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Structural Members
and Frames
The basis for this book is a set of lecture notes used in a graduate course with the same title. I have taught this course every year since 1960, when I inherited the privilege of teaching it from my own professor, Dr. Bruno Thürlimann. Originally a course on structural members and frames was developed at Lehigh University in the early 1940's by Dr. Bruce G. Johnston and has been taught there ever since. It has spread from Lehigh to other universities through the graduates of Lehigh's Fritz Engineering Laboratory, and its original content has been greatly modified by the research performed by students of the original teachers.

I would like to express my appreciation and gratitude to Dr. Bruno Thürlimann of the Swiss Federal Institute of Technology and to Dr. Robert L. Ketler of the State University of New York at Buffalo for introducing me to the mysteries of inelastic and unstable behavior and for inspiring me to do the research that has enabled me to write this book. I further acknowledge Dr. Thürlimann's course notes, which have served as the embryo for this book.

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The manuscript was typed by Miss Grace Mann and by Mrs. Ray Tide, and Mrs. Alice Bletch and Mrs. Lois Simons assisted with a variety of clerical tasks during the preparation of this book. Their help is appreciated. To my family for their patience I owe my gratitude.

THEODORE V. GALAMBOS

St. Louis

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NOMENCLATURE

$A$  Area of cross section
$A_0$  Numerical coefficient
$A_f$  Area enclosed by a sector
$A_g$  Effective area
$A_r$  Area of one flange of a wide-flange shape
$A_w$  Area of web of a wide-flange shape
$a$  Distance between a point on a cross section and the shear center
$A_{1}, A_{2}$  Plate dimension
$a_{1}, a_{2}$  Coefficients in a series
$B$  Numerical coefficient
$B_1$  Bending stiffness about $x$ axis
$B_2$  Bending stiffness about $y$ axis
$b$  Flange width of a wide-flange shape
$C$  Plate dimension
$C_1, C_2$  Width of rectangular cross section
$C_{x,y}$  Centroid of a cross section
$C_{x}$  Stability function defined by Eq. 4.79
$C_{y}$  Center of rotation
$C_{x}$  Equivalent moment ratio defined by Eq. 5.37
$C_{y}$  Approximate equivalent moment ratio defined by either Eq. 5.39 or Eq. 5.40
$C_{1}$  St. Venant torsional stiffness
$C_{2}$  Warping stiffness
$C_{1}, C_{2}, \ldots$  Constants of integration
$C_{1}$  Nondimensional coefficient defined by Eq. 3.79
$C_{1}$  Stability function defined by Eq. 4.75
$C_{1}$  Nondimensional coefficient defined by Eq. 3.46
$D_r$  Depth of a wide-flange section
$D_r$  Depth of a rectangular section
$E$  Modulus of elasticity
$E_{rt}$  Endpoint on a thin-walled section
$E_{t}$  Strain-hardening modulus
$E_t$  Tangent modulus
$E_{r}$  Reduced modulus
$e$  End eccentricity
$e_1$  Initial eccentricity
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<td>$\sigma_{sa}$</td>
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<tr>
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<td>Stress in loading zone of a column</td>
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<td>$\tau_{xy}$</td>
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<td>Curvature of column-deflection-curve segment</td>
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<td>Coefficient defining yield penetration</td>
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<td>Normalized unit warping</td>
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<td>Unit warping with respect to the shear center</td>
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<td>$\kappa$</td>
<td>End moment ratio</td>
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Introduction

1.1. CLASSIFICATIONS AND SCOPE

STRUCTURAL DESIGN

The design of a structure is an art in which the experience of past successful and unsuccessful construction, the laws of physics and mathematics, and the results of research are utilized to provide structures which can function efficiently and safely, which are economical to build and maintain, and which are aesthetically pleasing. This definition of structural design is a grossly abbreviated definition of an operation which, for a major project, may involve the cooperation among, and the pooling of the knowledge of, hundreds of experts from a variety of disciplines. One could not even attempt to place only the major phases of structural design within the covers of one book, and it would be impossible to find a person who would be an expert in all the fields of knowledge involved. The purpose in this book is to deal with
only one of the many facets of the design process, namely, the analysis of the strength of metal frames and their structural components.

One step in the design of structural frames is determining the geometrical configuration of the members comprising the load-carrying skeleton of the frame. The design is often a separate and important study. We shall not be concerned with this topic further except to define the types of loads acting on the structures to be analyzed.

Loads may be either static or dynamic. The weight of the structure, called the dead load, and certain specific fixed loads which do not change during the life of the structure are the only true static loads. In usual practice, however, the live loads due to occupancy and, in many instances, the wind loads, also, are treated as static loads. For static loads we can neglect the effects of inertia arising from the acceleration of the mass of the structure and the effects of rapid load changes on the material properties. For dynamic loading these effects may not be neglected; they may well play a predominant role. Dynamic loading arises from the acceleration caused by wind, earthquake, blast, or impact. In the past such loads have usually been considered as quasi-static, and in the analysis of the structure no distinction was made between their effects and those of true static loads. With the development of methods of dynamic analysis and the use of the computer, it is now possible to make an analysis for dynamic effects. Because this is a separate and important field of study of structural behavior, it will not be dealt with in this book.

With the exception of the weight of the structure, which usually remains constant during the life of the structure, the loads fluctuate. These load repetitions may lead to design considerations involving the fatigue cracking of the material and to failure due to successively larger deflections after each load repetition. We shall restrict ourselves here to nonrepetitive loads.

Thus the loads on the structures to be analyzed will be static and nonrepetitive. We shall further specify that some or all of the loads are related by a constant factor of proportionality (proportional loading) and that the loads will retain the same direction throughout the whole loading history. The reason for these latter restrictions is that in the inelastic range the response of the structure is dependent on the sequence in which the various loads acting on the structure are applied.

CLASSIFICATION OF STRUCTURES

Structures can be classified in many ways. For our purposes the subdivisions into shell and frame structures, as given in Ref. 19, is adequate. In shell structures, the load-carrying element also serves the functional requirements of enclosing space. The structural frame, or skeleton,
usually serves only to support the loads transmitted from the functional elements of the structure. We shall deal with frame structures only. Some examples of such structures are simple and continuous beams, rigid frames, trusses, and plate girders.

1.2. THE RESPONSE OF STRUCTURES TO LOADS

THE LOAD-DEFORMATION BEHAVIOR

The behavior of a frame under loads is best visualized from a curve which relates the load to the deflection of any characteristic point on the structure. For example, if the two-story rigid frame in Fig. 1.1 were subjected to the vertical loads \( P \) and the horizontal loads \( \alpha P \) (where \( \alpha \) is a constant factor of proportionality) and loading were started at \( P = 0 \), an experimenter would obtain a curve like the one shown in this figure for the relationship between \( P \) and the horizontal deflection \( \nu \) of the top of the structure.

The load-deflection relationship in Fig. 1.1 is typical of the response of frame structures to static proportional loading. As \( P \) increases from zero, the structure behaves elastically until the elastic limit is reached. For any load below this limit the structure is elastic, that is, it will return to its original undeformed position upon complete removal of the load.

Beyond the elastic limit some portions of the frame begin to yield. As a result, the frame members become less stiff, and increasingly larger deflections result from equal increments of load until finally a peak is reached on the curve. This is the maximum load which can be supported. Under some conditions the load may drop very sharply after the peak of the curve is reached, and in some instances this drop is very gradual, resulting in a flat plateau. With further deformation the load must decrease if static equilibrium is to be maintained. If the loads are removed anywhere in the inelastic region, then the structure will not return along its path of loading and a permanent deflection results when \( P \) is zero (see dashed line in Fig. 1.1). Subsequent reloading will follow approximately the unloading curve. It should be noted that even in the elastic region the deflection is not necessarily a linear function of the loading. This nonlinearity is introduced by the changes of the geometry of the deformed structure.

LIMITS OF STRUCTURAL USEFULNESS

From a load-deflection curve we can make several observations about the usefulness of the structure. The most obvious of these is the maximum load. If the load is due to dead weight, the structure will collapse when this load is reached. In design we must be certain that the working load (see Fig. 1.1), which is to be supported under service conditions, is substantially less than the maximum load. The ratio of the maximum load to the working load is called the load factor, and it is usually prescribed in structural specifications. For the type of structure shown in Fig. 1.1, for instance, the load factor prescribed by the 1963 AISC specification is 1.40 if the horizontal loads are due to wind.\(^{(1.20)}\)

Under certain conditions the use of the structure dictates deflection limitations under working loads. A load-deflection curve can also serve as a check on this condition. In fact, the load-deflection curve is a record of the history of the structure. If we have such a curve for our structure, we can check for various criteria of structural usefulness. When one such limit is reached, we have arrived at what we call the failure of the structure. Under static nonrepetitive loads we can have three important criteria of failure: (1) limiting deflection, (2) maximum load, and (3) the start of unstable behavior. Of these, the first criterion is often dictated by rather hard-to-define factors (such as plaster cracking), but the other two are real and definite limits to usefulness. Thus failure will generally mean that either the maximum load has been reached or that the load-deformation path has arrived at a point at which instability sets in.
1.3. INSTABILITY

INELASTIC INSTABILITY

The load-deflection curve in Fig. 1.1 represents the locus of points for which the structure is in equilibrium with the applied loads. This equilibrium may be either stable or unstable. The state of the equilibrium is of vital importance because we cannot tolerate excursions into the unstable range; we are particularly interested in the point at which it goes from the stable into the unstable condition, as this represents a real limit to the usefulness of the structure.

A structure is stable if it tends to return toward its original position after a small disturbance is applied to it and then removed. On the other hand, a structure is unstable if a small disturbance produces a further increase of deflection. In the first instance an addition of energy is required to produce the disturbance, and in the second instance energy is released.\(^{(1.51,1.62,1.30)}\)

In the mathematical treatment of stability problems the disturbance is usually virtual, that is, it does not change the existing force system.\(^{(1.57)}\) In an actual structure these disturbances are of course real, and their effect is reflected not only on the structure but also on the loading system. Thus we must consider the response of both the structure and the loading device for a test of stability.\(^{(1.43,1.64)}\)

Let us first consider the stability of a structure subjected to dead, or gravity, loads (Fig. 1.2). The addition of weight to the structure causes an increase in potential energy, and the load-deformation characteristics of the load system can be represented by a series of straight lines parallel to the deflection axis, as shown by the dashed lines in Fig. 1.2. Each line corresponds to a different weight or energy level defined by the intercept with the load axis.\(^{(1.24)}\)

The intersections between the load characteristics and the structure load-deflection curve correspond to equilibrium points. For example, the points \(A\) and \(B\) on the load characteristic \(CD\) in Fig. 1.2 are equilibrium situations. In order to check for stability we disturb the structure a small amount, displacing \(A\) to \(A'\) and \(B\) to \(B'\). For point \(A\) this disturbance requires an increase of energy, that is, the load characteristic tends toward a higher energy level \(A''\). The unbalanced force, representing the difference between the two characteristics, is directed toward the point \(A\). An increase of energy is required to make this disturbance, and thus point \(A\) is stable. For point \(B\) the disturbance tends toward a lower energy level \(B''\), and the unbalanced force is directed away from \(B\). The energy is released, and \(B\) is therefore unstable.

Similar tests on all points on the ascending portion of the curve will show that these are stable; on the other hand, the descending portion is unstable.

The boundary between the two states of equilibrium is at the peak of the load-deflection curve. This point, being neither stable nor unstable, is in neutral equilibrium. It represents the point at which the structure will collapse under dead loads.

Not all load characteristics are like those shown for dead loads in Fig. 1.2. Another type of loading, commonly encountered in screw-type testing machines and in loads transmitted from adjacent elastic structures and representing the elastic response of the load system, is shown as a series of parallel lines in Fig. 1.3.\(^{(1.24,1.23)}\) Applying the same test for stability as for the loading in Fig. 1.2, we find that points \(A\), \(B\), \(C\), \(F\), and \(G\) are stable and that point \(E\) is unstable. Neutral equilibrium exists at point \(D\), where the gradient of the load characteristic is equal to the gradient of the structure curve. It should be noted that point \(B\), which is at the peak of the curve, and point \(C\), which is already beyond it, are both stable.

Because disturbances are naturally present in any test, the structure curve will not follow its path from the start of instability at \(D\) through \(E\) and \(F\) to \(G\), where it is again stable, unless it is externally restrained to do so, but it
will rapidly pass from $D$ to $G$. This phenomenon is called the dynamic jump, and it usually involves large, if not catastrophic, changes in geometry and can in most cases not be tolerated for satisfactory structure performance.\(^1\)

According to our previous discussion then, a structure is stable if the gradient of the load-deflection curve of the structure $g_\varepsilon$ is larger than the gradient of the load characteristic $g_L$, or

$$g_\varepsilon > g_L \quad \text{stable equilibrium}$$

$$g_\varepsilon = g_L \quad \text{neutral equilibrium}$$

$$g_\varepsilon < g_L \quad \text{unstable equilibrium}$$

In metal frame structures of the type discussed here, the point of neutral

\(^1\) Stable points beyond the peak of the load-deflection curve have been frequently encountered in tests on structures (for example, see Ref. 1.26). The phenomenon of the dynamic jump is well known for shells and rigid bar systems,\(^1,23\) where it is called oil canning, or snap-through, and it has been observed for frame structures during tests.\(^1,17\) A theoretical treatment of the dynamic jump and a review of the literature is contained in Ref. 1.28.

...equilibrium will occur at the peak of the load-deflection curve or on the descending portion of it. Since these parts of the curve are already in the inelastic region, we shall call this form of instability inelastic instability. In other than laboratory tests we do not know the load characteristic very precisely. For our purposes we shall conservatively use the peak of the curve ($g_\varepsilon = 0$) as the point of neutral equilibrium, and so this failure load of the structure will also be its maximum load.\(^2\)

**BIFURCATION OF THE EQUILIBRIUM**

Inelastic instability may often be preceded by a phenomenon called variously buckling or bifurcation of the equilibrium (Fig. 1.4). As the load is increased from zero, the structure begins to deform in a pattern characteristic of the type of structure and loading. Basing our supposition on this initial deformation pattern, we can expect the structure to deform according to curve $OAB$. However, under some conditions it is possible that at a certain critical load the deformation configuration suddenly changes into a different pattern. The equilibrium is said to bifurcate (there is a fork at point $A$ in Fig. 1.4), or the structure buckles, into branch $AC$.

The actual load-deformation curve then consists of two stable branches: curve $OA$, the prebuckling branch, and curve $AC$, the postbuckling branch. Branch $AB$ is unstable and will not be followed by the structure. Buckling does not mean that the structure has necessarily failed; the postbuckling branch is still stable by our previous definition and failure will be due to inelastic instability, as discussed before. However, in many cases, and for reasons which will become evident in later chapters, the initiation of buckling (that is, the point of bifurcation) is considered the limit of usefulness of the structure. Buckling can occur anywhere along the original curve $OAB$: in the elastic region as well as beyond the peak. After its advent the course of the original curve usually is changed downward, resulting in a weakening effect.

