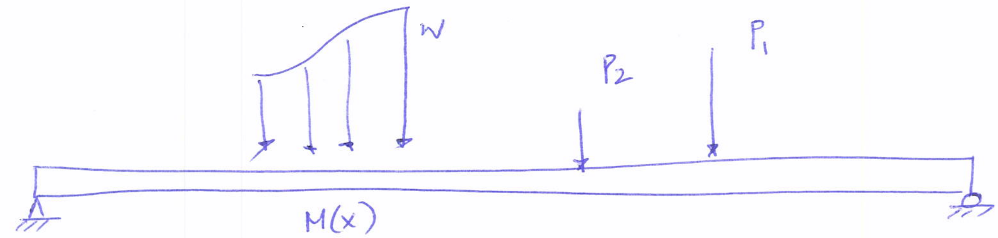
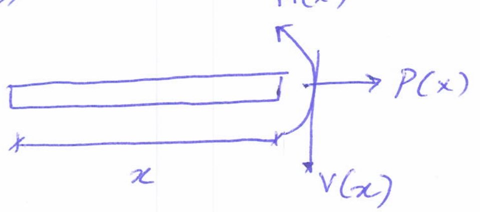
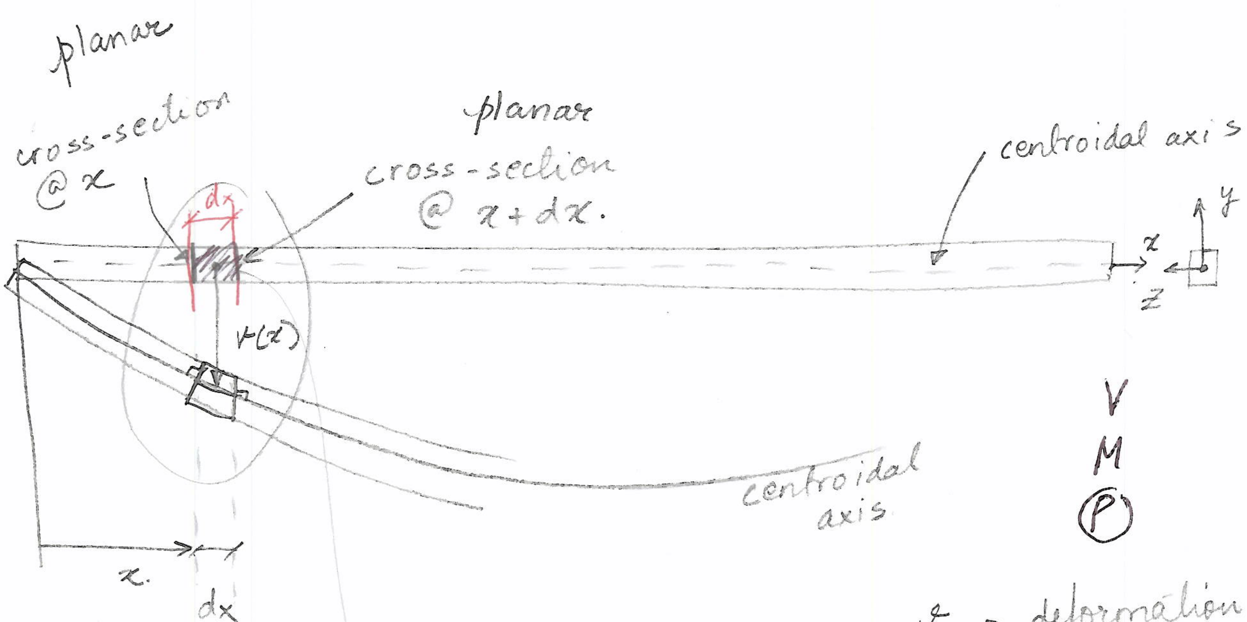

M
M

flexure or bending dominated bending



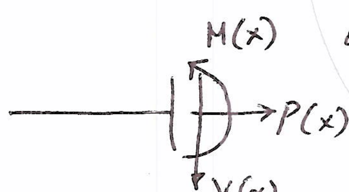


← Internal Forces.
Stress Resultants!

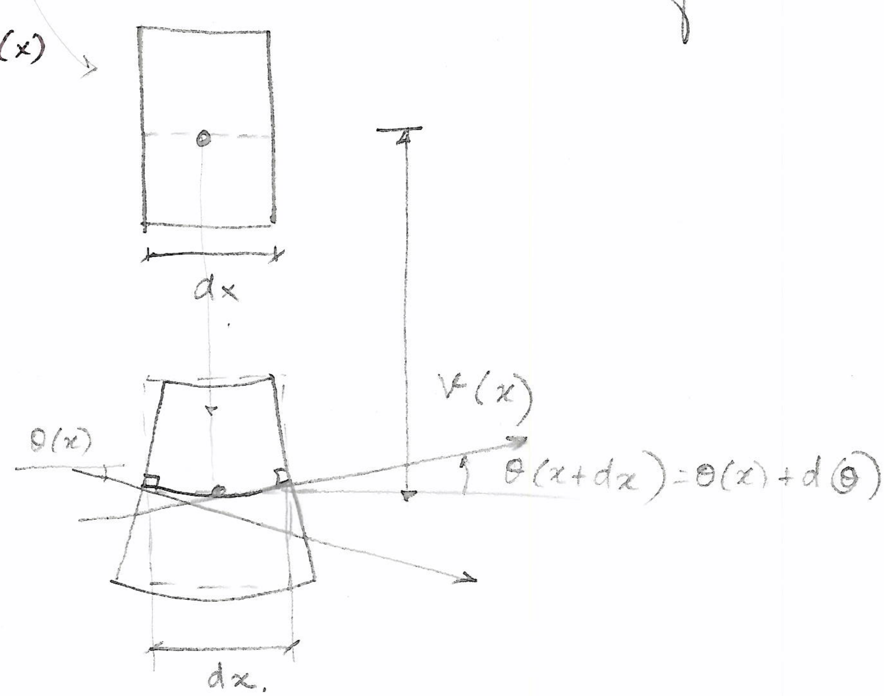


V
M
P

v → deformation of centroidal axis



as the internal forces

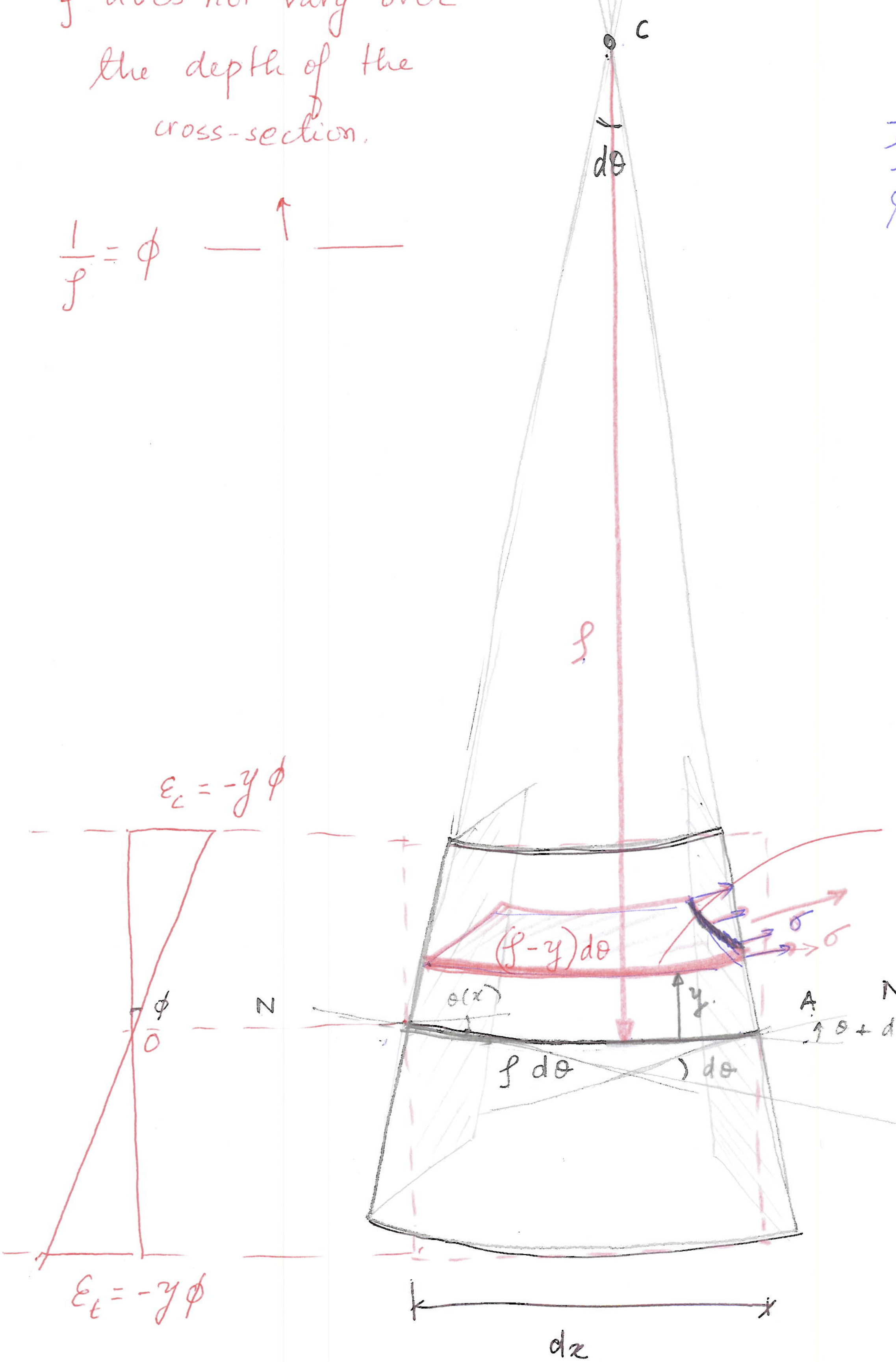


CURVATURE

curvature $= \dfrac{1}{\text{radius of curvature}}$

ρ does not vary over the depth of the cross-section.

$\dfrac{1}{\rho} = \phi$



Neutral Axis

$$\frac{1}{\rho} = \frac{d\theta}{dx}$$

$$\phi = \frac{d}{dx}\left(\frac{dv}{dx}\right) = \frac{d^2 v}{dx^2}$$

$$\rho\, d\theta = dx$$

$$\text{or } \frac{d\theta}{dx} = \frac{1}{\rho} = \phi$$

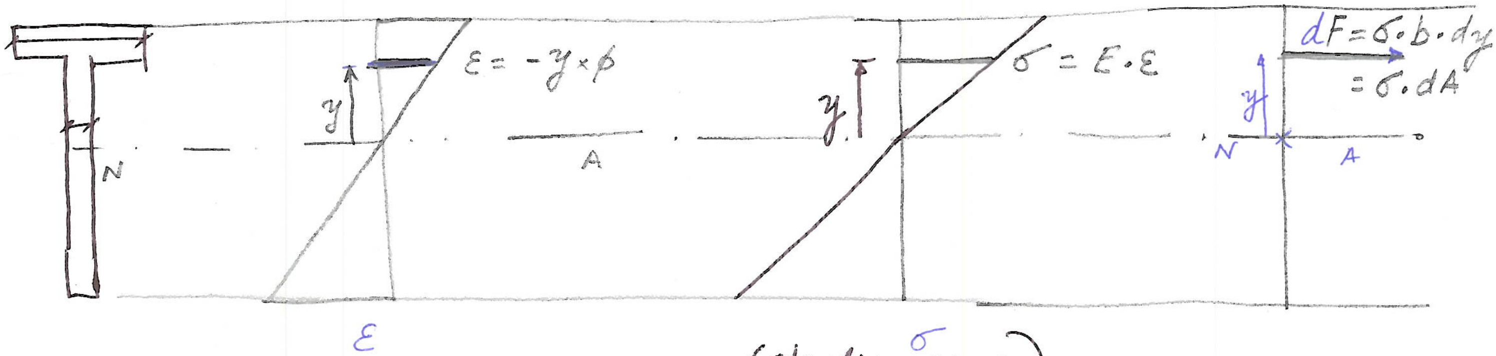
$$\varepsilon = \frac{\text{change in length}}{\text{original length}} = \frac{(\rho - y)\,d\theta - dx}{dx}$$

$$\varepsilon = \frac{\overset{dx}{\rho\, d\theta} - y\,d\theta - \cancel{dx}}{dx}$$

$$\varepsilon = -\frac{y\,d\theta}{dx}$$

$$\therefore\ -\frac{\varepsilon}{y} = \frac{d\theta}{dx} = \frac{1}{\rho} = \phi$$

$$\varepsilon = -y \times \phi$$



(elastic σ range)

$$\sum_A F = P = \int_A dF = \int_A \sigma \, dA = \int_A E \, \varepsilon \cdot dA$$

$$= \int_A E \, (-y\phi) \, dA$$

$$P = -E\phi \int_A y \, dA$$

If $P = 0$ $\quad \int_A y \, dA = 0 \longrightarrow$ elastic centroid

$NA \longrightarrow$ centroidal axis.

coincides with elastic centroid.

$$dM = dF \times y = \sigma \cdot dA \times y = \sigma \cdot b \cdot y \cdot dy$$

$$\int_A dM = M = \int_A \sigma \, b \, y \, dy = \int_A E \cdot \varepsilon \cdot b \, y \, dy$$

$$= \int E \, (-y\phi) \, b \, y \, dy$$

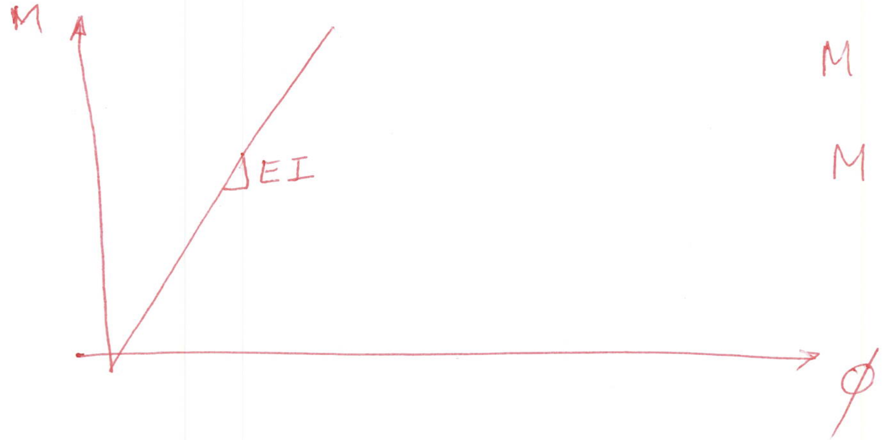
$$= -E\phi \int y^2 b \, dy$$

moment of inertia

$$M = -E\phi \times I$$

$$\phi = -\frac{M}{EI} \rightarrow$$ in the elastic world

→ represents ϕ



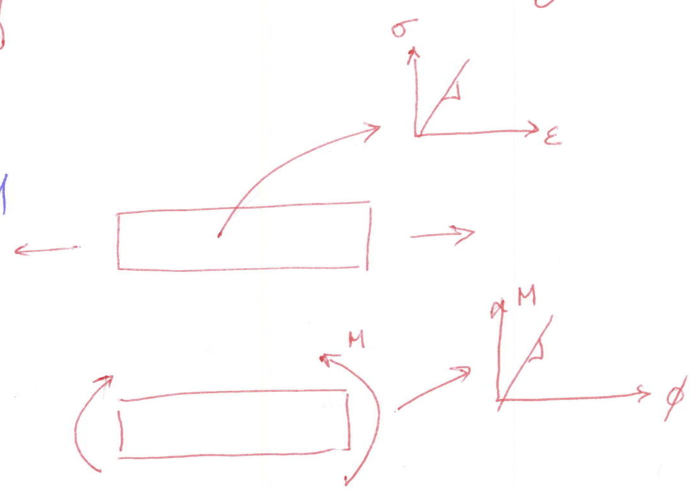
$M \propto \phi$

$M = -EI\phi$

fundamental behavior of all cross-sections along the length

generalized stress → M

generalized strain → ϕ

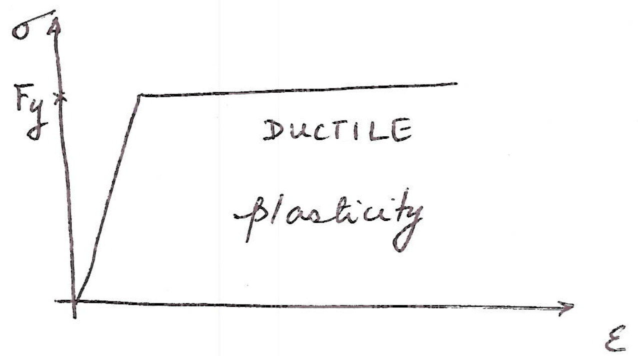


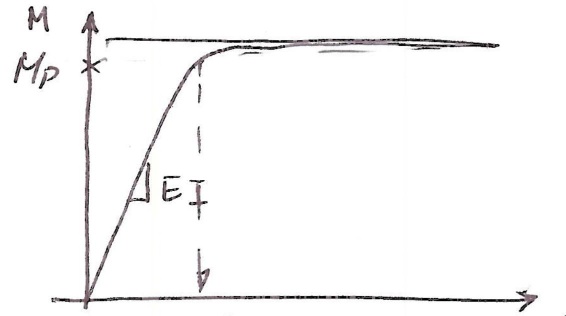
$$\phi = -\frac{M}{EI}$$

$$\therefore M = -E\phi I = -\frac{E \times \varepsilon}{y} I = -\frac{\sigma I}{y}$$

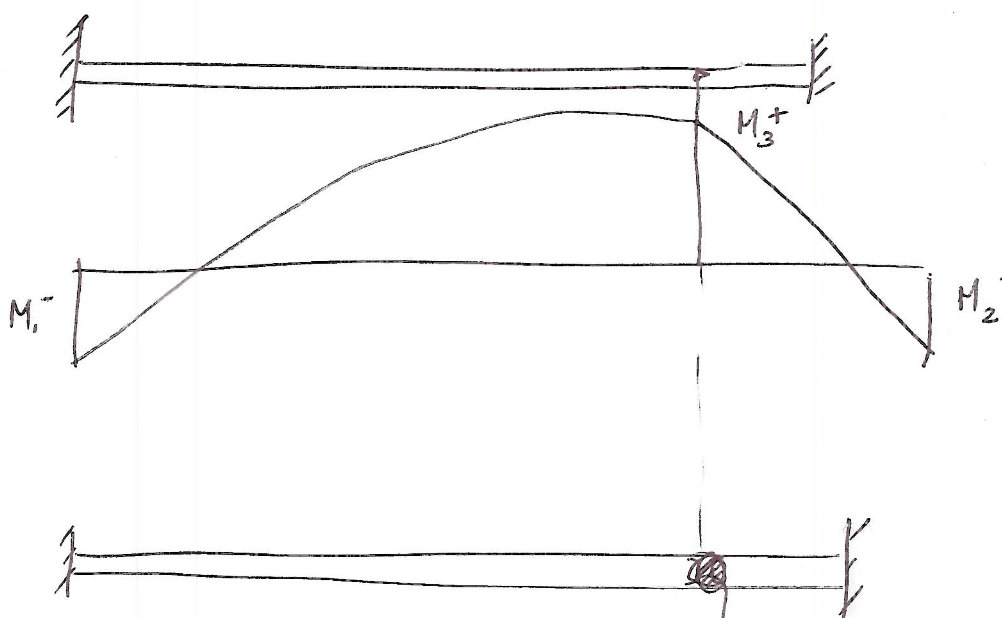
$$\sigma = -\frac{My}{I}$$

$$\sigma \propto y$$



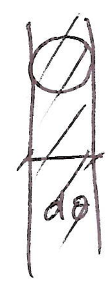


$P - \varepsilon - M - \phi$



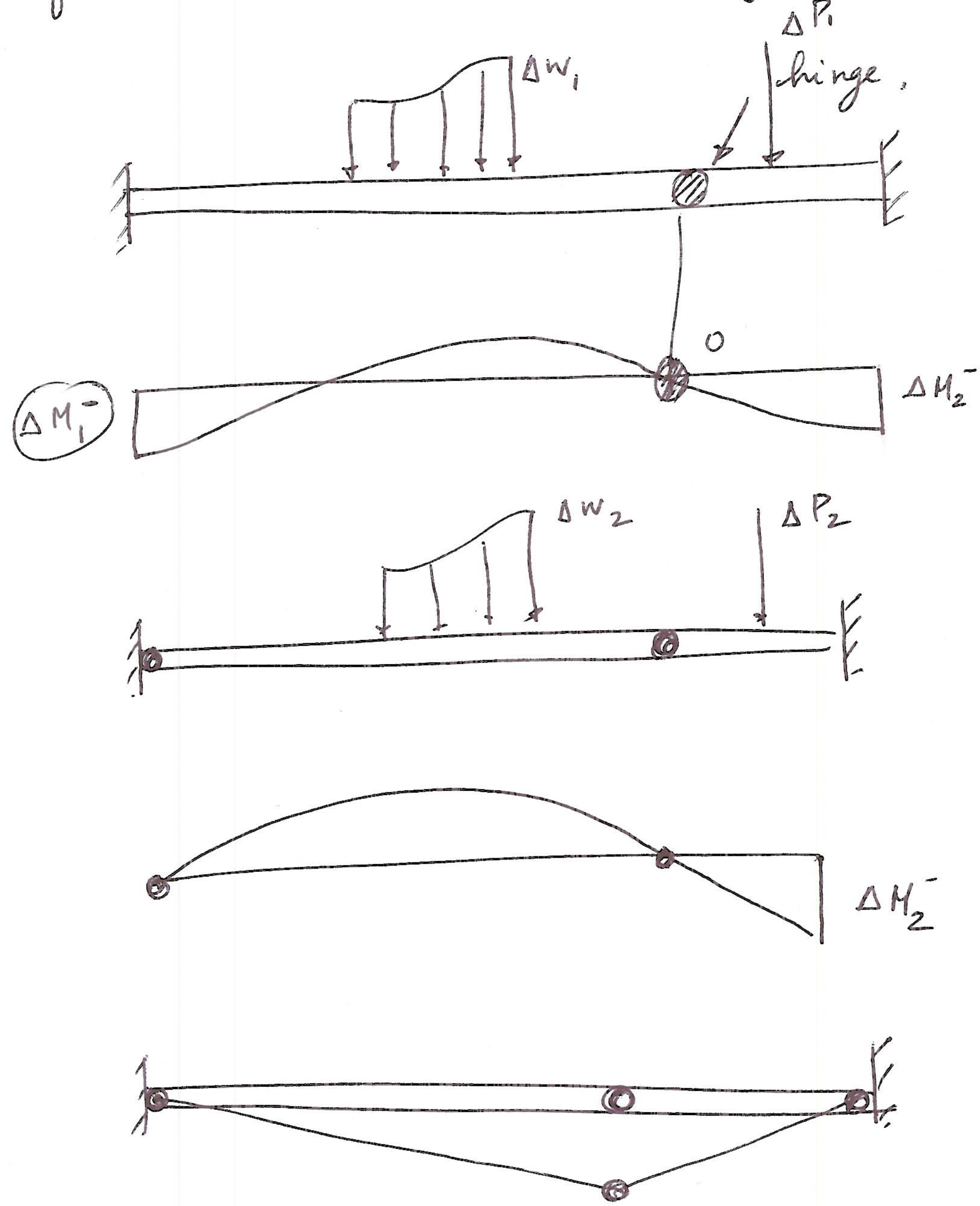
yielded plastic hinge.

what is rotation.



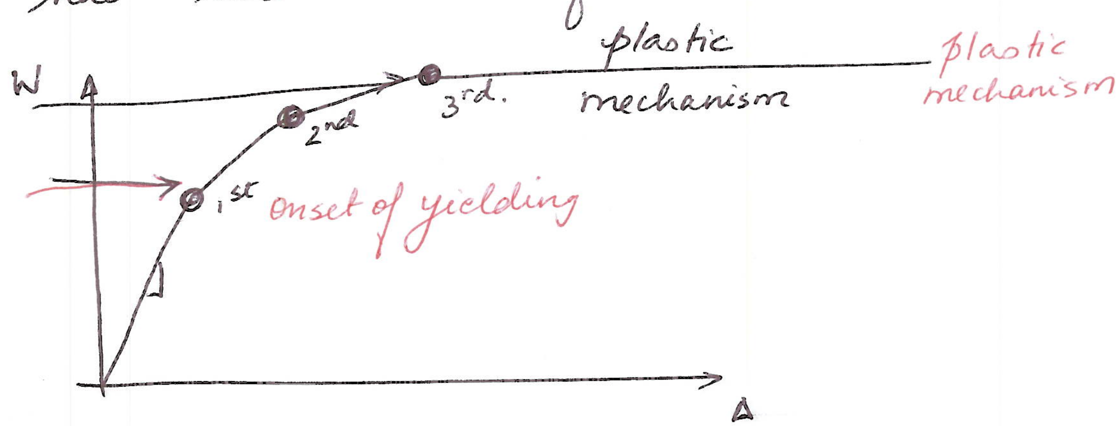
curvature increasing rapidly.

~~fail~~ for all incremental loading

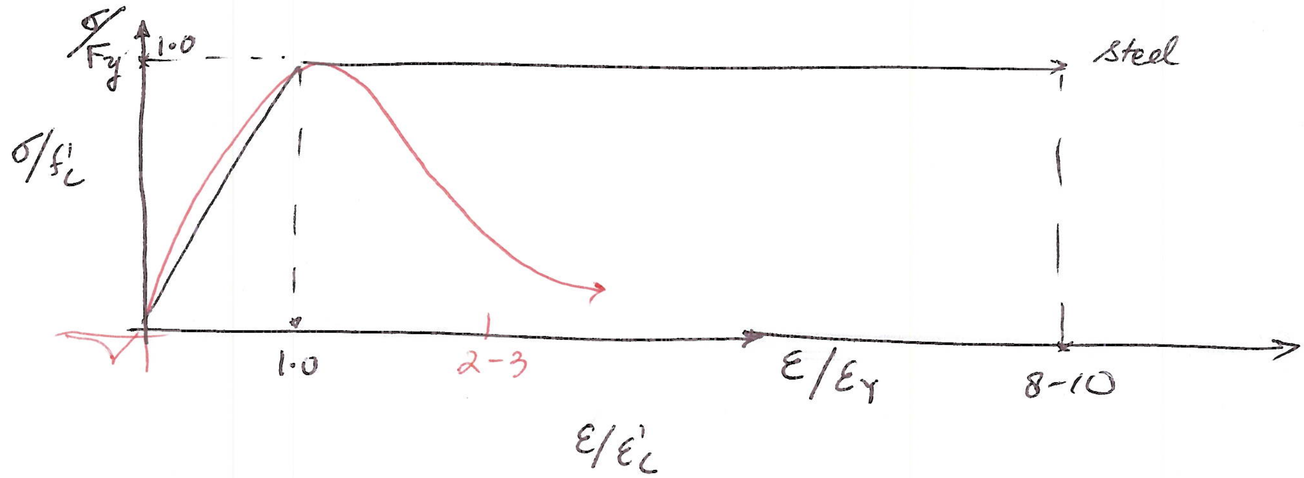


What is a mechanism?

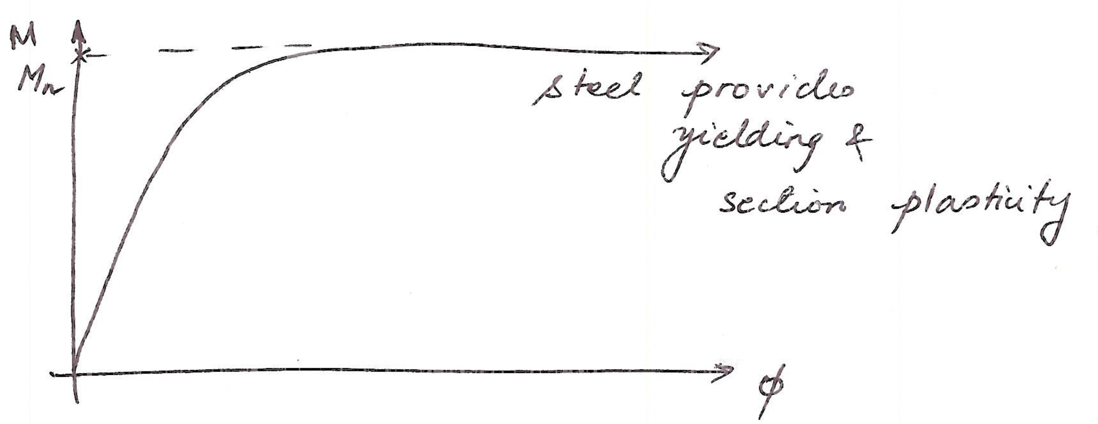
has less internal forces than reactions.

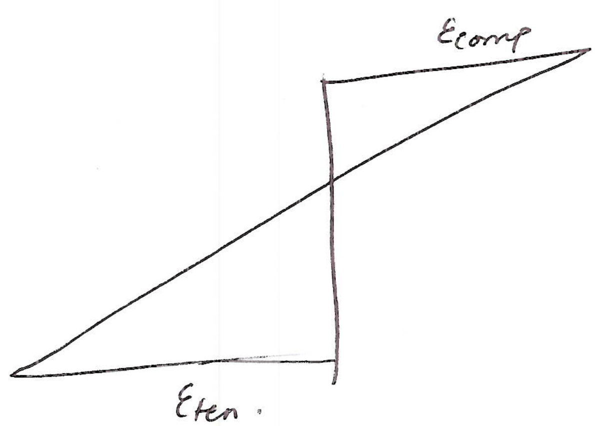


What about concrete?

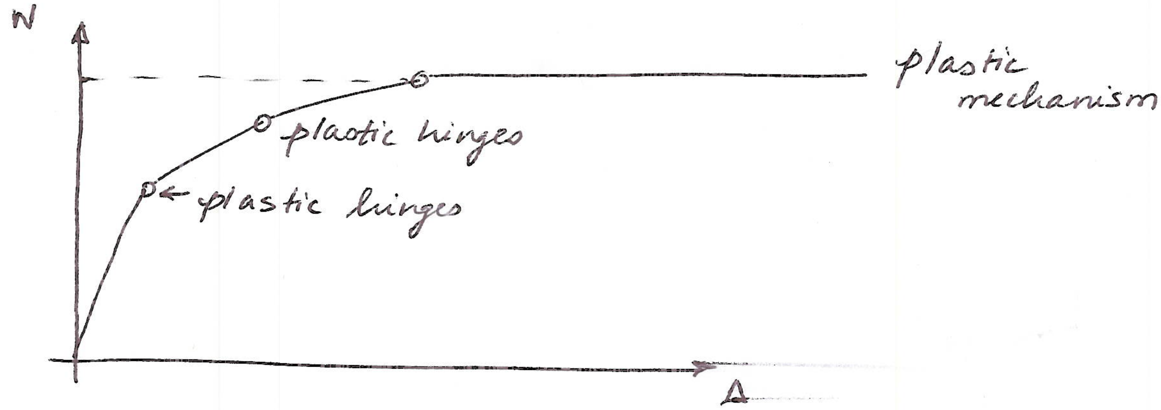


What about reinforced concrete



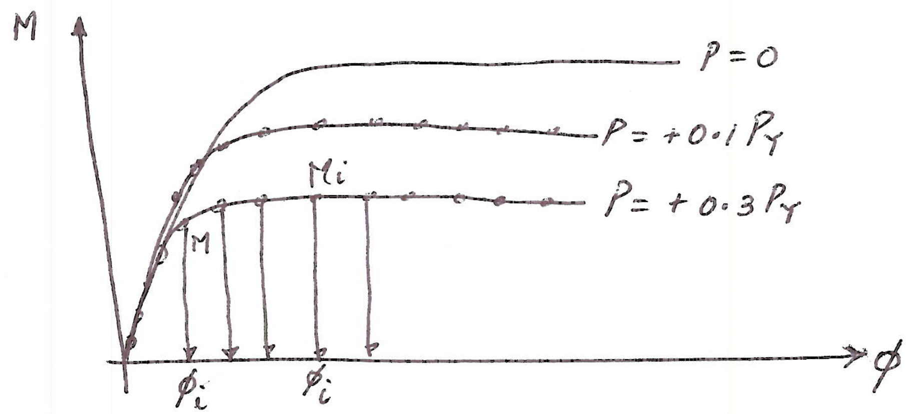


CE 592 PLASTIC DESIGN.



Section behavior $P - \varepsilon - M - \phi$ → key for Member plasticity

section $M-\phi$ behavior for different levels of axial force.



$\underbrace{\text{For each value of}}$ ϕ_i ⟶ M_i

while $P \rightarrow$ constant

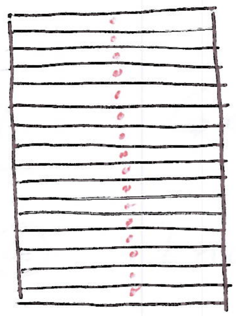
↓

increment curvature from zero to $\dfrac{?}{0.001 \text{ 1/in.}}$

Objective: Develop $P-\varepsilon-M-\phi$ relationships for cross-sections

Step 1. Discretize the cross-section into fibers.

uniaxial bending.



cross-section

How many fibers?

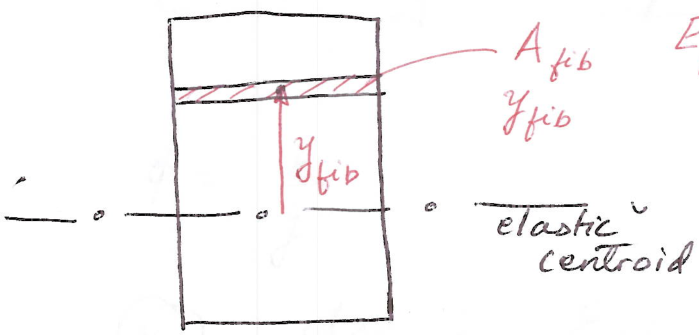
A. Difficult to answer.

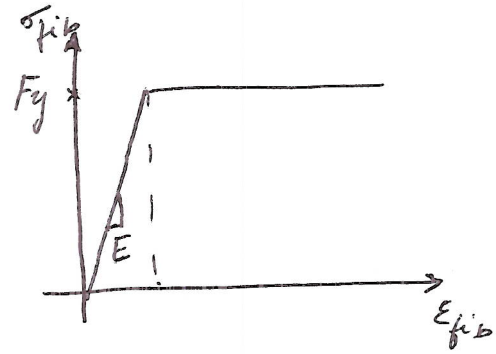
Through the depth @ least 20 or so fibers.

The number of fibers influences the numerical accuracy.