\(^2\) This applies if we consider the whole structure. We shall utilize the descending stable portions of the curve for individual members in later studies.
Examples of buckling are column buckling, lateral-torsional buckling of beams and beam-columns, local buckling, and frame instability under symmetric axial loads. Because of the importance of buckling we shall discuss it in considerable detail in later portions of this book.

1.4. FRAMES AND THEIR COMPONENTS

EQUILIBRIUM AND COMPATIBILITY

The preceding comments concerning the behavior of complete structures apply equally well to the components of the frame. In fact, when we set about to analyze a frame, we start with the analysis of components, building up the complete structure by maintaining equilibrium at the juncture of the components and observing the proper boundary conditions. In rigidly jointed structures we assume that the deformations at the joint are compatible with each other. We can thus synthesize the behavior of the complete frame from the knowledge of the behavior of its components.

FRAME COMPONENTS

We shall subdivide frames into the following components: members and connections. Members are elements which are much longer than they are deep, and connections are the devices used to connect two or more members so that the forces occurring at the ends of the members are transmitted and that, in rigidly jointed structures, continuity is maintained.

TYPES OF MEMBERS

We shall employ two kinds of classifications for members. One of these will be prescribed by the kind of loading, and the other will depend on the nature of the cross section.

We can classify members as beams, if bending predominates and the effect of axial load can be ignored; as beam-columns, if significant amounts of both bending and axial force are present; and as columns, if compressive axial forces predominate to the extent that bending can be neglected.

Members are also classified according to the geometry of the cross section into solid (or thick-walled) members and into thin-walled members. For the former all the dimensions of the cross section are of the same order of magnitude, whereas for the latter the thickness of the plate elements from which the cross section is made up is of a smaller order of magnitude than the depth and width of the member. Examples of solid members are reinforced concrete beams or heavy steel columns. Most shapes used in metal construction, on the other hand, are thin-walled. Because of their practical importance and the predominant role of instability we shall concentrate on thin-walled members in this book.

MATERIAL AND CROSS-SECTIONAL PROPERTIES

The behavior of members depends on their geometry, on the loads, and on the properties of the cross section. The cross-sectional properties in turn depend on their geometry, on the material properties, and on the residual strains present as a result of the fabrication process.

The material properties which will be of importance are obtained from a tension or compression test on a small coupon of the material from which the member is made. Such a test furnishes stress-strain curves, two of which are shown in Fig. 1.5. Only the initial portions of the curves are shown here, as we are not going to utilize the full curves up to rupture. The curve in Fig. 1.5(a) is typical of structural carbon steels; the curve in Fig. 1.5(b) is characteristic of structural aluminum.

For both types of curves the stress is proportional to the strain up to the yield stress $\sigma_y$ or the proportional limit $\sigma_p$. In this elastic range $\sigma = E\varepsilon$, where $E$ is the modulus of elasticity, $\sigma$ is the average stress $P/A$, and $\varepsilon$ is the corresponding strain. In structural carbon steel the curve exhibits a pronounced plastic plateau, where strain increases but the stress remains equal to $\sigma_y$. Strain hardening commences at a strain $\varepsilon_{yp}$; the slope of the curve upon further straining is $E_{yp}$, the strain-hardening modulus.

The solid curve in Fig. 1.5(a) represents a static condition. In reality it is not possible to test a coupon in this manner. If the coupon is strained in a testing machine at a slow and constant strain rate, then we obtain the dashed curve. This curve generally exhibits an upper yield point $\sigma_{yu}$ and a lower yield point $\sigma_{yl}$. The plastic range is a somewhat wavy line, and the static yield stress (or static yield level) is obtained by stopping the straining (see dip in the dashed curve). The dynamic yield stress $\sigma_{yd}$ is always higher than $\sigma_y$.\(^{(1.3)}\)

Since we have adopted static loading as the basis for our work here, we shall use the static yield level as the termination of elastic behavior.

If no pronounced yield level exists [Fig. 1.5(b)], then a yield strength $\sigma_{yo}$ is defined by the per cent offset method (see dashed line). For such a material we shall use this value as our limit of elastic behavior, calling it for convenience also $\sigma_y$.\(^{(3)}\)
Residual stresses are caused by a variety of factors, and in some cases they may be as large as the yield stress. They are a result of plastic deformation during the manufacturing process. For example, uneven cooling of steel shapes after hot rolling or welding, and cold straightening by gaging or rotarizing, all result in residual stresses. For hot-rolled wide-flange shapes the flange tips cool faster than the metal at the flange-web juncture. As the slower-cooling portions finally cool and contract, they induce compressive residual stresses on the parts which are already cold. Thus the flange tips have compressive residual stresses, whereas the flange centers are in tension.

Residual stresses can be greatly reduced by stress-relieving the member, but this is usually done only in exceptional cases because of the great expense involved. We must therefore assume that steel members will have substantial residual stresses. The distribution and magnitude of residual stresses is a function of a variety of factors. In Refs. 1.34 through 1.36 there are shown a great number of diagrams with the measured residual stresses, caused by hot rolling, welding, and flame cutting, on steel cross sections of various shapes and sizes. The residual stresses due to cold straightening are usually localized and can in most cases be neglected.

The diagram in Fig. 1.6 shows an idealization of the residual stress distribution in a hot-rolled wide-flange shape. This pattern has been used in a number of determinations of the inelastic strength of beam-columns.
and we shall adopt it throughout this book to show the effect of residual stresses in various inelastic studies. The pattern closely approximates the residual stresses measured on ASTM-A7 steel 8WF31 sections. In Fig. 1.6 the web has tensile stresses $\sigma_t$, which are constant, and the stresses in the flanges are assumed to vary linearly from $\sigma_{rf}$ at the flange center to a compressive stress $\sigma_{re}$ at the four flange tips. The stresses are also assumed to be constant across the thickness of the plates.

Since no external forces exist, equilibrium requires that the sum of the stresses over the whole cross section must be zero. Thus, from Fig. 1.6,

$$\int_A \sigma_r \, dA = 0 = \sigma_{rf}(d - 2t)w + 4t \left( \frac{\sigma_r}{2} \right) \left( \frac{b}{2} - \frac{b \sigma_{re}}{2 \sigma_{re} + 2 \sigma_{rf}} \right) - 4t \left( \frac{\sigma_{rf}}{2} \right) \left( \frac{b \sigma_{re}}{2 \sigma_{re} + 2 \sigma_{rf}} \right)$$

where $\sigma_r$ is the residual stress, $dA$ is an area element, and integration is performed over the whole area $A$. After some algebraic manipulation we find that

$$\sigma_{rf} = \left[ \frac{bt}{bt + (d - 2t)w} \right] \sigma_{ro}$$  \hspace{1cm} (1.1)

We shall now show how residual stresses can affect the load-deformation behavior of the cross section. In Chapter 4 we shall find that the relationship between a compressive axial load and the resulting axial strain is of importance for the determination of the critical load of columns. In the short wide-flange column of Fig. 1.7 the axial stress $\sigma$ is shown superimposed on the residual stresses. Since $\sigma$ is a compressive stress, the maximum combined stress in the member will be at the flange tips, where the compressive residual stress is largest. When $\sigma = \sigma_t - \sigma_{re}$, yielding will start at the four flange tips; the plastic zones will penetrate toward the center of the flange until the cross section is completely plastified.

The stress condition in the flanges of the wide-flange member of Fig. 1.7 is shown more clearly in Fig. 1.8. Equilibrium over the whole cross section requires that

$$P = \int_A \sigma \, dA = A \sigma - 4t \left( \frac{\bar{\sigma}}{2} \right) \left( \frac{b}{2} - \bar{\sigma}b \right)$$  \hspace{1cm} (1.2)

where $\bar{\sigma}$ is a coefficient defining the extent of yield penetration in the flanges. From Fig. 1.8 we see that $\bar{\sigma} = \sigma + \sigma_{re} - \sigma_t$; it can be also proved from the geometry of similar triangles that

$$\sigma = (1 - 2\bar{\sigma})(\sigma_{re} + \sigma_t) + \sigma_t - \sigma_{re}$$  \hspace{1cm} (1.3)

With $\bar{\sigma}$ and $\sigma$ substituted into Eq. (1.2), we obtain the following formula for the average stress $\sigma_d$:

$$\sigma_d = \sigma_d = \frac{P}{A} = (1 - 2\bar{\sigma})(\sigma_{re} + \sigma_t) + \sigma_t - \sigma_{re} - \frac{bt}{A} (1 - 2\bar{\sigma})(\sigma_{re} + \sigma_t)$$  \hspace{1cm} (1.4)
The axial shortening of the member is determined by the strains in the elastic core of the column, or

$$e = \frac{\sigma}{E}$$  \hspace{1cm} (1.5)

If we define a *yield strain* at the hypothetical start of yielding in the absence of residual stresses as

$$e_Y = \frac{\sigma_Y}{E}$$  \hspace{1cm} (1.6)

and substitute $\sigma$ from Eq. (1.3) into Eq. (1.5), we find that

$$\bar{\alpha} = \frac{\sigma_{rl}}{\sigma_Y} + \frac{e}{e_Y} - \frac{(e/e_Y)\sigma_Y}{2(\sigma_{re} + \sigma_{rl})}$$  \hspace{1cm} (1.7)

If we now substitute $\bar{\alpha}$ into Eq. (1.4), we obtain the relationship

$$\frac{\sigma_d}{\sigma_Y} = \frac{e}{e_Y} - \left[ \frac{ht\sigma_Y}{A(\sigma_{re} + \sigma_{rl})} \right] \left( \frac{e}{e_Y} - 1 + \frac{\sigma_{re}}{\sigma_Y} \right)^2$$  \hspace{1cm} (1.8)

Equation (1.8) represents the relationship between the average stress $P/A$ and the resulting axial strain $e$. Since the member is very short, this is also a property of each cross section. It is valid in the range between initiation of yielding and full plastification, that is, $1 - \sigma_{re}/\sigma_Y \leq e/e_Y \leq 1 + \sigma_{rl}/\sigma_Y$. In the elastic range $\sigma_d/\sigma_Y = e/e_Y$, and after full yielding $\sigma_d/\sigma_Y = 1.0$, regardless of the value of $e$, provided we neglect the effect of strain hardening [Fig. 1.5(a)].

The curve showing the relations discussed above for an 8WF31 rolled shape with $\sigma_{re} = 0.3 \sigma_Y$ is given in Fig. 1.9.\(^4\) The dashed lines give the situation in which no residual stresses are present. The effect of the residual stresses is to bring about a rounding of the curve due to earlier yielding which results in a reduced stiffness; the residual stresses do not, however, inhibit the column from reaching its fully yielded condition when $\sigma_d = \sigma_Y$.

The preceding discussion pertained to a very simple case of loading. Similar relationships for beams and beam-columns will be discussed in later chapters. For these members the significant relationship is between the moment acting at a cross section and the resulting curvature. The conclusions reached from this simple example, however, remain valid: Residual stresses will cause yielding earlier than is expected if they are neglected, and they cause a reduction in the stiffness of the member. The stiffness of the cross section is here defined as the slope of the relevant force-deformation relationship. Now from Fig. 1.9 one could superficially conclude that the effect of the residual stresses is not very great. However, we should realize that the behavior of the whole member is obtained by the integration of the characteristics of each cross section along the whole length of the member, and we shall see that the inelastic behavior of these members cannot be properly predicted unless residual stresses are accounted for. We shall also see that buckling is sensitive to member stiffness and that therefore residual stresses play a significant role in a buckling analysis.

1.5. ORGANIZATION OF THE BOOK

The material in this book deals with that step in the structural design process when the assumed structure has been analyzed and its internal force distribution is known and it is desired to determine whether the forces will exceed the limits of structural usefulness. These limits of usefulness will be those of maximum capacity or instability, and they will be defined in terms of the load-deformation relationship of the individual members or of the whole frame. The emphasis will be on structural behavior.

This book is organized in the following manner: Chapter 2 will introduce certain topics on the elastic behavior of prismatic members of a thin-walled open cross section. In this chapter we shall study the problem of torsion, and we shall set up the differential equations of combined bending and torsion. These elastic equations will serve as the starting point for each of the following three chapters: Beams (Chapter 3), Columns (Chapter 4), and Beam-Columns (Chapter 5). In each of these chapters, the general behavior of the
member is described first. This is followed by a study of elastic buckling, and the differential equations developed at the end of Chapter 2 will serve as the starting point. The elastic studies will be followed by a consideration of inelastic behavior and inelastic instability. The final portions of each of these chapters will deal with structural specifications and their relationships to the actual limits of structural usefulness. The final chapter (Chapter 6) will be concerned with the overall behavior of frames. The assumptions underlying the various structural analyses will be examined, and elastic as well as inelastic frame behavior will be discussed. Finally the provisions of the relevant structural specifications will be presented and examined in the light of what we have learned about the behavior of frames.

Before entering into these studies, it will be well to consider briefly the sources of our knowledge on structural behavior. These sources are: (1) successful construction experience, (2) unsuccessful experience (that is, failures during fabrication, erection, or in service), and (3) structural research. The benefits of the first two sources have been available to our builders from the beginning of history, and they form an important basis for the design of all future structures. Research has been a relatively recent source of knowledge, starting with Galileo's famous experiments on beams, and assuming a really vital role since the second half of the 19th century. Research now no longer follows new developments in construction, as was the case in the latter part of the 19th century (witness, for example, Tetmaier's experiments on columns, which were motivated by the failure of the compression chords of railroad trusses), but it precedes construction. A classic example of this is the acceptance of plastic design by the steel construction industry in the 1950's. This method was accepted, in terms of specifications and buildings designed, only after research showed it to be both as safe as, and more economical than, the previously used elastic design methods.

In the 1960's it is becoming evident that one of the important design considerations is the maximum strength of the structure. Structural research in the 1945–1965 period has led to a thorough understanding of the behavior of structures under known loads. This knowledge must yet be integrated with parallel efforts which consider the probabilistic aspects of the design problem to provide a truly realistic means of designing structures.

REFERENCES


**PROBLEM**

1.1. Develop expressions for the $\epsilon / \epsilon_y$ versus $\sigma / \sigma_y$ relationships for an ideal wide-flange cross section with residual stress (Fig. 1.6) and for which the axial force is in tension. Plot the curve and compare it with the curve for axial compression (Fig. 1.9).
Elastic Behavior of Members

2.1. ELASTIC BEHAVIOR

In the initial stages of loading and often throughout a substantial portion of the loading history the behavior of structures can be represented by assuming elastic behavior. This means that the structure will return to its original undeflected position upon the removal of the loads (Fig. 1.1).

We shall start the study of structural behavior by examining some topics about the elastic response of prismatic members, because this serves as an initial step in understanding, not only the elastic behavior of members and frames, but also their later inelastic response. In this chapter we shall discuss the stresses and deformations resulting from the separate effects of bending moment, axial force, shear force, and twisting moment. The final section is devoted to the development of the differential equations of bending for the case in which the changes of geometry due to deformation cannot be neglected and bending, torsion, and compression interact with each other.

It is assumed in this chapter that the material is elastic, homogeneous (that is, it has the same elastic properties everywhere in the bar), and isotropic (that is, the elastic properties are the same in every direction). Furthermore, we assume that the strains and deformations are relatively small. These are the usual assumptions of the theory of elasticity.\(^{(2,4)}\)

2.2. RESPONSE TO BENDING MOMENT AND AXIAL FORCE

Consider the cross section of a bar as shown in Fig. 2.1(a). This bar is of an arbitrary shape. Any cross section can be located along the longitudinal axis by the coordinate \(z\). Any point in a section can be identified from the \(x-y\) coordinate system. A force \(P\) acting parallel to the \(z\) axis and in its positive direction is applied at a distance \(e_x\) and \(e_y\) from the origin of the \(x-y\) system. This force can be resolved into a force \(P\) through the origin and into two bending moments \(M_x = Pe_y\) and \(M_y = -Pe_x\). These three forces are shown in Fig. 2.1(b); the bending moments are shown as vectors with directions defined by the right-hand rule (see inset). The vectors in Fig. 2.1(b) define the sign convention: Tensile forces are positive, and moments are positive according to the right-hand rule.

The forces \(P\), \(M_x\), and \(M_y\) will cause stresses in the \(z\) direction (positive in tension). From elementary strength of materials it can be shown that normal stress is equal to\(^{(2,4)}\)

\[
\sigma = \frac{P}{A} + \frac{M_y y}{I_x} - \frac{M_x x}{I_y} \tag{2.1}
\]

provided that the \(x-y\) axis-system is a centroidal and principal axis system, that is,

\[
\int_A y \, dA = \int_A x \, dA - \int_A xy \, dA = 0 \tag{2.2}
\]

and

\[
\int_A x \, dA = A; \quad \int_A x^2 \, dA = I_x; \quad \int_A y^2 \, dA = I_y \tag{2.3}
\]

where \(dA\) is an area element \(dy \, dx\), \(A\) is the cross-sectional area, and \(I_x\) and \(I_y\) are the moments of inertia about the \(x\) and the \(y\) axes, respectively. The corresponding strain is

\[
\varepsilon = \frac{P}{AE} + \frac{M_y y}{EI_x} - \frac{M_x x}{EI_y} \tag{2.4}
\]

From Eqs. (2.1) and (2.4) we see that the stresses and strains caused by
Fig. 2.1. Cross section of a bar subjected to bending and axial force

the three forces $P$, $M_y$, and $M_z$ can be separated. The axial force $P$ will elongate a small length element $dz$ (Fig. 2.1) by the amount $\epsilon \ dz = P \ dz / AE$. Since the strains due to $M_x$ and $M_y$ are zero at $x = y = 0$, the centroidal axis of the element will not change its length if $P = 0$. But away from the neutral axis the strains vary as the distance from it and the element will bend. The measure of this bending is the curvature $\Phi_y$, which denotes the change in the slope of the centroidal axis between two points $dz$ apart. From Fig. 2.2 we see that in the $y$-$z$ plane

$$\tan \Phi_y = \frac{\epsilon}{y} \quad (2.5)$$

We now make the usual assumption that the strains are small, $\tan \Phi_y \approx \Phi_y$, and so

$$\Phi_y = \frac{\epsilon}{y} \quad (2.6)$$

From Eq. (2.4) we see that for $P = M_y = 0, \epsilon = M_x y / EI_y$, and thus

$$M_z = EI_y \Phi_y \quad (2.7)$$

Similarly it can be shown that

$$M_y = -EI_y \Phi_y \quad (2.8)$$

2.3. SHEAR STRESSES DUE TO BENDING

ASSUMPTIONS

Bending moments are usually accompanied by shear forces unless the moments do not vary along the length of the bar. The determination of the shear stresses is complicated for a general cross section (see Chap. 12 of Ref. 2.1), and so we shall introduce further stipulations to arrive at a simpler
solution. The first of these is that we restrict the discussion to thin-walled members having an open cross section.

A cross section is thin-walled if its thickness is of a smaller order of magnitude than its other dimensions; it is open if the middle line does not contain a closed loop. Such a section is shown in Fig. 2.3. We can define the position of any general point \( Q(x, y) \) on the middle line by the centroidal principal axis system \( x, y, z \), and also by its distance \( s \) from point \( O(x, y) \) at the edge of the cross section. The thickness \( t \) is assumed to be a function of \( s \). The point \( S \) in this figure is the shear center.

The choice of a thin-walled section was prompted by the fact that the shear stress \( \tau \) can be assumed to be uniform across the thickness of a thin plate;\(^{3,3} \) the choice of an open section was dictated by considerations which will become obvious later. A great many practical metal sections, such as the wide flange, the tee, or the angle, can be treated as thin-walled open sections.

We shall also make the assumptions that (1) the member is straight and prismatic (that is, the thickness does not vary with \( z \)) and (2) the cross section will retain its shape. This last assumption is an extension of the earlier assumption that the strains are small. Since the deformations are related to the strains, statement 2 above means that we are formulating the equilibrium equations on the undeformed element.

**STRESSES**

The general thin-walled cross section shown in Fig. 2.4(a) is assumed to be subjected to a positive shear force \( V_y \) acting parallel to the \( y \) axis through the shear center \( S \). The shear stress is distributed uniformly across the plate thickness \( t \), resulting in a shear flow \( \tau t \) acting at the middle line. Also shown in Fig. 2.4(a) are the forces acting on an element \( ds \) of the cross section. The longitudinal stresses \( \sigma \) are due to the bending moment \( M_x \), which is also present but is not shown (\( M_y \) and \( P \) are assumed to be zero). Equilibrium of the forces acting on this element and manipulations familiar from elementary strength of materials give the following expression for the shear flow:

\[
\tau t = -\frac{V_y}{I_x} \int_0^t y \, ds
\]

(2.9)

Equation (2.9) reflects the fact that the shear stress at the edge \( (s = 0) \) is zero.\(^1 \)

\(^1\) The application of Eq. (2.9) is straightforward if we have an open cross section. However, if the cross-sectional shape contains one or more closed cells, we do not know where \( \tau t = 0 \) for the starting point of the integration. We have, in fact, a statically indeterminate problem, and we must invoke the condition of the compatibility of the shear deformations. The shear stress distribution of multicellular cross sections is discussed further in Ref. 2.4, for example.
The term \( \int_{y}^{t} y t \, ds \) is the statical moment of the area taken at the point where \( t \) is desired. At the other edge of the cross section (point \( E \), Fig. 2.3) the shear flow is also zero, since \( \int_{y}^{t} y t \, ds = \int_{x}^{y} y dA = 0 \) [Eq. (2.2)], \( t \, ds = dA \) and the integral over the region \( OE \) denotes integration along the middle line from one edge to the other. The units of the shear flow are force per unit length of the middle line.

**THE SHEAR CENTER**

We define the shear center as the point \( S(x_s, y_s) \) in the plane of the cross section through which the shear force \( V_y \) must act if no twisting of the section is to take place.\(^{2,3}\) This means that the resultant torsional moment about \( C \) or any other point in the plane of the cross section must be zero. For the situation shown in Fig. 2.4(a) and (b), for example,

\[
\int_{S}^{t} \rho t \, ds - x_s V_y = 0
\]

where \( \rho \) is the lever arm of the shear flow \( t \) and \( x_s \) is the lever arm of the shear force \( V_y \). The negative sign for \( V_y \) was used to denote the reaction rather than the resultant shear force, as shown in Fig. 2.4(b). Rearrangement of this expression and substitution of the formula for \( t \) [Eq. (2.9)] gives

\[
x_s = -\frac{1}{I_x} \int_{S}^{t} \rho \left( \int_{S}^{t} y t \, ds \right) \, ds
\]

(2.10)

We shall now manipulate Eq. (2.10) to make it coincide with concepts to be introduced when we discuss torsion. From Fig. 2.5 we see that the incremental area \( dA_o \) enclosed by the cross-hatched sector is \( dA_o = \frac{1}{2} \rho \, ds \). The total area from \( O \) to \( s \) is then \( A_o = \frac{1}{2} \int_{S}^{t} \rho \, ds \). We define a new term \( \omega \), having units of area, as the double sectorial area or the unit warping with respect to the centeroid

\[
\omega = 2A_o = \int_{S}^{t} \rho \, ds
\]

(2.11)

With this new definition \( \rho \, ds = d\omega \) and so Eq. (2.10) can be written as

\[
x_s = -\frac{1}{I_x} \int_{S}^{t} \rho \, ds \, ds
\]

(2.12)

We now integrate by parts, that is, \( \int_{S}^{t} \rho \, d\omega = \int_{S}^{t} \rho \, d\tilde{\omega} - \int_{S}^{t} \tilde{\omega} \, d\rho \). With \( \tilde{\omega} = \int_{S}^{t} y t \, ds \) and \( d\tilde{\omega} = d\omega, d\tilde{\omega} = y t \, ds \) and \( d\rho = d\omega, d\rho = y t \, ds \) and \( \tilde{\omega} = \omega, \) or

\[
x_s = -\frac{1}{I_x} \left( \omega \int_{S}^{t} y t \, ds - \int_{S}^{t} \omega y t \, ds \right)
\]

2.4. THE DIFFERENTIAL EQUATIONS OF BENDING

**DEFLECTIONS**

In the previous sections we have shown how to compute the longitudinal stresses \( \sigma \) and the shear stresses \( \tau \) for prismatic members having a general thin-walled open cross section which is subjected to an axial force \( P \), bending moments \( M_x \) and \( M_y \), and shear forces \( V_x \) and \( V_y \). We shall now show how these forces can be determined, and we shall develop expressions whereby the deflection due to bending can be computed.

In accordance with the principle of superposition we shall treat the forces and deformations in the \( y-z \) and the \( x-z \) plane separately. An element of a beam of length \( dz \) is shown in Fig. 2.6(a). The end forces on this element consist of shear forces \( V_x \) and \( V_y + dV_y \) and bending moments \( M_x \) and \( M_y + dM_y \). The moments are represented by vectors with two arrows in this figure; their direction is determined in accordance with the right-hand rule. In addition to the end forces a uniformly distributed load \( q_y \) is also present (positive in the positive \( y \) direction). Both the end shears and \( q_y \) act in a plane parallel to the \( y-z \) plane through the shear center \( S \), and so no twisting mo-
the remaining terms, are neglected:

\[
\frac{dM_y}{dz} = V_y
\]  

(2.19)

If we differentiate Eq. (2.19) once and substitute into Eq. (2.18), we obtain the differential equation of bending

\[
\frac{d^2M_y}{dz^2} = -q_y
\]  

(2.20)

This equation can be also written [with Eq. (2.7)] as

\[
EI_y \Phi'' = -q_y
\]  

(2.21)

where the primes indicate differentiation with respect to \( z \). The curvature of a plane curve can be expressed mathematically in terms of the deflection \( v \) as

\[
\Phi_y = \frac{-q''}{[1 + (\Phi')^2]^{3/2}}
\]  

(2.22)

where the primes again represent differentiation with respect to \( z \). Since the deflections are assumed to be small, the term \( (\Phi')^3 \ll 1.0 \), and therefore

\[
\Phi_y \approx -v''
\]  

(2.23)

and setting Eq. (2.23) into Eq. (2.21),

\[
EI_y v'' = q_y
\]  

(2.24)

In the case that the beam is not uniform, \( I_z \) is also a function of \( z \), and then the differential equation becomes

\[
\frac{d^2}{dz^2} (EI_y v'') = q_y
\]  

(2.25)

The deflection \( v \) is obtained by integrating the differential equation with respect to \( z \).

By an identical process we can also develop the differential equation of bending in the \( x-z \) plane (with forces \( q_x \), \( V_x \), and \( M_x \))

\[
EI_x \mu'' = q_x
\]  

(2.26)

In Eq. (2.26) \( \mu \) is the deflection in the positive \( x \) direction, \( q_x \) and \( V_x \) are positive when acting in the positive \( x \) direction, and \( M_x \) is positive as shown in Fig. 2.1.

**SUMMARY**

From the differential equations of bending [Eqs. (2.24) and (2.26)] the deflections \( v \) and \( \mu \) in the \( y \) and \( x \) directions, respectively, and the slopes \( \phi \) and \( \psi \) can be determined. We are also able to find the forces \( V_x, V_y, M_x, \) and \( M_y \) at any location \( z \) along the longitudinal axis from the relationships

\[
M = -EI_y \phi, \quad M = +EI_x \psi
\]  

(2.27)
And

\[ V_y = -EI\alpha''', \quad V_z = +EI\alpha'''' \]  \hspace{1cm} (2.28)

The stresses at any location in the cross section can then be determined from Eqs. (2.1), (2.9), and (2.15). The forces acting at any cross section are shown in their positive direction in Fig. 2.7.

Before going on to discuss stresses due to torsion we shall examine the assumptions underlying the derivations of the foregoing equations. In order to apply all these equations with confidence, it is necessary that the following assumptions hold: (1) the material is elastic, (2) the members are prismatic and straight, (3) the cross sections are thin-walled and open, (4) plane sections remain plane, (5) the deformations are small, (6) shear deformations are neglected, and (7) the shape of the cross section remains unchanged. Assumptions (1) and (5) permit us to use superposition, to neglect higher-order terms, and to formulate the equilibrium on the undeformed element. We have also neglected the effect of shear on the curvature since it is small compared with unity. This is a very reasonable assumption for all except very short members.

This long list of assumptions would seem to be restrictive. Fortunately, this is not so; a great many practical problems can be solved with this theory, and as we know, the theory is used in many applications in determining the strength of materials and in the theory of statically determinate and indeterminate structures. Since the deformations are assumed to have no effect on the magnitude of the internal forces, we call this type of an analysis a first-order analysis.

### 2.5. Torsional Stresses and Deformations

#### General Comments

On Torsion Problems

One of the principal distinguishing features of the response of members to torsion is that sections which were originally plane are no longer so after the twisting moment is applied, that is, the cross section will warp. Exceptions to this rule are solid or tubular circular sections and thin-walled sections for which all elements intersect at a point, such as the cruciform, angle, and tee sections. These sections do not warp under torsion. Depending on whether a cross section is free to warp or whether warping is restrained, we distinguish between uniform (or pure, or St. Venant) or nonuniform (or warping) torsion, respectively. In general both types of torsion will be present; their effects can be separated if we remain within the limits of our previously stated assumptions.