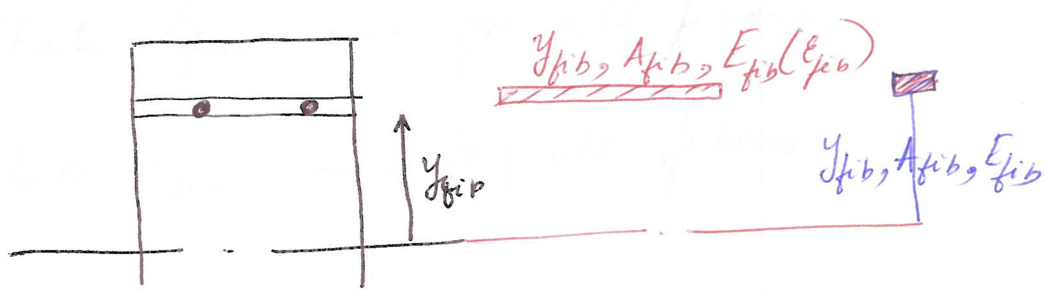
Step 2. Bunch of Book-keeping

elastic centroid of the cross-section

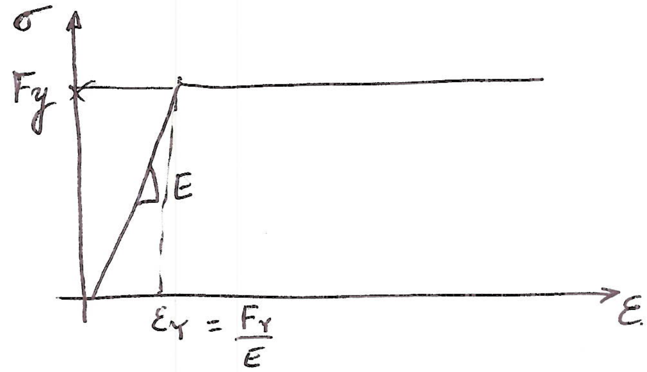




more than one material at each y_{fib} is possible



How to calculate σ_{fib}:

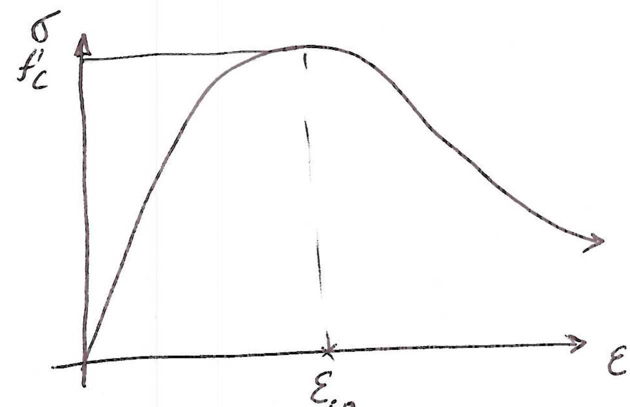


Is $|\varepsilon_{fib}| > \varepsilon_Y$?

N → $\sigma_{fib} = E \cdot \varepsilon_{fib}$

Y → $\sigma_{fib} = F_Y \times \frac{\varepsilon_{fib}}{|\varepsilon_{fib}|}$

$$\text{sign} = \frac{\varepsilon_{fib}}{|\varepsilon_{fib}|}$$



Popovic's

Hogenstad

$\sigma_c \rightarrow \varepsilon_c$

Is $\varepsilon_{fib} > 0$?

N → Popovic's equation for stress-strain

Y → $\sigma_{fib} = 0$

(e) Calculate $F_{fib} \longrightarrow \sigma_{fib} \times A_{fib}$

(f) Calculate $M_{fib} \longrightarrow \sigma_{fib} \times A_{fib} \times y_{fib}$

(g) $\sum_{i=1}^{n} F_{fib} - P = P_{unbal}$

(h) Is P_{unbal} within tolerances?

Tolerance of 0.1% is adequate for most situations.

Correct the original

$$\varepsilon_{cen}^{new} = \varepsilon_{cen}^{original}\left(1 - \frac{P_{unbal}}{P}\right)$$

just one way to correct the original.

(i) Go to (c)

After convergence

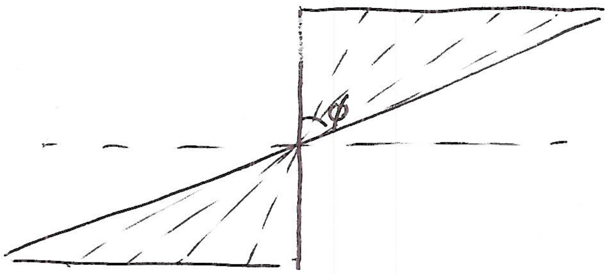
(j) $\phi_i \longrightarrow M_i \qquad M = \sum_{i=1}^{n} M_{fib}.$

(k) Increment ϕ & try to find M_i

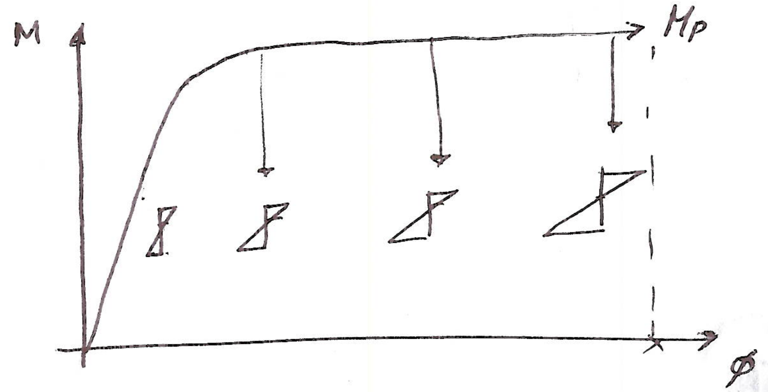
After convergence

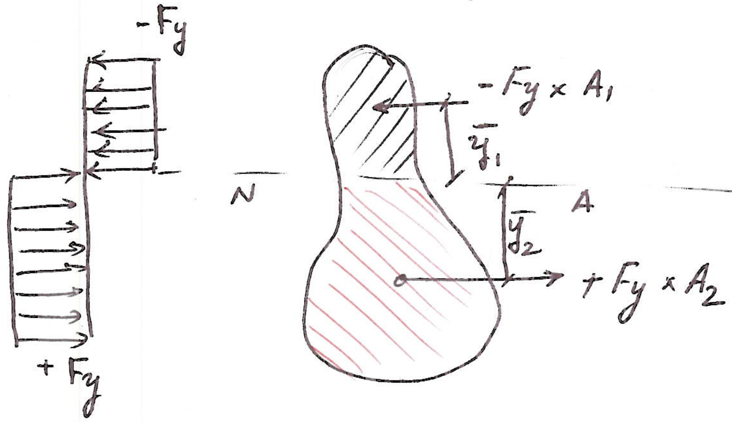
ε_{cen}, ϕ_i, ε_{fib}, σ_{fib}, P, M.

How to calculate M_p → plastic Moment.



ϕ is very large.





$\sum F = 0$

$P = 0$

$\therefore -F_y \times A_1 + F_y \times A_2 = 0$

$$A_1 = A_2$$

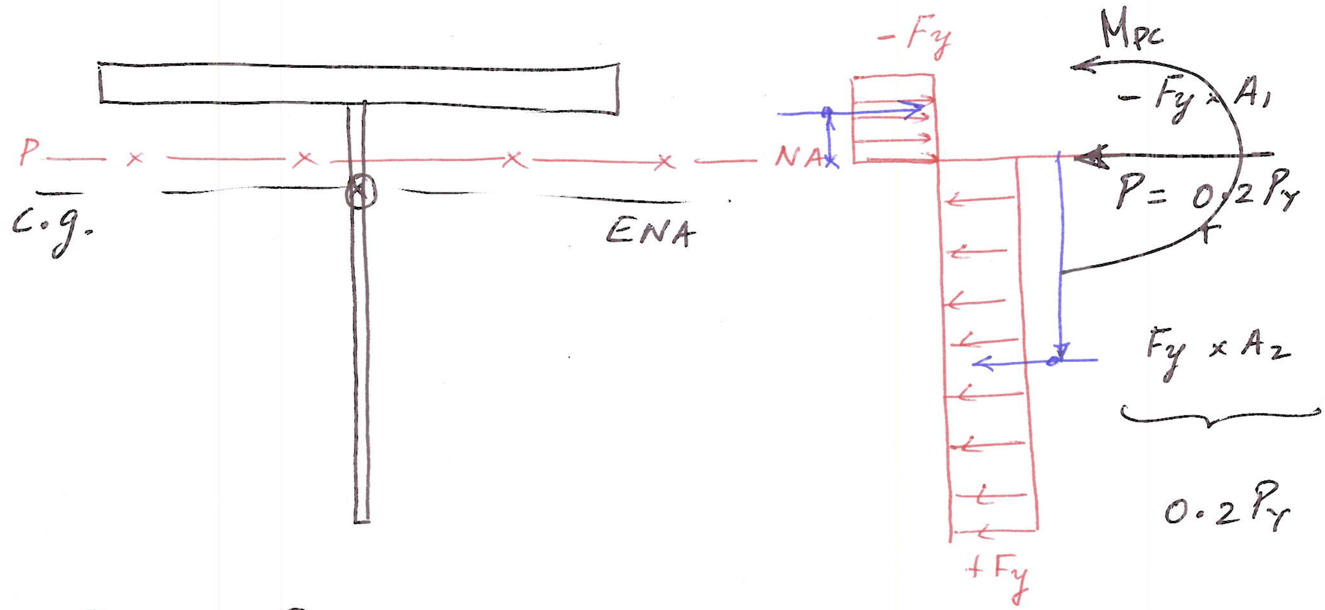
The neutral axis will be such that it divides the cross-section into two equal area.

$$M_p = F_y \times A_1 \times \bar{y}_1 + F_y A_2 \bar{y}_2$$

$$= F_y \underbrace{\left[\frac{A}{2}(\bar{y}_1 + \bar{y}_2)\right]}_{\text{section plastic modulus} = Z}$$

$$M_p = Z F_y$$

What will happen if I have a singly symmetric cross-section subjected to a non-zero axial force?

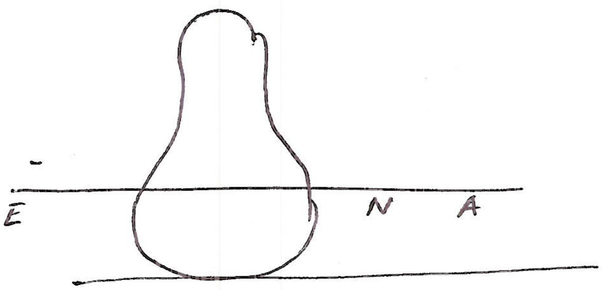


$P = +0.2\,P_Y$

PNA $\longrightarrow$ no longer divides the cross-section into two equal halves.

How will I calculate M_{pc}?

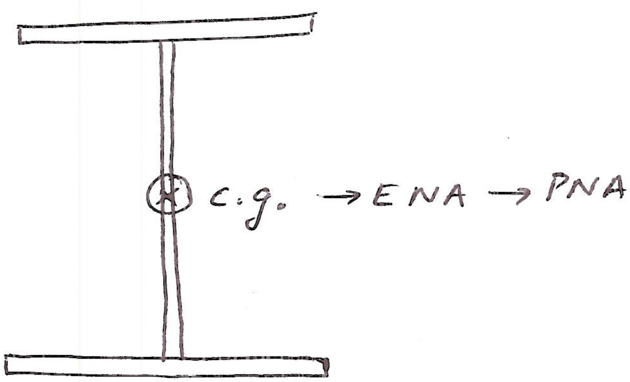
Elastic Neutral Axis = ?

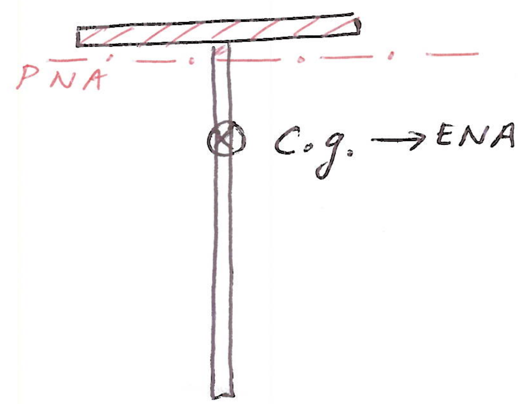


$$\frac{\sum A_i y_i}{\sum A_i} = \bar{y}$$

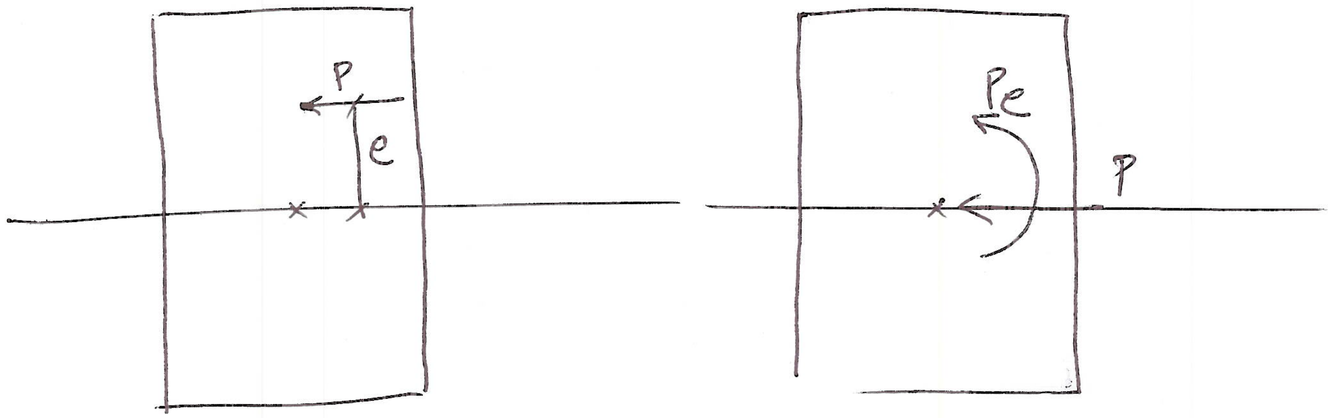
Not the same as the plastic neutral axis.

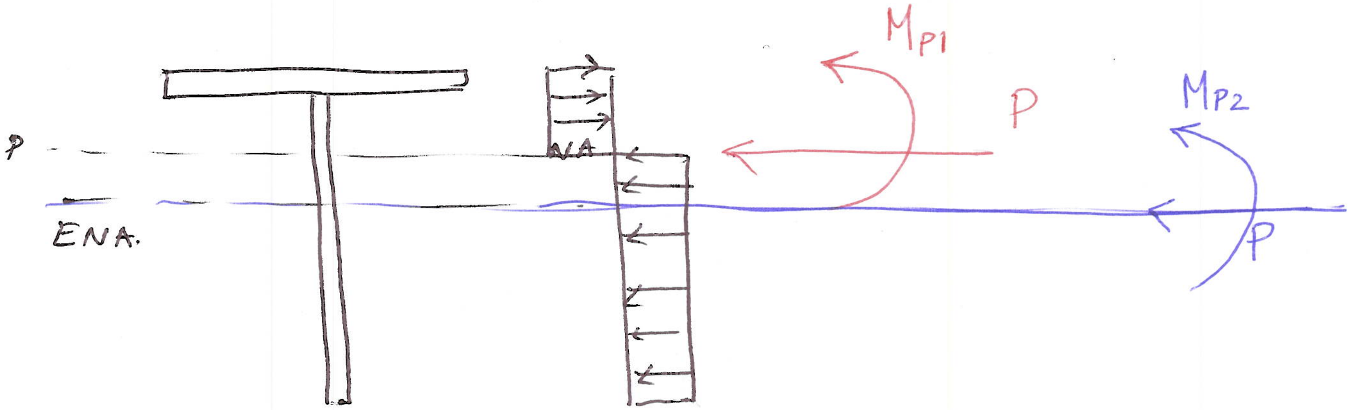
ENA $\neq$ PNA $\rightarrow$ unless the cross-section is doubly symmetric

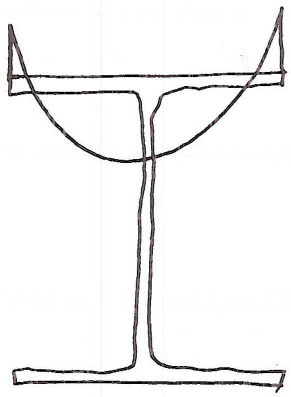




not fundamentally the same.


P
e
Pe
P

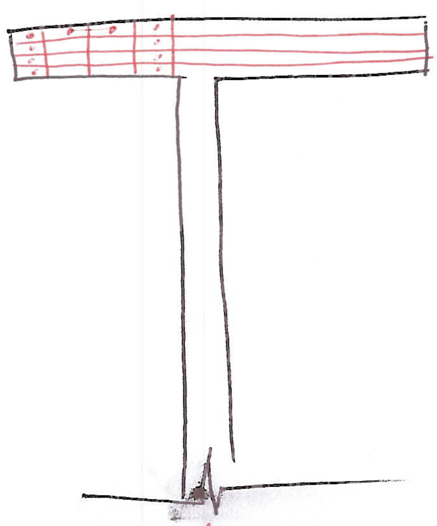

P
ENA.
NA
M_{P1}
P
M_{P2}
P



more precise distribution using residual strains gages.

Ans. Yes I need to worry about it when I go from $\varepsilon_{fib} \longrightarrow \sigma_{fib}$

because there will be a stress already present

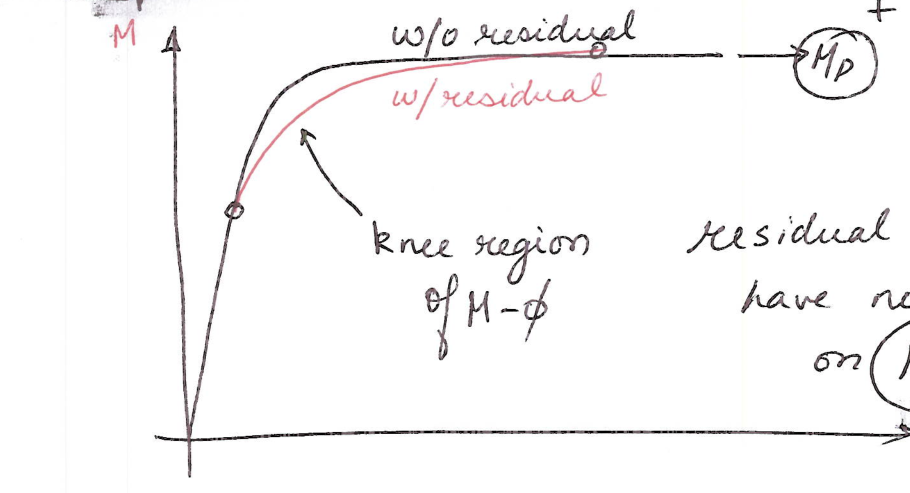


$$\frac{\sigma_{res.}}{E} \rightarrow (\varepsilon_{res}).$$

$\downarrow$

add it to $\varepsilon_{fib} \rightarrow \phi y$.

$$\varepsilon_{fib} = \varepsilon_{cen} + \phi\, y_{fib} + \varepsilon_{res}.$$



residual stresses have no influence on Mp

Special Conditions.

1. ENA & elastic centroid → are they always the same.

2. residual stress → what do they do?

3. local buckling & strain hardening

4. cyclic loading → stress states

Q.) Does the elastic neutral axis always pass through the centroidal axis?

$$\int \sigma \, dA = 0 = \int E \cdot \varepsilon \, dA = \int E \phi \cdot y \, dA = E\phi \int y \, dA = 0$$

Ans: For axial force = 0

If there is only one E through the depth.

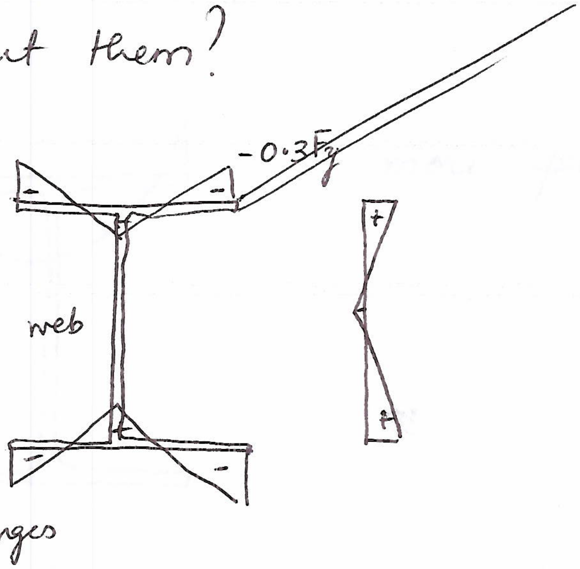
If there are two materials $\longrightarrow$ concrete & steel.

Elastic neutral axis $\longrightarrow$ needs to be determined

Q.) Which axis is going to be my reference axis for section P-M behavior.

Ans: c.g. is the best choice. for developing P-M curve.

Q:) What are residual stresses? Do I need to worry about them?

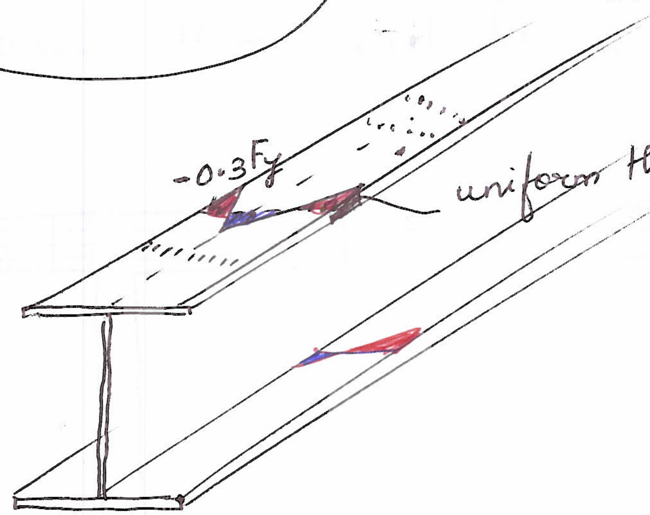


Simplified residual stress patterns through the section.

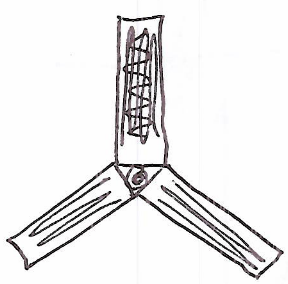
$F_r|_{max} = \pm 0.3 F_y$

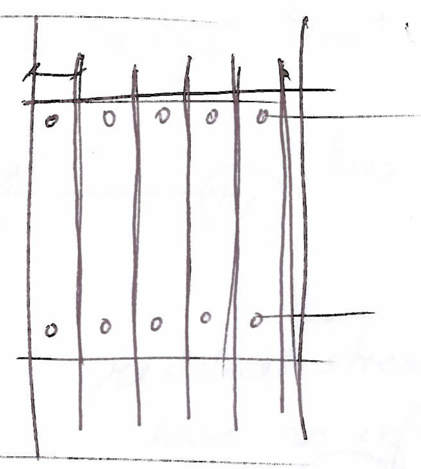
$$\int \sigma_r \, dA = 0.$$

residual stress pattern needs to be self-equilibrating



longitudinal or axial stresses.

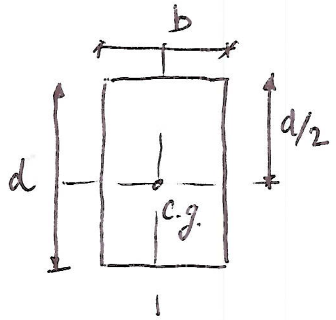


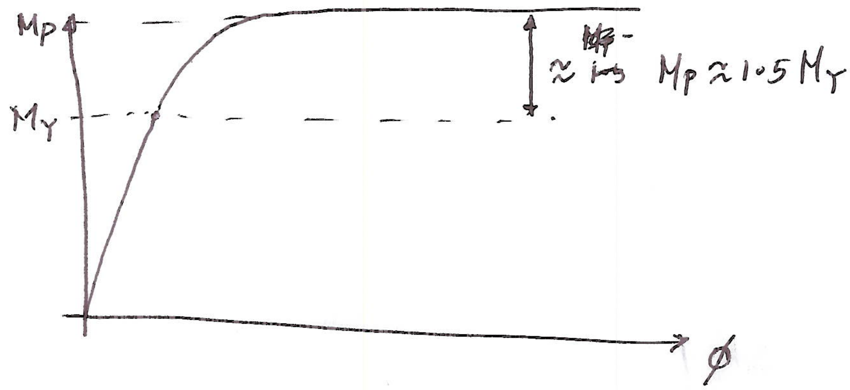


Effects of Unloading, Reloading or Cyclic Behavior.

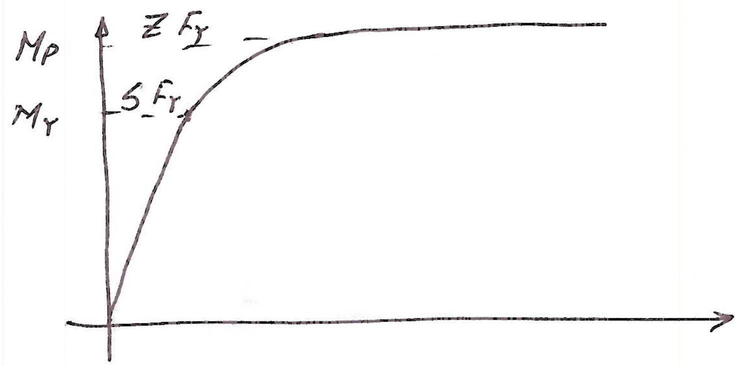
on behavior of cross-section.

rectangular cross-section:





What is shape factor.



$$f = \frac{M_P}{M_Y} = \frac{Z}{S}$$

$f = 1.11$ for WF

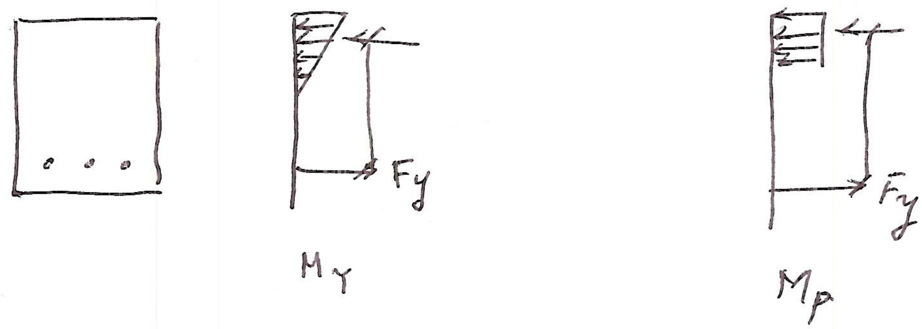
$= 1.5$ for ▯

$= 2.0$ for ◇

Why is the shape factor limited $\quad f \leq 1.5$

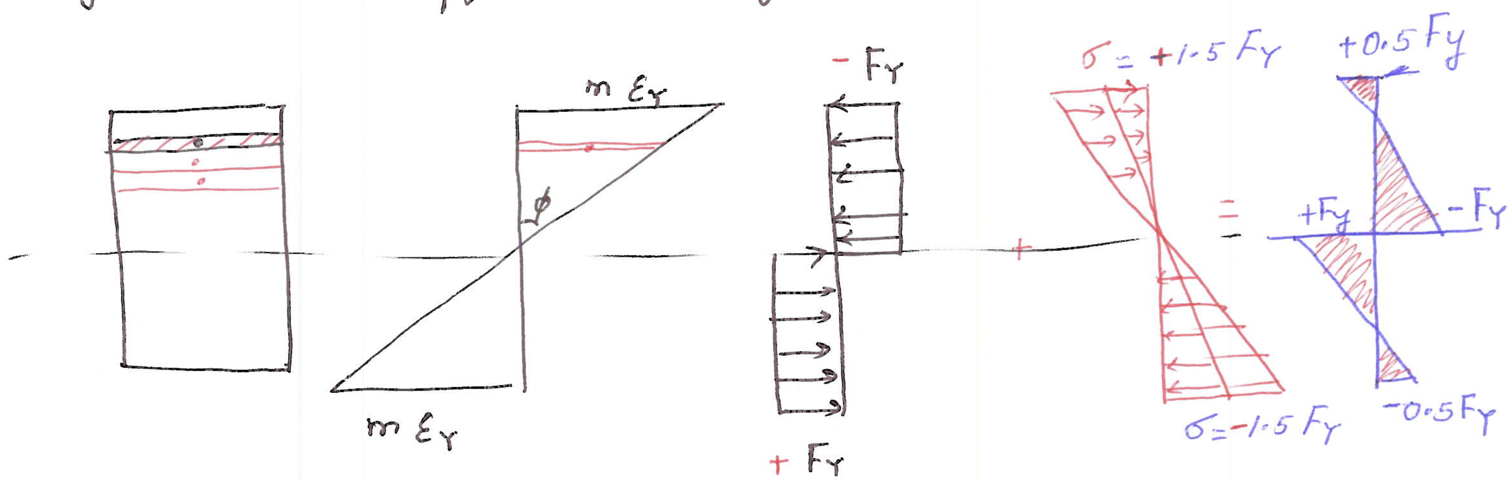
Design for $\underline{\underline{1.2D + 1.6L}} \longrightarrow M_P$

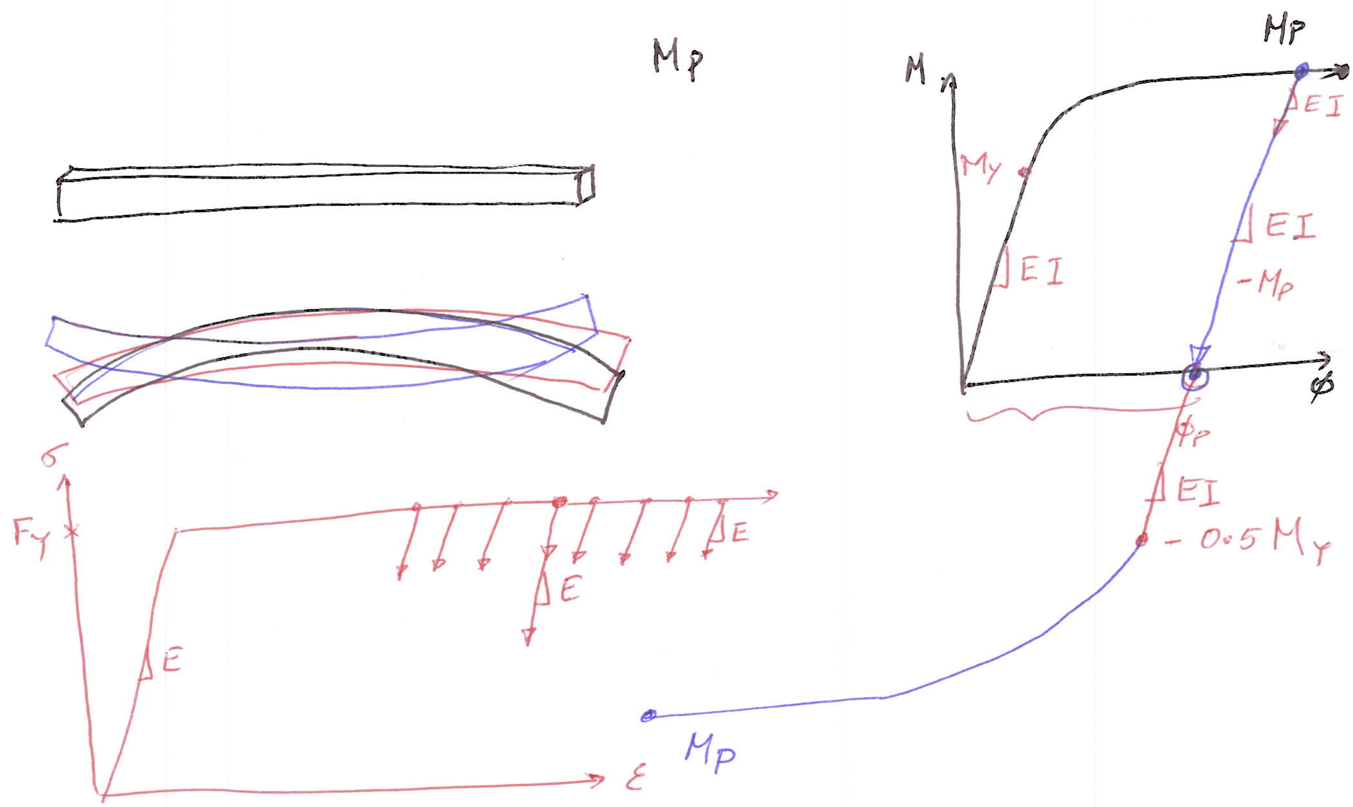
$M_y < M_P/1.5 \longrightarrow$ so that yielding does not service-level loads.

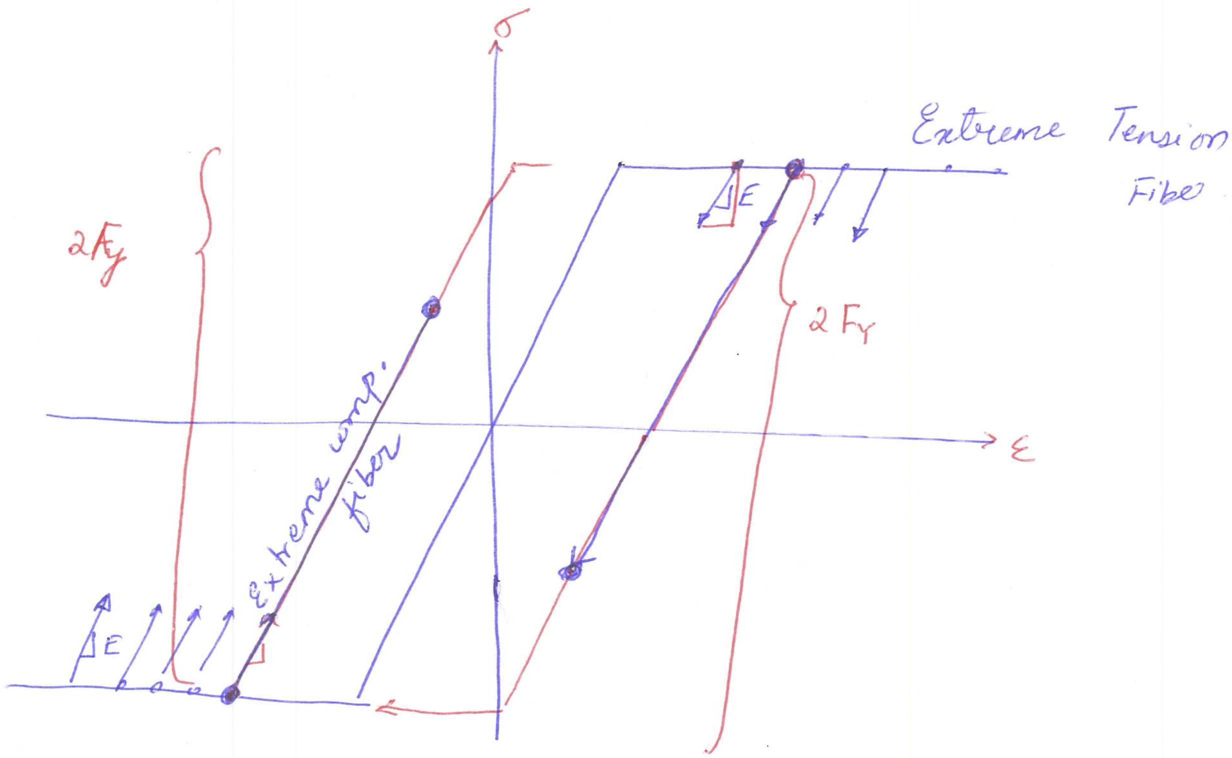


Shape factor for RC sections is usually 1.1 – 1.2

Cyclic loading effect on rectangular section.