The study of the torsion of general cross sections is beyond the scope of this book and belongs to topics usually studied in the theory of elasticity. Because of its later relevance we shall concentrate here on the torsion of thin-walled members of open cross section. Open sections are very inefficient in torsion, and are susceptible to lateral-torsional buckling which involves torsion even though no intentional torsional loading is applied.

#### Uniform Torsion

The study of the uniform torsion of thin-walled open cross sections is greatly simplified by the fortuitous fact that certain relationships exist between the torsion problem and the deformations of a membrane stretched across an opening equal in size and shape to the cross section for which the torsional properties are desired. This leads to the membrane analogy, with which the reader is assumed to be acquainted from his study of the strength of materials. Table 2.1 lists these analogous relationships and explains the notation used. This table states that (1) the stress function \( \Psi \), which is a measure of the torsional deformations, is analogous to the deflection \( \xi \) of the membrane, (2) the shear stresses due to torsion correspond to the slopes of the membrane, and (3) the volume under the membrane is related to the twisting moment.

We shall first consider the torsion of a thin rectangular element subjected

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2 Torsion problems in general are covered in texts on the theory of elasticity (for example, Ref. 2.1). The torsion of closed thin-walled single and multicell members is treated in most aircraft structure texts; Refs. 2.5 and 2.6 as well as Chap. IV in Ref. 1.18 are a limited sampling of other available literature on this subject.

3 See, for example, Chap. 11, Ref. 2.1, or Chap. 9, Ref. 2.2.
Table 2.1. Membrane-Torsion Analogy

<table>
<thead>
<tr>
<th>Membrane</th>
<th>Torsion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Differential equation: ( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{q}{F} )</td>
<td>Differential equation: ( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2G\phi' )</td>
</tr>
<tr>
<td>Deflection: ( z )</td>
<td>Stress function ( \psi )</td>
</tr>
<tr>
<td>( \frac{q}{F} )</td>
<td>( 2G\phi' )</td>
</tr>
<tr>
<td>Slope ( \frac{\partial z}{\partial y}, \frac{\partial z}{\partial x} )</td>
<td>Stresses ( \tau_{xy}, \tau_{yx} )</td>
</tr>
<tr>
<td>Volume ( V ) = ( \int \int z , dx , dy )</td>
<td>Moment ( M_{sy} ) = ( 2 \int \int \psi , dx , dy )</td>
</tr>
<tr>
<td>( V ) = ( \frac{M_{sy}}{2} )</td>
<td></td>
</tr>
</tbody>
</table>

Notation. \( x, y \): coordinates in plane of cross section  
\( q \): pressure perpendicular to membrane  
\( F \): circumferential force  
\( \phi' \): angle of twist per unit length  
\( G \): shear modulus

to a torsional moment \( M_{sy} \) as shown in Fig 2.8(a). The subscript \( SV \) identifies \( M \) as a twisting force causing uniform or St. Venant torsion. We assume that a membrane is stretched across an opening \( bt \), where \( b \) and \( t \) are the depth and thickness of the cross section, respectively [Fig. 2.8(b)]. Except at the very ends, the membrane will deform into a parabolic shape under a pressure \( q \) [Fig. 2.8(c)]. We shall assume that the element is thin-walled, and so \( b \gg t \); therefore we shall neglect the end effects and assume that the membrane deforms everywhere into the same shape. The height of the membrane at the middle line is \( z_0 \), and the equation of the parabola is

\[ z = \frac{4z_0 y^2}{t^2} \]

The slope of this parabola is

\[ \frac{dz}{dy} = \frac{8z_0 y}{t^2} \]

This is a linear function of \( y \), and thus the slope of the membrane as well as the shear stress \( \tau \) varies as shown in Fig. 2.8(c). The stress is zero at the middle line and maximum at the outside faces of the plate. Thus at \( y = t/2 \)

\[ (\tau_{sy})_{\text{max}} = \frac{4z_0}{t} \]

The equilibrium of the forces on the membrane is \( qtb - 2bF \sin \alpha = 0 \). But for small membrane deformations \( 4 \sin \alpha \approx \alpha = 4z_0/t \), and thus

\[ q \approx \frac{8z_0}{t^2} \]

(2.29)

The volume under the membrane, neglecting the end effect, is equal to

\[ V = \frac{2t^2 b}{3} \]

and this in turn is equal to \( \frac{1}{2}M_{sy} \) from membrane analogy (Table 2.1). Thus

\[ M_{sy} = \frac{4z_0 b}{3} \]

(2.30)

Also, from membrane analogy

\[ \frac{q}{F} = 2G\phi' \]

But from Eq. (2.30), \( z_0 = 3M_{sy}/4tb \), and therefore [using Eq. (2.29)]

\[ M_{sy} = \frac{t^2 b G\phi'}{3} \]

(2.31)
The angle of twist per unit length $\phi'$ is a constant as long as no warping torsion is present and if $M_{SV}$ does not vary along the $z$ axis. If we introduce the torsion constant

$$K_T = \frac{t^2b}{3}$$

(2.32)

the relationship between the applied twisting moment $M_{SV}$ and the resulting twisting angle per unit length becomes

$$M_{SV} = G K_T \phi'$$

(2.33)

The maximum shear stress $(\tau_{SV})_{max}$ is then

$$(\tau_{SV})_{max} = \frac{tM_{SV}}{K_T}$$

(2.34)

We can proceed with the same reasoning as was used for the rectangular strip in Fig. 2.8 to a more complicated cross section [Fig. 2.9(a)] by replacing it approximately by a convenient number of straight elements [Fig. 2.9(b)] each having a constant thickness $t_{ij}$ and a length $b_{ij}$. For this section the torsion constant is equal to the sum of the $K_T$ values of each element

$$K_T = \frac{1}{n} \sum_{i=1}^{n} b_{ij} t_{ij}$$

(2.35)

The maximum shear stress for each element is at its edge, and is equal to

$$(\tau_{SV}) = \frac{t_{ij} M_{SV}}{K_T}$$

(2.36)

For example, $K_T$ for a wide-flange shape is [Fig. 2.9(c)]

$$K_T = \frac{1}{2} [2br^2 + (d - x)w]^2$$

(2.37)

It should be noted that in Eq. (2.37) we used the distances between the intersections of the middle lines. We also show in Fig. 2.9(c) the distribution of the shear stresses and the maximum stress in the flange.

It is emphasized that the torsion constant $K_T$ determined by Eq. (2.35) is only valid for thin-walled open sections. We also should note that the maximum stress $(\tau_{SV})_{max}$ ([Eq. (2.36)]) is reliably predicted only along the straight portions of the plate elements. At the ends and at the fillets these stresses can be considerably different. At very sharp corners severe stress raisers can exist; however, the fillets on rolled shapes are smooth enough so that the stress is of the same order of magnitude as it is in the straight portions.\(^{(2.8)}\)

Equation (2.35) neglects the effects of the ends of the plate element where the membrane flattens out and the effects of the intersections; similarly, the influence of sloping flanges is neglected. These effects have been considered for common rolled shapes by Johnston\(^{(2.8,19)}\) who gives a series of correction terms to Eq. (2.35). These corrections have been included in the tabulation of $K_T$ for all rolled wide-flange shapes produced by the Bethlehem Steel Company.\(^{(1.10)}\) For extruded aluminum shapes $K_T$ is tabulated in Refs. 2.11 and 2.12. In later examples we shall use these tabulated values instead of the values computed by Eq. (2.35).

### WARping DEFORMATIONS

The next step is to determine the deformations $w$ due to uniform torsion in the direction of the $z$ axis. These deformations will distort the originally plane $x-y$ plane into a warped surface. In Fig. 2.10 the middle line of a general open cross section is shown. We have already defined $\rho$ as the perpendicular

\(^{3}\) These values are also tabulated in the Appendix of Ref. 1.19. Modifications for riveted built-up girders are given in Ref. 2.13.
distance from a tangent line passing through a general point \( Q(x, y) \) to the centroid \( C \). A distance \( \rho \) is defined as the distance between the tangent and the torsion center \( S(x_0, y_0) \). The torsion center is defined as a point in the original plane of the cross section about which twisting takes place. We shall see later that the location of the torsion center and the previously defined shear center is the same. We define \( \rho \) and \( \rho_0 \) to be positive if \( C \) and \( S \), respectively, are to the left of an observer standing at \( Q \) and looking toward the positive direction of the tangent. The positive directions of the twisting moment \( M_{sr} \) and the total twisting angle \( \phi \) are shown also.

In the following derivation we shall retain all the previously stated assumptions and use again the coordinate system \( x, y, z \) as a principal centroidal axis system. The only stresses acting on the section are the St. Venant shear stresses \([\text{Eq. (2.36)}]\). There are no shear stresses at the middle line, and so any rectangular element \( ds \, dz \) in the \( s-z \) plane will remain rectangular after twist is applied.

Figure 2.11(a) shows a strip cut from the bar. The location of the strip in the member is given in Fig. 2.11(b). This strip is of width \( dz \), and in its undeformed position it is shown over part of the cross section in solid lines. Also shown in this three-dimensional view [Fig. 2.11(a)] are the tangent line, the distances \( \rho_0 \) and \( a \) (where \( \rho_0 \) is the perpendicular distance to the tangent and \( a \) is the distance between \( S \) and the point \( Q \) where the element \( ds \, dz \) is located), and a length \( dz \) of an axis passing through \( S \). An element \( ds \, dz \) is delineated by the points \( A, B, C, \) and \( D \). After the application of a torsional couple the cross section \( z + dz \) will be deformed through an angle \( d\phi \) with respect to its neighboring cross section at \( z \). Twisting will be about the torsion center \( S \). Due to this twist the element is displaced into the position \( AB'D'C' \) (dashed lines). The displacement of \( D' \) with respect to \( C' \) (or \( B' \) with respect to \( A \)) in the direction of the \( z \) axis is the desired differential warping deformation \( dw \). This deformation \( w \) is assumed positive in the direction of the positive \( z \) axis (that is, elongation is positive).

The element \( ds \, dz \) of Fig. 2.11(a) is shown again in Fig. 2.12 in the \( x-y \) plane and in a \( 90^\circ \) projection in the \( s-z \) plane. The heavy solid lines show the position of the element before twisting, and the dashed lines show its position after the deformation \( d\phi \) is applied. As shown in Fig. 2.12, the rectangle will deform as a rigid body through an angle \( a \sin \beta \, d\phi/dz \) if we neglect deformations involving products of infinitesimal quantities. The differential warping
SEC. 2.5

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where \( w \) is the warping deformation of any point on the middle line a distance \( s \) from the edge \( O \), and \( \omega_s \) is the deformation of point \( O \). If we define

\[
\omega_s = \int_0^s \rho_0 \, ds
\]  

(2.40)

as the double sectorial area or the unit warping with respect to the torsion center \( S \), we have finally (with \( d\phi/dz \) abbreviated as \( \phi' \))

\[
w = w_0 - \phi' \omega_s
\]

(2.41)

We should note here that no restraint against warping has been assumed, and therefore no axial strains and stresses are developed. The term \( \omega_s \) is determined in the same way as \( \omega \) is [Fig. 2.5 and Eq. (2.11)], except that the apex of the sector is \( S \) instead of \( C \). Another useful expression for \( \omega_s \) can be obtained from the geometric relationships given in Fig. 2.13. We can show that

\[
\rho = fe = fd - ed = x \sin \gamma - y \cos \gamma = x \frac{dy}{ds} - y \frac{dx}{ds}
\]

Fig. 2.12. Geometric relationships for determining the differential warping

\[
d\omega = -(a \sin \beta \frac{d\phi}{dz}) \, ds
\]

The negative sign is due to the fact that \( d\omega \) is in the negative direction of \( z \). If we note that \( a \sin \beta = \rho_0 \) (Fig. 2.12),

\[
d\omega = -\rho_0 \left( \frac{d\phi}{dz} \right) \, ds
\]

(2.38)

After integrating we obtain

\[
w = w_0 - \frac{d\phi}{dz} \int_0^s \rho_0 \, ds
\]

(2.39)

Fig. 2.13. Geometric relationships for determining the tangential distances \( \rho \) and \( \rho_0 \)
and
\[ \rho_0 = Ql = (Qg + gh) \sin \gamma = [(x - x_0) + (y - y_0) \cot \gamma] \sin \gamma \]
\[ = x \frac{dy}{ds} - x_0 \frac{dy}{ds} - y \frac{dx}{ds} + y_0 \frac{dx}{ds} = \rho + y_0 \frac{dx}{ds} - x_0 \frac{dy}{ds} \]

Integration of this expression from \( O \) to \( s \) (noting from Fig. 2.3 that the coordinates of \( O \) are \( x_1 \) and \( y_1 \)) gives
\[ \omega_0 = \int_0^s \rho_0 \, ds = \int_0^s \rho \, ds + y_0 \int_{s_1}^s \, dx - x_0 \int_{y_1}^y \, dy \]
or
\[ \omega_0 = \omega + y_0 x - y_0 x_1 - x_0 y + x_0 y_1 \quad (2.42) \]

The relative warping of one point on a cross section with respect to any other point is thus proportional to \( \omega_0 \), as can be seen by setting \( w_0 = 0 \) in Eq. (2.41) and observing that \( \phi \) is independent of \( s \); \( \omega_0 \) is a property of the entire cross section, and it denotes a rigid body translation in the \( z \) direction.

**NONUNIFORM TORSION**

If the warping deformations according to Eq. (2.41) are in any way constrained by end conditions (for example, a heavy plate welded across the ends of a member could completely prohibit warping) or by variation of the applied twisting moment \( M_t \) along the \( z \) axis, then in addition to the St. Venant shear stress \( \tau_{yw} \) [Eq. (2.36)] we have longitudinal stresses \( \sigma_w \) and shear stresses \( \tau_w \). These stresses result from the fact that the warping deformations \( w \) cannot fully develop. In this case we have **nonuniform or warping torsion**.

The longitudinal stress \( \sigma_w \) is equal to
\[ \sigma_w = E \epsilon_w \quad (2.43) \]
where \( \epsilon_w \) is the longitudinal strain due to restrained warping. This strain is defined as
\[ \epsilon_w = \frac{dw}{dz} \quad (2.44) \]

Substituting Eqs. (2.41) and (2.44) into Eq. (2.43), we get
\[ \sigma_w = E \omega_0 - E \omega \phi'' \quad (2.45) \]
where the primes indicate differentiation with respect to \( z \).

The shear stress \( \tau_w \) can be obtained from the equilibrium of the strip \( s \, dz \) in Fig. 2.14. At the edge \( O \) the shear stress is zero, and so from equilibrium considerations,
\[ \tau_w t \, dz + \int_0^s t(\sigma_w + d\sigma_w) \, ds - \int_0^s t\sigma_w \, ds = 0 \]
or
\[ \tau_w t = -\int_0^s t \frac{d\sigma_w}{dz} \, ds \quad (2.46) \]

Before we can further develop expressions for the stresses we must consider some additional equilibrium conditions. Since only a twisting moment \( M_t \) is applied (axial force \( P \) and bending moments \( M_x \) and \( M_y \) are zero), the resultant axial force and the bending moments due to \( \sigma_w \) must be zero at any cross section. That is,
\[ P = 0 = \int_0^s \sigma_w t \, ds = E \int_0^s (w_0 - \omega \phi'') t \, ds \quad (2.47) \]
\[ M_x = 0 = \int_0^s \rho t \, ds = E \int_0^s (w_0 - \omega \phi'') y t \, ds \quad (2.48) \]
\[ M_y = 0 = \int_0^s \rho t \, ds = E \int_0^s (w_0 - \omega \phi'') y t \, ds \quad (2.49) \]

If we substitute the expression for \( \omega_0 \) from Eq. (2.42) into Eqs. (2.47) and (2.49) and note that \( \int_0^s x t \, ds = \int_0^s y t \, ds = \int_0^s y t \, ds = 0 \) [Eq. (2.2)], \( \int_0^s x t \, ds = A_s \), \( \int_0^s x y t \, ds = I_{xy} \), \( \int_0^s y t \, ds = I_{tx} \), \( \int_0^s y t \, ds = I_{ty} \), \( \int_0^s w t \, ds = I_{wz} \), \( \int_0^s u t \, ds = I_{ux} \), we obtain the following two equations:
\[ \phi'' - x_0 I_{tx} + I_{ux} = 0 \]
\[ \phi'' - y_0 I_{ty} + I_{ux} = 0 \]

Since \( \phi'' \neq 0 \), we get the coordinates of the torsion center \( S \) as
\[ x_0 = \frac{I_{tx}}{I_x} \]
\[ y_0 = -\frac{I_{ty}}{I_y} \quad (2.50) \]

As can be seen, the torsion center by Eqs. (2.50) and (2.51) and the shear center, as defined for bending without torsion, are identical. That is, twisting
takes place about the shear center if the axial load and bending moments are zero. We shall refer to \( S \) from here on as simply the shear center.

From Eq. (2.47) we find that since \( w_0' \) is not a function of \( s \),

\[
w_0' \int_0^s t \, ds = \phi'' \int_0^s \omega_0 t \, ds
\]

or

\[
w_0' = \phi'' \int_0^s \omega_0 t \, ds \quad (2.52)
\]

We now substitute Eq. (2.52) into Eq. (2.45), that is,

\[
\sigma_w = E\phi'''\left(\frac{1}{A} \int_0^s \omega_0 t \, ds - \omega_0\right)
\]

(2.53)

Defining a new cross-sectional property \( \omega_0 \), the normalized unit warping, as

\[
\omega_0 = \frac{1}{A} \int_0^s \omega_0 t \, ds - \omega_0 \quad (2.54)
\]

we can express the longitudinal stress due to restrained warping as

\[
\sigma_w = E\phi''
\]

(2.55)

The shear flow from Eq. (2.46) is now

\[
\tau_{w't} = -\int_0^s tE\omega_0 \phi''^{' '} \, ds
\]

(2.56)

By again defining a new term \( S_0 \) as the warping statical moment

\[
S_0 = \int_0^s \omega_0 t \, ds
\]

(2.57)

the shear flow equation becomes equal to

\[
\tau_{w't} = -ES_0 \phi''
\]

(2.58)

**THE DIFFERENTIAL EQUATIONS OF TORSION**

The contribution of the warping shear flow \( \tau_{w't} \) to the total torsional moment \( M_z \) is equal to

\[
M_w = \int_0^s \tau_{w't} \rho_0 \, ds
\]

(2.59)

The term \( \rho_0 \) is the lever arm of the force \( \tau_{w't} \rho_0 \) with respect to \( S \). Substitution of Eq. (2.56) into Eq. (2.59) results in the following expression:

\[
M_w = -E\phi'''' \int_0^s \rho_0 \left(\int_0^s \omega_0 t \, ds\right) \, ds
\]

We shall now integrate by parts, letting \( \bar{u} = \int_0^s \omega_0 t \, ds \) and \( \bar{\phi} = \rho_0 \). With

\[
d\bar{u} = \omega_0 t \, ds \quad \text{and} \quad d\bar{\phi} = \rho_0 \, ds
\]

we obtain

\[
M_w = -E\phi'''' \int_0^s \omega_0 t \, ds - \int_0^s \omega_0 \omega_0 t \, ds
\]

(2.60)

Noting from Eq. (2.54) that the first term in the \( \omega_0 \) equation is a constant, we get for the first term

\[
\int_0^s \omega_0 t \, ds = \left(\frac{1}{A} \int_0^s \omega_0 t \, ds\right) \int_0^s t \, ds - \int_0^s \omega_0 t \, ds = 0
\]

(2.61)

The second term yields

\[
\int_0^s \omega_0 \omega_0 t \, ds = \left(\frac{1}{A} \int_0^s \omega_0 t \, ds\right) \int_0^s \omega_0 t \, ds - \int_0^s \omega_0 \omega_0 t \, ds = 0
\]

Since it was shown above that \( \int_0^s \omega_0 t \, ds = 0 \)

\[
M_w = -E\phi'''' \int_0^s \omega_0 \omega_0 t \, ds
\]

(2.62)

Introducing a new term \( I_0 \) the warping moment of inertia

\[
I_0 = \int_0^s \omega_0 \omega_0 t \, ds
\]

we finally obtain

\[
M_w = -EI_0 \phi''''
\]

(2.63)

The total twisting moment \( M_z \) is the sum of the warping contribution and the St. Venant contribution, that is,

\[
M_z = M_{sv} + M_w
\]

(2.64)

From Eq. (2.33) we see that \( M_{sv} = G K_0 \phi' \), and thus

\[
M_z = G K_0 \phi' - EI_0 \phi''''
\]

(2.65)

This differential equation applies at a location where a concentrated torque is applied. We can find a corresponding equation for a distributed torque \( m_z \) (Fig. 2.15) from the equilibrium of moments on an element of length \( dz \)

\[
-M_z + m_z dz + M_z + dM_z = 0
\]

from which

\[
m_z = -\frac{dM_z}{dz}
\]

(2.66)

Thus the differential equation for a distributed torque is obtained by differentiating Eq. (2.65)

\[
G K_0 \phi'' + EI_0 \phi''''' = -m_z
\]

(2.67)
Elastic Behavior of Members

Equations (2.65) and (2.67) can be written as

\[ \phi''' - \lambda^2 \phi' = -\frac{M_x}{EI_u} \]  (2.68)

and

\[ \phi'' - \lambda^3 \phi'' = \frac{m_z}{EI_u} \]  (2.69)

where

\[ \lambda^2 = \frac{GK_F}{EI_u} \]  (2.70)

The solution of Eq. (2.68) can be written either in the form\(^{34}\)

\[ \phi = C_1 + C_2 e^{\lambda x} + C_3 e^{-\lambda x} + \frac{M_x z}{\lambda^3 EI_u} \]  (2.71)
or

\[ \phi = C_1 + C_2 \cosh \lambda x + C_3 \sinh \lambda x + \frac{M_x z}{\lambda^3 EI_u} \]  (2.72)

For the uniformly distributed twist the solution is

\[ \phi = C_1 + C_2 x + C_3 \cosh \lambda x + C_4 \sinh \lambda x - \frac{m_z x^2}{2GK_F} \]  (2.73)

The coefficients \(C_1, C_2, C_3, C_4, C_5, C_6\), and \(C_7\) are constants of integration and are determined from the boundary conditions of the twisted member. If at any boundary \(\phi = 0\), the section cannot twist; if \(\phi' = 0\), no warping can take place ["fixed" end; see Eq. (2.41)]; if \(\phi''' = 0\), warping is not restrained ["pinned" end; \(\sigma_w = 0\) from Eq. (2.55)]; and if \(\phi'''' = 0\), the shear flow due to warping is zero ["free" end]. The sketches in Fig. 2.16 show two types of practical boundary conditions.\(^6\)

2.6. Summary of the First-Order Elastic Solutions

Combined Stresses

In Secs. 2.2 through 2.5 we discussed the individual effects of axial force, bending moments, and torsion, respectively. We introduced a considerable number of new cross-sectional properties and many formulas. The most important of these are summarized in Table 2.2 for easy reference. This table also lists the assumptions which underlie all the formulas, so that it will be easier to resist the temptation to use them where they do not apply. There are of course many more situations which could have been examined, but they are not relevant to the later portions of this book, and the student can refer to the cited literature for further study.

As already noted, the cross-sectional forces will produce longitudinal stresses \(\sigma\) and shear stresses \(\tau\). Each of these will be composed of the sum of the stresses due to axial force, bending, and torsion. Within the limits of the assumptions we can superimpose these stresses. This first-order analysis

\(^6\)The practical design of welds for such connections is discussed in Ref. 2.15. The problem of restrained warping at the ends (that is, continuous beams under torsion) is treated by Goldberg in Ref. 2.16.
leads to excellent results in many practical situations, and as will be seen later, is considerably simpler than a second-order analysis in which equilibrium is formulated on the deformed structure.

Table 2.2. SUMMARY OF FORMULAS FOR AXIAL FORCE, BENDING, AND TORSION

1. Assumptions
(a) The material is elastic.
(b) x and y are principal centroidal coordinates.
(c) Member is straight and prismatic.
(d) Cross section is thin-walled and open.
(e) Cross section does not change shape.
(f) Equilibrium is formulated for the undeformed member.
(g) Deflections are small.

2. Cross-sectional properties
(a) Properties dependent on s

\[ \omega = \int_0^s \rho \, ds \]

\[ \omega_0 = \int_0^s \rho_0 \, ds = \omega + y_0 x - y_0 x_0 - x_0 y_0 \]

\[ \omega_n = \frac{1}{A} \int_0^s \omega_0 t \, ds \]

\[ S_n = \int_0^s \omega_0 t \, ds \]

(b) Properties of the cross section

\[ \int_0^s x t \, ds = \int_0^s x t \, ds = C \]

\[ \int_0^s t \, ds = A; \int_0^s y t \, ds = I_x; \int_0^s y t \, ds = I_y; \int_0^s \omega_0 x t \, ds = I_{x0}; \int_0^s \omega_0 x t \, ds = I_{y0}; \int_0^s \omega_0 x t \, ds = I_{x0}; \]

\[ x_0 = \frac{I_{x0}}{I_x}; \quad y_0 = -\frac{I_{y0}}{I_y} \]

(c) Units

(Length): x, y, x, y_0, y_0

(Length): a, a_0, a_0, A

(Length): S_n, I_n, I_y

(Length): I_{x0}, I_{y0}

(Length): I_0

3. Differential equations
(a) Axial force: \[ \frac{d^2 u}{dz^2} = \frac{P}{A} \]

(b) Bending about x axis: \[ \frac{d^2 \phi}{dz^2} (EI \phi') = q_y \]

(c) Bending about y axis: \[ \frac{d^2 \phi}{dz^2} (EI \phi') = q_x \]

(d) Concentrated torque: \[ G K \phi' - EI \phi''' = M_z \]

\[ \phi = C_1 + C_2 \cosh \lambda z + C_3 \sinh \lambda z + \frac{M_z \lambda^2}{2K} \]

(e) Uniformly distributed torque: \[ G K \phi'' - EI \phi''' = -m_z \]

\[ \phi = C_4 + C_5 x + C_6 \cosh \lambda z + C_7 \sinh \lambda z - \frac{m_z \lambda^2}{2K} \]

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4. Stresses
(a) Due to axial force: \[ \sigma_P = \frac{P}{A} \]

(b) Due to bending and shear: \[ \sigma_{x0} = \frac{M_x y}{I_y}; \quad \sigma_{y0} = -\frac{M_y x}{I_x} \]

\[ \tau_{x0} = -\frac{V_y}{I_x} \int_0^s x t \, ds; \quad \tau_{y0} = -\frac{V_x}{I_y} \int_0^s y t \, ds \]

(c) Due to torsion:

\[ \tau_{x0} = \frac{V_x}{I_x} \]

The longitudinal stress is made up as follows (tensile stresses are positive):

\[ \sigma = \frac{P}{A} + \frac{M_x y}{I_y} - \frac{M_y x}{I_x} - E \omega_0 \phi'' \]

(2.74)

The corresponding equation for the shear stresses is

\[ \tau = -\frac{V_x}{I_x} \int_0^s x t \, ds; \quad \tau = \frac{V_y}{I_y} \int_0^s y t \, ds; \quad \tau = -\frac{E S \omega_0 \phi'''}{I_y} + G t \phi' \]

(2.75)

Each term in Eqs. (2.74) and (2.75) consists of two parts: One part refers to the variation of the stress along the longitudinal axis z (P, M_x, M_y, \phi', V_x, V_y, \phi''', \phi'), and the other part defines the variation of the stress in the cross section (x, y, \omega_0, (1/t) \int y t \, ds, (1/t) \int x t \, ds, S_n, t). The analysis problem then consists in determining the variation of these two types of quantities and locating the positions where the combined stresses are maximum. We are already familiar with this procedure from drawing shear and moment diagrams. For the torsional components it is necessary to know the variation of \phi', \phi''', and \phi'''. Because of the more complicated nature of the differential equation this could be a rather laborious procedure, and it will be illustrated by several examples in Sec. 2.7. Charts for twelve frequently occurring torsional loadings have been published \(^{(10)}\) and with the help of these the maximum values of \phi', \phi''', and \phi''', and their locations can be obtained very rapidly. These charts are presented in a nondimensional form and are applicable for any material and cross section.

The computation of the torsional section properties (\omega_0, I_n, etc.) for cross sections made of plate elements [Fig. 2.9(b)] is greatly simplified by the fact that between points of intersection the unit warping properties \omega, \omega_0, and \omega_n vary linearly. In Fig. 2.17 one such plate element \( \theta \) is chosen. The variation of \omega is shown in this figure. From geometry we can determine that

\[ \omega = \omega_1 + \frac{(\omega_0 - \omega_1)(x - x_0)}{x_j - x_i} \]
where \( x_i \) and \( x_j \) are the \( x \) coordinates of the ends of the element, and \( \omega_i \) and \( \omega_j \) are the corresponding values of \( \omega \). If we now wish to calculate \( I_{oz} = \int_0^b \alpha x dx \), we can write

\[
I_{oz} = \sum_{i=1}^{n} \frac{t_{ij} \int_{x_i}^{x_j} \left[ \omega_x + (\omega_y - \omega_x)(x - x_i) \right] dx}{x_j - x_i}
\]

In this expression we used the relationship (from Fig. 2.17) \( ds = dx/\cos \alpha_{ij} \).