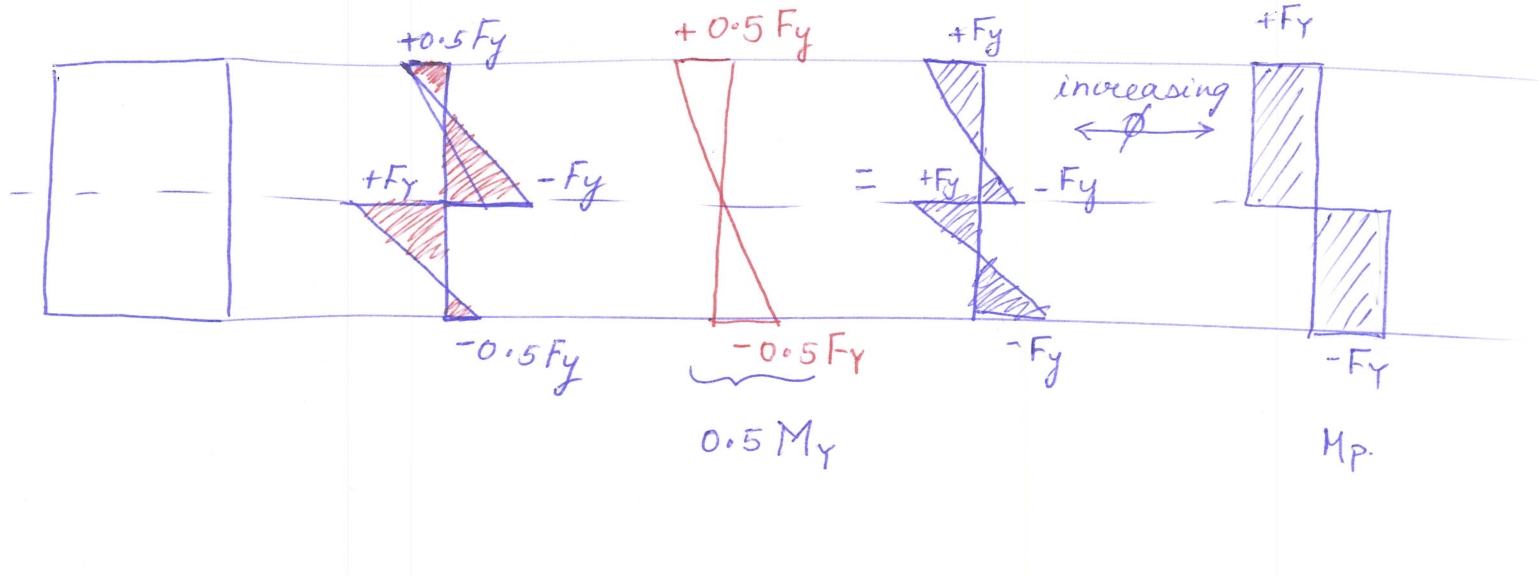


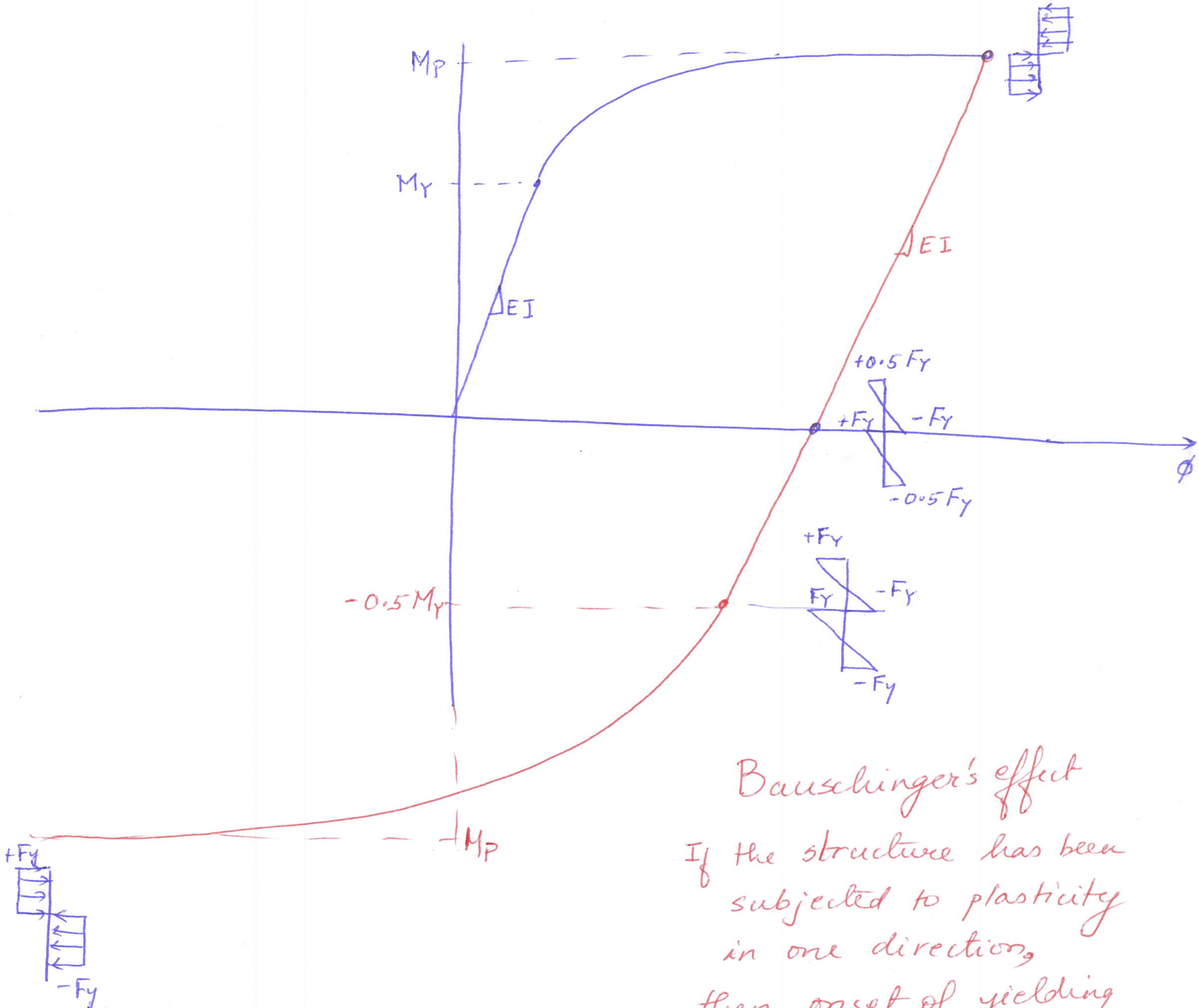


How to apply $-M_p$ → to unload the cross-section
↓
elastic stress blocks.

$$M_{unloading} = S \cdot \sigma = -M_p$$

$$\therefore \ \sigma = -\frac{M_p}{S} = \frac{-Z F_y}{S} = -1.5 F_Y$$





Bauschinger's effect

If the structure has been subjected to plasticity in one direction, then onset of yielding in the other direction will occur sooner.

- Section $P-\varepsilon-M-\phi$ relationship

  Steel section

  composite / RC section

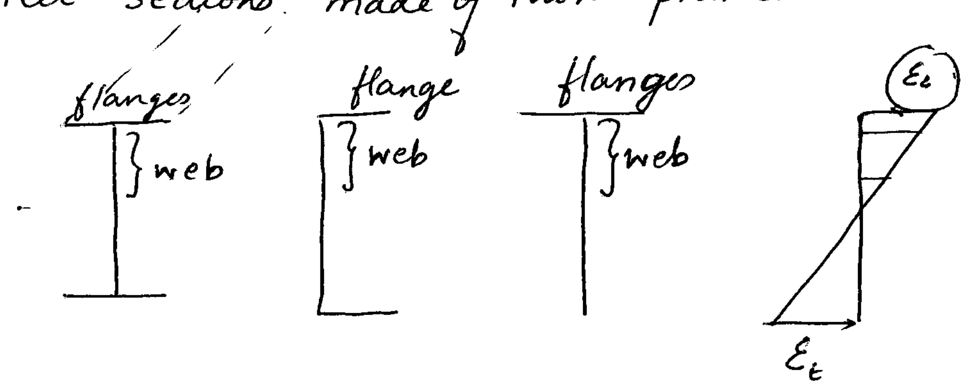
- Effects of Axial force on the $M-\phi$ behavior

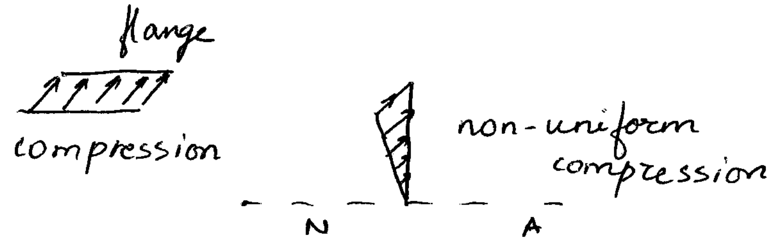
- Effects of residual stresses → effects of hot rolling & fabrication

- Effects of load reversal → residual stresses due to cold bending etc.

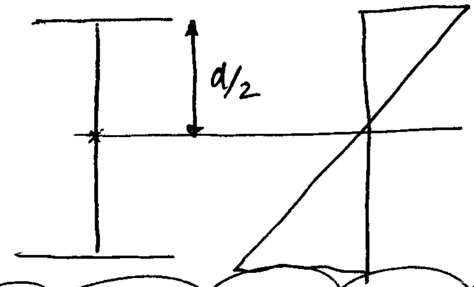
- Effects of local buckling

  Steel sections made of thin plates.



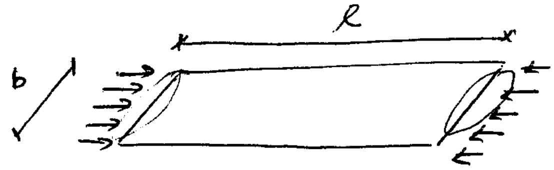


Thin plates subjected to compressive strains <u>will</u> undergo local buckling.



$\varepsilon_c \approx > 0.01$

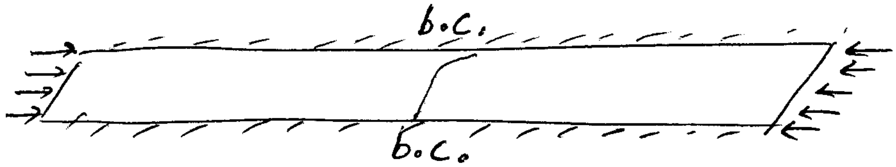
How to deal with local buckling?



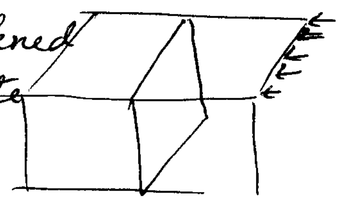
$\ell > 3b$ does not matter what ℓ is.

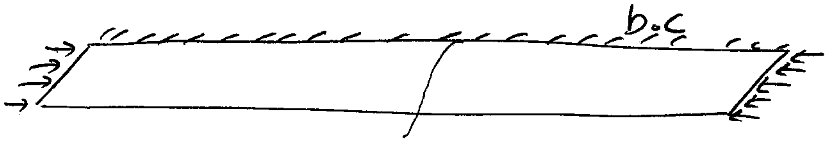
① ✓ primary parameter = b

②

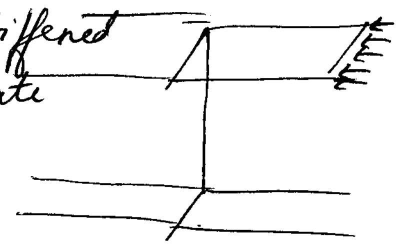


stiffened plate





unstiffened plate

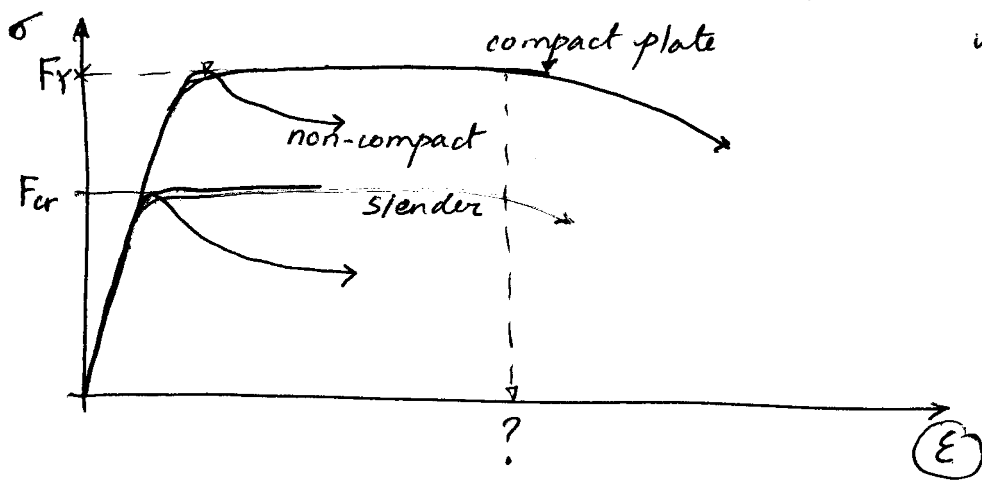


~~slender~~ ③ plate thickness, t

④ F_y of steel

$$\underbrace{\frac{b}{t} \times \sqrt{\frac{F_y}{E}}}_{\text{holds the key to local buckling.}} \longrightarrow \text{normalized plate slenderness coefficient}$$

Understand the compression behavior of plates?



with different λ values.

As λ increases, the plate becomes more slender.

λ_p λ_r are the limits that distinguish the plate behavior

When $\lambda > \lambda_r \longrightarrow$ slender plate
buckle for $F_{cr} < F_y$

when $\lambda_p < \lambda < \lambda_r \longrightarrow$ noncompact plate
buckle after reaching F_y

when $\lambda < \lambda_p \longrightarrow$ compact plate
develop F_y in compression & sustain for large ε_c values before local buckling.

$\underbrace{\lambda_r \text{ and } \lambda_p}$ are given in AISC Specification.

How did they get there? <u>Ans</u>: Combination of mechanics, stability analyses and experimental results

Applies to compact sections only.

$\lambda \leq \lambda_p$ for all the plate elements in the cross-section.

λ → ~~no~~ normalized slenderness ratio

λ_p → stiffened or unstiffened
→ ~~ben~~ flexure or axial comp + flexure.

λ_{pd} ⟶ ~~fa~~ compactness criteria for plastic design

λ_p for seismic design → "highly ductile"
⟶ AISC 341-10 (Seismic provisions)

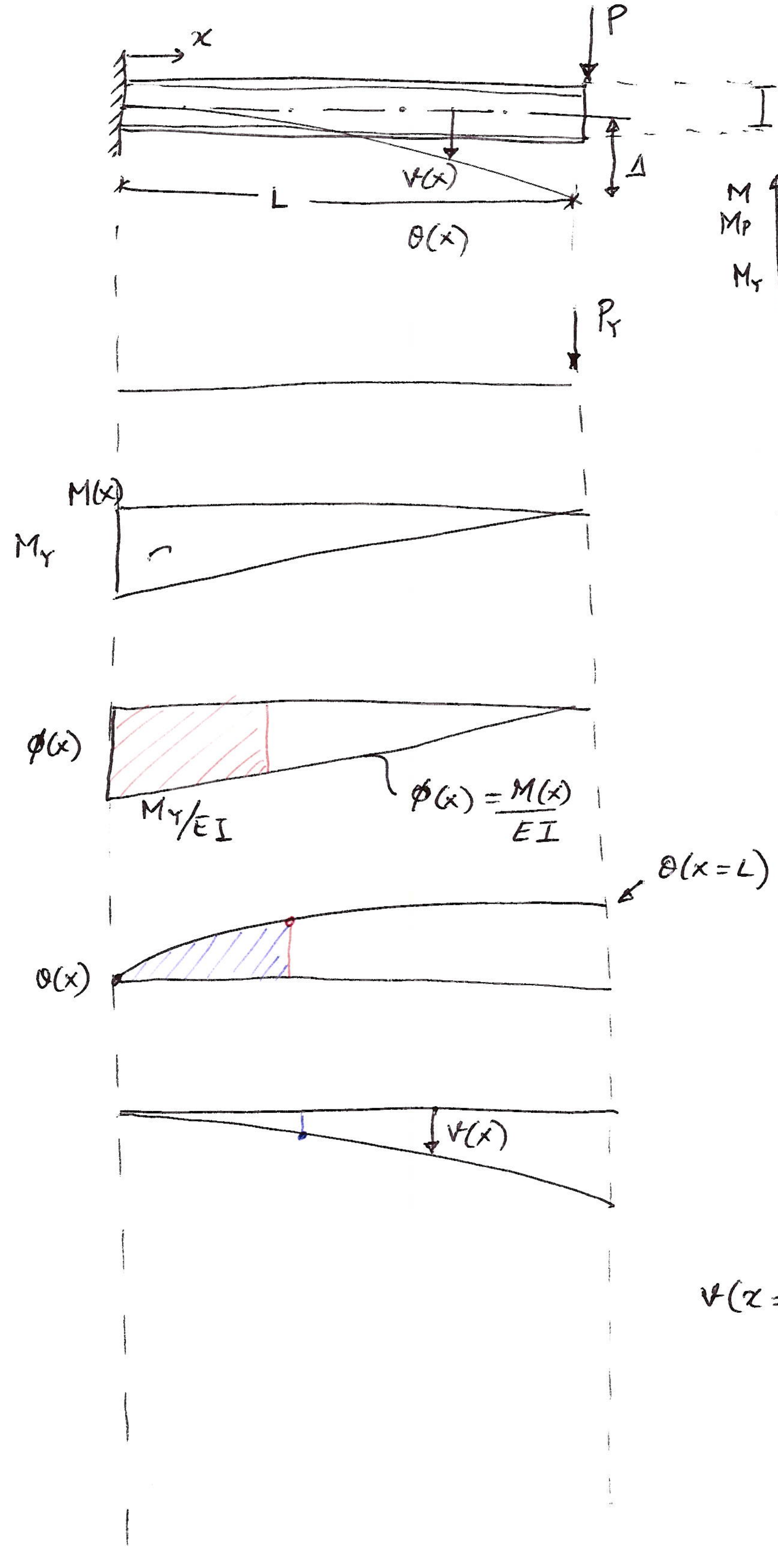
Thresholding → we setup a threshold to follow & prevent local buckling from highjacking the ductility or plastic behavior.

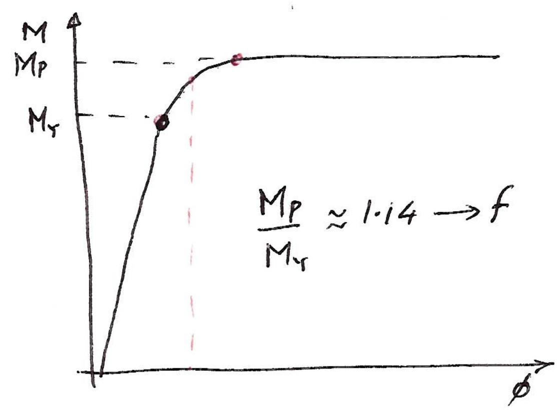
"Special MRFs" ⟶ highly ductile sections & plastic design behavior will be very useful.

——x—— Completed Section Plasticity ——

Plastic Design

Cantilever beam



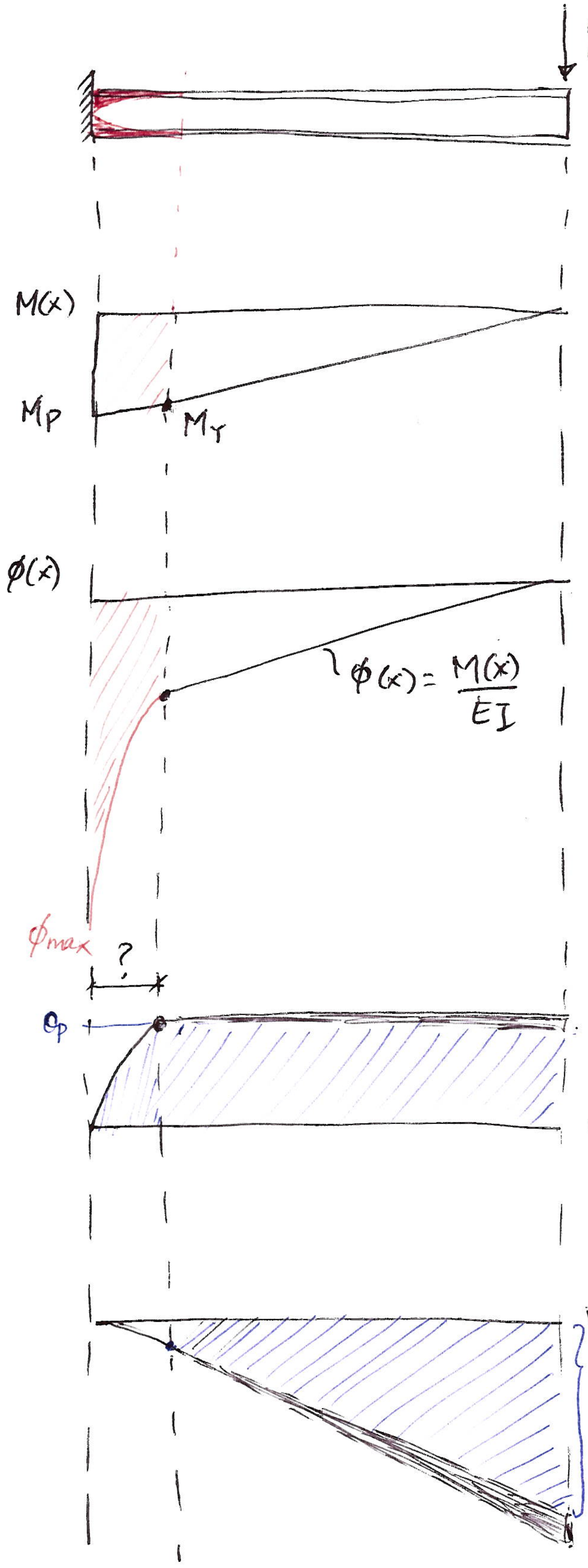


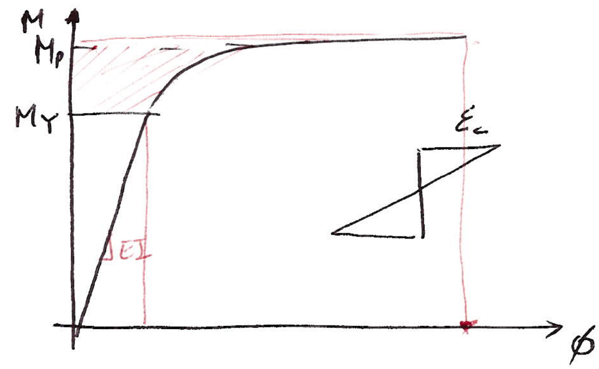
$$\theta(x) = \int_0^x \phi \, dx$$

$$v(x) = \int_0^x \theta(x) \, dx$$

$$v(x=L) = \Delta = \frac{M_Y L^2}{3EI}$$

$$\Delta = \frac{P_Y \times L^3}{3EI}$$





$$\theta(x) = \int_0^x \phi \, dx$$

$$v(x) = \int_0^x \theta \, dx$$

⊙ What will be the length of yielded portion?

If $M_{max} = M_p = P \cdot L$

$L_y \rightarrow$ yielded length

$$= \frac{M_p - M_Y}{M_p} \cdot L$$

$$= \frac{f-1}{f} \cdot L$$

If $f = 1.14 \longrightarrow L_y = 0.2L$

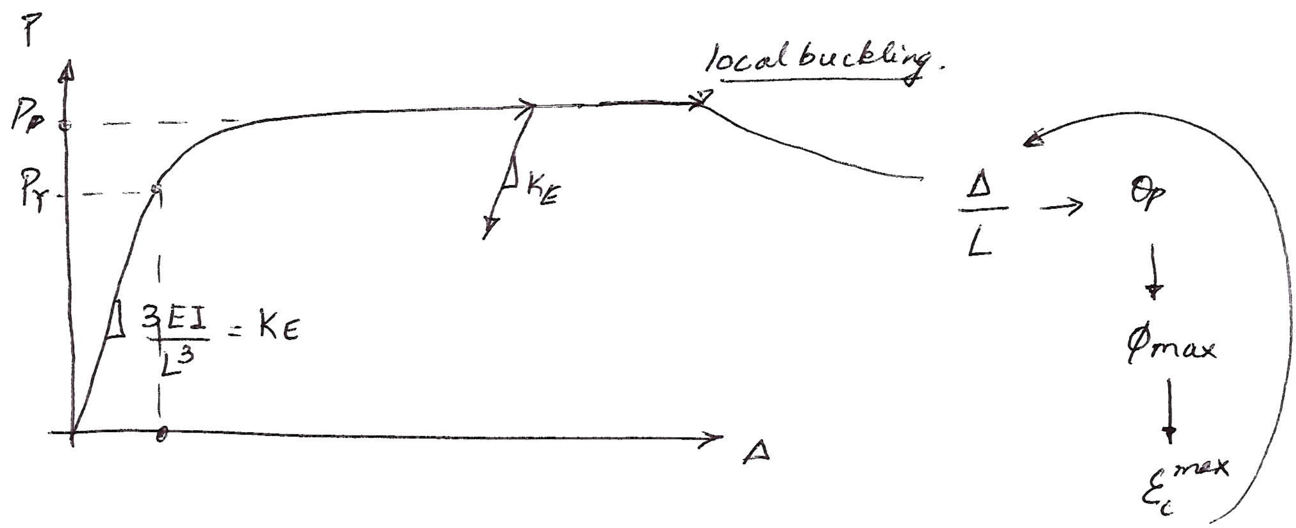
$f = 1.5 \longrightarrow L_y = 0.33L$

$f = 1.0 \longrightarrow L_y = 0$

The tip displacement is largely due to the curvature concentration in the yielded length.

PLASTIC HINGE $\longrightarrow$ yielded length of beam

- significant curvature concentration over plastic hinge
- significant rotation θ_p over hinge
- displacements associated with θ_p



compactness values designed to permit plastic hinge rotation θ_p upto 0.03 rad.

$P_p \longrightarrow$ plastic limit load for the member.

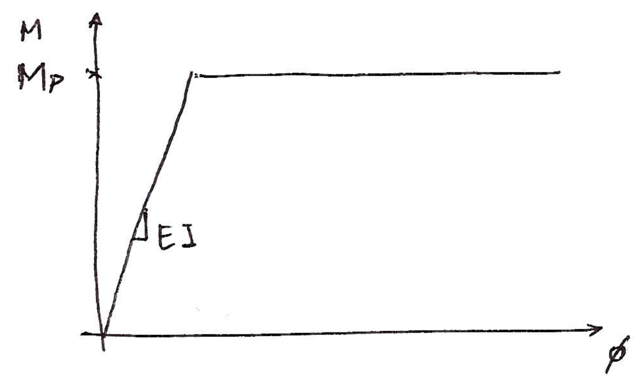
Idealized Simplified Behavior.

- Assume $f = 1.0$

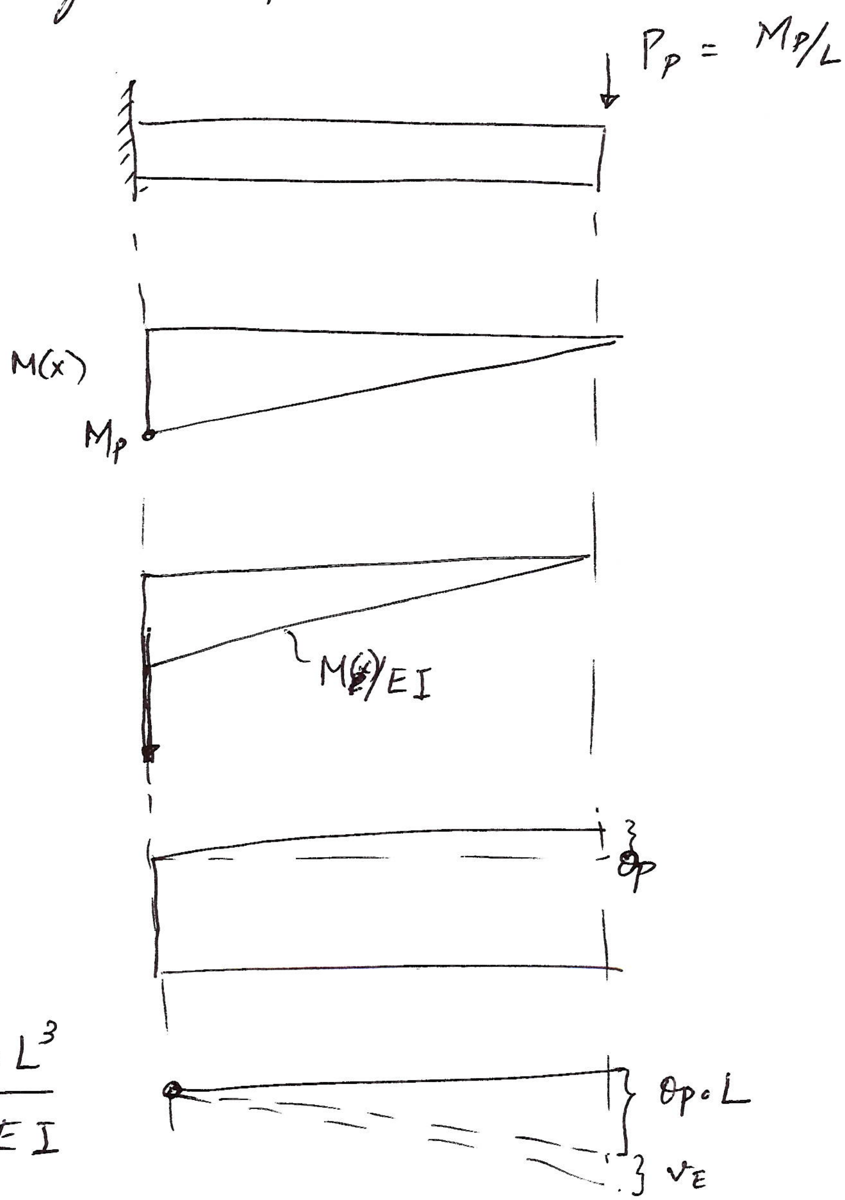
  yielding concentrated at a point $L_y = 0$

  elastic - plastic

  $M - \phi$ response

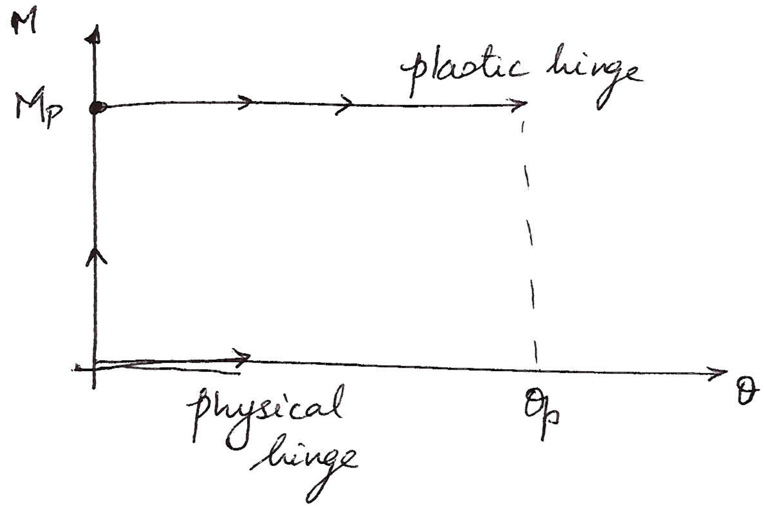


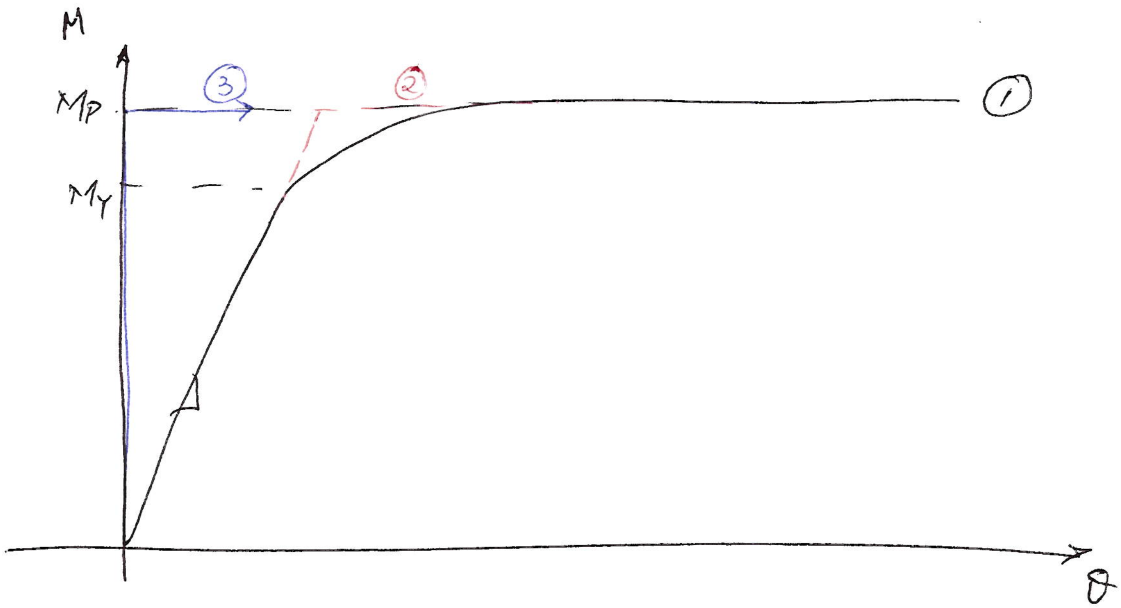
- plastic hinge: idealized concept with zero length and plastic rotation θ_p occurring at a point.



$$\Delta = \theta_p L + \frac{P_p L^3}{3EI}$$

Rigid - Plastic Idealization.





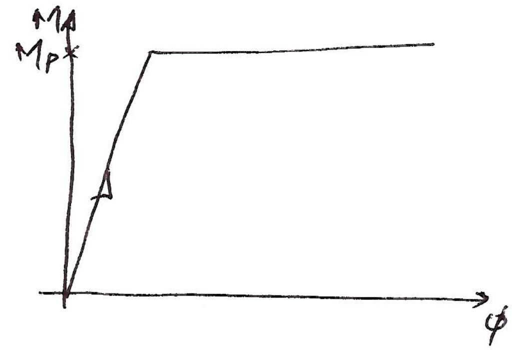
3 possibilites.

2 & 3 $\longrightarrow$ keep around

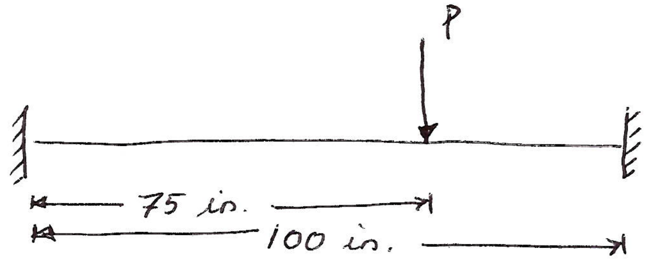
1 $\longrightarrow$ ABAQUS.

CE592: PLASTIC DESIGN

idealized zero length plastic hinge



Example:



$E = 30{,}000$ ksi

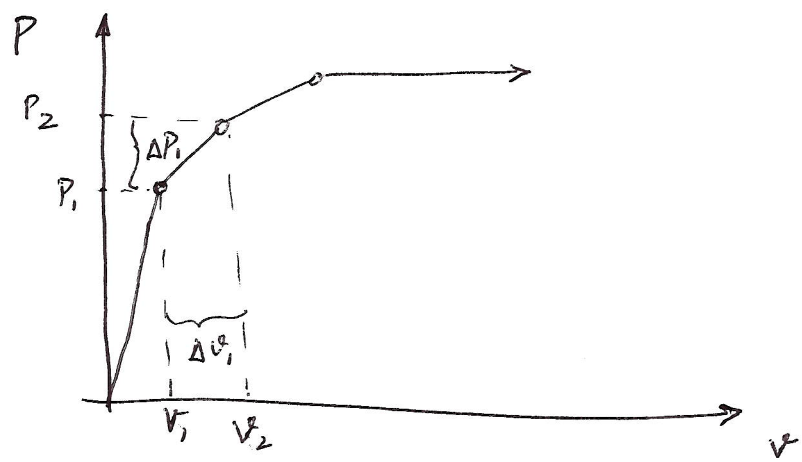
$I = 250\ in^4$

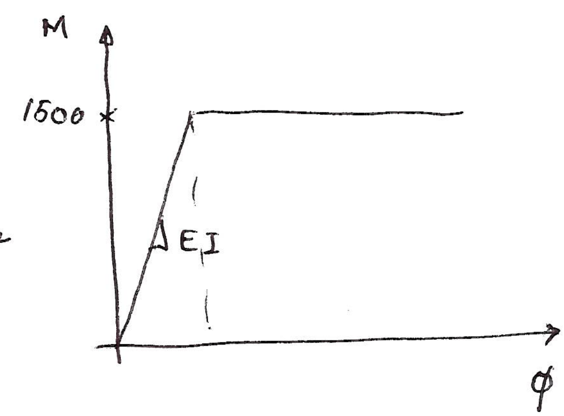
W14x26 beam

$M_p = 1500$ k-in.

$F_y = 36$ ksi

Obj: Complete analysis to collapse



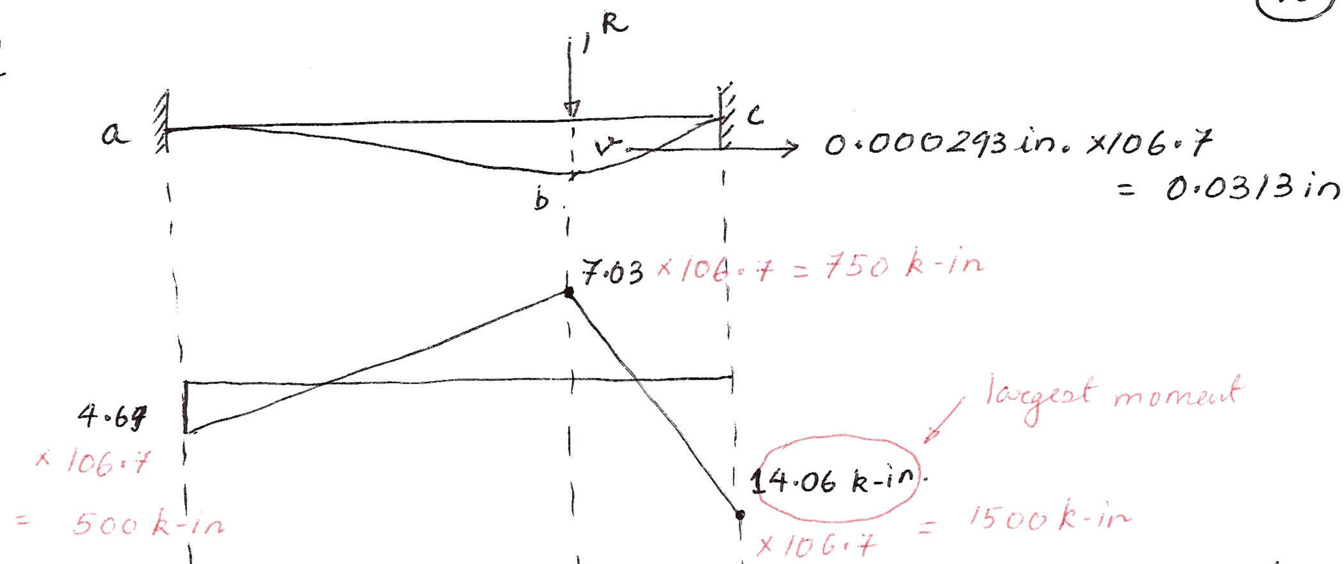


P_i, M_i, v_i, θ_{pi}

& ΔP_i, Δv_i, ΔM_i etc

} notation

Step ①



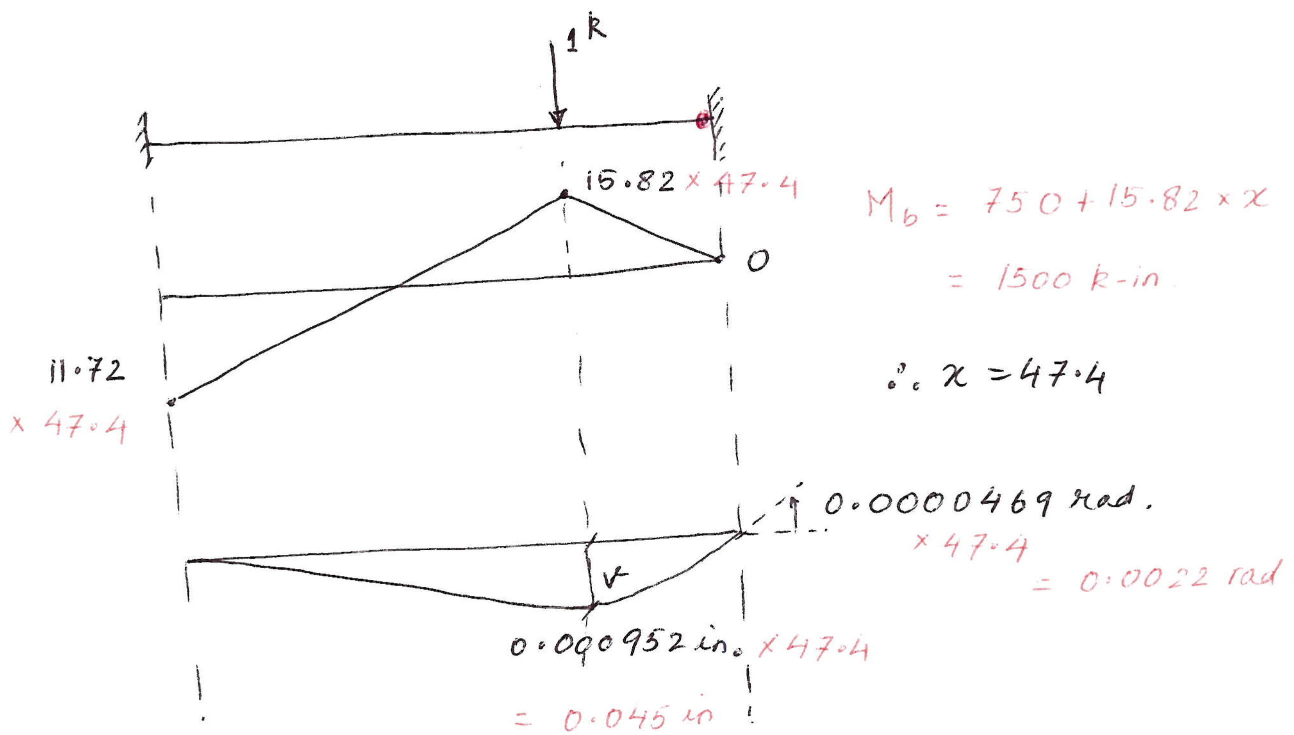
$$\frac{M_P}{14.06} = \frac{1500}{14.06} \times 1^k$$

$$= 106.7^k$$

scaled the response by 106.7^k

1st plastic hinge forms @ c

Step ② Modify the model for incremental analysis

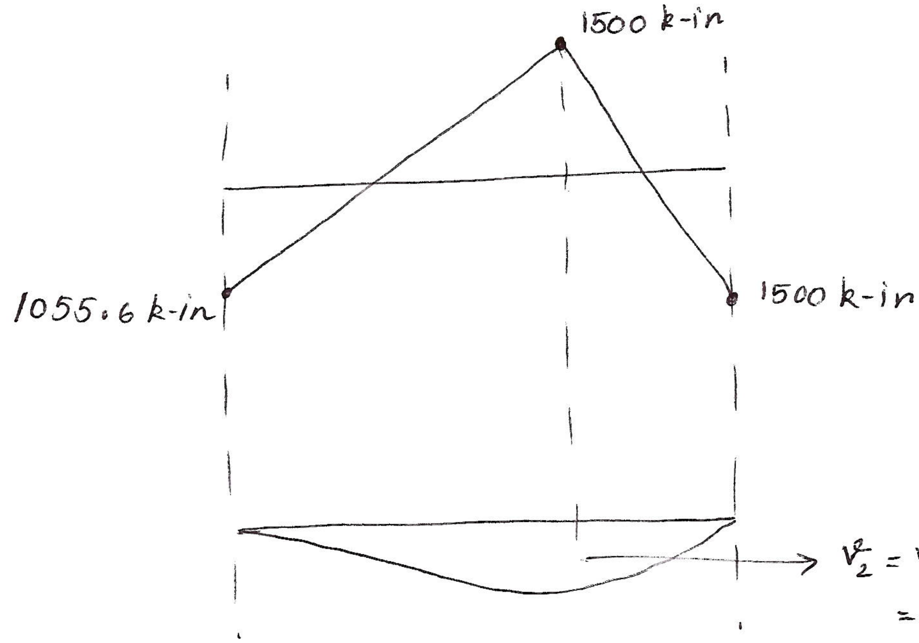


$$M_b = 750 + 15.82 \times x$$

$$= 1500 \text{ k-in}$$

$$\therefore x = 47.4$$

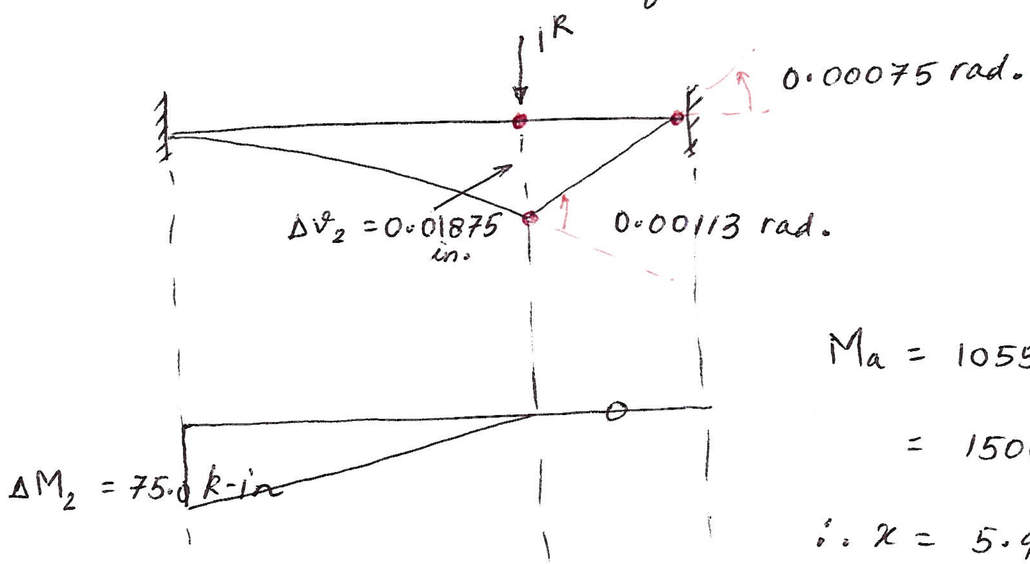
∴ Total response after 2^{nd} step



$v_2 = v_1 + \Delta v_1$

$= 0.0764$ in.

Step (3) Incremental response after 2^{nd} hinge

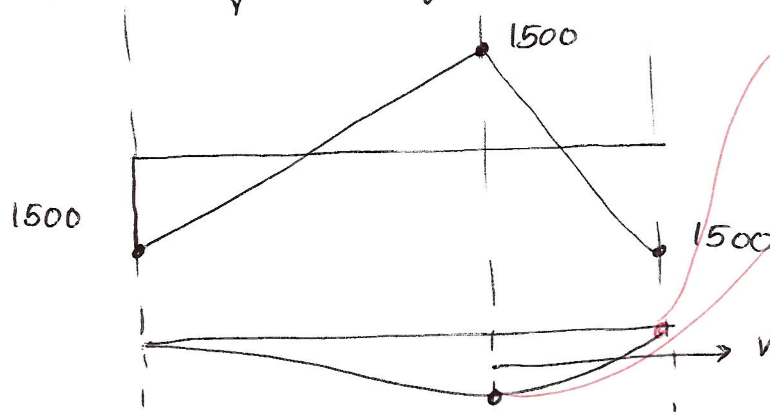


$M_a = 1055.6 + 75x$

$= 1500$

$\therefore x = 5.93^k$

Total response after hinge at (a)



$\theta_{p3} = \theta_{p2} + \Delta\theta_{p2}$

$= 0.0066$ rad.

$\theta_{p3} = \theta_{p2} + \Delta\theta_{p2}$

$= 0 + 0.00113 \times 5.93$

$= 0.0066$ rad.

$v_3 = v_2 + \Delta v_2$

$= 0.0764 + 5.93 \times$

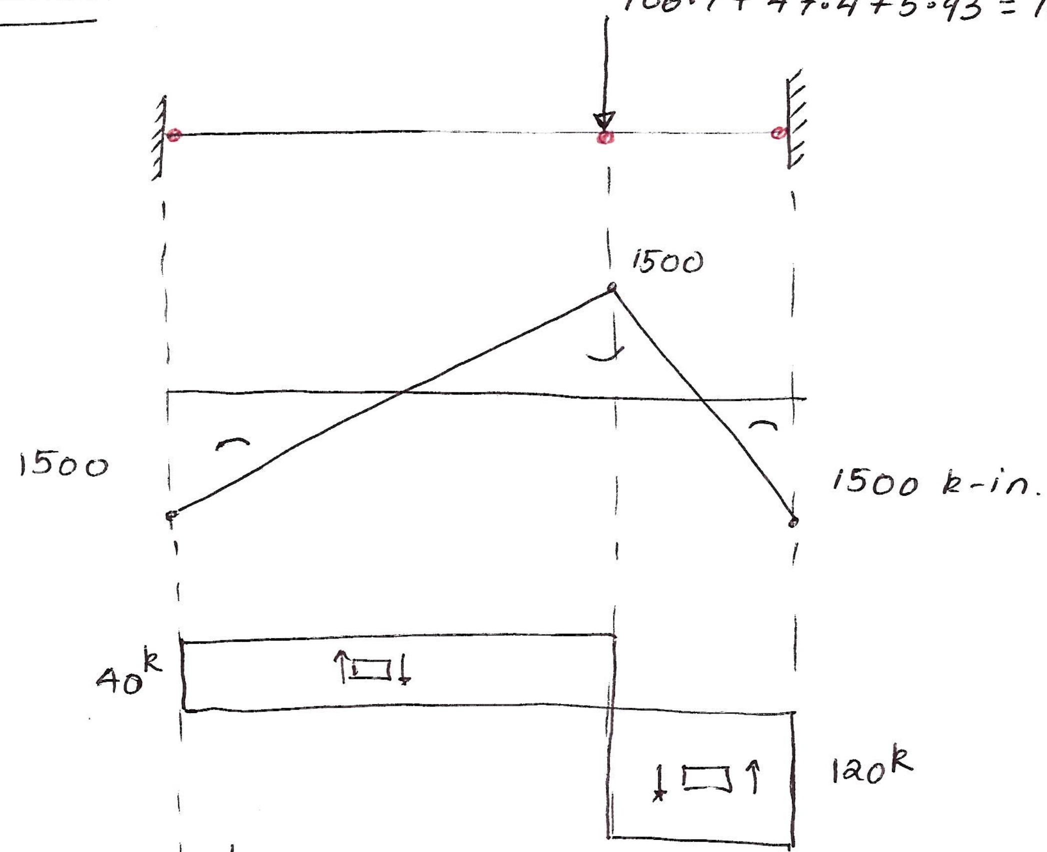
Step 4 Incremental response after plastic hinge at 'a'

mechanism & $\Delta P_3 = 0$

Total response:

$106.7 + 47.4 + 5.93 = 160^k$

(i)



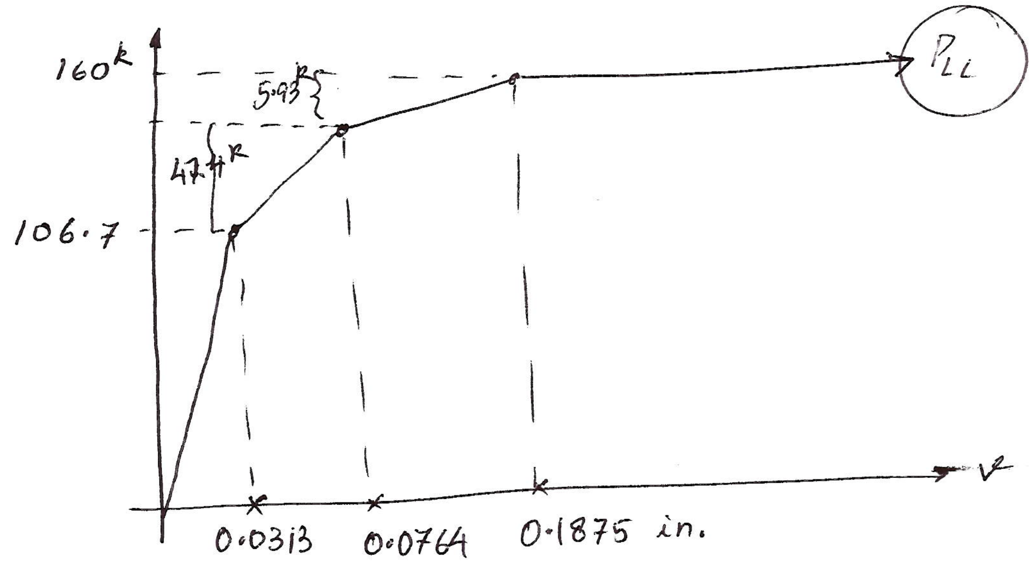
① EQUILIBRIUM !

② $M \leq M_p$ everywhere

③ Enough hinges to form a mechanism

Structure is at its plastic limit load.

Total load displacement curve:



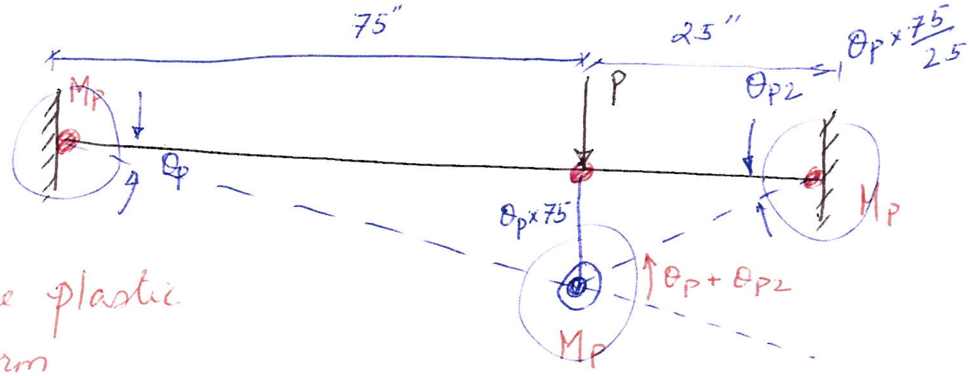
Event-to-Event Analysis

1. conducting elastic analysis
2. scaling the results so that first plastic hinge(s) forms.
3. Adding the hinge at location with Mp. Change model and conduct incremental analysis
4. Conduct elastic analysis for incremental loads.
5. Scale results so that 2nd plastic hinge(s) form
6. Go to 3 until plastic mechanism forms.

Pursue 2 different paths

1. Event-to-event analysis → using computer
2. Calculate directly plastic limit load by mechanism analysis.

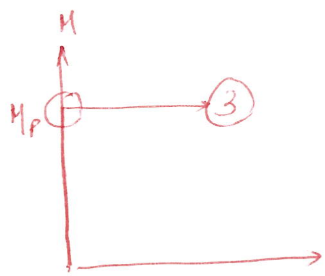
PLASTIC MECHANISM ANALYSIS



① Assume where plastic hinges form

$$W_{ext} = P \times \Delta = P \times \theta_p \times 75$$

$$W_{int} = M_p \times \theta_p + M_p(\theta_p + \theta_{p2}) + M_p \times \theta_{p2}.$$



$$\therefore \quad W_{ext} = W_{int}$$

$$P \times \theta_p \times 75 = M_p \times \theta_p + M_p \times 4\theta_p + M_p \times 3\theta_p$$

$$P \times \cancel{\theta_p} \times 75 = 8\, M_p\, \cancel{\theta_p}$$

$$\therefore P = \frac{8 \times M_p}{75\,in} = \frac{8 \times \cancel{1500}^{\,20}\ k\text{-}\cancel{in}}{\cancel{75}\ \cancel{in}}$$

$$= 160\ kip$$

KINEMATIC METHOD or Mechanism Method

Definition: Estimate the plastic limit load of a structure by applying virtual displacements to an assumed mechanism, and equating internal and external virtual work.

Method:
- Satisfies equilibrium
- sufficient number of hinges to form a mechanism

may violate the requirement $|M| \leq M_p$

upper bound method →

plastic limit load by mechanism approach $\geq$ actual plastic limit load

Advantages:
- Gives you estimate of plastic limit load directly. No need for event-to-event analysis
- Generally easiest method for simple flexural systems (beams, frames, plates)

Disadvantage: If incorrect mechanism is used, then the plastic limit load will be unconservative.

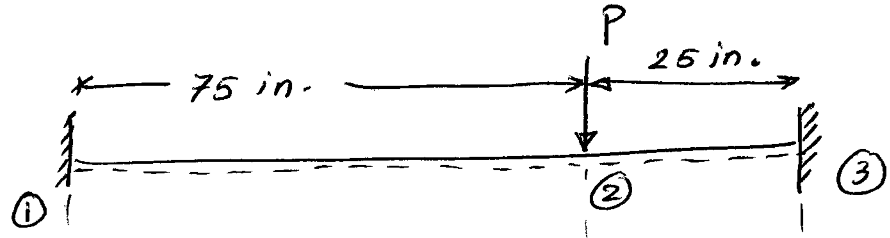
Procedure:

You can assume different mechanisms and calculate plastic limit load.

The lower plastic limit load and corresponding mechanism will always govern.

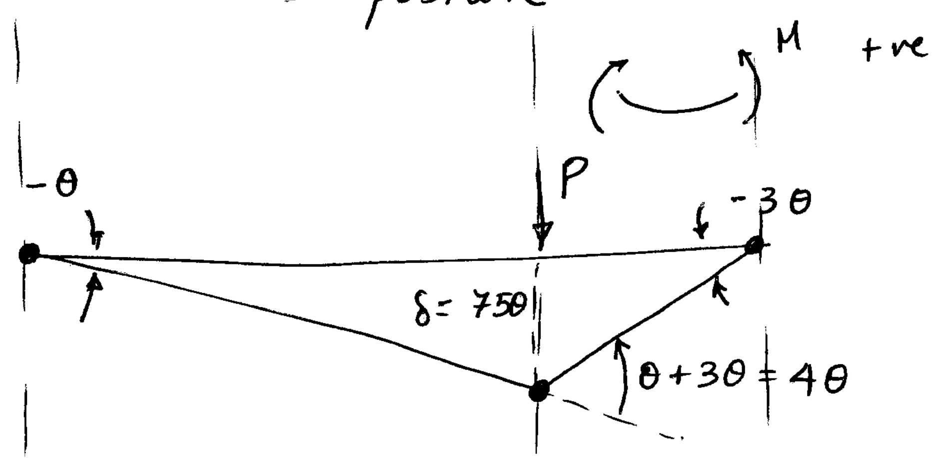
This method does not generate information regarding deflections, rotations, load-deflection curves, order in which hinges form etc.

Example:



$M_p = 1500$ k-in

M & θ that produce tension on - - - side = positive



Apply virtual displacement at hinges only

members move as rigid bodies.

$$W_{EXT} = W_{INT}$$

rigid-plastic idealization

In general;

$$\sum \underset{\text{real loads}}{P}\,\underset{\text{virtual disp}}{\bar{\delta}} = \sum \underset{\text{real moments}}{M}\,\underset{\text{virtual rotation}}{\bar{\Theta}} + \cancel{\int \underset{\text{real moment}}{M}\,\underset{\text{virtual } \phi}{\bar{\phi}}\,dx}$$

$$\therefore \; W_{EXT} = P \times 75\,\theta$$

$$W_{INT} = -\theta \times M_1 + 4\theta\, M_2 - 3\theta\, M_3$$

$$\therefore \; 75\,\cancel{\phi} \times P = \cancel{\phi}\left(-M_1 + 4M_2 - 3M_3\right)$$

$$\therefore \; \underbrace{P = \frac{-M_1 + 4M_2 - 3M_3}{75}}_{\text{equilibrium equation for the beam}}$$

$$\left.\begin{aligned} M_1 &= -M_P \\ M_3 &= -M_P \end{aligned}\right\} \quad \begin{aligned} W_{INT} &= \theta_P \times M_P + 4\theta_P M_P + 3 M_P \theta_P \\ &= 8 M_P \theta_P \end{aligned}$$

W_{INT} will always be positive

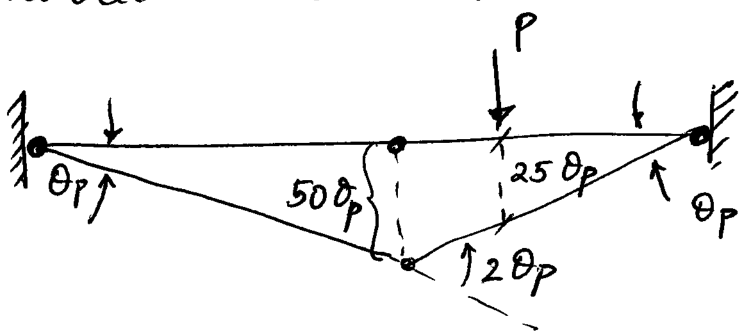
no need to track signs of M, θ for

internal work

$$75\, P\, \cancel{\theta_p} = 8 M_p\, \cancel{\theta_p}$$

$$\therefore P_p = 8 \times \frac{M_p}{75} = \frac{8 \times 1500}{75} = 160^k$$

If, incorrect mechanism



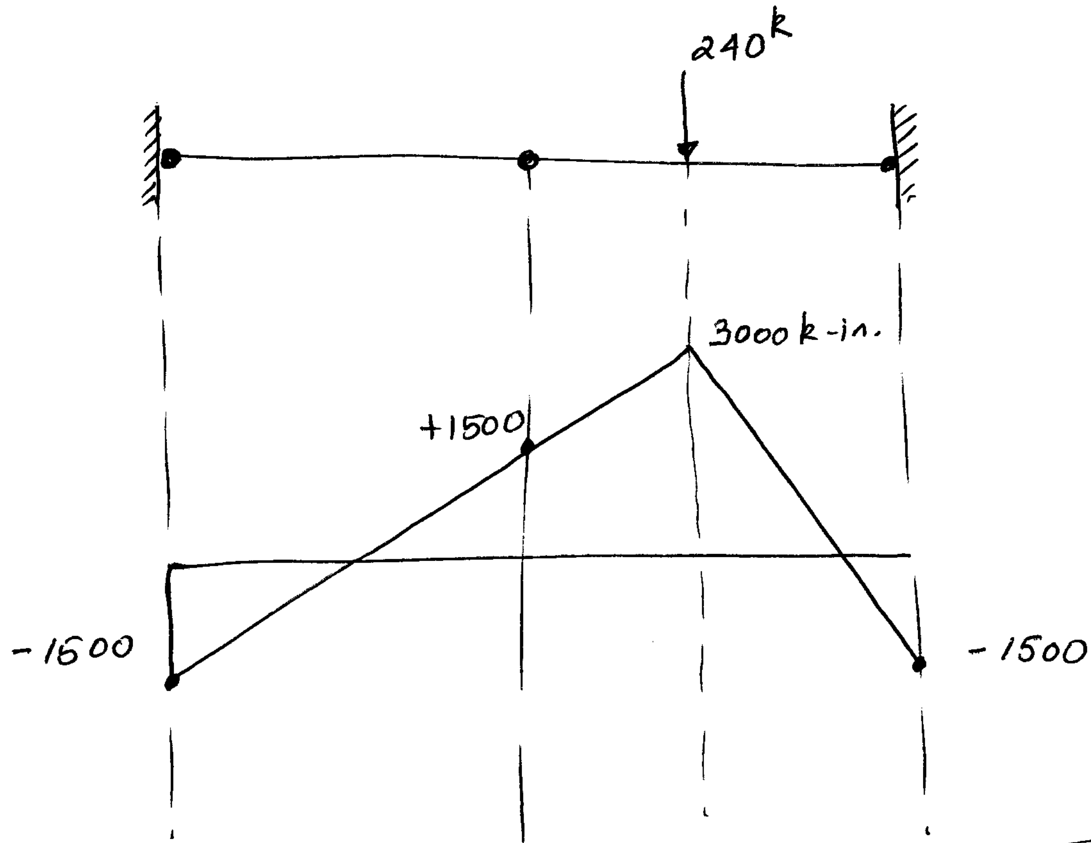
$$W_E = P \times 25\, \theta_p$$

$$W_{INT} = M_p\theta_p + M_p \times 2\theta_p + M_p\theta_p = 4M_p\theta_p$$

$$\therefore P_p = \frac{4M_p}{25} = \frac{4 \times 1500}{25} = 240^k$$

$>$ true plastic limit load.

But, I could check the resulting bending moment diagram to see if $|M| > M_p$ anywhere



$M_p = 1500$ k-in

$|M| \geq M_p$

Mechanism or kinematic method

→ satisfied equilibrium

→ sufficient # of hinges for mechanism

but M can exceed M_p if the incorrect mechanism is used.

If the correct mechanism is assumed,

$|M| = M_p$ at the hinges

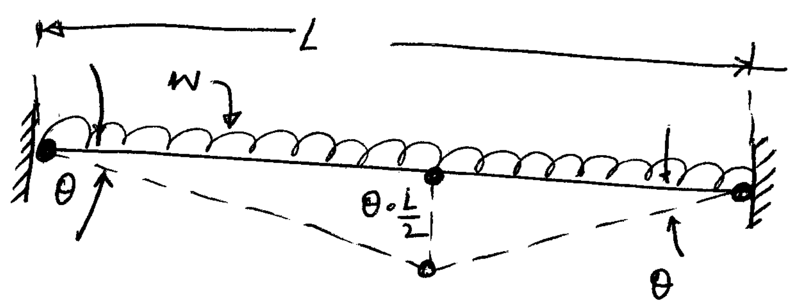
& $\leq M_p$ everywhere else

Upper Bound theorem of plasticity

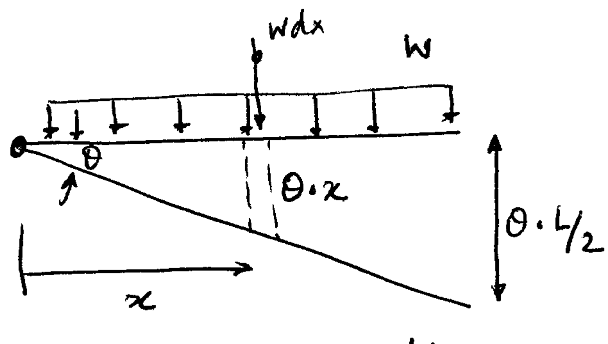
A limit load computed using an assumed mechanism $\geq$ true plastic limit load

If several mechanisms are considered, then the one with the lowest limit load is closer to the true plastic limit load.

Example ②



$$W_{EXT} = W_{INT}$$



$$dW_e = (w \cdot dx)(\theta \cdot x)$$

$$W_E = \left[\int_0^{L/2} dW_e\right] \times 2$$

$$\therefore W_E = 2 \times \left[\int_0^{L/2} w\,\theta\,x\,dx\right]$$

$$= 2 \times w\theta \times \frac{x^2}{2}\Bigg|_0^{L/2} = w \cdot \theta\,\frac{L^2}{8} \times 2$$

$$W_{Ext} = 2\left[\frac{wL}{2}\cdot\frac{1}{2}\,\theta\,\frac{L}{2}\right] = \underline{\underline{wL}} \times \left[\frac{1}{2}\cdot\theta\,\frac{L}{2}\right]$$

Total load on beam

$\frac{1}{2}$ of virtual displacement at hinge (valid for uniform load only)

Internal Work:

$$W_I = M_p\theta + M_p(2\theta) + M_p(\theta) = 4M_p\theta$$

$$W_E = W_I$$

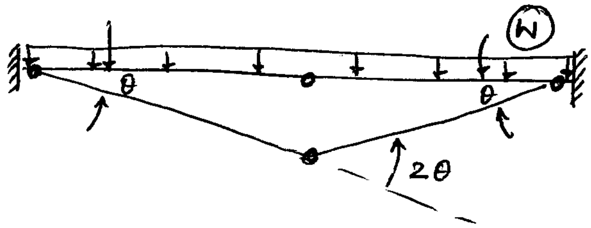
$$wL \times \frac{1}{2}\cancel{\theta}\frac{L}{2} = 4M_p\cancel{\theta}$$

$$w = \frac{16M_p}{L^2}$$

← plastic limit load.

Check by developing BMD & SFD.