---

**Fig. 2.17.** Distribution of \( \omega \) on a plate element

---

**Table 2.3. Formulas for Sections with Straight Plates**

[As shown in Fig. 2.9(b)]

- \( \omega_{x_{ij}} = \sum_{i=1}^{n} \rho_i \beta_{ij} \)
- \( \omega_{y_{ij}} = \sum_{i=1}^{n} \rho_i \theta_{ij} \)
- \( \omega_{x_{ij}} = \frac{1}{A} \left[ \frac{1}{2} \sum_{i=1}^{n} (\omega_{x_i} + \omega_{y_i}) \gamma_{ij} \beta_{ij} \right] - \omega_{ij} \)
- \( I_{o_{ij}} = \frac{1}{3} \sum_{i=1}^{n} (\omega_{x_i} x_i + \omega_{y_i} y_i) \gamma_{ij} \beta_{ij} + \frac{1}{6} \sum_{i=1}^{n} (\omega_{y_i} x_i + \omega_{x_i} y_i) \gamma_{ij} \beta_{ij} \)
- \( I_{o_{ij}} = \frac{1}{3} \sum_{i=1}^{n} (\omega_{x_i} x_i + \omega_{y_i} y_i) \gamma_{ij} \beta_{ij} + \frac{1}{6} \sum_{i=1}^{n} (\omega_{y_i} x_i + \omega_{x_i} y_i) \gamma_{ij} \beta_{ij} \)
- \( I_{oz} = \frac{1}{3} \sum_{i=1}^{n} (\omega_{x_i} x_i + \omega_{y_i} y_i) \gamma_{ij} \beta_{ij} + \frac{1}{6} \sum_{i=1}^{n} (\omega_{y_i} x_i + \omega_{x_i} y_i) \gamma_{ij} \beta_{ij} \)
- \( K_{oz} = \frac{1}{3} \sum_{i=1}^{n} t_i \beta_{ij} \)

---

**Table 2.4. Warping Torsional Properties**

*Wide Flange Shape*

- \( \omega_{x_1} = d^3 b \alpha / 2 \)
- \( \omega_{x_2} = d^3 (1 - \alpha) / 2 \)
- \( S_{oz_1} = d^3 b^2 / 8 \)
- \( S_{oz_2} = d^3 b^2 (1 - \alpha) / 8 \)

*Channel*

- \( x_0 = \left[ h + \frac{b}{3} \alpha \right] \)
- \( \alpha = \frac{1}{2 + \frac{d'w}{b} / \beta} \)
- \( I_x = \frac{d''(d''/d')^3}{2 + \frac{d''w}{b}} \left[ \frac{1 - \frac{3}{6} \alpha + \frac{\alpha^2}{2} (1 + \frac{d''w}{6b})^2}{1 - \frac{3}{6} \alpha + \frac{\alpha^2}{2} (1 + \frac{d''w}{6b})^2} \right] \)
- \( \omega_{x_1} = d'b \alpha / 2 \)
- \( \omega_{x_2} = d'b (1 - \alpha) / 2 \)
- \( S_{oz_1} = d''(d''/2)^2 (1 - \alpha) / 2 \)
- \( S_{oz_2} = d''(d''/2)^2 (1 - \alpha) / 2 \)
prestressed concrete elements. For most of these members the cross section will retain its shape, except for some light-gage members or thin shells in which the deformations must be considered. \(2(3,9,10,11)\)

The assumption of small deflections is generally excellent for members in framed structures, even in the inelastic range. The members must not be too short or be subjected to very high shear forces, because the effect of shear on the deformations has been neglected. The effect of shear is not too important if the shear span is more than three or four times the depth of the member.

In Sec. 2.5 we assumed that the element \(ds\, dz\) in Fig. 2.11 will remain rectangular. We neglected the effect of the warping shear which would distort this rectangle. For closed sections this can lead to serious discrepancies, \(2,13\) but it is not a serious neglect for open sections provided the member is not very short. \(2,11\) We also assumed that the warping shear stress is uniform across the thickness of the element. This is not precisely so, but its effect is again small. Bleich\(^7\) gives formulas for \(I_o\) which include this effect for tee and angle shapes, for which our theory gives \(I_o = 0\).

The assumption of formulating equilibrium on the undeformed member is usually good except for beam-columns with compressive axial forces. This assumption also leads to a theory which cannot predict buckling. For the inner reason we shall formulate equilibrium in Sec. 2.8 for the deformed member, thus abandoning the very convenient principle of superposition.

### 2.7. EXAMPLES

#### STRESSES IN A SEMICIRCULAR BEAM

As our first example we shall compute the stresses in a thin-walled semicircular cantilever beam loaded at its end by a vertical force \(Q\) which passes through the origin of the circle [Fig. 2.18(a)]. A bending moment \(M_s\), a shear force \(V_s\), and a twisting moment \(M_t = Px\) will act at any cross section. The following quantities are given for this problem in terms of nondimensional ratios: \(G/E = 0.383, r/t = 24\) (where \(r\) is the radius of the circle to the middle line and \(t\) is the uniform thickness), and \(I/r = 20\).

We shall first determine the necessary cross-sectional properties. With the aid of Fig. 2.18(b) we can determine the following geometric properties:

\[
\begin{align*}
    ds &= r\, d\beta, & x_1 &= \frac{2r}{\pi}, & y_1 &= r \\
    x &= \frac{2r}{\pi} - r\sin\beta, & y &= r\cos\beta \\
    p &= r - \frac{2r\sin\beta}{\pi}
\end{align*}
\]

\(7\) Chapter IV in Ref. 1.34, in which a more elaborate discussion is also given on the problem of nonuniform torsion.
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\[ \omega_0 = \omega + y_0 x - y_0 x_1 + x_0 y - x_0 y = r^2 \left[ \beta - \frac{4}{\pi} (1 - \cos \beta) \right] \]

\[ \omega_n = \frac{1}{A} \int_0^r \omega_0 ds - \omega_0 = r^2 \left[ \frac{\beta}{2} - \beta - \frac{4 \cos \beta}{\pi} \right] \]

\[ S_n = \int_0^r \omega_0 ds = r^2 \left[ \frac{\beta}{2} (\pi - \beta) - \frac{4 \sin \beta}{\pi} \right] \]

\[ I_n = \int_0^r \omega_0 ds = r^2 \left[ \frac{\pi^2}{12} - \frac{\beta}{8} \right] \]

\[ \int_0^r y_0 ds = r^2 \sin \beta \]

\[ K = \frac{\pi r^3}{2} \]

The properties needed for the later stress calculations are plotted in the nondimensional graph shown in Fig. 2.19. The functions \( \omega_0/r^2 \), \( y/r \), \( 10S_n/r^2 t \) and \( \int_0^r y_0 ds/r^2 t \) are plotted against the angle \( \beta \).

The shear and moment diagram for the problem is shown in Fig. 2.20 along with the variation of the first, second, and third derivative of the angle of twist. These latter properties were obtained from the formula for the deformed shape of a twisted member [Eq. (2.68)]

\[ \phi = C_1 + C_2 \cosh \lambda z + C_3 \sinh \lambda z + \frac{M_z}{\lambda^2 EI_z} \]

The boundary conditions at end \( z = 0 \) are \( \phi = \phi' = 0 \) ("fixed" end, see Fig. 2.16); at the end \( z = L \) the section is free to warp \( (\sigma_0 = 0) \), and so from Eq. (2.55), \( \phi'' = 0 \). With these boundary conditions we find that

\[ \phi = \frac{M_z r^2}{\lambda^2 EI_z} (z - \sinh \lambda z + \tanh \lambda L (\cosh \lambda z - 1)) \]

Fig. 2.18. Semicircular cantilever beam

The cross-sectional properties are now computed from the formulas in Table 2.2 and are given below.

\[ A = \int_0^r t ds = \pi r t \]

\[ I_2 = \int_0^r y^2 ds = \frac{\pi}{2} (r^2 t) \]

\[ \omega = \int_0^r p ds = \int_0^r \left( r - 2r \sin \frac{\beta}{\pi} \right) d\beta = r^2 \left[ \beta - \frac{2}{\pi} (1 - \cos \beta) \right] \]

\[ I_{2z} = \int_0^r \omega y ds = -r^3 t \]

\[ x_0 = \frac{I_{2x}}{I_2} = -\frac{2r}{\pi}, \quad y_0 = 0 \quad \text{by inspection} \]
nondimensional derivatives are plotted against $z$. For our given situation \( \lambda L = L \sqrt{K/G} = 2.730 \).

The information given in Figs. 2.19 and 2.20 permits us now to calculate the stresses. The longitudinal stresses are

\[
\sigma_x = \frac{M_x}{I_x} \text{ and } \sigma_y = E \omega_x \phi''
\]

Both the bending moment $M_x$ and the property $\phi''$ are maximum at the fixed end

\[
(M_x)_{\text{max}} = -QL \text{ and } (\phi'')_{\text{max}} = \frac{0.992 M_x}{\lambda E I_o}
\]

The twisting moment $M_z$ is $Q$ times the distance from its line of action to $S$

\[
M_z = (x_1 + |x_0|)Q = \frac{4Qr}{\pi}
\]

Substituting $M_x$, $\phi''$, and $I_z$ into the stress equations, we get

\[
\frac{\sigma_{yy}}{Q} = -306 \frac{Y}{r} \text{ and } \frac{\sigma_{zz}}{Q} = 5941 \frac{a_o}{r}
\]

The distribution of these stresses and their sum across the cross section at $z = 0$ is shown in Fig. 2.21. The major share of the stress is contributed by restrained warping. The maximum longitudinal stresses are at the extreme fibers of the cross section and are equal to

\[
\sigma_{\text{max}} = 1465 \frac{Q}{r^{\frac{1}{2}}}
\]

The maximum St. Venant shear stress is $\tau_{SV} = Gr\phi'$, and it is the same everywhere in the cross section since $t$ is constant. The value of $\phi'$ is maximum at the free end where $z = L$ (Fig. 2.20). With

\[
(\phi')_{\text{max}} = 0.870 \frac{M_x}{\lambda E I_o}
\]

we get for the maximum St. Venant shear stress

\[
(\tau_{SV})_{\text{max}} = \frac{6070Q}{r^{\frac{1}{2}}}
\]

The shear force $V_0$ is constant along the length of the member, and so the bending shear stress is equal to, everywhere along $z$,

\[
\tau_b = -\frac{V_0}{I_z} \int_0^z yt \, dt \, ds
\]

or, after substitution of $V_0 = Q$ and $I_z = \pi r^4 / 4$

\[
\frac{\tau_{SS}}{Q} = -\frac{150}{r^4} \int_0^z yt \, ds
\]
At the other end \((z = L)\), \(\phi''' = -0.130M_s/EI_w\), and thus
\[
\frac{\tau_{w}r^2}{Q} = 106 S_w r L/
\]

The maximum value of \(S_w = 0.0524r^4t\) (from Fig. 2.19), and therefore
\[
(\tau_w)_{\text{max}} = \frac{43Q}{r^2} \quad \text{at} \quad z = 0 \quad \text{and} \quad (\tau_w)_{\text{max}} = \frac{6Q}{r^4} \quad \text{at} \quad z = L
\]

These quantities are again small compared with the bending stresses at \(z = 0\) and the St. Venant shear stress at \(z = L\). By inspection of Fig. 2.20 we see that it is not necessary to consider sections other than the ends, and either \((\sigma)_{\text{max}}\) at \(z = 0\) or \((\tau)_{\text{max}}\) at \(z = L\) will govern. For example, with the maximum allowable value of \(\sigma = 30\) ksi and \(\tau = 20\) ksi and for \(r = 6\) in. the maximum value of \(Q\) permitted is equal to 0.737 kip. In this case \((\sigma)_{\text{max}}\) was controlling. With these same allowable stresses, but by neglecting the torsional stresses, we would have obtained \(Q = 3.53\) kip. In this case the neglect of torsional stresses would have resulted in premature failure.

**STRESSES IN A WIDE-FLANGE BEAM**

As the next example we shall consider the simply supported wide-flange beam shown in Fig. 2.22. The central concentrated load \(Q\) is applied with an...
The sketch in Fig. 2.23 shows the direction of integration by arrows, and the sign of \( \rho \); \( \rho \) is the perpendicular distance of the element from \( C \), and it is positive when \( C \) is to the left of an observer who faces in the direction of the arrow. The direction and sequence of integration are arbitrary. Finally the same \( \omega \) diagram is obtained regardless of the order of integration; however, \( \omega \) and \( \omega_a \) will be dependent on the order of integration. The first column in the table of Fig. 2.23 shows the points \( i \) on the cross section, the second column gives the values of \( \rho = \rho_i \), and the third column is the length of the element \( ij \). The fourth column shows \( \rho_i \delta ij \) for each plate element. The summation is started at point 1 and carried on through to point 4, as indicated by arrows in the table. The summation for the branches 5–2 and 6–3 is started with the values of \( \omega_0 \) computed in the upper half of the table. The final values of \( \omega_a \) are then given in the fifth column. We can now compute \( \omega_a \) by the formula (from Table 2.3)

\[
\omega_{aj} = \frac{1}{2A} \sum_{j=1}^{n} (\omega_0 + \omega_{0j}) \delta ij - \omega_{0j}
\]

\[
= \frac{1}{2A} \left[ \left( 0 + \frac{d' b}{4} \right) \left( \frac{tb}{2} \right) + \left( \frac{d' b}{4} + \frac{d' b}{4} \right) \left( \frac{td'}{4} \right) \right]
\]

The expression can be simplified to

\[
\omega_{aj} = \frac{d' b}{4} - \omega_{0j}
\]

if we note that the area \( A \) is approximately equal to

\[
A = 2bt + d'w
\]

The corresponding values of \( \omega_a \) are listed in the last column of the table in Fig. 2.23, and the distribution of \( \omega_a \) over the whole cross section is shown in Fig. 2.24(a). The value of \( \omega_a \) is zero in the web. Since the warping stress \( \sigma_w \) is proportional to \( \omega_a \) [Eq. (2.55)], we see that the twisting moment will cause no longitudinal stresses in the web. The distribution of \( \sigma_w \) in the flanges is identical with that which would result from a bending moment about the \( y \) axis of the flanges, each acting in opposite directions to the other. This provides a simple physical picture of warping torsion for wide-flange shapes.

The \( S_\omega \) diagram [Fig. 2.24(b)] is constructed directly from the \( \omega_a \) diagram by using the relationship of Eq. (2.57), that is,

\[
S_\omega = \int_0^\infty \omega_a t \, ds
\]

We know that \( \tau_w \) and thus \( S_\tau_w \) which is proportional to \( \tau_w \) [Eq. (2.58)], is zero at the tips of the flanges (Fig. 2.14), and so we can start the summation process at either of the flange tips. We again start, arbitrarily, at point 1. Here both \( \omega_a \) [Fig. 2.24(a)] and the direction of integration (sketch in Fig.

---

**Table:**

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \rho_i )</th>
<th>( \rho )</th>
<th>( \omega = \omega )</th>
<th>( \omega_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( \frac{d' b}{4} )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( b/2 )</td>
<td>( \frac{d' b}{4} )</td>
<td>( 0 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( d' )</td>
<td>( 0 )</td>
<td>( \frac{d' b}{4} )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( b/2 )</td>
<td>( \frac{d' b}{4} )</td>
<td>( 0 )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( -d' )</td>
<td>( -\frac{d' b}{4} )</td>
<td>( 0 )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( b/2 )</td>
<td>( -\frac{d' b}{4} )</td>
<td>( 0 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( d' )</td>
<td>( \frac{d' b}{4} )</td>
<td>( 0 )</td>
<td></td>
</tr>
</tbody>
</table>

**Fig. 2.23:** Determination of \( \omega_a, \omega_0, \) and \( \omega \) for a wide-flange cross section

eccentricity \( e \) so as to produce a twisting moment \( M_t = Q e \) about the shear center. This twist will produce additional stresses \( \sigma_r, \tau_r \), and \( \tau_w \) which must be superimposed on the longitudinal stress due to bending \( \sigma_t \) and the shear stress due to bending \( \tau_t \).

The first step in the solution will be the determination of the torsional cross-sectional properties. The table in Fig. 2.23 illustrates the computation of the unit warping terms \( \omega \) and \( \omega_a \), which in this case are equal since the shear center \( S \) and the centroid \( C \) coincide for a doubly symmetric section. From Eq. (2.11) or from Table 2.3 we know that

\[
\omega = \int_0^l \rho \, ds = \sum_{i=1}^{n} \rho_i \delta ij
\]
2.23) are positive. Thus the resulting shear flow \( S_{\omega} \) is positive. This shear flow is positive if it will cause a counterclockwise moment about the shear center, as shown by the small arrow at location 1 in Fig. 2.24(b). At point 2 the shear flow from branch 1–2 is simply the area under the \( \omega_{\omega} \) diagram times the thickness of the element, or \( S_{\omega} = t d b / 16 \). This amount of \( S_{\omega} \) enters joint 2, and the same amount must also flow out, either into branch 2–5 or 2–3, or into both. Since the direction of integration is toward joint 3, we must pick up the contribution of branch 2–5. At point 5 both \( \omega_{\omega} \) and the direction of integration are negative, so again a positive shear flow results. Integrating from 5 to 2, we find that \( S_{\omega} \) at 2 is again \( t d b / 16 \). The two shear flows entering joint 2 are in the same direction and of equal magnitude, and so all the flow from branch 1–2 flows into branch 2–5, and no \( S_{\omega} \) flows into the web. This is a special situation existing only for doubly symmetric shapes. In general, flow would be into both branches. Since \( \omega_{\omega} \) is also zero in the web, no shear flow exists in the web. A similar procedure gives the distribution of \( S_{\omega} \) in the top flange. The maximum value of \( S_{\omega} \) thus equal to \( t d b / 16 \), and it occurs at the center of the flanges. The arrows in Fig. 2.24(b) show the direction of the flow \( S_{\omega} \). This is not necessarily the direction of the shear stress \( \tau_{\omega\omega} \) since this is also proportional to \(-E \phi''\) [Eq. (2.58)].

The remaining torsional property to be computed is \( I_u \). From Table 2.3

\[
I_u = \frac{1}{3} \sum_{i=0}^{n-1} \left( \omega_{\omega i} + \omega_{\omega i} \omega_{\omega j} + \omega_{\omega j} \right)_{i} \frac{d}{b}
\]

From Fig. 2.24(a) we now have

\[
I_u = \frac{1}{3} \left\{ \left[ \frac{(d'b)^2}{4} \right] + \left( \frac{d'b}{4} \right) \left[ - \frac{d'b}{4} \right] \right\} \frac{d}{b}
\]

We now have the necessary information to determine the torsional stresses at the cross section if we also know the distribution of \( \phi', \phi'', \) and \( \phi''' \) along the \( z \) axis. This information is obtained from the equation of the angle to twist [Eq. (2.72)]. The boundary conditions for the half-length of the beam \( 0 \leq z \leq L/2 \) are 

1. for the simply supported, \( \phi(0) = \phi''(0) = 0 \) and 
2. for the center, \( \phi(L/2) = 0 \). This latter condition is due to the symmetry of the deformed shape about the center of the beam where \( M_e = Qe \) is applied. The deformed shape and its derivatives are, from Eq. (2.72) and the three boundary conditions,

\[
\phi' \left( \frac{2GK_x}{M_x} \right) = \frac{z}{L} - \frac{\sinh \lambda z}{\lambda L \cosh (\lambda L/2)}
\]

\[
\phi'' \left( \frac{2GK_x}{M_x} \right) = 1 - \frac{\cosh \lambda z}{\lambda \cosh (\lambda L/2)}
\]

\[
\phi''' \left( \frac{2GK_x}{\lambda M_x} \right) = -\frac{\sinh \lambda z}{\lambda \cosh (\lambda L/2)}
\]

The curves representing these quantities are given in Fig. 2.25 for \( \lambda L = 4.65 \). This value of \( \lambda L \) is for a 12WF50 steel shape, for which the following quantities are given or computed:

- \( L = 240 \text{ in.} \)
- \( G = 11,500 \text{ ksi} \)
- \( E = 30,000 \text{ ksi} \)
- \( I_o = 934.5 \text{ in}^4 \)
- \( S_{\omega} = 64.7 \text{ in}^3 \)
- \( \omega_{\omega} = 1.981 \text{ in}^4 \)
- \( \omega_{\omega \omega} = 30.18 \text{ in}^4 \)
- \( \omega_{\omega \omega \omega} = 23.32 \text{ in}^6 \)
- \( d = 11.5 \text{ in.} \)
- \( d' = 8.1 \text{ in.} \)
- \( \lambda^2 = \frac{G K_x}{E I_o} = 0.000371 \text{ in}^{-2} \)
- \( K_x = 1.82 \text{ in}^2 \)
- \( \lambda L = 4.65 \)

The quantities \( \phi, \phi'', \) and \( \phi''' \) are maximum at \( z = L/2 \), and \( \phi' \) is maximum at the ends \( z = 0 \) and \( z = L \). With these properties we can now compute the stresses due to the twisting moment \( M_e = Qe \).

1. St. Venant shear stress at \( z = 0 \). The St. Venant shear stress is \( \tau_{\omega\omega} = G t \phi' \), and from Fig. 2.25 \( \phi'(0) = 0.805(M_x/2GK_x) \), and therefore

\[
\tau_{\omega\omega} = \frac{0.805}{2K_x} (Qe) = 0.221 Qe
\]

In the flanges \( t = 0.641 \text{ in.} \), and so \( \tau_{\omega\omega} = 0.1418 Qe \); in the web \( t = 0.371 \text{ in.} \), and \( \tau_{\omega\omega} = 0.0820 Qe \). These stresses are distributed as shown in Fig. 2.9(c), and they are the maximum values at the outside face of the cross section.
the maximum value of $\sigma_y$ is

$$\sigma_y = -0.316Qe$$

To these stresses we must now add the stresses due to bending. From the shear and moment diagram in Fig. 2.26 we see that the shear $V_y = Q/2$ is uniform across the whole half-span $0 \leq z \leq L/2$, and that the maximum moment $M_y = QL/4$ occurs at the center of the span.

1. **Longitudinal stress due to bending at $z = L/2$.** The maximum bending stress at this location is at the outside fibers of the flanges, and this stress is equal to

$$\sigma_y = \frac{M_y}{S_y} = \frac{QL}{4S_y} = 0.927Q$$

where $S_y$ is the section modulus about the $x$ axis.

2. **Shear stress due to bending.** The shear stress in the half-span is defined by Eq. (2.4), or

$$\tau_y = -\frac{V_y}{I_x} \int_0^z y^2 \, ds$$

The $\int_0^z y^2 \, ds$ diagram for a wide-flange shape is shown in Fig. 2.27. The important locations of the shear stress are at the center of the flanges and at

![Fig. 2.25. Variation of $\phi$ and its derivatives along the length of the beam](image)

![Fig. 2.26. Shear and bending moment diagram for beam of Fig. 2.22](image)

![Fig. 2.27. Distribution of $\int_0^z y^2 \, ds$ for a wide-flange cross section](image)
the center of the web. At the flange center \( \int_{b_0}^{b} yt \, ds = d'bt/4; \, t = 0.641 \text{ in.} \), and therefore

\[
(\tau_{s})_{\text{FLANGE}} = 0.0295Q
\]

At the center of the web \( \int_{b_0}^{b} yt \, ds = (d'bt/2) + [\nu(d')^2/8]; \, w = 0.371 \text{ in.} \), and thus

\[
(\tau_{s})_{\text{WEB}} = 0.1233Q
\]

The longitudinal stresses are maximum at the center of the beam. At this location the bending stresses and the warping stresses are distributed as shown in Figs. 2.28(a) and (b), respectively. The combined longitudinal stresses are given in Fig. 2.28(c). The maximum stresses occur at the right flange tips, and they are equal to 1.243\(Q\) for the case of \( e = 1 \text{ in.} \). This 1-in. eccentricity can introduce warping stresses which increase the maximum stresses by more than 30 per cent, thereby showing the great sensitivity of wide-flange shapes to torsional moments.

The various shear stresses at locations \( z = 0 \) and \( z = L/2 \) are shown in Fig. 2.29. The torsional stresses are for an eccentricity of \( e = 1.0 \text{ in.} \). The direction and the magnitude of the stresses are shown at the center of the flanges (equal but opposing each other in the top and the bottom flange) and at the center of the web. The maximum values of the stresses are

\[
\begin{align*}
&z = 0, \quad \tau_{\text{FLANGE}} = (0.0295 + 0.1418 + 0.0024)Q = 0.174Q \\
&z = 0, \quad \tau_{\text{WEB}} = (0.1233 + 0.0820)Q = 0.205Q \\
&z = \frac{L}{2}, \quad \tau_{\text{FLANGE}} = (0.0295 + 0.0125)Q = 0.042Q \\
&z = \frac{L}{2}, \quad \tau_{\text{WEB}} = 0.123Q
\end{align*}
\]

![Fig. 2.29. Shear stresses at \( z = 0 \) and \( z = L/2 \)]

Even the largest shear stress is but a small percentage of the maximum longitudinal stress of 1.243\(Q\), and so in this problem the latter will govern the design.

An even more thorough analysis than was presented could involve the development of normal stress trajectories along the whole length of the beam. However, in most cases it is sufficient to determine the critical locations by visual inspection, as was done in this example.

### 2.8. THE SECOND-ORDER DIFFERENTIAL EQUATIONS

**GENERAL COMMENTS**

The differential equations\(^8\) to be derived next will be formulated on the already deformed member. Since the deformations and the internal forces are no longer independent of each other, we cannot consider each effect

\(^8\) The derivation of the differential equations is based on geometric considerations, following Timoshenko\(^{1,10,12,13}\) or Vlasov (Chap. 5, Ref. 1.18). These equations could also be derived from energy principles, as was done by Bleich (Chap. IV, Ref. 1.30).

![Fig. 2.28. Normal stress distribution at \( z = L/2 \)]
separately, and we must therefore combine all effects into one formulation. Because of the lengthy and complicated expressions which would result if we were to include every conceivable kind of loading, we shall restrict ourselves to the relatively simple case shown in Fig. 2.30. This model contains most of the loading cases with which we shall later be concerned.

The member in Fig. 2.30 is initially straight and prismatic. Its ends are pinned and are prevented from translating with respect to each other. Forces are only applied at its ends. The forces consist of a compressive axial force $P$, which is assumed to retain its original direction throughout the loading history, and bending moments $M_{y}x$, $M_{x}y$, $M_{xy}$, and $M_{y}x$. These moments are shown positive according to the right-hand rule. Reactions $R_x$ and $R_y$ also exist owing to the couples resisting the bending moments.

We shall continue to follow the same assumptions discussed before, notably the assumption of elastic behavior and small deflections, except that here we shall include the deformations. We shall continue to specify that the shape of the cross section does not change.

**DISPLACEMENTS OF THE CROSS SECTION**

The middle line of a general thin-walled open cross section is shown in Fig. 2.31(a). The $x$ and $y$ coordinates are principal coordinates passing through the centroid $C$. The point $Q$ is any location on the middle line. The displacements at a point $S$ are $u$ and $v$, positive as shown; in addition, the whole cross section rotates about $S$ through an angle $\phi$. As $S$ moves to $S'$ through displacements $u$ and $v$ [Fig. 2.31(b)], $Q$ moves to $Q'$ through $\phi$. From the geometric relationships shown in Fig. 2.31(b), we find that the displacements of $Q$ in the $x$ and $y$ direction are

$$u_q = u + a\phi \sin \alpha \quad \text{and} \quad v_q = v - a\phi \cos \alpha$$

\footnote{In our earlier discussion we considered a tensile axial force to be positive (Figs. 2.1 and 2.7). Since in most cases to be discussed in the later chapters a compressive axial force is more important, we will from here on consider $P$ to be positive if it is compressive.}

**EQUILIBRIUM EQUATIONS**

The forces acting in the $x$-$x$ and the $x$-$y$ plane are shown in Figs. 2.32(a) and (b), respectively. The internal resisting moments at a distance $z$ from the lower end of the member are

$$M_x = -M_{xx} + R_s x + P v_0$$
$$M_y = -M_{yy} + R_s x - P u_0$$

where $a$ is the distance between $Q$ and $S$.

But $\sin \alpha = (y - y)/a$ and $\cos \alpha = (x - x)/a$, and so

$$u_q = u + \phi(y - y) \quad \text{and} \quad v_q = v - \phi(x - x) \quad (2.77)$$

The displacements of the centroid are $(x = y = 0)$

$$u_o = u + \phi y_0 \quad \text{and} \quad v_o = v - \phi x_0 \quad (2.78)$$

\[\text{Fig. 2.31. Displacement of a point } q \text{ in a cross section}\]
These moments are positive according to the right-hand rule. Substituting $\psi_c$ and $u_c$ from Eqs. (2.78), and noting that

$$R_x = \frac{M_{Rx}}{L}, \quad R_y = \frac{M_{Ry}}{L}$$

we get

$$M_x = -M_{Rx} + \frac{z}{L}(M_{Rx} + M_{Ry})$$

$$M_y = -M_{Ry} + \frac{z}{L}(M_{Rx} + M_{Ry}) - P(u - \phi u_c)$$

Within the span of the member the cross sections will no longer be in the original undeformed $x$-$y$-$z$ coordinate system after the deformations have taken place. The cross section will translate and rotate as shown in Fig. 2.33 so that the principal axes are now a new set of rectangular coordinates $\xi$ and $\eta$. We must transform the moments $M_x$ and $M_y$ to these new axes. From Fig. 2.34 we get by vector addition

$$M_t = M_x + \phi M_y \quad \text{and} \quad M_v = M_y - \phi M_x$$

(2.82)

Since the angle $\phi$ is small, we have used the relationships $\sin \phi = \phi$, $\cos \phi = 1$.