Example



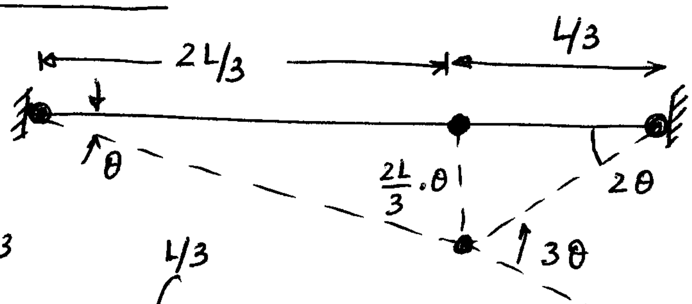
$$W_{EXT} = \frac{wL}{2} \times \left[\frac{1}{2} \times \theta \frac{L}{2}\right]$$

$$W_{INT} = M_p\,\theta + M_p\,2\theta + M_p\,\theta$$

$W_{EXT} = W_{INT}$

$$w = 16\frac{M_p}{L^2}$$

Incorrect Mechanism:



$$W_{EXT} = \int_0^{2L/3} dW_e + \int_0^{L/3} dW_e$$

$$= wL \times \left[\frac{1}{2} \times \frac{2L}{3}\,\theta\right]$$

total load

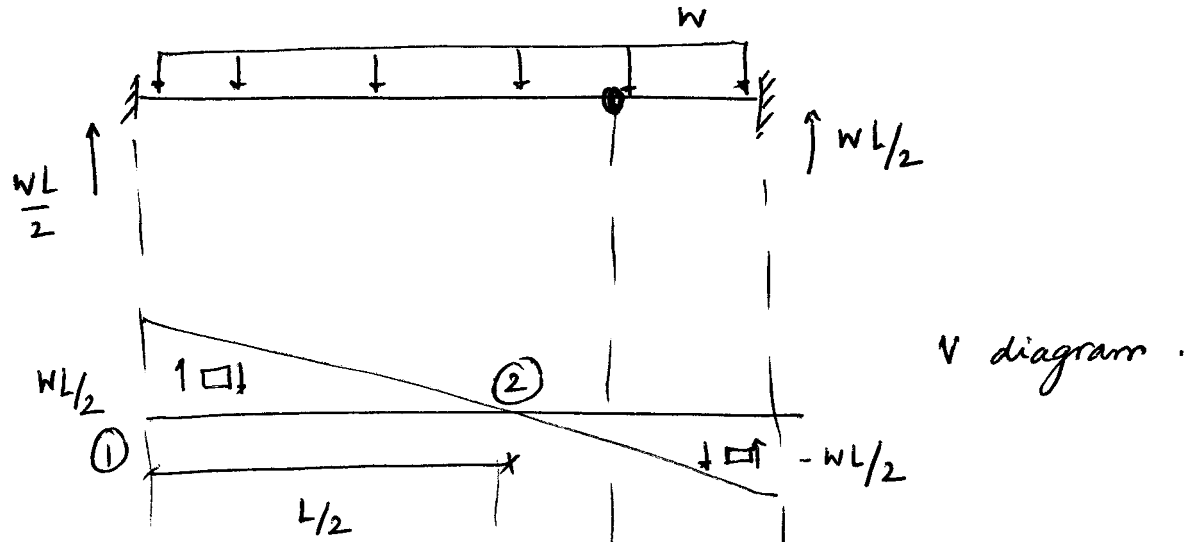
displacement at hinge

$$W_{INT} = M_p\,\theta + M_p\,2\theta + M_p\,3\theta = 6\,M_p\,\theta$$

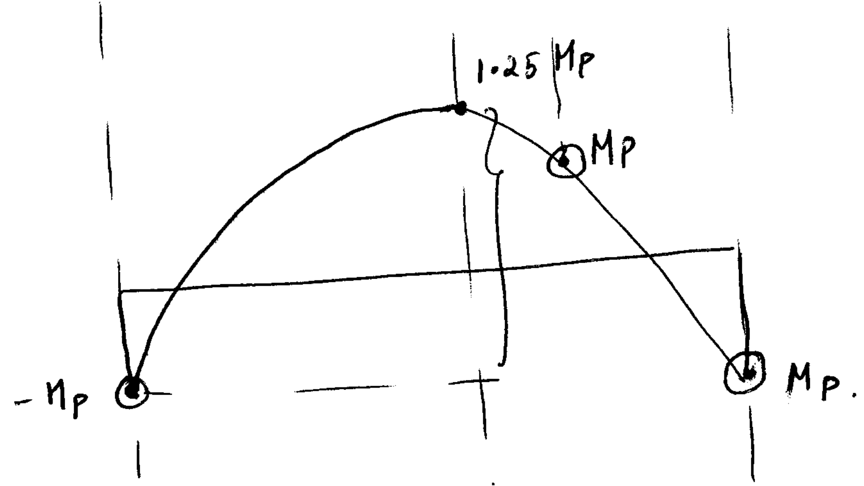
$W_{EXT} = W_{INT}$

$$\omega = 18\,\frac{M_p}{L^2}$$

Use incorrect mechanism to check moment diagram



$$\Delta M = \frac{1}{2} \times \frac{wL}{2} \times \frac{L}{2} = \frac{wL^2}{8}$$



Slope of $M \equiv V$

$$-M_p + \Delta M = -M_p + \frac{wL^2}{8}$$

$$= -M_p + \frac{\cancel{18}^{\,2.25} M_p}{\cancel{L^2}} \times \frac{\cancel{L^2}}{\cancel{8}}$$

$$= 1.25\,M_p$$

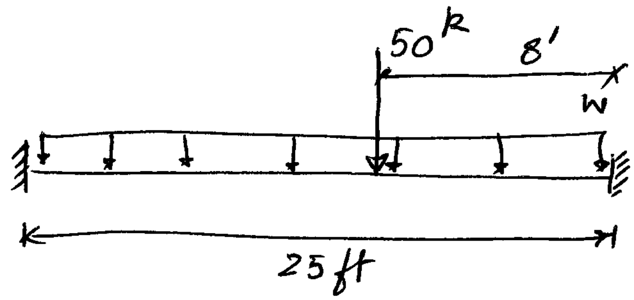
$$\underbrace{|M| \geq M_p}$$

somewhere for incorrect mechanism

General Approach for Kinematic Method.

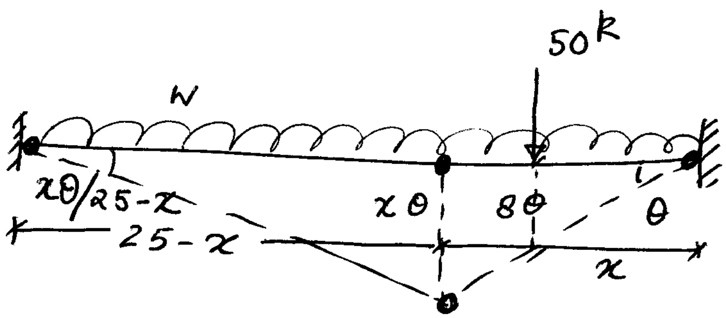
- Assume mechanism
- Compute limit load $W_{EXT} = W_{INT}$
- Draw the equilibrium moment diagram using the partial information →
  - (a) All the loads & locations
  - (b) All the hinges & location
  - (c) M_p at hinges
  - (d) plastic limit load
- If $|M| \leq M_p$ everywhere, then the correct mechanism & calculated true plastic limit load.

Example.



Find the plastic limit load for (w)

Assume hinge forms along the length



$$W_{EXT} = 50 \times 8\theta + \underbrace{(25\,w)}_{\text{total load}} \times \left(\frac{1}{2}\,\underbrace{x\theta}_{\text{displacement @ hinge}}\right)$$

$$W_{INT} = M_p \times \theta + M_p \times \frac{x\theta}{25-x} + M_p\left(\theta + \frac{\theta x}{25-x}\right)$$

$$= M_p\,\theta\left[1 + \frac{x}{25-x} + 1 + \frac{x}{25-x}\right]$$

$$W_{INT} = M_p\,\theta \times \frac{50}{25-x}$$

$$W_{EXT} = W_{INT}$$

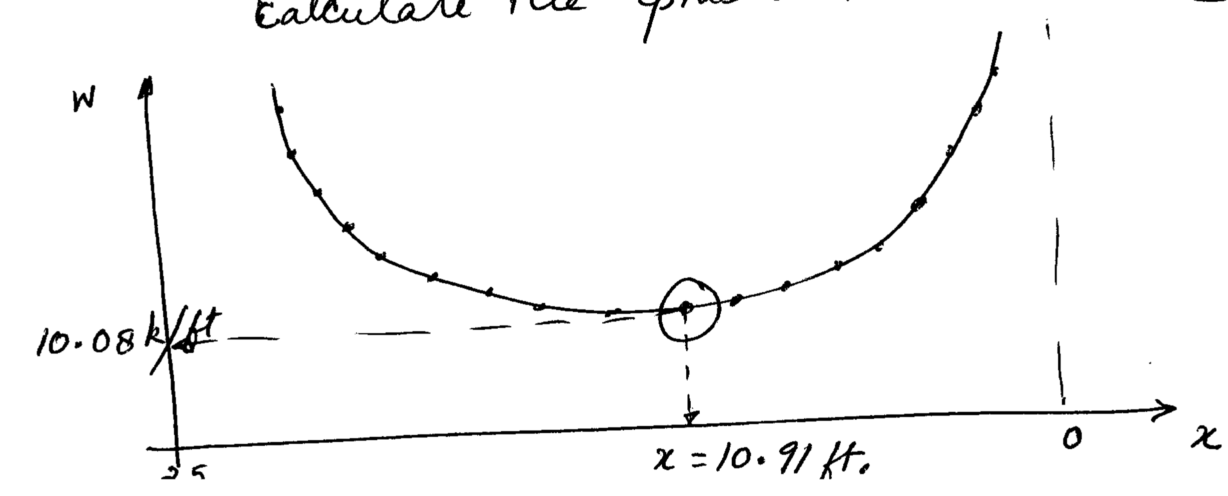
$$\Rightarrow \quad 50 \times 8\theta + 25\,W\left(\frac{1}{2}\,x\theta\right) = M_p\,\theta \times \frac{50}{25-x}$$

Given: $M_p = 500$ k-ft

$$\therefore \quad 400\,\cancel{\theta} + 12.5\,W x\,\cancel{\theta} = \frac{25000\,\cancel{\theta}}{25-x}$$

$$W = \frac{2000}{x(25-x)} - \frac{32}{x}$$

Depending on the value of x assumed, calculate the plastic limit load (W)

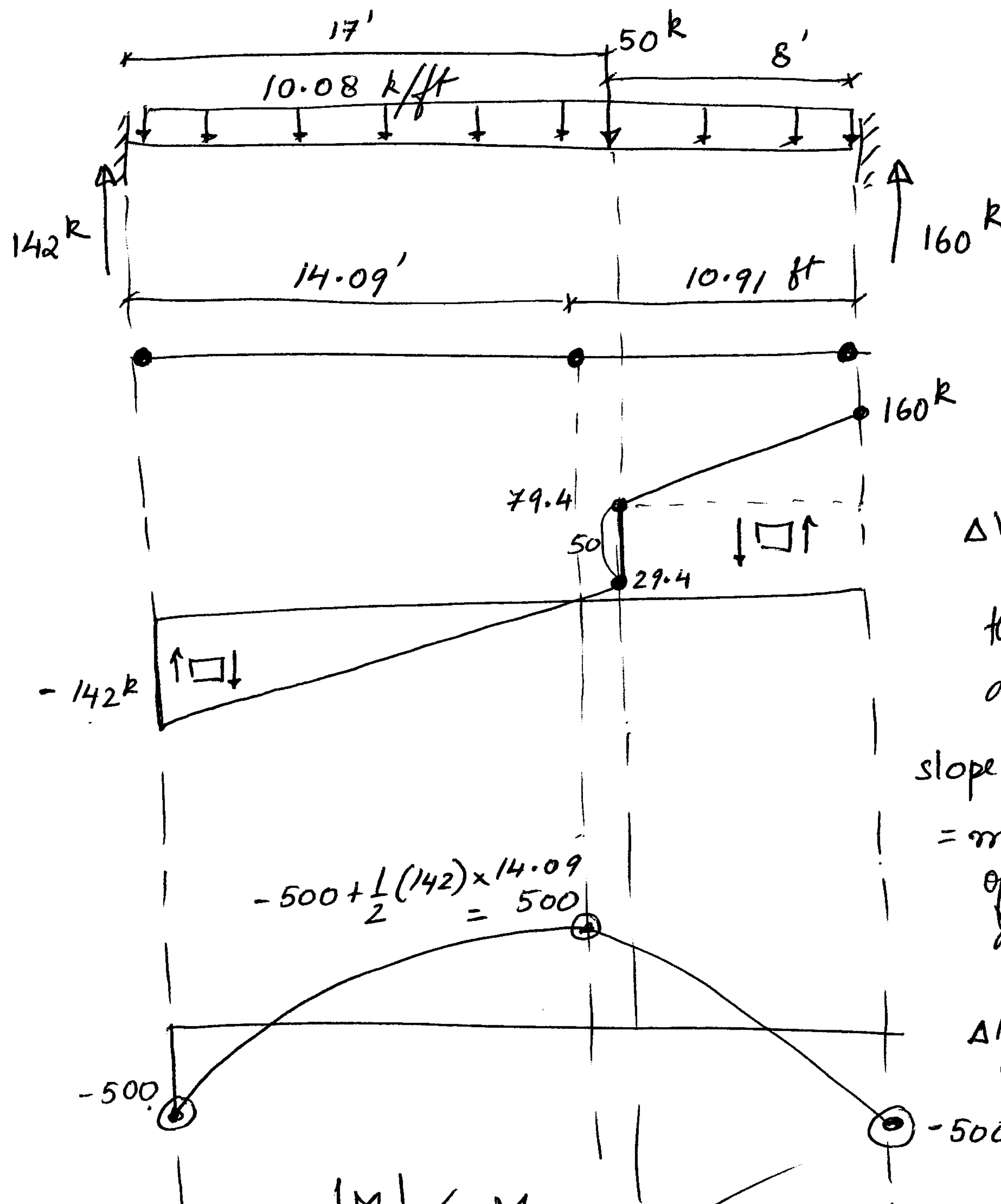


By trial & error or by graphing $w(x)$

the smallest collapse load $w = 10.08$ k/ft

& the hinge forms @ 10.91 ft

Check the BMD:



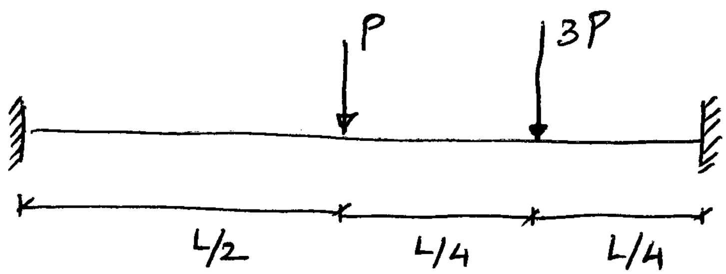
ΔV = area under the loading diagram

slope of SFD = magnitude of loading diagram

ΔM = area und SFD.

$|M| \leq M_p$

Example.

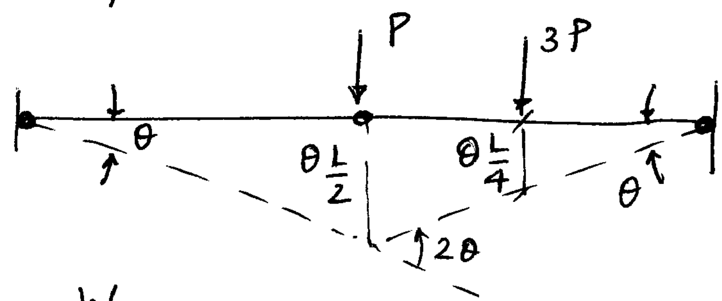


3 hinges to form the mechanism!

controlling mechanism not obvious.

Consider 2 possible mechanism

Mechanism (I)

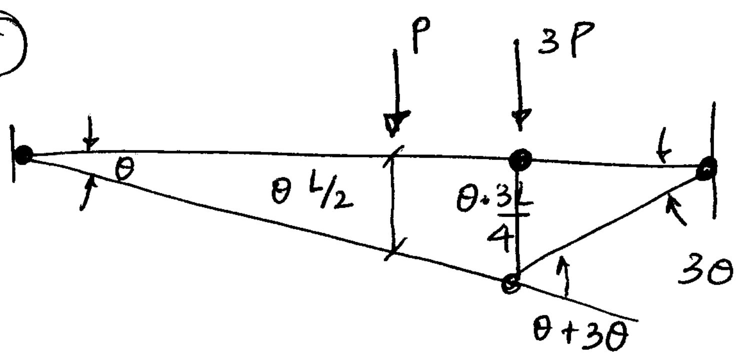


$$W_{EXT} = W_{INT}$$

$$\therefore P\theta\frac{L}{2} + 3P\theta\frac{L}{4} = M_p\theta + M_p\theta + M_p 2\theta$$

$$\therefore P_p = \frac{16}{5}\frac{M_p}{L} = 3.2\ M_p/L$$

Mechanism (II)



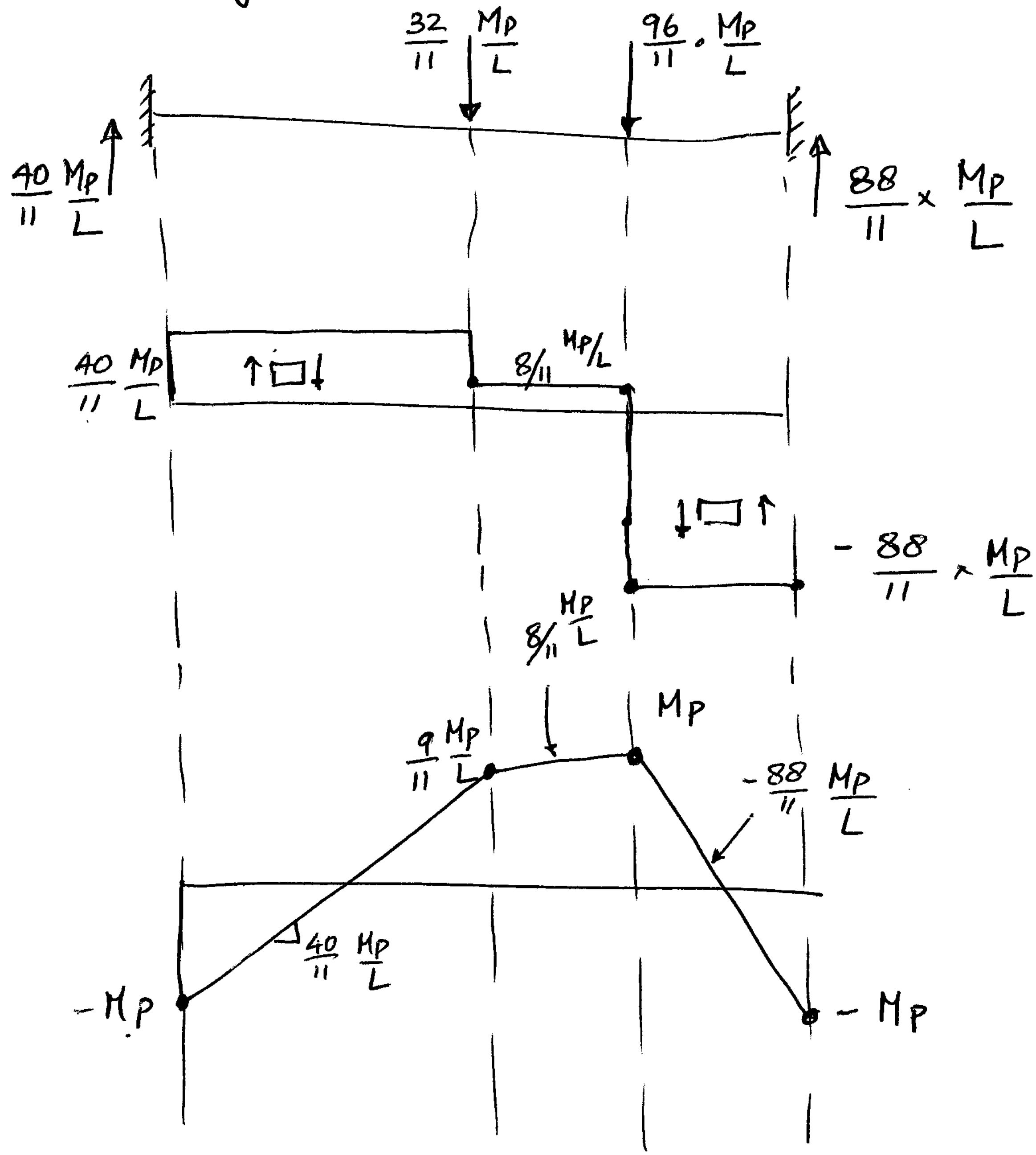
$$W_{EXT} = W_{INT}$$

$$\therefore P.\theta\frac{L}{2} + 3P\times\theta\times\frac{3L}{4} = M_p\theta + M_p 3\theta + M_p 4\theta$$

$$P_p = \frac{32}{11}\frac{M_p}{L} \approx 2.91\ M_p/L$$

Mechanism II provides lower value & governs

$\therefore$ Bending Moment Diagram for Mechanism II

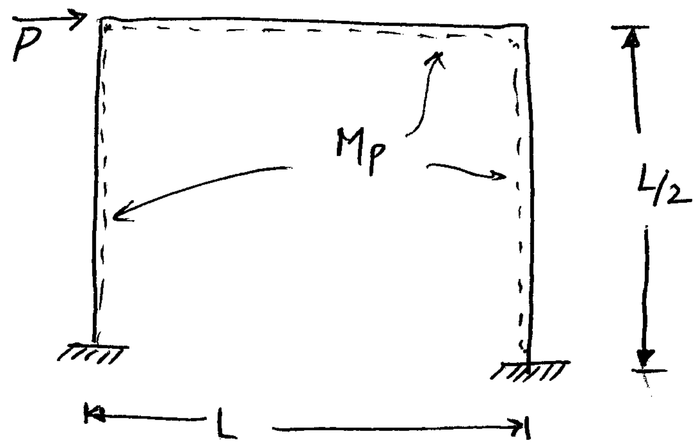


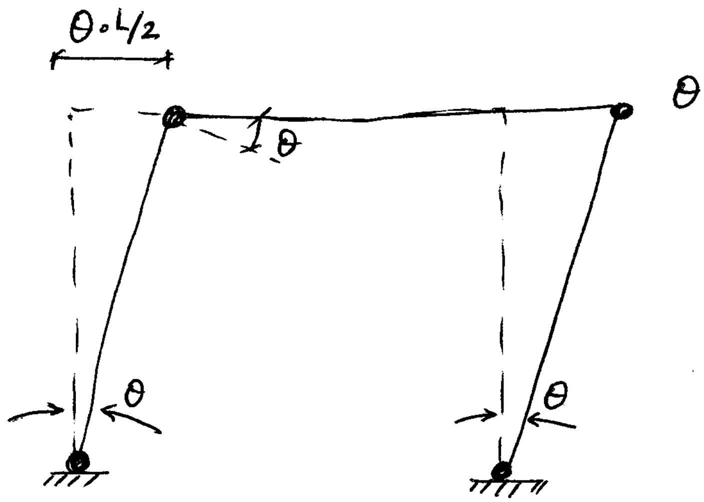
Slope of $M \longrightarrow$ magnitude of SFD

$\Delta M \longrightarrow$ area under SFD.

slope of $V \longrightarrow$ magnitude of loading

Example





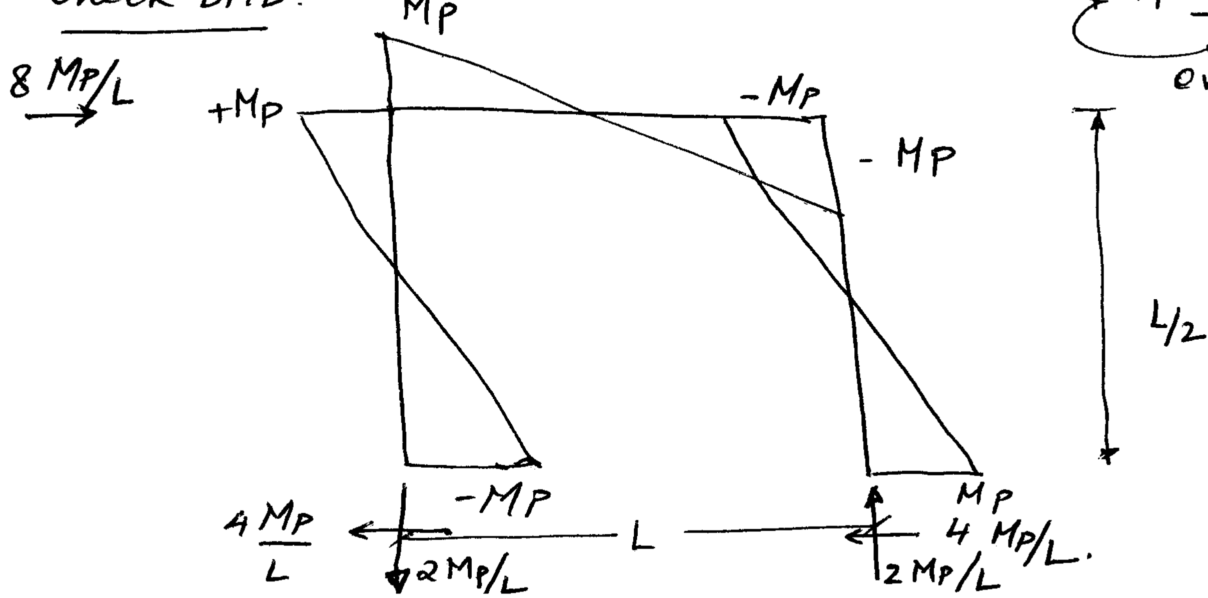
$$W_{EXT} = P \cdot \theta \cdot \frac{L}{2}$$

$$W_{INT} = M_p \times 4 \times \theta$$

$$\Bigg\} \quad P_p = 8\,\frac{M_p}{L}$$

correct plastic limit load!

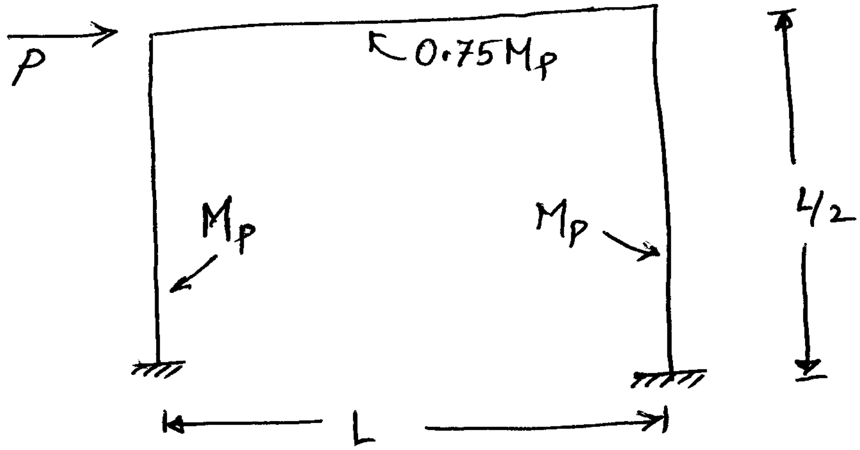
Check BMD:

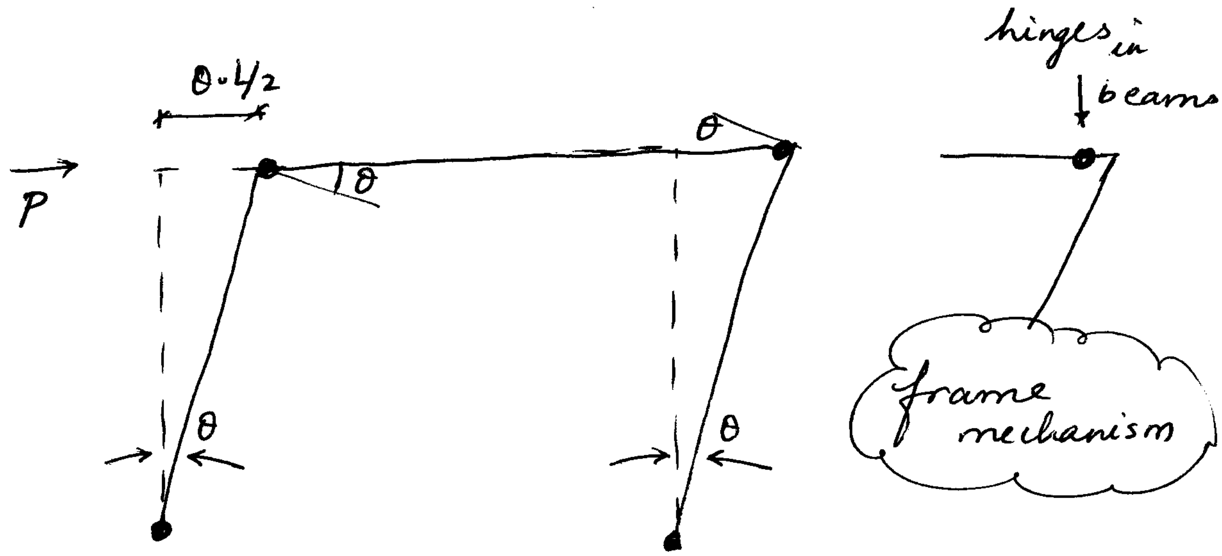


$|M| \leq M_p$

everywhere ✓

Example:

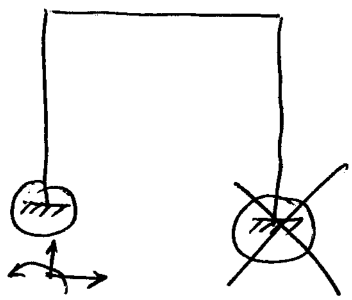
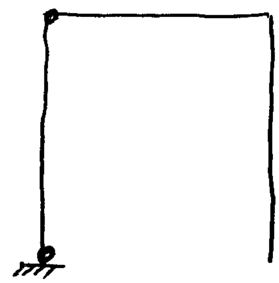




$$\therefore P \times \theta \times \frac{L}{2} = M_p\theta + M_p\theta + (0.75\, M_p\, \theta) \times 2$$

$$\boxed{P_p = 7\frac{M_p}{L}}$$ ✓

slight difference in the plastic limit load!

degree of redundancy $= r$

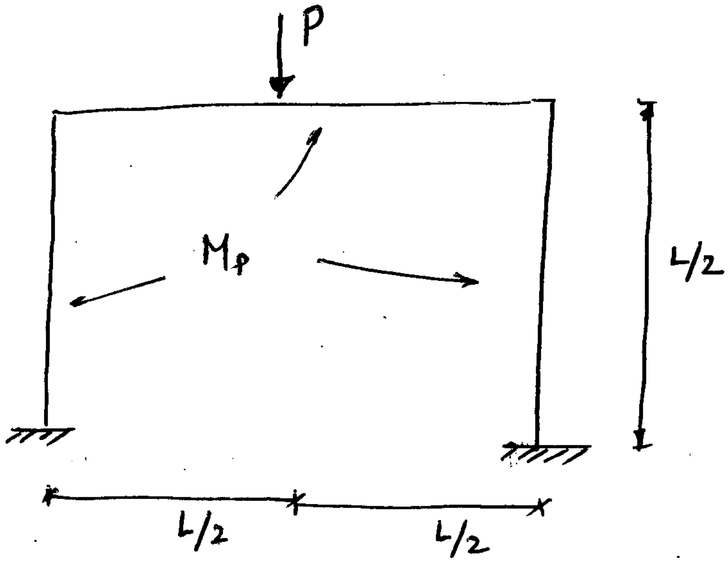
no. of hinges needed for collapse mechanism are typically $= r+1$

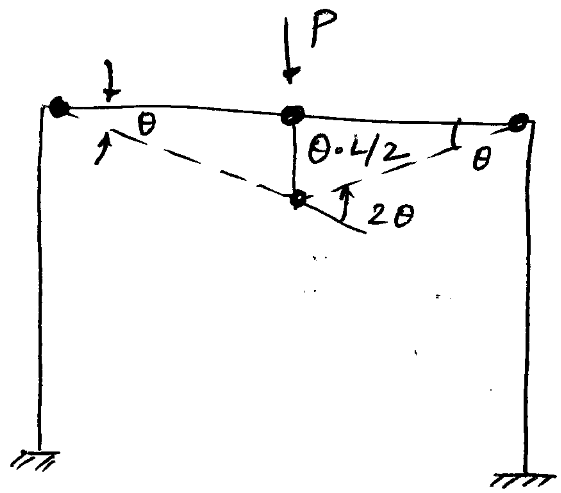
overall collapse mechanism!

There are local mechanisms that may be possible for particular structure and loading.

Need to be considered because they may govern!

Example





beam mechanism

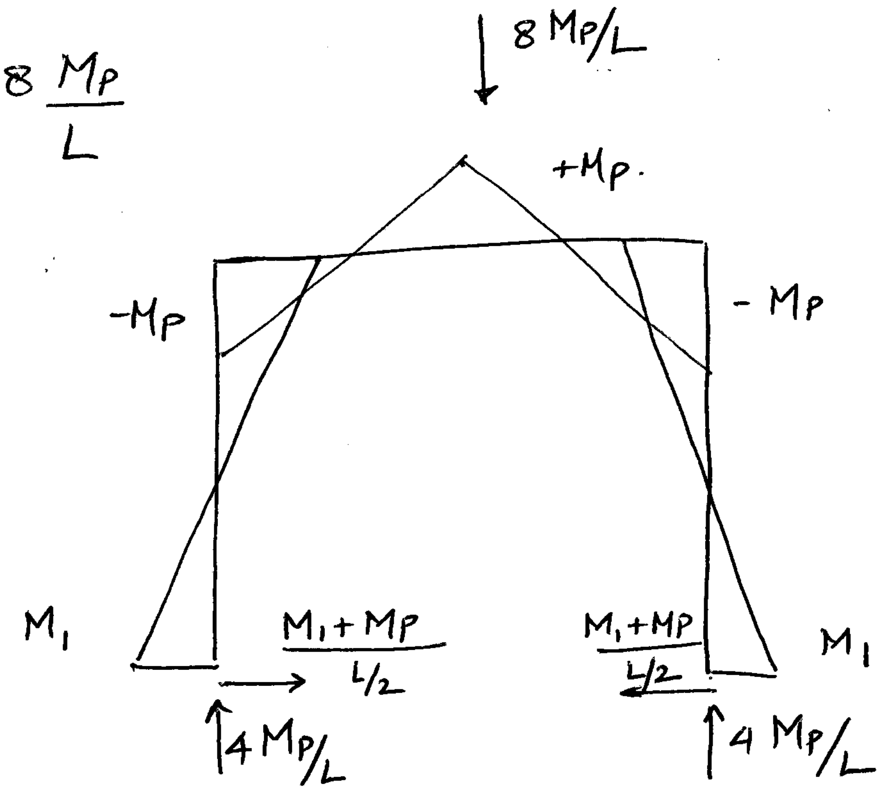
local mechanism

or partial collapse mechanism.

$$P \times \theta \times \frac{L}{2} = 4 M_p \theta$$

$$\therefore P_p = 8 \frac{M_p}{L}$$

Draw BMD:



M_1 cannot be determined from statics (due to partial collapse)

M_1 value $\leq$ M_p $\longrightarrow$ still provide equilibrium!

any value of M_1 satisfied equilibrium.

Choose $\leq M_p \longrightarrow P_p \rightarrow$ true plastic limit load!

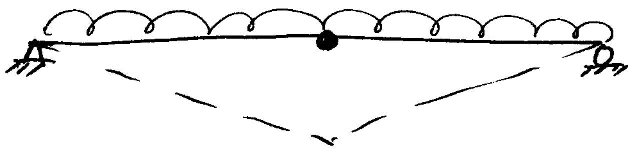
$$P_p = 8\frac{M_p}{L}$$

When the number of hinges

$< r+1 \longrightarrow$ partial mechanism

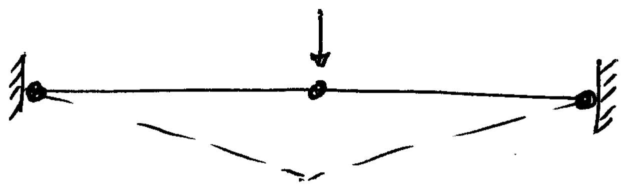
$\geq r+1 \longrightarrow$ complete mechanism

Examples of Complete Mechanisms:



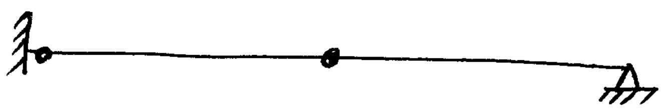
$r = 0$

$h = 1$



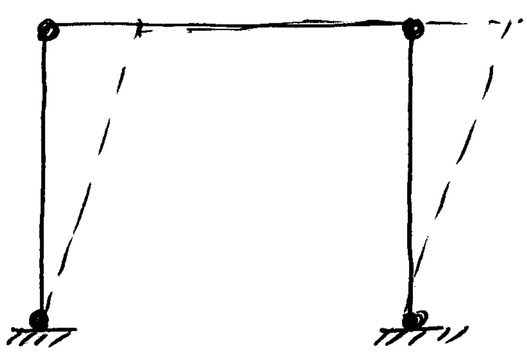
$r = 2$

$h = 3$



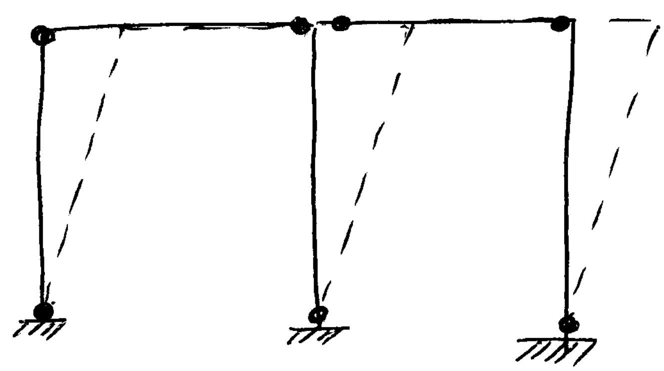
$r = 1$

$h = 2$



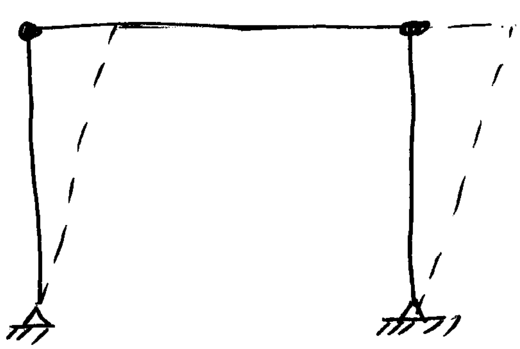
$r = 3$

$h = 4$



$r = 6$

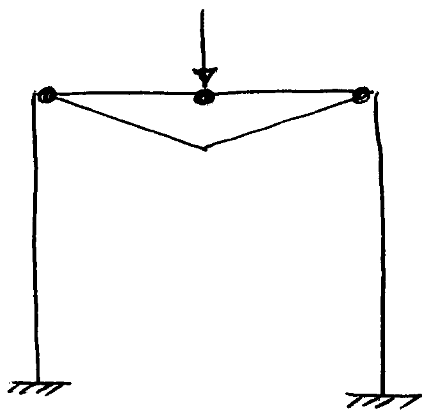
$h = 7$



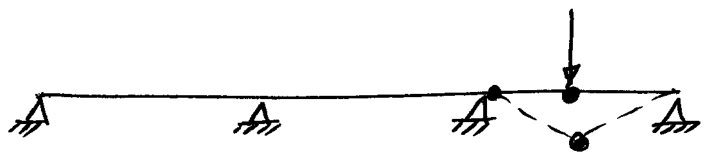
$r = 1$

$h = 2$

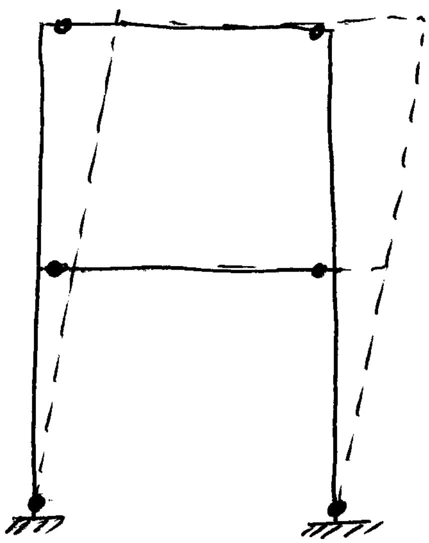
Examples of Partial Mechanisms:



r=3
h=3



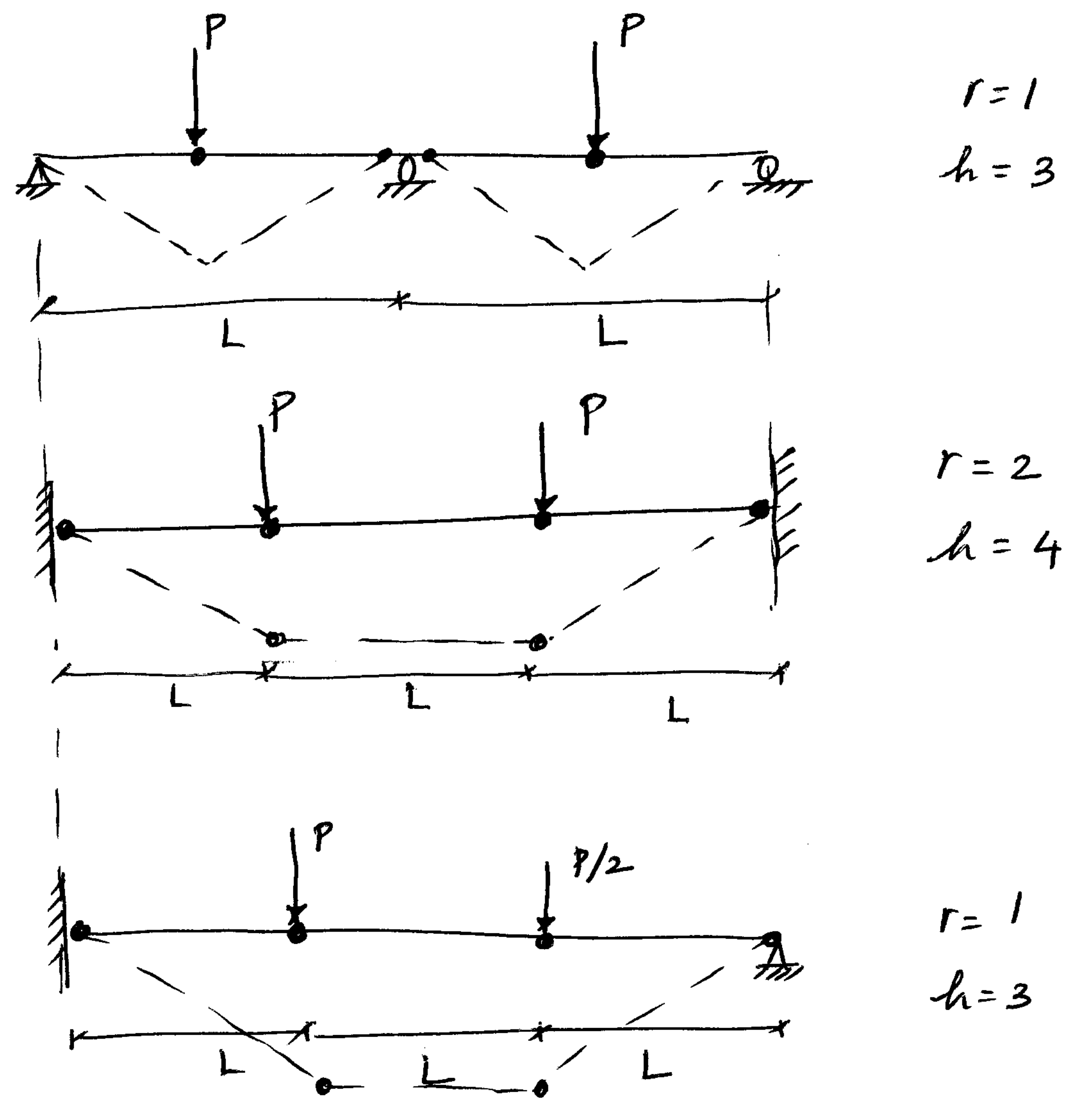
r = 2
h = 2



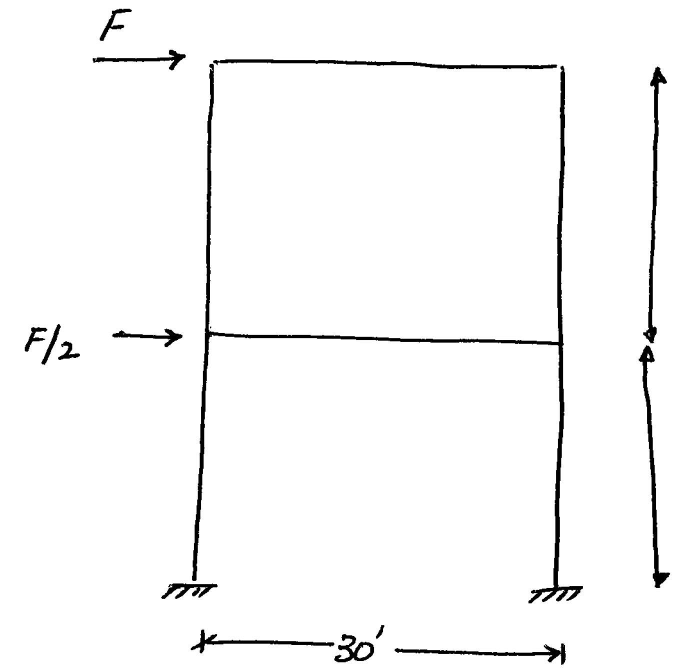
r=6
h=6

sway mechanism!

Examples of Overcomplete Mechanisms:



Example

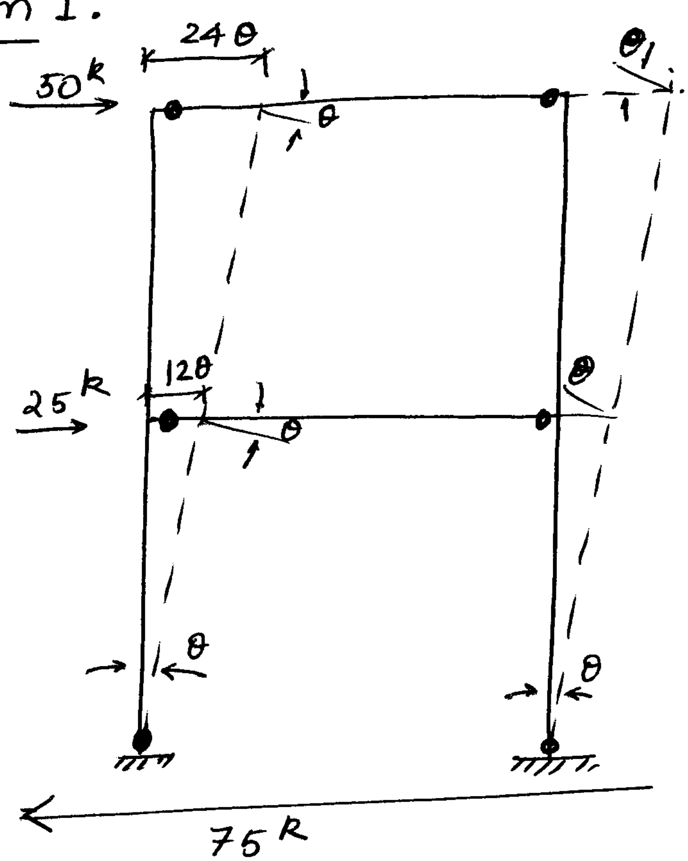


$M_p^{beam} = 225 \text{ k-ft}$

$M_p^{column} = 300 \text{ k-ft}$

$r = 6$, $h = 7$ } complete mechanism

Mechanism I:



$W_{EXT} = \frac{F}{2} \times 12\theta$

$+ F \times 24\theta$

$= 30 F\theta$

$W_{INT} = 300 \times \theta \times 2$

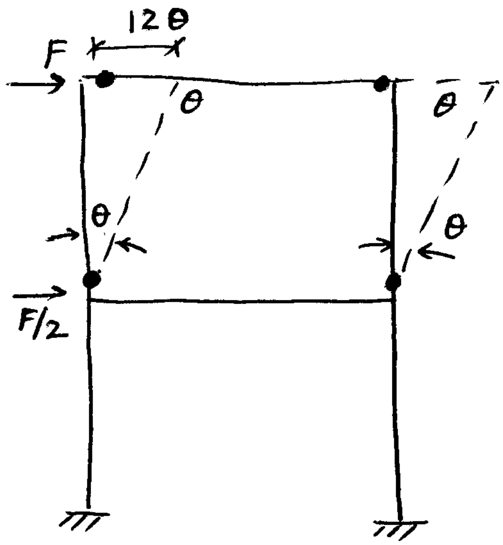
$+ 225 \times \theta \times 4$

$= 1500\,\theta$

$\therefore 30 F\theta = 1500\,\theta$

$\therefore F = 50^k$

MECHANISM II

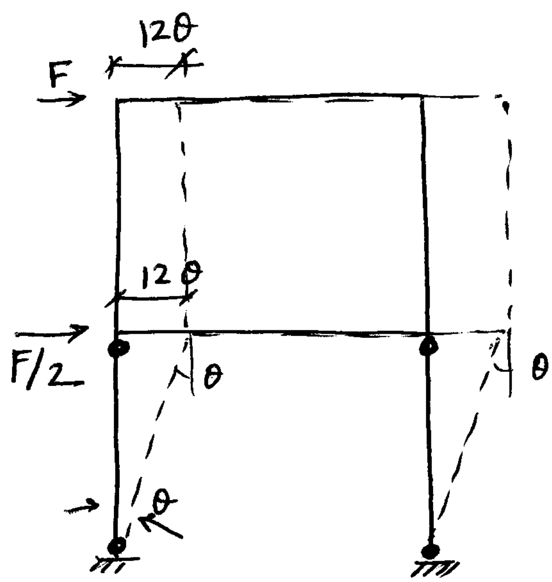


story or portal mechanism
(local or partial collapse mechanism)

$F \times 12\theta = 300 \times \theta \times 2 + 225 \times \theta \times 2$

$\therefore F = 87.5^{k}$

MECHANISM III



$W_{EXT} = \frac{F}{2} \times 12\theta + F \times 12\theta$

& $W_{INT} = 4 \times 300 \times \theta$

$\therefore F = 67^{k}$

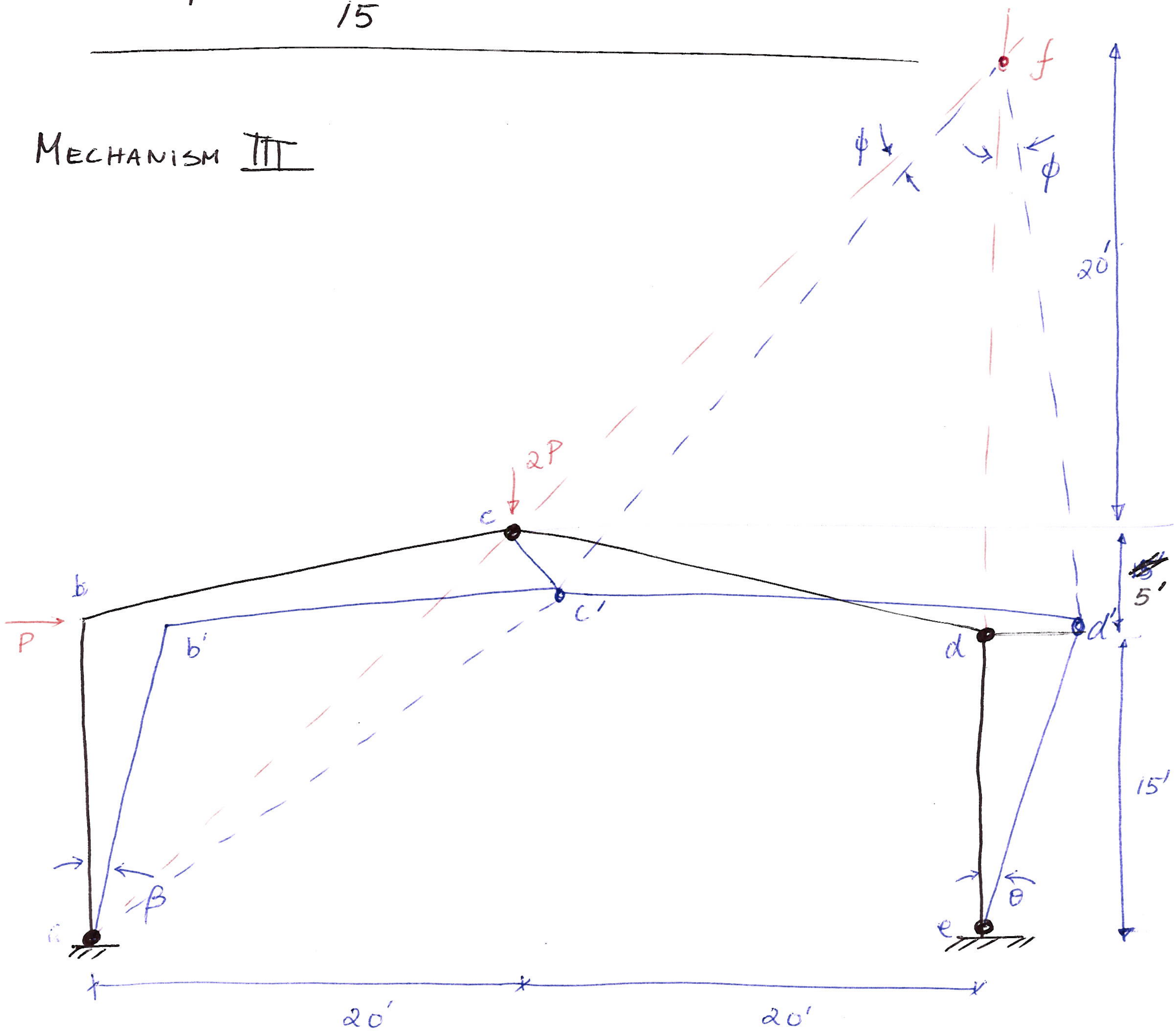
Mechanism I governs so far!

$$W_{EXT} = W_{INT}$$

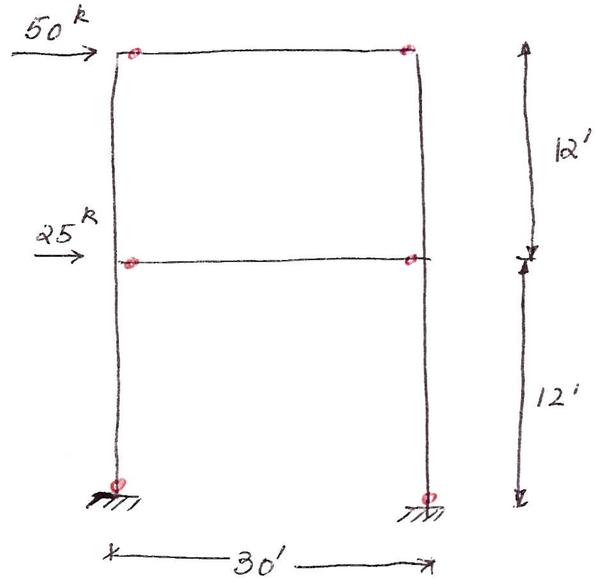
$$2P(20\beta) + M_p \times (1.5\theta + 3\theta + 2.5\theta + \theta)$$

$$\therefore \quad P_p = \frac{2M_p}{15} = 26.67^k$$

MECHANISM $\underline{III}$



Example:

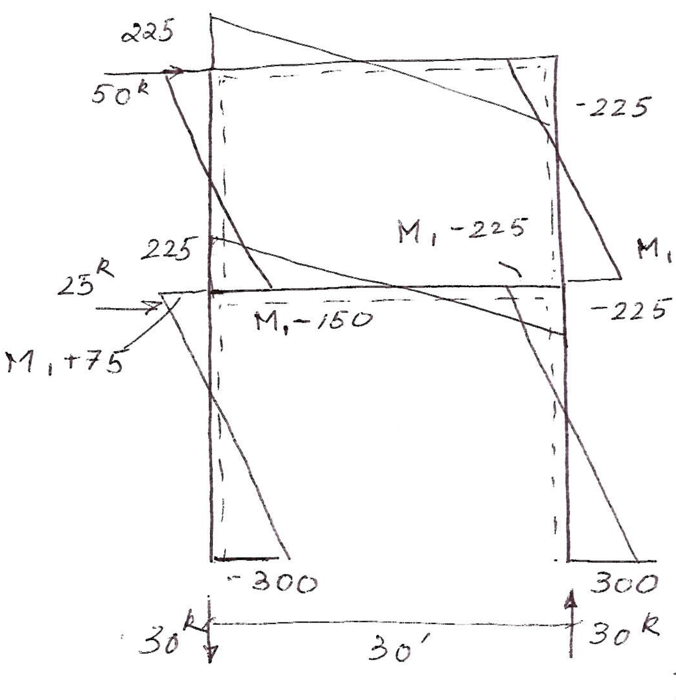


$r = 6$

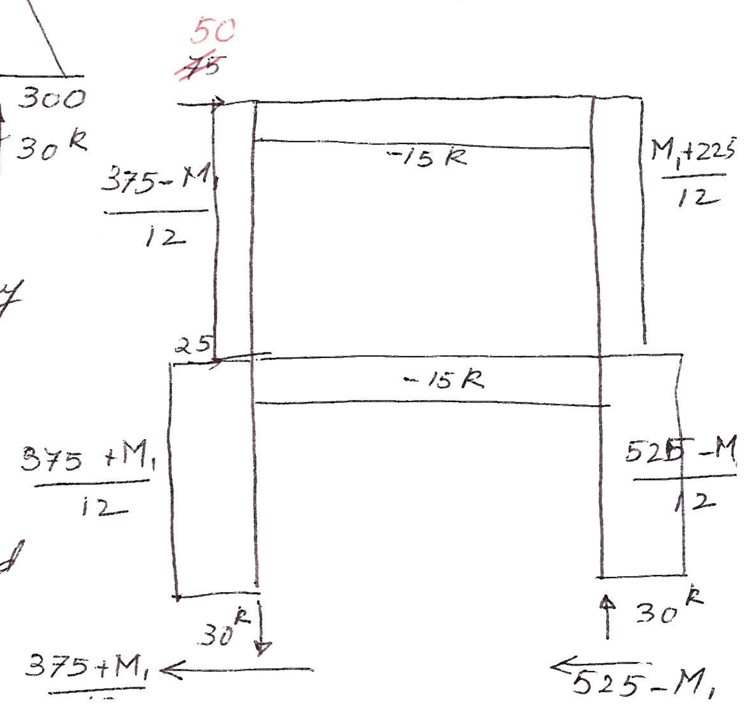
$M_p = 225$ k-ft

$M_p = 300$ k-ft

Check this mechanism → look at bending moment diagram



Cannot determine M_1 from Statics

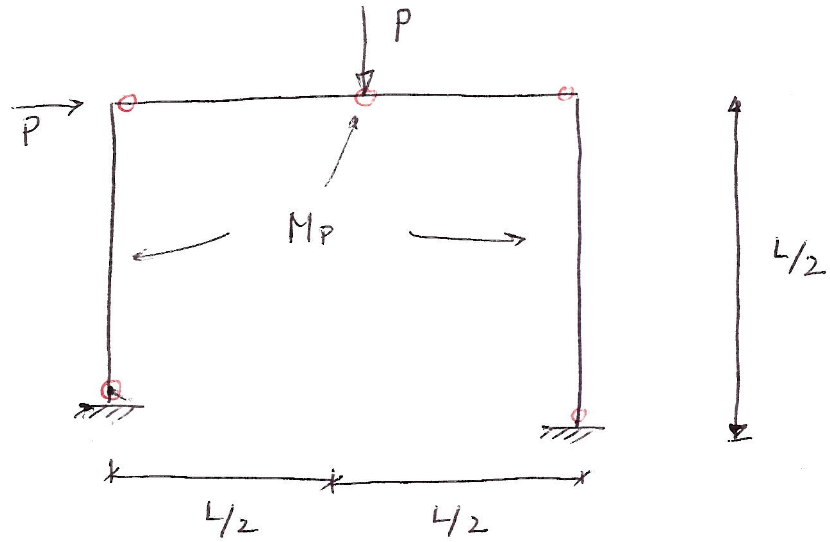


Equilibrium is possible for any value of M_1

$\therefore$ Assume $|M_1| \le M_p$

$\therefore F = 50^k \rightarrow$ plastic limit load actual!

Example:



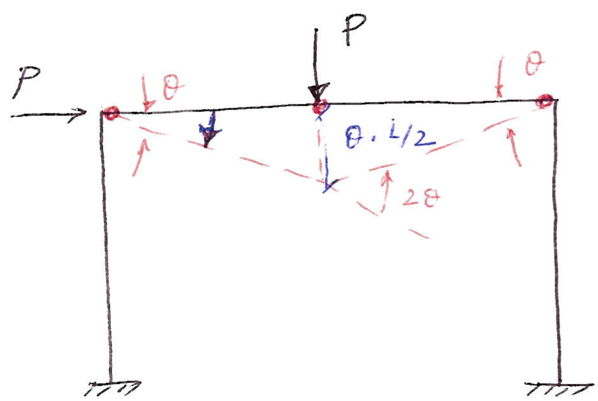
$r = 3$

$h = 4$ for a complete mechanism

5 possible hinge locations!

Consider different Mechanisms:

Mechanism I ⟶ Beam Mechanism

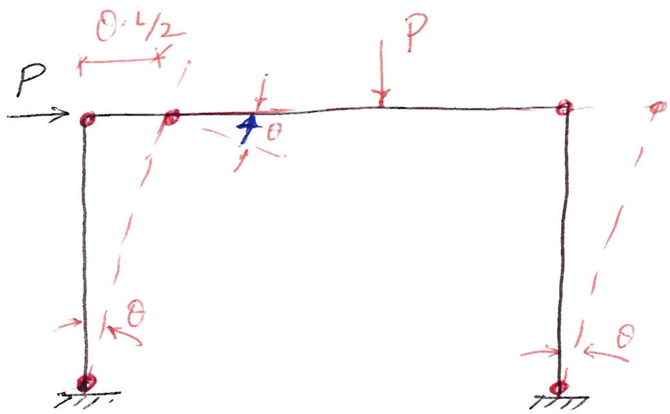


$W_{ext} = P \cdot \theta \cdot \frac{L}{2}$

$W_{INT} = M_p\theta + M_p\theta + M_p \cdot 2\theta = 4 M_p \theta$

$P_p = \frac{8 M_p}{L}$

Mechanism II ⟶ Sway Mechanism

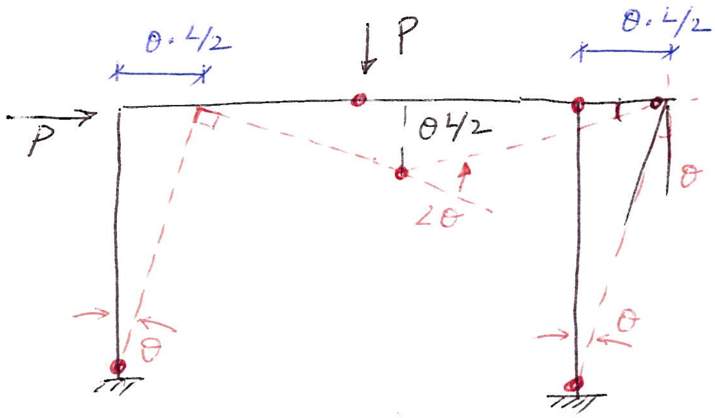


$W_{ext} = P \cdot \theta \cdot \frac{L}{2} + 0$

$W_{INT} = M_p \theta \times 4 = 4 M_p \theta$

$P_p = \frac{8 M_p}{L}$

Mechanism III   Combined Mechanism



$$W_{EXT} = P \cdot \theta \frac{L}{2} + P \cdot \theta \frac{L}{2}$$

$$W_{INT} = M_p \theta + M_p 2\theta + M_p \theta + M_p \theta + M_p \theta$$

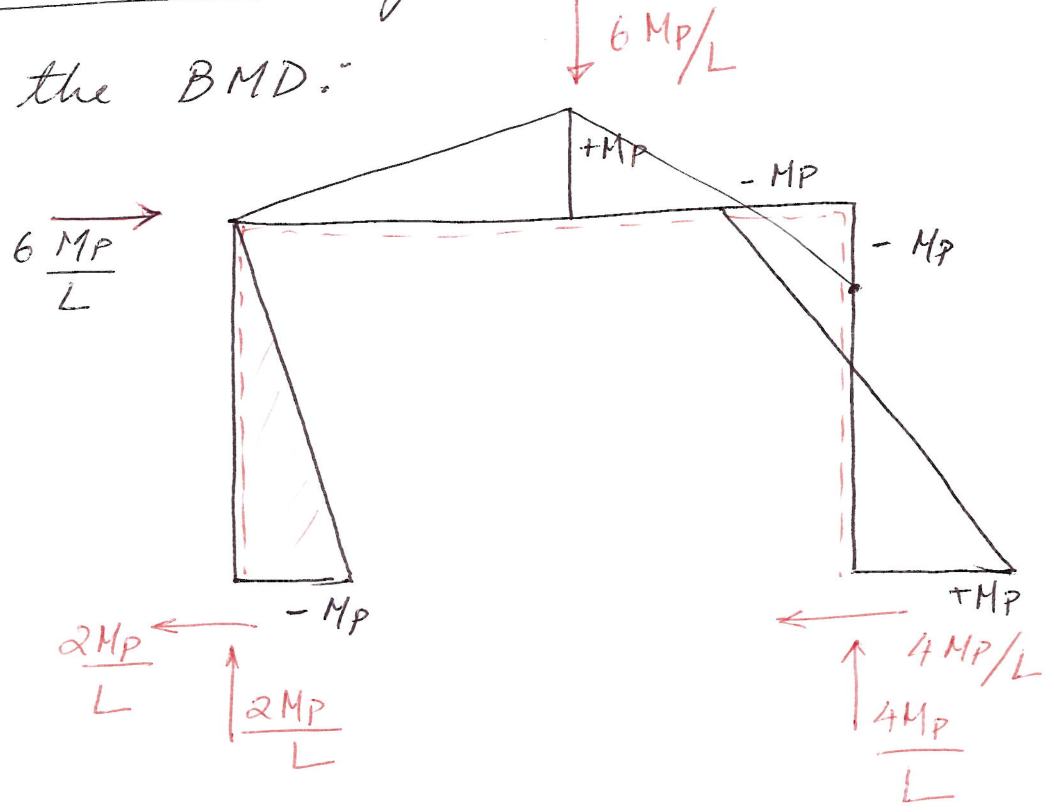
$$P \cancel{\theta} L = 6 M_p \cancel{\theta}$$

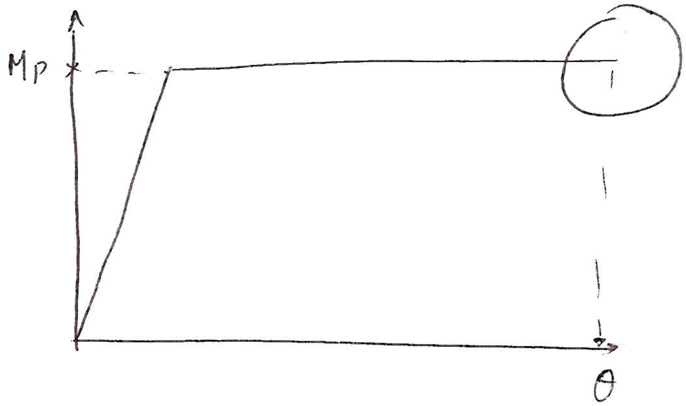
$$P_p = \frac{6 M_p}{L}$$

→ smaller than both fundamental mechanism

Combined Mechanism governs.

Check the BMD:





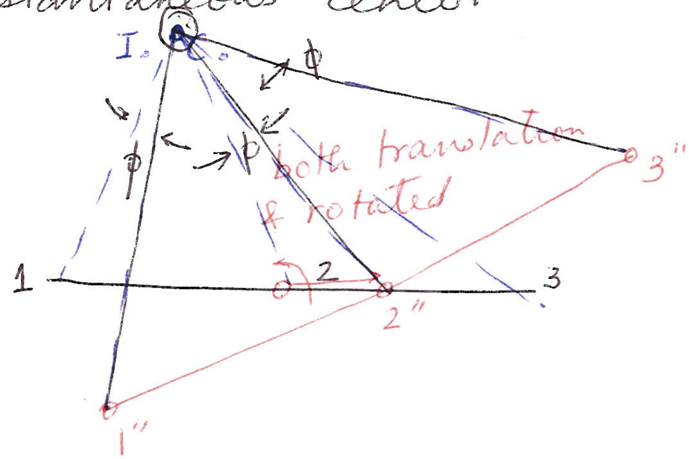
inelatic distortions.
local buckling
inelatic lateral torsional buckling
connection rotation capacities

INSTANTANEOUS CENTER OF ROTATION:

→ Useful tool for evaluating geometry of mechanism.

For small displacements, rigid body movements can be represented by rotations about the I.C. of rotation

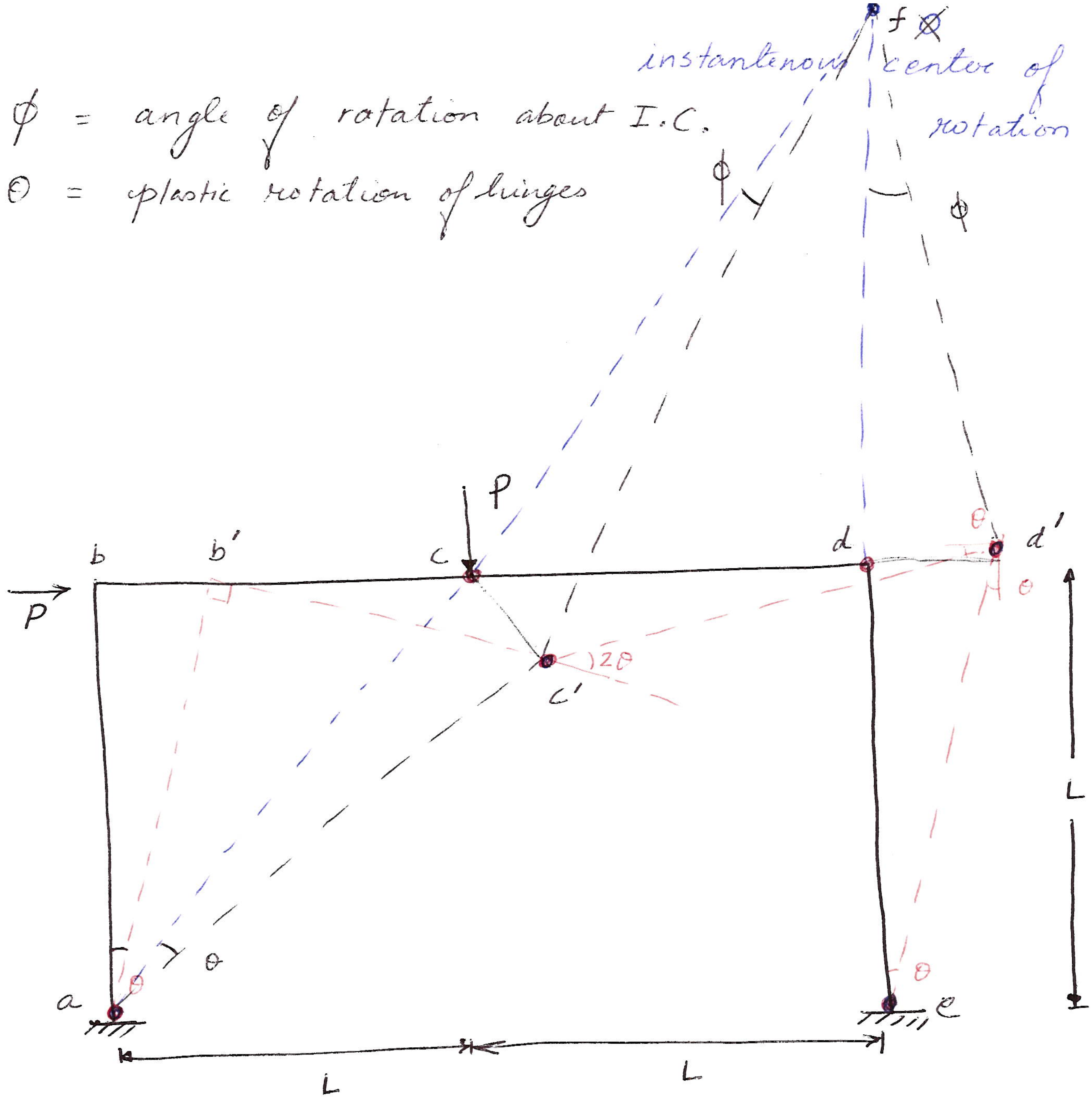
Every point on the riged moves perpendicular to the instantaneous center



129

ϕ = angle of rotation about I.C.

θ = plastic rotation of hinges



difference between mechanism & plastic mechanism

no P required for rotation

only P_p will cause plastic rotation.

$$\overline{dd'} = \theta \cdot L$$

$$\phi = \frac{dd'}{L} = \frac{\theta L}{L} = \theta$$

$$\overline{cc'} = \phi\, \overline{cf}$$

$$= \theta \times \overline{ac}$$

Since $\phi = \theta$

$$\therefore \overline{cf} = \overline{ac}$$

Rotation at $a = \theta$

Rotation at $c = \theta + \theta = 2\theta$

Rotation at $d = \theta + \theta = 2\theta$

Rotation at $e = \theta$

horizontal displacement at $b = \theta \times L$

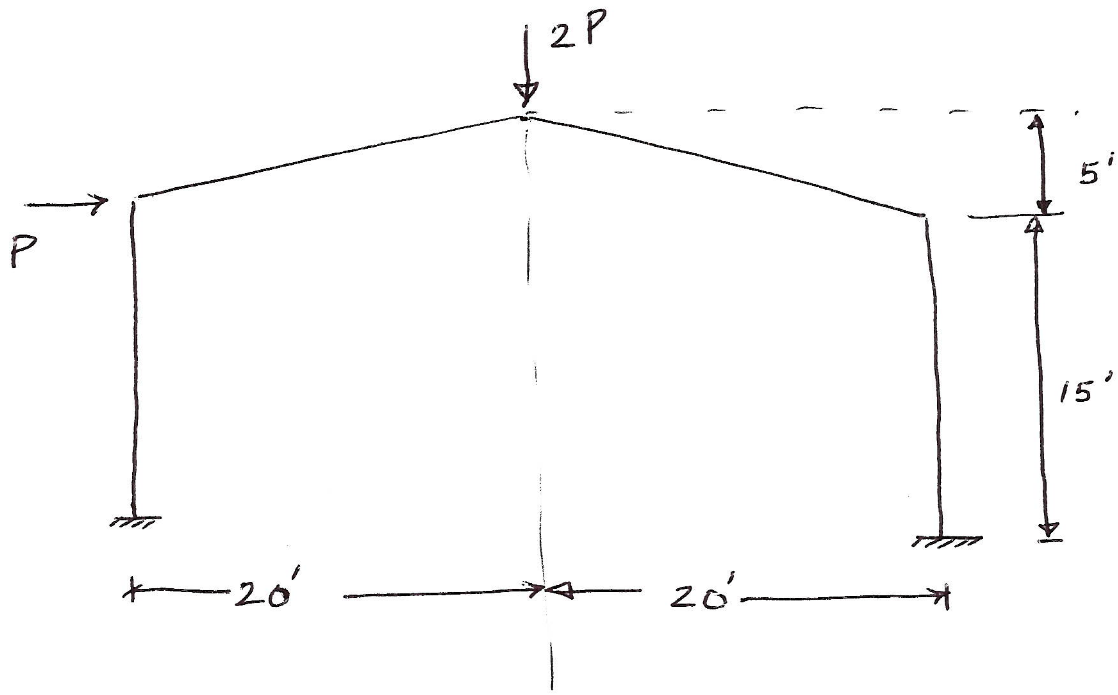
vertical displacement at $c = \theta \times L$

$$\therefore W_{EXT} = W_{INT}$$

$$\cancel{P} \times \theta L + P\theta L = M_p \theta + M_p 2\theta + M_p 2\theta + M_p \theta$$

$$\therefore P_p = \frac{\overset{3}{\cancel{6}} M_p \cancel{\theta}}{\cancel{2} \cancel{\theta} L} = \frac{3M_p}{L}$$

Example Gable Frame: $\longrightarrow$ Plastic limit load.

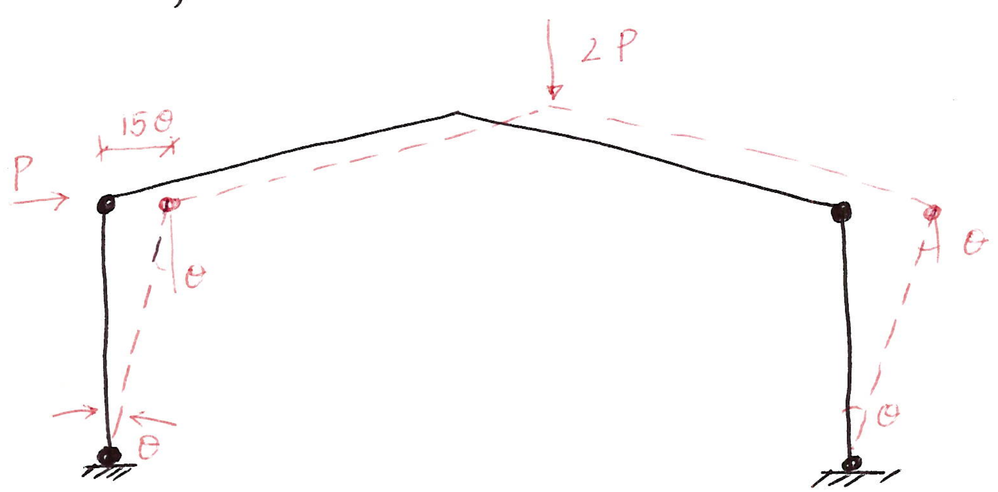


$M_p = 200$ k-ft for all members

$r = 3$

$h = 4$ for a complete mechanism.

3 complete mechanisms.



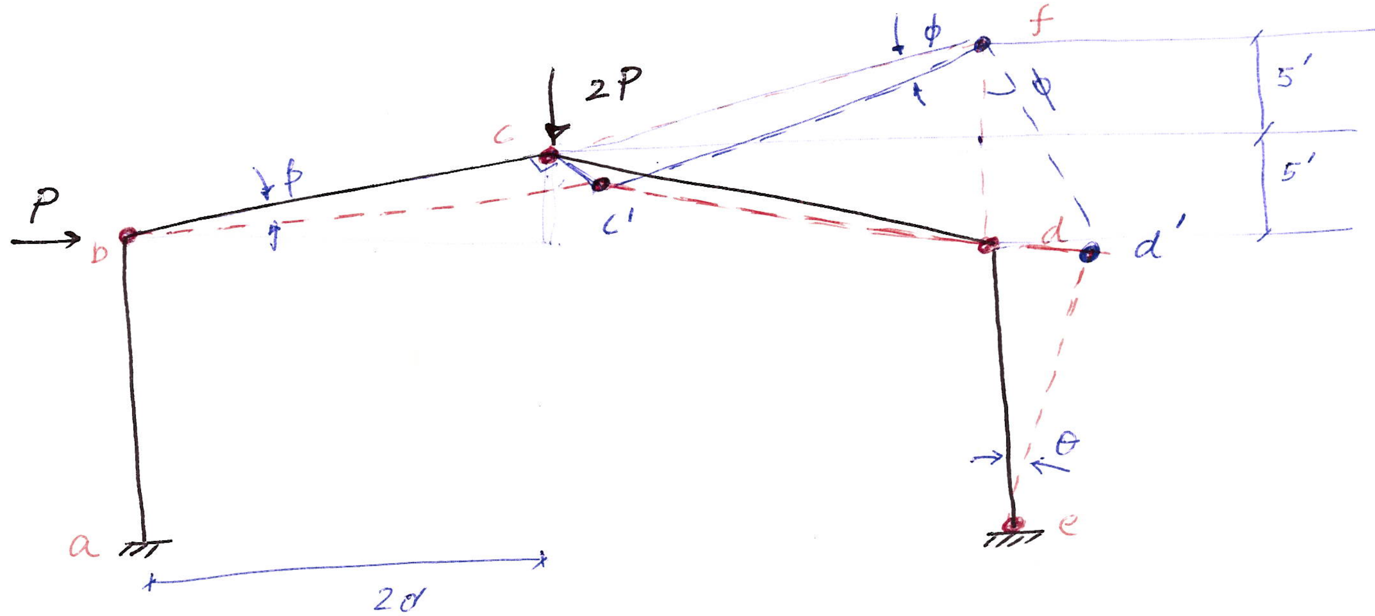
mechanism I

$W_{ext} = W_{int}$

$\therefore P \times 15\theta = M_p \theta \times 4$ $\qquad \therefore P_{p1} = \dfrac{4M_p}{15} = \dfrac{4 \times 200^{k\text{-}ft}}{15}$

$= 53.3^k$

Example.



$\theta \rightarrow$ rotation of de

$\phi \rightarrow$ rotation of cd

$\beta \rightarrow$ rotation of bc

$$\overline{dd'} = 15\,\theta = \phi \times 10 \longrightarrow \phi = 1.5\,\theta$$

$$\overline{cc'} = \phi \times \overline{cf} = \beta \times \overline{bc}$$

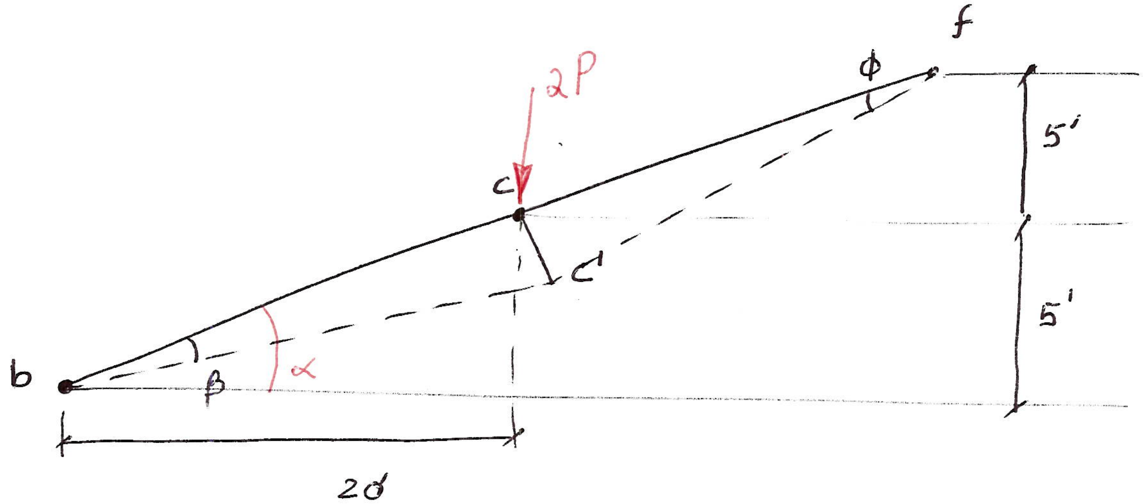
$$\therefore \beta = \frac{\overline{cf}}{\overline{bc}} \cdot \phi = \phi = 1.5\,\theta$$

$$W_{EXT} = P \times 0 + 2P \times \underline{\underline{cc'}}\Big|_{\text{vertical component!}}$$

$$= 2P \times (20\beta)$$

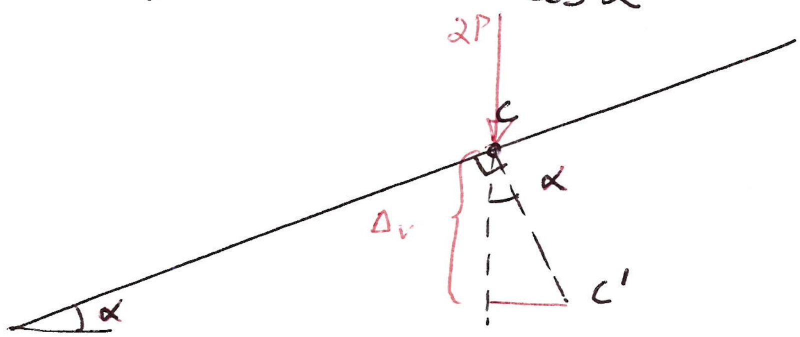
$$= 40\,P\beta$$

$$W_{INT} = M_p\,\beta\Big|_b + M_p\,(\beta + \phi)\Big|_c + M_p\,(\theta + \phi)\Big|_d + M_p\,\theta\Big|_e$$



Angle of rafter bc = α with respect to horizontal.

Length of rafter bc $= \dfrac{20\,ft}{\cos\alpha}$



$\Delta_v = \overline{cc'} \times \cos\alpha = \beta \times \overline{bc} \times \cos\alpha$

$= \beta \times \dfrac{20\,ft}{\cancel{\cos\alpha}} \times \cancel{\cos\alpha}$

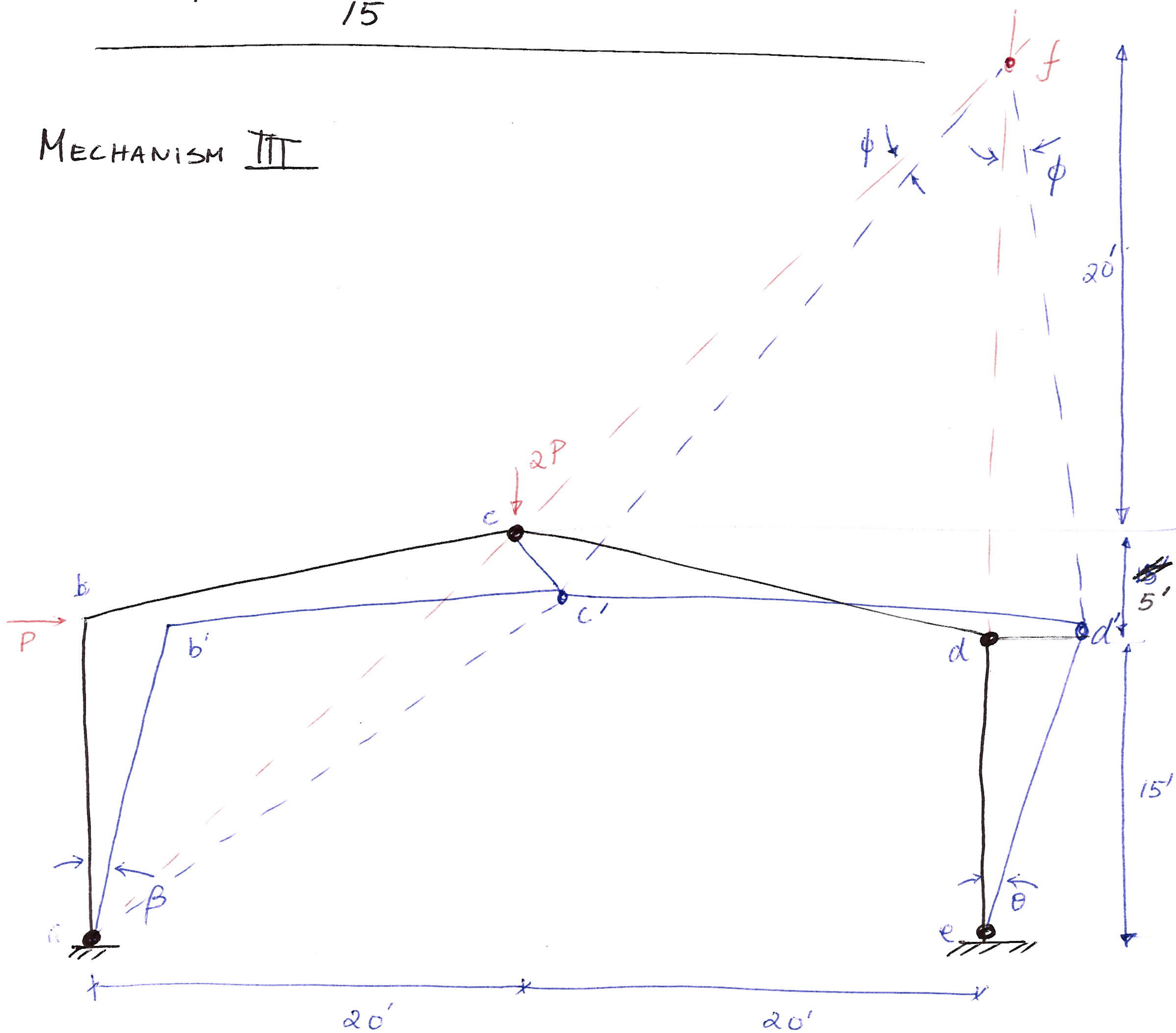
$\Delta_v = \beta \times 20\,ft.$

$$W_{EXT} = W_{INT}$$

$$2P(20\beta) + M_P \times (1.5\theta + 3\theta + 2.5\theta + \theta)$$

$$\therefore \quad P_P = \frac{2M_P}{15} = 26.67^k$$

MECHANISM $\underline{III}$



$$\overline{dd'} = \theta \times \overline{ed} = \phi\ \overline{df}$$

$$\therefore \theta \times 15' = \phi \times 25'$$

$$\therefore \phi = \frac{25}{15} \times \theta \qquad \phi = \frac{15}{25} \times \theta = \frac{3}{5}\theta$$

$$\beta \times \overline{ac} = \phi\ \overline{cf}$$

$$\boxed{\beta = \phi = \frac{3}{5}\theta.} \longrightarrow \text{rotation of frame about IC.}$$

horizontal displacement @ B = $15' \times \beta$

vertical displacement @ C = $\underline{\underline{20' \times \beta}}$

$$W_{EXT} = W_{INT}$$

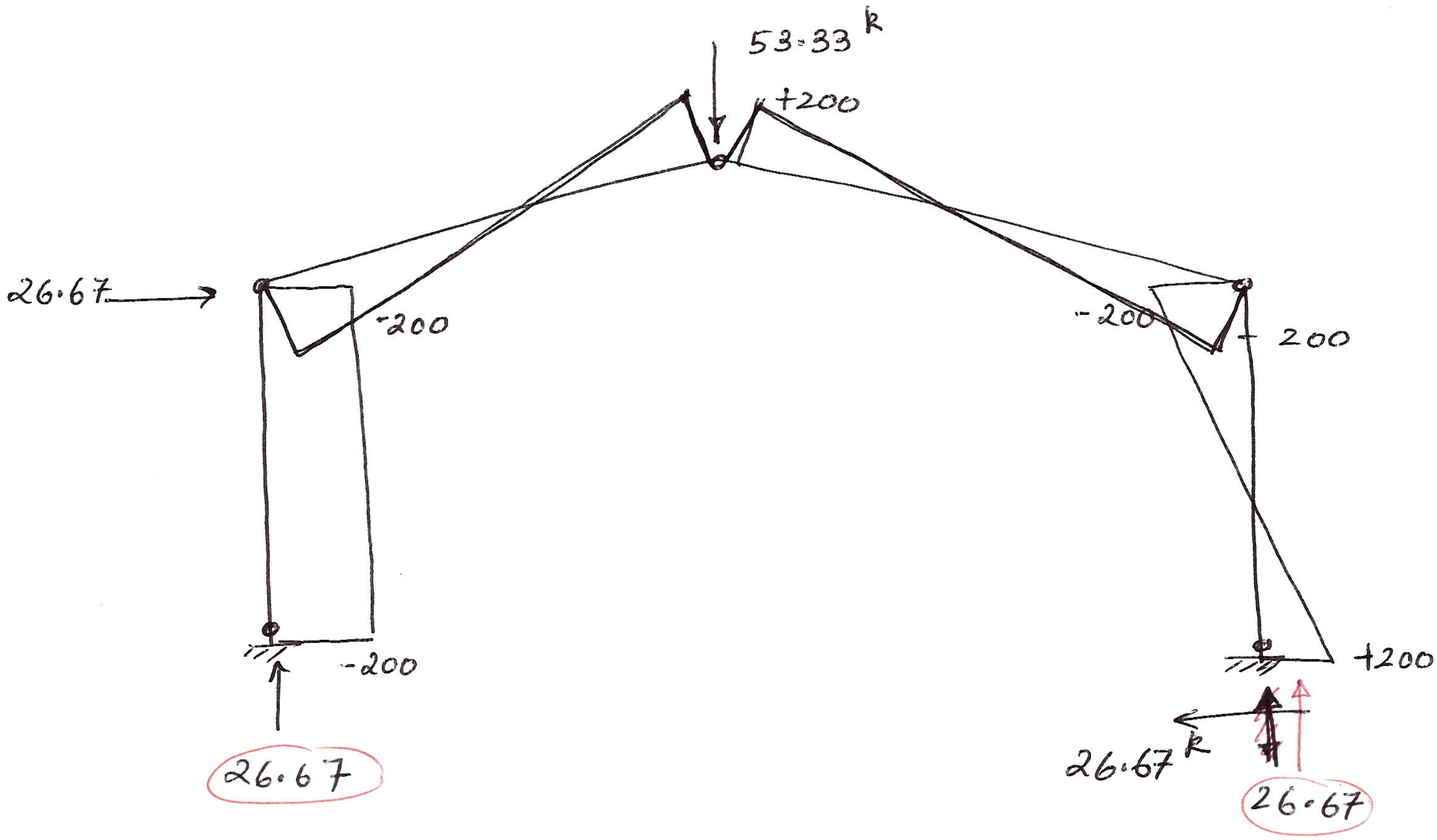
$$P \times 15\beta + 2P(20\beta) = M_p\Big[\underset{a}{(\beta)} + \underset{c}{(\beta+\phi)} + \underset{d}{(\phi+\theta)} + \underset{e}{\theta}\Big]$$

substitute $\phi = \beta = \frac{3}{5}\theta$ & simplify

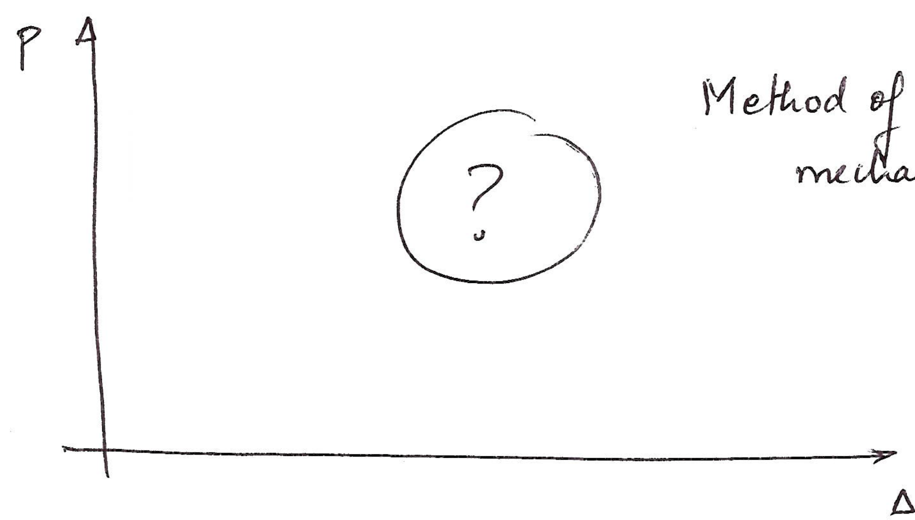
$$P = \frac{2M_p}{15} = \underline{\underline{26.67^k}}$$

Both mechanisms II & III are lower and governing!

Check BMD:



$|M| \leq M_p$ OK.



Method of mechanism!

Methods of Simple Plastic Analysis.

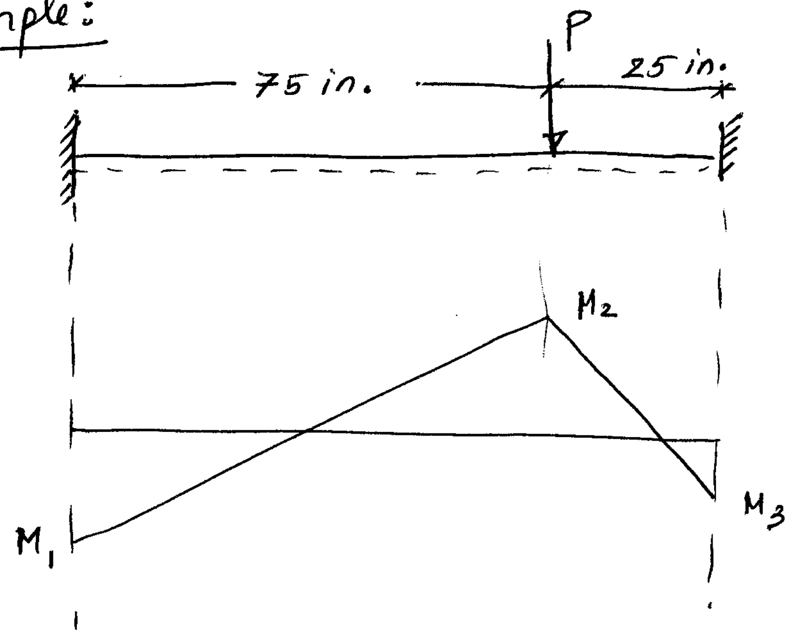
1. KINEMATIC METHOD or the approach of mechanisms.

   - Upper bound method
   - Consider all possible mechanisms until you can determine with the lowest collapse load
   - Check local mechanisms or partial mechanisms
   - Establish BMD to check is $|M| < M_p$ everywhere
   - Cumbersome, tedious $\longrightarrow$ not useful for design

2. Equilibrium Method

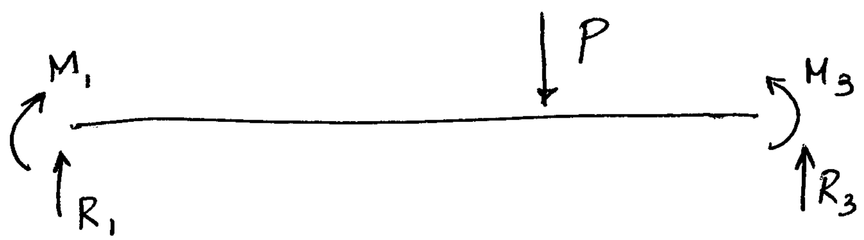
   - Lower bound method.
   - Estimate the plastic limit load by satisfying equilibrium & material strength limits.
   - Does not consider mechanisms or kinematic method.
   - Advantage: $\longrightarrow$ always gives a conservative answer (unless there is a non-ductile component)
   - Disadvantage: $\longrightarrow$ even more difficult than the kinematic method to apply!

Example:



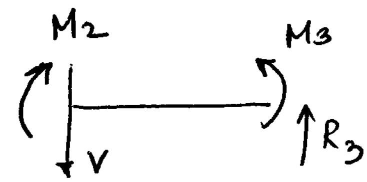
$M_p = 1500$ k-in.

The moment diagram is completely defined by M_1, M_2 & M_3



Write the equation of equilibrium

$$\overset{+}{\circlearrowleft} \sum M_{①} = -M_1 + M_3 - 75P + 100R_3 = 0$$



$$\overset{+}{\circlearrowleft} \sum M_{②} = -M_2 + M_3 + 25R_3 = 0$$

Combine and eliminate R_3

$$\boxed{P = \frac{-M_1 + 4M_2 - 3M_3}{75}} \longrightarrow \text{Equation 1}$$

$$P = \frac{-M_1 + 4M_2 - 3M_3}{75} \longrightarrow$$

given any choice of M_1, M_2 & M_3, the resulting value of P will always satisfy equilibrium.

0 Choose M_1, M_2 & M_3 $\leq M_p$

↳ corresponding plastic limit load.

1 $M_1 = -500$ k-in, $M_2 = 750$ k-in, $M_3 = -1500$ k-in. $\Big\}$ $P = 106.7^k$

2 $M_1 = M_3 = -1000$ k-in, $M_2 = 1000$ k-in $\Big\}$ $P = 106.7^k$

3 $M_1 = M_3 = 0$, $M_2 = 500$ k-in $\longrightarrow P = 26.7^k$

4 $M_1 = -1500$, $M_2 = 1500$, $M_3 = -1500$ k-in $\Big\} \longrightarrow P_p = 160^k$

5 $M_1 = -1500$, $M_2 = 750$, $M_3 = 1500$ $\Big\}$ $P = 0$

- There are infinite number of moment diagrams that satisfy the equilibrium equation

- If you guess the wrong moment diagram
  → you get the wrong plastic limit load.

- All the incorrect solutions → do not provide enough hinges to get a mechanism

LOWER BOUND THEOREM.

For flexural systems:

A limit load is computed based on moment diagram that satisfied
→ equilibrium
→ $|M| \leq M_p$ everywhere

$\leq$ True Plastic limit load.

For any other structural system of component

A limit load bases on internal stress or force distribution satisfying:

① → Equilibrium

② Material plastic strength limit
($M \leq M_p$, $P \leq P_Y$, $V \leq V_p$

$\leq$ True Plastic limit load. $\sigma \leq F_y$)

EQUILIBRIUM METHOD

- Any moment diagram in equilibrium with externally applied loads, and for which the moments at any point does not exceed the moment capacities will provide an estimate of the plastic limit load.

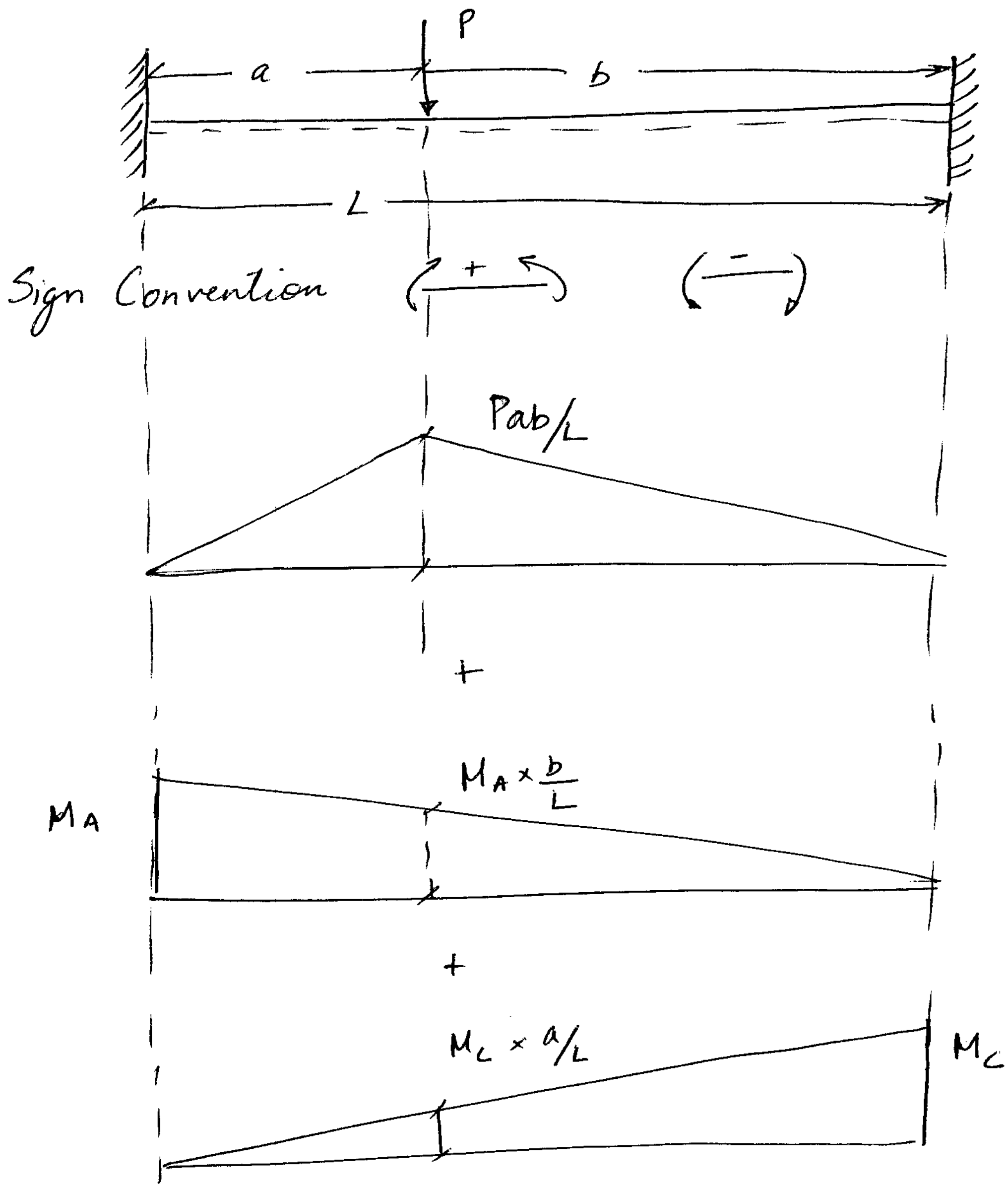
- The estimate will be the ~~a~~ true plastic load if the moment diagrams show a sufficient number of hinges to form a plastic collapse mechanism.

- If sufficient hinges are not found, the collapse load is ~~not~~ ~~con~~ a conservative estimate.

Systematic Approach.

1. Eliminate redundancies $\longrightarrow$ Make structure determinate
2. Draw BMD for determinate structure
3. Draw BMD associated with introduction of each redundancy
4. Construct composite BMD $\rightarrow$ combine BMD from 2 & 3
5. From composite BMD $\longrightarrow$ establish equilibrium equations
6. Establish locations where M reaches $\rightarrow$ M_p such that sufficient number of hinges will exist to form a plastic collapse mechanism & integrate this information into equilibrium equation
7. Solve for plastic collapse load using equilibrium equations

Example



$$M_B = \underbrace{\frac{Pab}{L} + M_A \times \frac{b}{L} + M_C \times \frac{a}{L}}_{\text{Equilibrium equation}}$$

$M_B = +M_P$

M_A & $M_C \neq +M_P \longrightarrow$ because then $P = 0$

$\therefore\ M_A$ & $M_C = -M_P$

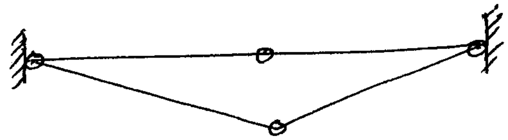
$$\therefore\ +M_P = \frac{Pab}{L} - M_P\left(\frac{b}{L} + \frac{a}{L}\right)$$

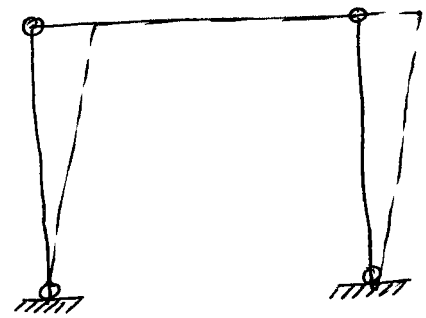
$$\therefore\ P_p = \frac{2M_P \times L}{ab}$$

Answer

Basic Mechanisms

1. Beam (Member) Mechanism
2. Panel (Frame) Mechanism
3. Gable Mechanism
4. Joint Mechanism





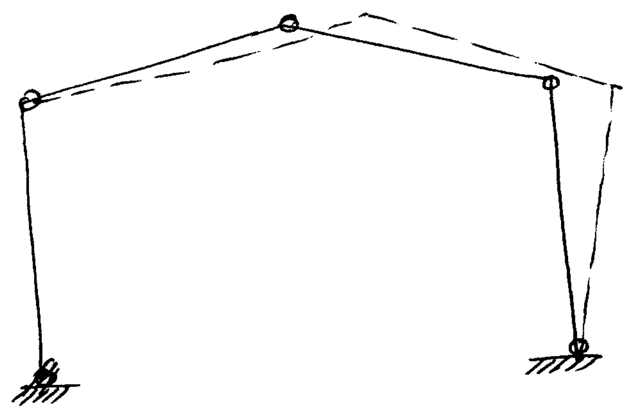
↓

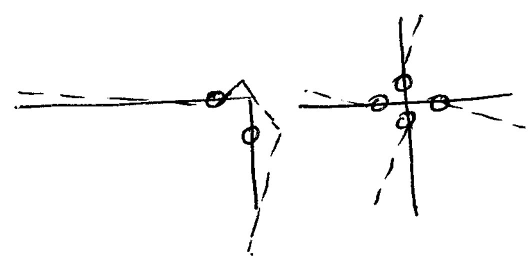
rotation of a joint with plastic hinges being developed in all members framing into the joint

local rotation at joint only

no external work

convenient to reduce internal work



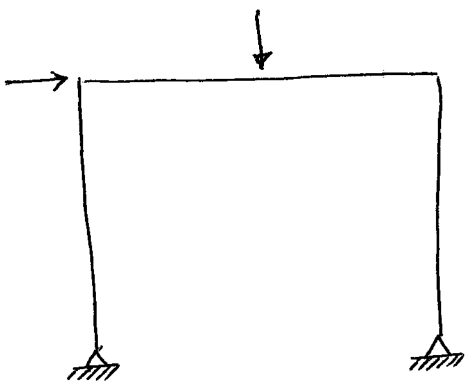


$n = N - X$

N: potential hinge locations

X: degree of indeterminacy

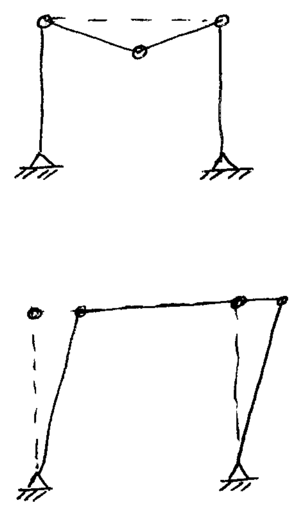
n: # of possible mechanisms.

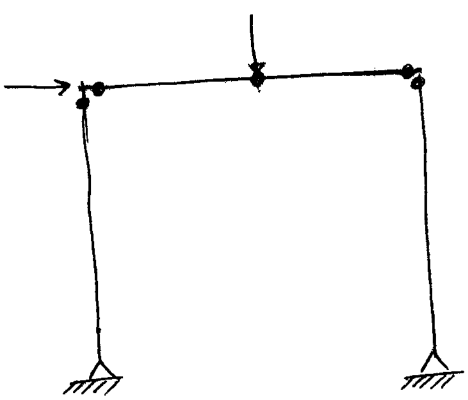


$N = 3$

$X = 1$

$n = 2$

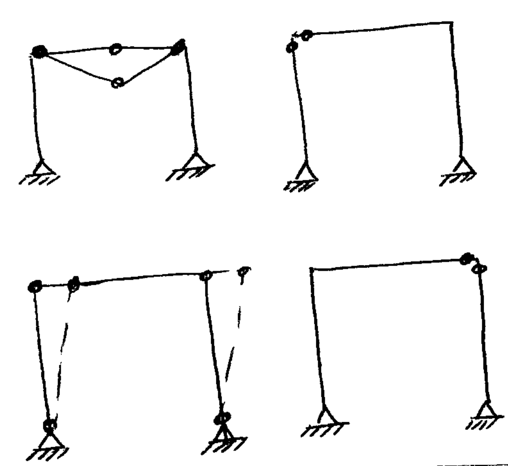


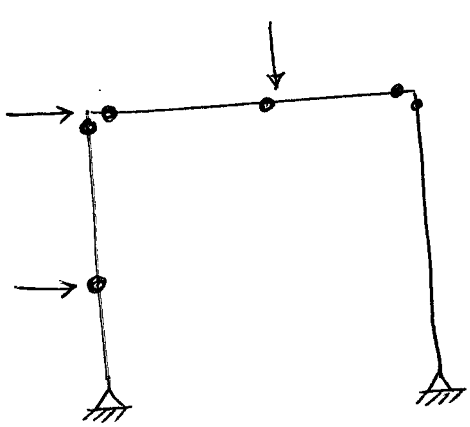


$N = 5$

$X = 1$

$n = 4$

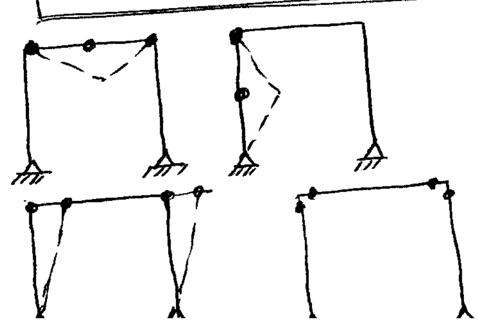


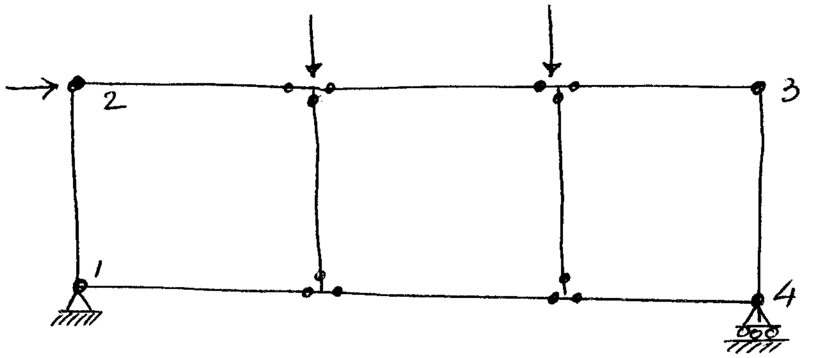


$N = 6$

$X = 1$

$n = 5$

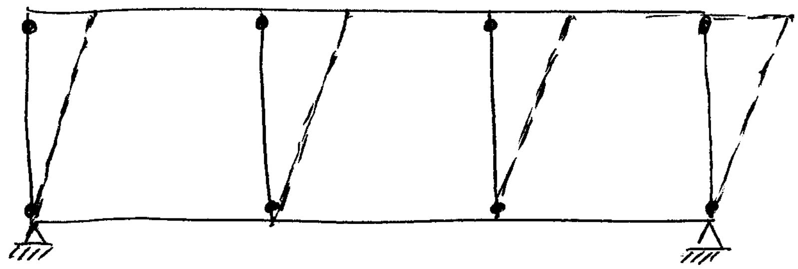


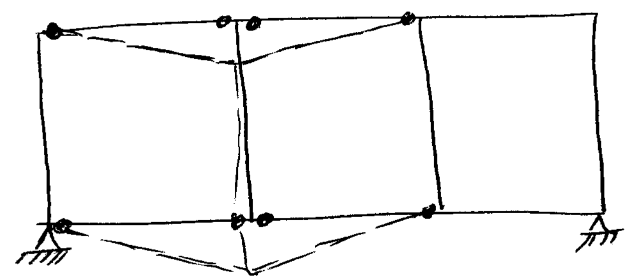


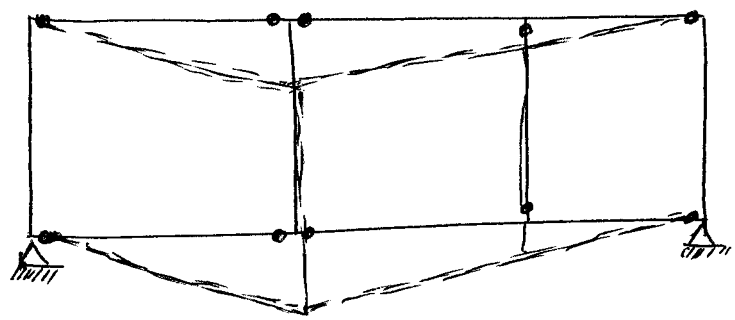
$N = 16$

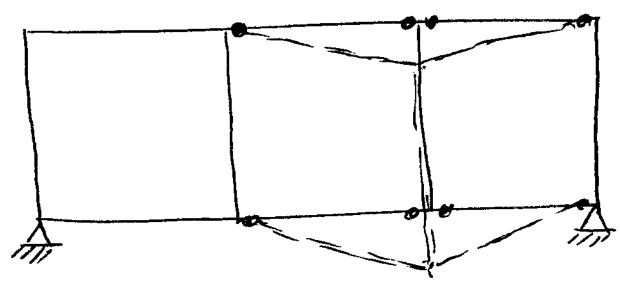
$X = 3 \times 3 = 9$

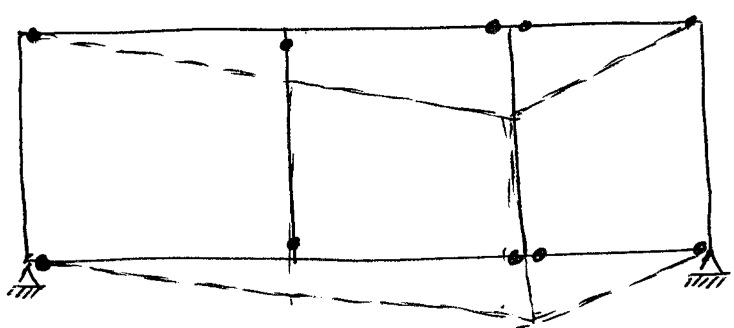
$n = 7$











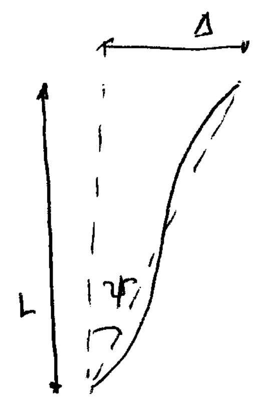
What should we do Now?

$$\{F\} = [K]\{\Delta\}$$

assembled from local $[K_e]$ using compatibility & equilibrium

$$\{F_e\} = [K_e]\{\Delta_e\}.$$

→ assembling the slope-deflection equations



Where did the slope-deflection equations come from?

Moment-Area, Conjugate beam, Virtual Work method

↓

solution of the 2nd order d.e. $\dfrac{d^2v}{dx^2} = \dfrac{M(x)}{EI}$

or $\dfrac{d^4v}{dx^4} = \dfrac{w(x)}{EI}$ ← applied loading.

$$\frac{d^2v}{dx^2} = \frac{M(x)}{EI}$$ → what do you see?

Fear, Anguish, Frustration Emotional

$$\frac{d^2v}{dx^2} = \frac{M(x)}{EI}$$

I see my Kreyzig book → solution that involve $e^{\lambda x}$

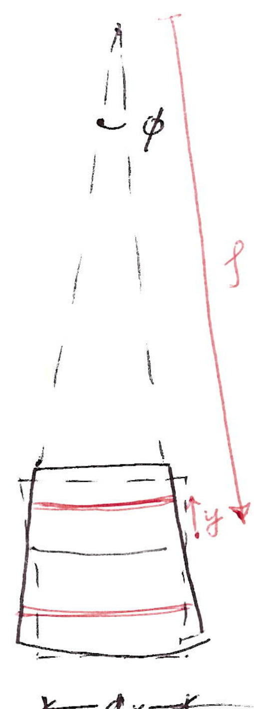
Mathematical response.

$v(x) =$ _ _ _ hermetian cubic polynomial.

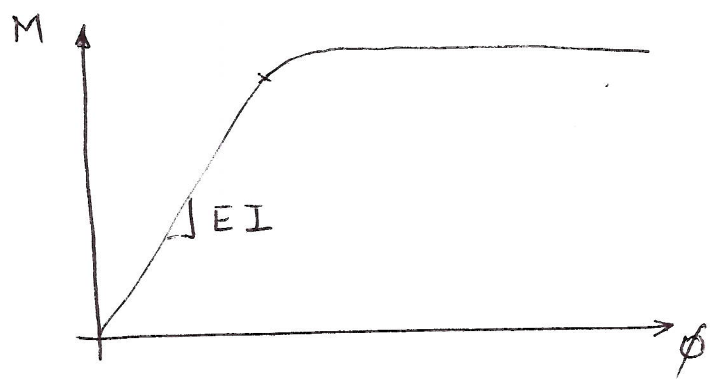
What do you see? Beauty!

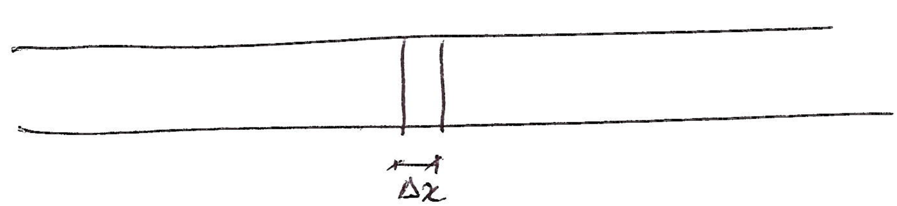
$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

$$\phi = \frac{M}{EI}$$



differential element





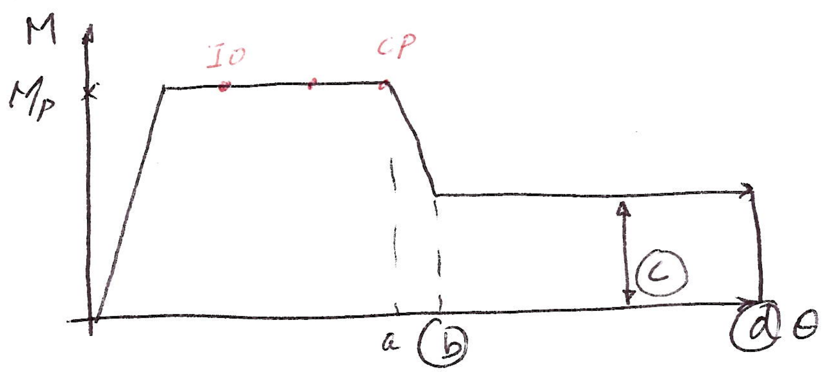
$$\left.\begin{array}{c}[K_e] \\ \downarrow \\ [K]\end{array}\right\} \longrightarrow \text{elastic behavior no matter what!}$$

Lower bound & Upper bount

Event-to-event solution:

would like to know the order of hinge

the plastic rotation demands being placed on the hinges.



plastic hinge

ASCE 41 -

performance
- IO
- CP
- Safety

comparing the rotation demands $\longrightarrow$ accepted as rotation capacities.

Need to perform nonlinear inelastic analysis of the frame for the applied loads.

→ Need an inelastic beam-column finite element

How to build it?

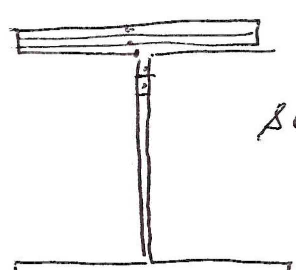
---

① Concentrated plasticity based finite element

② Distributed plasticity based finite element

both are effective
which computer program is being used.

- goes back to the first priciple & fixes it
- discretizing the cross-section into fibers, assuming σ-ε behavior that is ienelastic



section P-ε-M-ϕ relations

- going to get complicated in state determination

- require the use of force-based method & flexibility formulations to be more accurate

- Finite elements are typically displacement based & stiffness formulations

X

$$[K_e] = \int [B]^T [D] [B] \, dV$$

? ? ?

derivative of shape functions

material elasticity

shape functions $\begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} = \begin{bmatrix} \text{Shape} \\ \text{Functions} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$

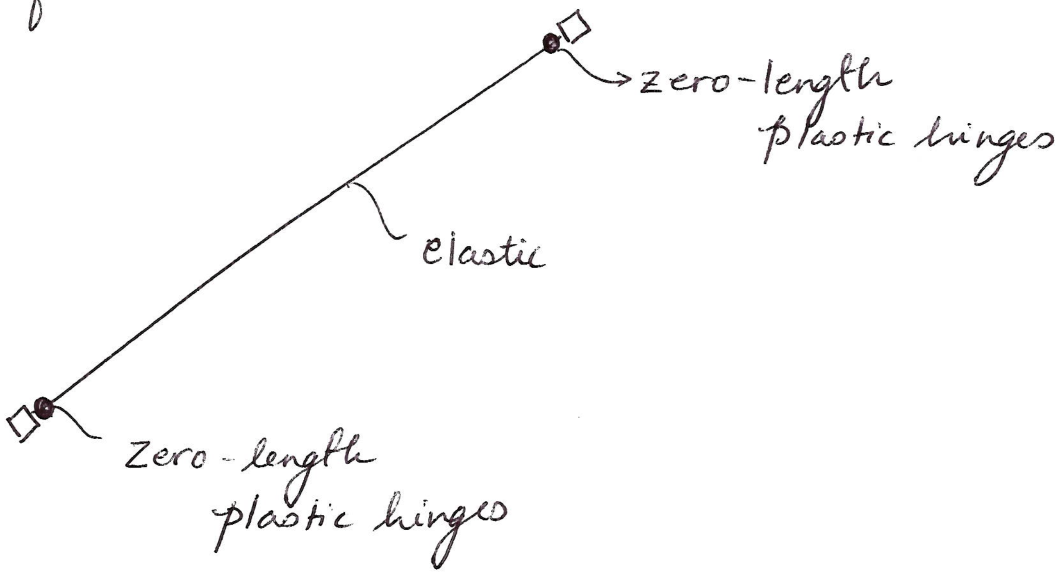
$$[F_e] = \int [B]^T [D] [B] \, dV$$

derivative shape functions

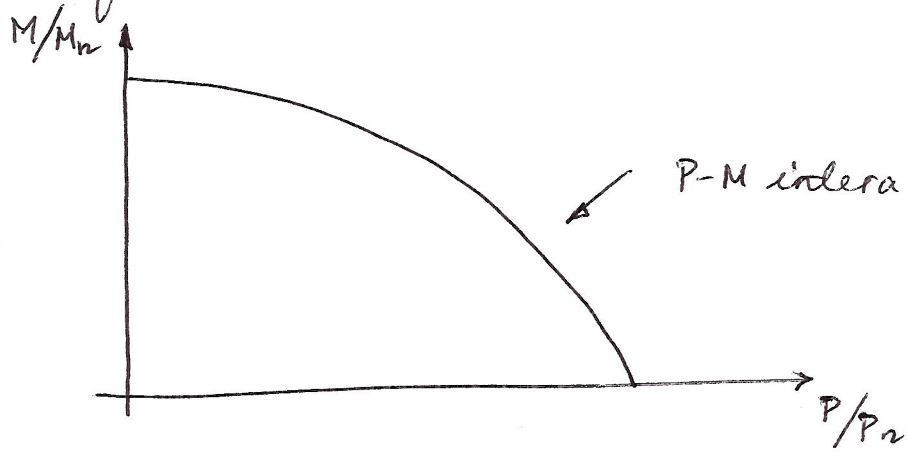
Shape function $\begin{Bmatrix} P(x) \\ M(x) \end{Bmatrix} = \begin{bmatrix} \text{Shape} \\ \text{functions} \end{bmatrix} \cdot \begin{Bmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{Bmatrix}$

Concentrated Plasticity Formulation:

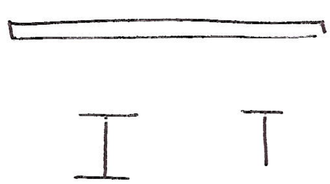
fixes it at the end

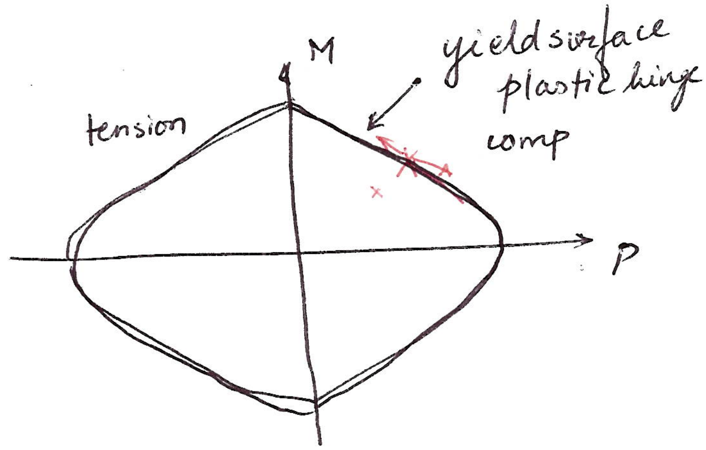


develop the fundamental P-M interaction



different cross-sections





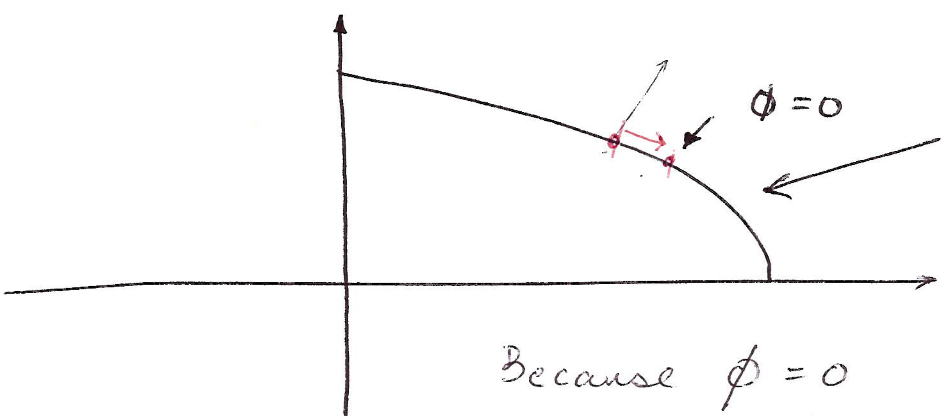
$\{P_1, P_2 \; ------ \; P_n\}$ stress resultants!

Yield surface $\quad \phi(P_1, P_2 \; ...... \; P_n) = 0$

$\phi = 0 \longrightarrow$ yield surface & yield condition

After yielding, my state will remain on the yield surface

apply incremental loading $\{\Delta P_1, \Delta P_2 \; ... \; \Delta P_n\}$.



curve

continuous equation
convex.

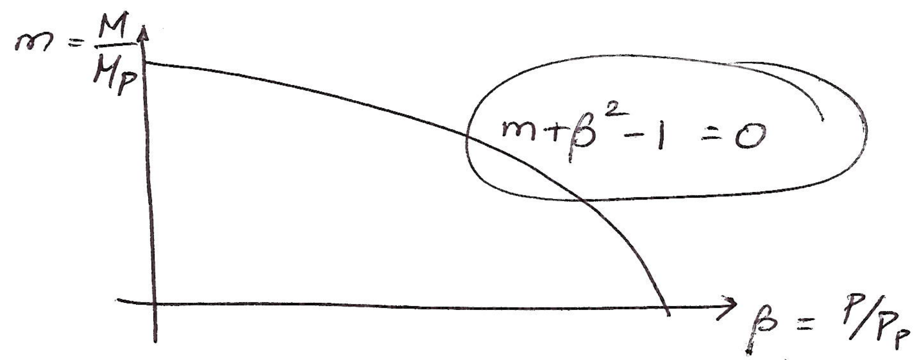
Because $\phi = 0$
& point stays on the surface!

$$\Delta\phi = 0$$

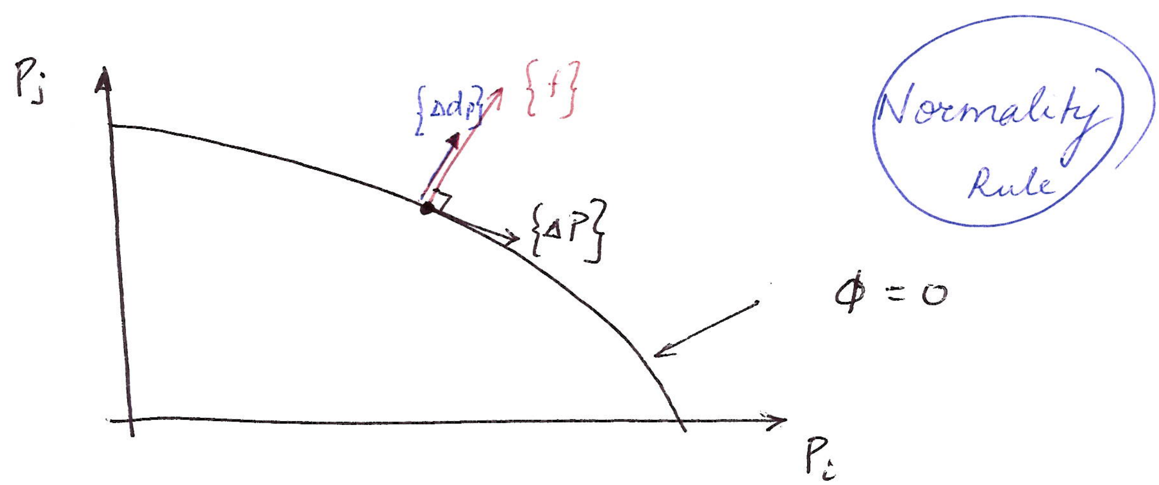
$$= \frac{\partial\phi}{\partial P_1} \cdot \Delta P_1 + \frac{\partial\phi}{\Delta P_2} \cdot \Delta P_2 + ------ + \frac{\partial\phi}{\Delta P_n} \cdot \Delta P_n$$

$$\therefore \; \Delta\phi = \{f\}^T \{\Delta P\} = 0$$

where $\{f\}^T \longrightarrow$ contains $f_i \longrightarrow \dfrac{\partial\phi}{\partial P_i}$



$$\Delta\phi = \{f\}^T\{\Delta P\} = 0$$ ⟶ consistency condition!



Flow Rule:

$$\{\Delta d_p\}^T\{\Delta P\} = 0$$ ⟶ no work done by the increment of plastic strains on the stress increment

$\{\Delta d_p\}$: vector of plastic deformation increments

$$\{\Delta d_p\} = \lambda\{f\}$$

$\{f\}$: gradient or normal to the yield surface

λ: scalar / plastic multiplies.

Associated Flow Rule

$$\Delta W = \{P\}^T \{\Delta d_p\} = \lambda \{P\}^T \{f\}$$

scalar → λ governs the ΔW

Elastic State → $\lambda = 0$ & $\phi < 0$

Plastic loading [PS1 to PS2] → $\lambda \geq 0$, $\phi = 0$, $\Delta\phi = 0$

Unloading Plastic State to Elastic State → $\lambda < 0$, $\phi < 0$, $\Delta\phi < 0$

General Elasto-Plastic Stiffness Matrix

→ In incremental form

What is that? $\{P\} = [K_e]\{d_e\}$

Incremental Form of elastic behavior $\{\Delta P\} = [K_e]\{\Delta d_e\}$

$$\{\Delta P\} = [K_e]\left\{\{\Delta d\} - \{\Delta d_p\}\right\}$$

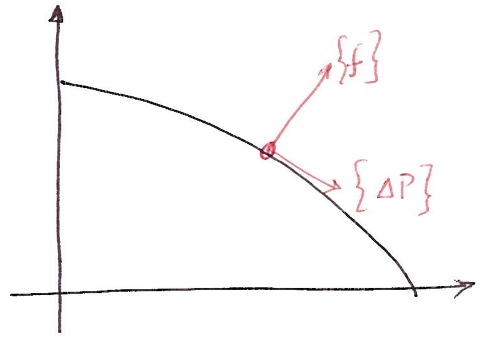
$$\{\Delta P\} = [K_e]\Big\{\{\Delta d\} - \lambda\{f\}\Big\}$$

What is λ → scalar that needs to be determined!

Use the consistency condition to determine it

$$\Delta\phi = 0$$

$$\therefore \{f\}^T\{\Delta P\} = 0$$



$$\{f\}^T [K_e]\Big\{\{\Delta d\} - \lambda\{f\}\Big\} = 0$$

$$\therefore \{f\}^T[K_e]\{\Delta d\} - \lambda\{f\}^T[K_e]\{f\} = 0$$

$$\therefore \lambda = \frac{\{f\}^T[K_e]}{\{f\}^T[K_e]\{f\}}\{d\Delta\}$$

$$\{\Delta P\} = \underbrace{[K_{pe}]}_{\text{plastic stiffness reduction matrix}} \{\Delta d\}$$

$$[K_{pe}] = [K_e] - \frac{[K_e]\{f\}\{f\}^T[K_e]}{\{f\}^T[K_e]\{f\}}$$

full of only $[K_e]$ & $\{f\}$ → gradient to the yield surface.

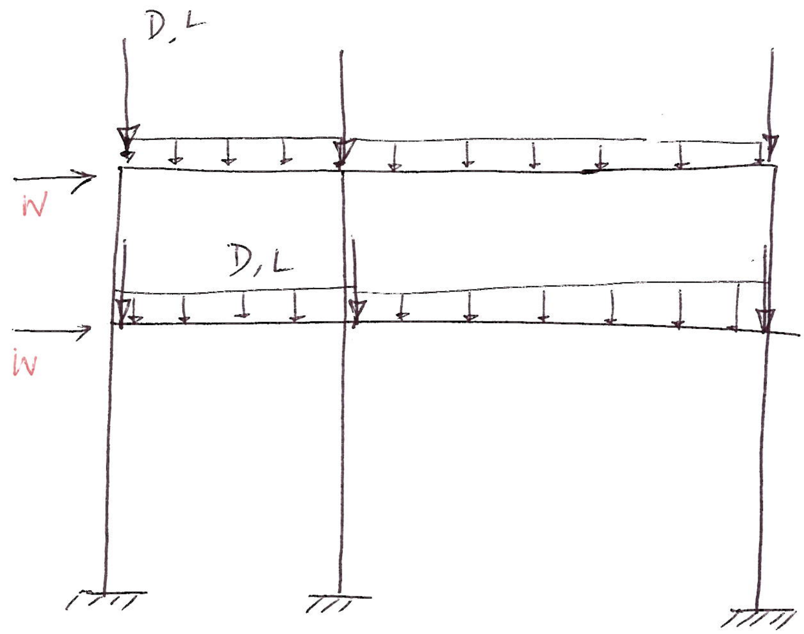
depends on how the yield surface was reached & whether one or both ends have hit the yield surface

---

- Concentrated plasticity based beam-column element!

Sufficient # of elements → work fine!

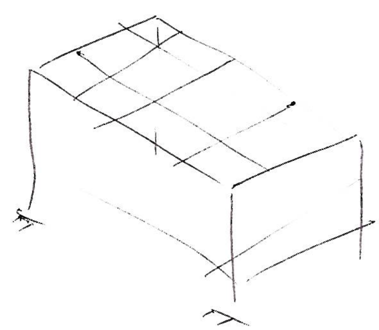
No $P-\Delta$ effects or $P-\delta$ effects.



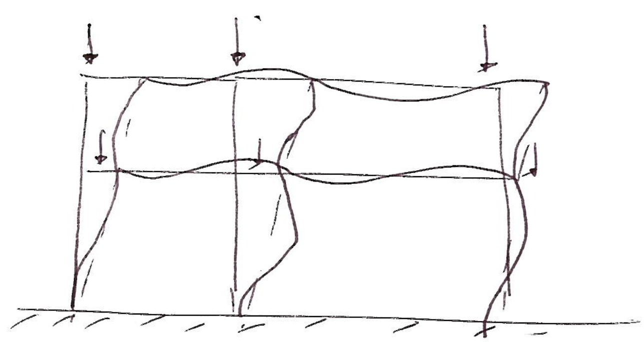
ASCE 7
minimum loads
& combinations

factored loads!

1st order elastic Analysis
→ stiffness matrix from CE 474
$[K_e]$



For this frame → perform a frame buckling analysis



frame buckling load.

2nd order elastic Analysis
→ stiffness matrix from CE474
→ geometric stiffness matrix $[K_G]$ from CE 579
→ 2nd order effects $P-\Delta$ & $P-\delta$ effects
→ but no plasticity

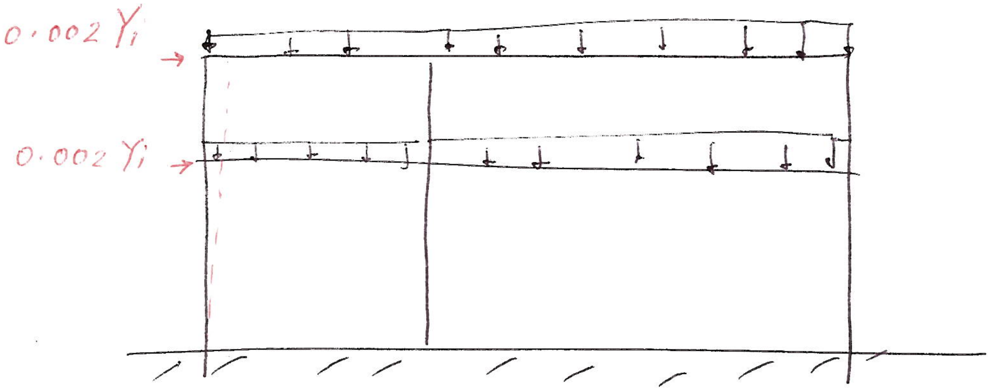
Chapter C of AISC 360-10

→ direct analysis method

2nd order analysis

with notional lateral loads

& stiffness reduction

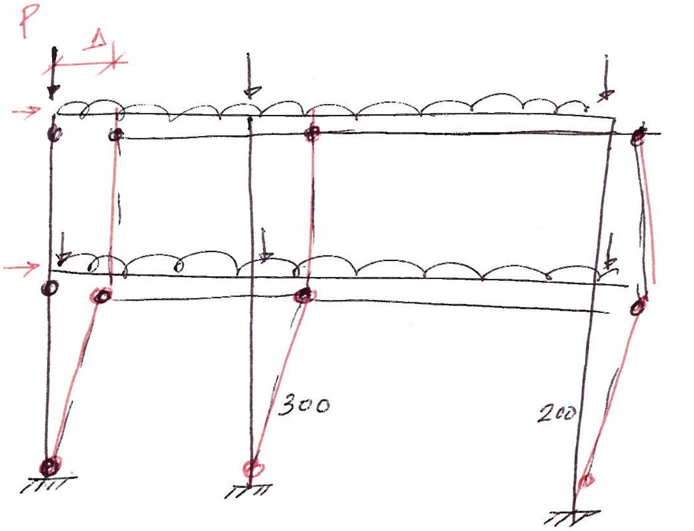


initiates P-Δ & P-δ effects.

residual stresses cause yielding to occur earlier

1st order plastic analysis

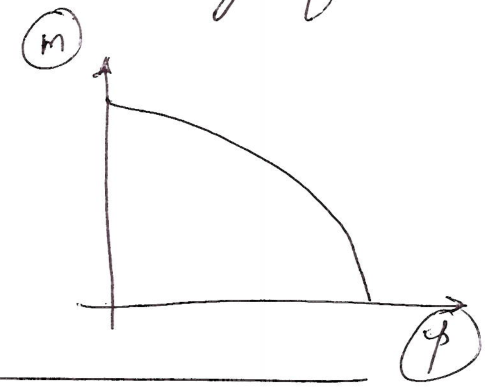
$[K_{pe}]$ → plastic stiffness reduction matrix



plastic analysis leading to the formation of mechanism

$$[K_{pe}]$$

$[K_e]$ & introducing hinges as they form



plastic mechanism analysis to calculate

$P_p \rightarrow$ plastic limit load

2^{nd} order plastic analysis

$$[K_e] - [K_g] - [K_p]$$

elastic stiffness matrix

geometric stiffness of matrix

concentrated plastic hinges.

$P - \Delta$

$P - \delta$

hinges

order

event to event analysis.

1st order beam-column finite element!

Distributed Plasticity-Based Formulation!

Orders of Analysis!