In addition to the components of $M_x$ and $M_y$ along the $\xi$ and $\eta$ axes, they also have components along the $\zeta$ axis which is perpendicular to the cross section and is inclined from the $z$ axis (Fig. 2.35). These components will result in a twisting moment [Figs. 2.35(a) and (b)].

$$M_{t\zeta} = M_x \frac{du}{dz} + M_y \frac{dv}{dz}$$

(2.83)

where we have again used the relationships $\sin \frac{du}{dz} = \frac{du}{dz}$ and $\sin \frac{dv}{dz} = \frac{dv}{dz}$.

In addition to the twist due to components of $M_x$ and $M_y$, we have other factors which contribute further torque components. One of these is due to the fact that $P$ retains its original direction. In the $x$-$y$ plane, therefore, $P$ has a component $P(\frac{du}{dz})$ which acts through the centroid [Fig. 2.35(c)]. Together with the component of $P$ in the $z$-$\zeta$ plane [Fig. 2.35(d)] we have thus a twisting moment about the shear center which is equal to (see Fig. 2.36)

$$M_{t\zeta} = P\left(\frac{v}{z} - \phi \frac{dx}{dz}\right)$$

(2.84)

A third contribution to $M_{t\zeta}$ is caused by the fact that two cross sections $d\zeta$ apart will warp with respect to each other (Fig. 2.37), and therefore the stress element $\sigma \, dA$ (positive in tension) is inclined by the angle $a (\frac{d\phi}{d\zeta})$ to the $\zeta$ axis. The component of this stress element is $\sigma \, dA (a \, \frac{d\phi}{d\zeta})$, and it
Fig. 2.35. Twisting due to components of \( \mathbf{M}_x \), \( \mathbf{M}_y \), and \( P \)

causes a twist about the shear center equal to

\[
dM_{1s} = -\sigma(\sigma \, dA)\left(\frac{d\phi}{dz}\right)
\]

Integrating over the whole cross section, we obtain

\[
M_{1s} = -\frac{d\phi}{dz} \int_A \sigma^2 \, dA
\tag{2.85}
\]

Letting

\[
\int_A \sigma^2 \, dA = \bar{K}
\tag{2.86}
\]

and noting that \( d\phi \approx dz \) if we neglect terms of higher order of magnitude, we get

\[
M_{1s} = -\bar{K} \frac{d\phi}{dz}
\tag{2.87}
\]

A fourth and final contribution is due to the end shears (Fig. 2.38), that is,

\[
M_{1s} + R_x \, u + R_y \, v = 0
\]

Using the values of \( R_x \) and \( R_y \) from Eq. (2.79), we find that

\[
M_{1s} = -\frac{v}{L} (M_{1y} + M_{1m})
\]

\[
-\frac{u}{L} (M_{1x} + M_{1m})
\tag{2.88}
\]

The total twisting moment is the sum of the four components [Eqs. (2.83), (2.84), (2.87), and (2.88)], or
ELASTIC BEHAVIOR OF MEMBERS

\[ M_t = M_u + M_yu + P_yu' - P_xu' - \frac{u}{L}(M_{o} + M_{m}) \] (2.89)

We now know the moments which are acting at any location along the z axis. These are \( M_z, M_y, \) and \( M_t. \) We can equate them to the internal resistance of the member. From Eq. (2.27) we get

\[ M_t = -EIz'' \quad \text{and} \quad M_y = +EIu'' \]

and from Eq. (2.65) the twisting moment is equal to

\[ M_t = GKP\phi' - EI\phi''' \]

If we define the bending stiffnesses as

\[ B_z = EI_z \quad \text{and} \quad B_y = EI_y \] (2.90)

and the St. Venant torsional stiffness as

\[ C_y = GKP \] (2.91)

and the warping stiffness as

\[ C_y = EI_z \] (2.92)

we can, after some rearrangement and by substituting \( M_z \) and \( M_y \) from Eqs. (2.80) and (2.81) into Eqs. (2.82) and (2.89), write the final differential equations. These are

\[ B_zu'' + Pu - \phi \left[ M_{by} - \frac{z}{L}(M_{zy} + M_{by}) + P_{x} \right] = M_bz - \frac{z}{L}(M_{zy} + M_{by}) \] (2.93)

\[ B_yu'' + Pu - \phi \left[ M_{by} - \frac{z}{L}(M_{zy} + M_{by}) - P_y \right] = -M_by + \frac{z}{L}(M_{zy} + M_{by}) \] (2.94)

\[ C_y\phi''' - (C_z + \phi)y' + u\left[ -M_{by} + \frac{z}{L}(M_{zy} + M_{by}) + P_x \right] - v\left[ M_{by} - \frac{z}{L}(M_{zy} + M_{by}) + P_x \right] - \frac{u}{L}(M_{zy} + M_{by}) - \frac{u}{L}(M_{zy} + M_{by}) = 0 \] (2.95)

These three differential equations describe the equilibrium of any cross section in the member shown in Fig. 2.30. In accordance with the assumption of small deflections we have neglected all terms involving the products of the quantities \( u, v, \phi, \) etc., and thus the equations are linear with respect to the three deformations \( u, v, \) and \( \phi \) and their derivatives. The equations are not independent of each other. Equation (2.93), which represents bending about

the strong axis, for example, depends also on the angle to twist \( \phi \) which was associated with torsion only in Sec. 2.4.

With these three differential equations we shall begin our examination of the behavior of beams, columns, and beam-columns in the succeeding three chapters. By giving up the prerogative of using the principle of superposition, with these equations we shall be able to tell more about the behavior of these members than we could with the earlier independent equations. These equations will permit an examination of problems of elastic buckling, and they will also introduce and lead to an understanding of problems related to buckling in the inelastic range.

REFERENCES


2.3. S. P. Timoshenko, "Theory of Bending, Torsion, and Buckling of Thin-Walled Members of Open Cross Section," Journal of the Franklin Institute, Vol. 239, Nos. 3, 4, 5, March, April, May 1945.


2.5. R. Dabrowski, "Torsion Bending of Thin-Walled Members with Non-Deformable Closed Cross Section" (Columbia University, Department of Civil Engineering and Mechanics, 1963).


PROBLEMS

2.1. Determine the normal stress $\sigma$ and the shear flow $t \tau$ for a thin-walled open cross section subjected to forces $P$, $M_x$, $M_y$, $V_x$, $V_y$ (Fig. 2.7) if the $x$-$y$ coordinates are centroidal but not principal axes.

2.2. Determine the coordinates of the shear center location ($x_s$, $y_s$ in Fig. 2.3) if the $x$-$y$ coordinates are centroidal but not principal axes.

2.3. Derive the equations given in Table 2.3.

2.4. Derive the relationships given in Table 2.4.

2.5. See Fig. P. 2.5. Determine the location of the shear center, draw the $S_m$, $S_{mz}$, $\int x^2 \, dt$ and $\int y^2 \, dt$ diagrams and compute $I_s$ for the following thin-walled open cross sections. Except for cross section $g$ the thickness of the plate is the same everywhere in the cross section.

2.6. See Fig. P. 2.6. (a) Determine the maximum value of $m_s$ if the allowable normal stress is 15 ksi and the allowable shear stress is 10 ksi. Check answer with Ref. 2.10.

(b) What is the corresponding value of $m_s$ if the cross section is a closed circular shape?

(c) Discuss the comparison of the answers from (a) and (b).

2.7. See Fig. P. 2.7. (a) Derive expressions for the angle of twist $\phi$.

(b) Determine the maximum normal stress as a function of $M_z$ and $m_w$, respectively, for the cross section of problem 2.5(g). $L = 300$ in.; $E = 30,000$ ksi; $G = 11,500$ ksi. Use a computer to develop values of $\phi''$ along the z axis.

2.8. See Fig. P. 2.8. A uniformly varying distributed load is applied with an eccentricity of 1 in. to a 16 WF58 steel beam with simply supported ends (both with respect to bending and torsion). Determine the maximum allowable value of $q_v$. Check answer with Ref. 2.10.
2.6. Cross section

\[ r = 5 \text{ in.} \]
\[ f = 0.25 \text{ in.} \]
\[ L = 120 \text{ in.} \]
\[ E = 10,000 \text{ ksi} \]
\[ G = 3800 \text{ ksi} \]

2.7. \( m_2 = m_2 \sin \frac{2\pi L}{L} \)

2.8. \( E = 30,000 \text{ ksi} \)
\[ v = 11,500 \text{ ksi} \]
\[ \sigma = 20 \text{ ksi} \]
\[ \tau = 13 \text{ ksi} \]

2.9. See Fig. P. 2.9. (a) The load \( Q \) acts over one of the webs of the cross section in problem 2.5(f) (point A). Determine the locations and magnitudes of the maximum normal stress \( \sigma \) and the maximum shear stress \( \tau \). The ends of the beam are simply supported with respect to bending \((\psi = \psi'' = 0)\) and fixed with respect to torsion \((\phi = \phi' = 0)\).

(b) What is the horizontal and the vertical deflection of the flange end [point \( B \), problem 2.5(f)] at the center of the beam?

2.10. See Fig. P. 2.10. Determine the maximum allowable value of \( q \) if \( q \) acts through the centroid of the channel. The ends are flexurally and torsionally simply supported.
Beams

3.1. THE RESPONSE OF BEAMS TO LOAD

Beams are members in which the applied loads produce principally bending and shear, and in which the axial force is negligibly small ($P = 0$ in Fig. 2.30). Members which can be idealized in design as beams comprise a large portion of the members in framed structures.

Most beams are designed to carry loads which produce bending about the major principal axis of the cross section only [that is, loads act in a plane parallel to the $y$-$z$ axis which passes through the shear center $S$—see Fig. 2.6(a)]. We shall call this in-plane bending. Open sections are not very efficient in resisting torsion. Even though in-plane bending does not introduce torsion intentionally, it does exist from the very beginning of loading because of unavoidable initial imperfections in beam geometry and the unintentional small eccentricity of the loads. When the out-of-plane deformations in the direction of the $x$ axis (Fig. 2.7) become magnified to the extent that they terminate the usefulness of the beam, we have lateral-torsional buckling.

Thin-walled beams are made up of relatively thin plate elements. Under some conditions these plates experience local buckling. Local buckling in combination with lateral buckling often is the cause of failure of steel beams.

When bending about both axes as well as torsion is introduced intentionally, we have biaxial bending. In this chapter we shall examine each of the effects enumerated above. But first we shall examine the test performance of a beam to the failure point.

TEST ON A STEEL BEAM

The photograph in Fig. 3.1 shows the overall experimental setup. A 10-in. wide-flange beam was suspended at the third-points from an overhead girder. Two equal vertical loads were applied with hydraulic jacks at the beam ends. This loading produced a moment distribution as shown in the diagram in the inset of Fig. 3.2. The central third of the beam was under uniform moment.

Lateral support was provided by knife-edge guides at the support rods and at the ends (Fig. 3.1). These guides prevented deflection in the lateral, or $x$, direction and also twisting at these four locations (see inset in Fig. 3.3).

The performance of this beam can be studied from curves which relate the deflections in the $y$-$z$ plane and in the $x$-$z$ plane with the load $Q$ as the in-plane deflection is slowly increased from zero to its final value at the end of
The performance in the plane of bending is shown in Fig. 3.2, where the moment $M = QL$ is plotted against the vertical deflection $v_0$ at the center of the beam. Out-of-plane behavior is illustrated in Fig. 3.3, where the curves give the moment versus lateral deflection $u_0$ relationship at the center of the beam for both the tension and compression flange.

At first the response of the beam was elastic, as can be seen by the initially linear $M$-$v_0$ relationship in Fig. 3.2.\footnote{No units are given in Figs. 3.2 and 3.3, since we are considering only general behavior. For the actual values of the deformations and forces, as well as for other details of this test, see Ref. 3.1.} Elastic behavior was terminated when the sum of the bending stress and the residual stress first reached $\sigma_y$. Yielding was first observed from flaking mill scale at the horizontal arrow in Fig. 3.2. In this beam, yielding started at the tips of the compression flange in the central third of the beam. As more and more of the material in the uniform moment region yielded, the resistance of the beam to further load increases was reduced, so that finally no additional load could be carried. This occurred when the $M$-$v_0$ curve became horizontal at a moment equal to $M_p$, the \textit{plastic moment}.\footnote{No units are given in Figs. 3.2 and 3.3, since we are considering only general behavior. For the actual values of the deformations and forces, as well as for other details of this test, see Ref. 3.1.} Beyond this point the deformation increased...
local and lateral buckling occur. This is an ideal condition seldom reached with practical beams. The situation described above for the test beam is given by curve OAC. Load deflection curves are often idealized by the elastic portion OAD and a plastic hinge region DE.

The curve OAIJK is typical of beams with varying moment gradient (such as a simply supported beam with a central concentrated load). For such beams strain hardening exists (Fig. 1.5) at the region of the maximum moment, and thus the \( M-v \) curve rises above \( M_p \). This upward rise is checked and finally reversed by local and lateral buckling.

Curves OAFG and OAHI represent situations in which lateral-torsional or local buckling influences the in-plane behavior after some portions of the beam have yielded; curve OLM shows the same influences, but here they occur while the beam is still elastic.

In Fig. 3.6 we have shown the best possible performance of beams (curve OAB), and we also have shown how actual beams fall short of this ideal. In the subsequent articles of this chapter we shall examine these shortcomings in more detail, and we shall see how they can be considered in the design of beams. We shall start our examination with elastic beams.

### 3.2. ELASTIC BEAMS

#### THE DIFFERENTIAL EQUATIONS

The differential equations for elastic prismatic members subjected to end forces were developed in Chapter 2, and we shall make use of them here in the form in which they apply to beams. Among the forces shown in Fig.

---

**Fig. 3.5. Local buckling of the compression flange of a wide-flange beam**

without an appreciable change in the load over a range many times larger than the deflection at initial yielding.

As soon as \( M_p \) was sensibly attained, we observed a bowing out of the compression flange into the \( x \) direction (open circles in Fig. 3.3). This lateral deflection continued to increase with the vertical deflection \( v_0 \), while the lateral deflection of the tension flange (filled-in circles) remained very small. During this twofold deformation, that is, the beam moving as a whole in the \( y \) direction and the compression flange moving in the \( x \) direction, the initial shape of the cross section was distorted as shown in the sketch in Fig. 3.3.\(^{1,2,3}\) Unloading was finally triggered by local buckling in the most strained half-flange in the central region of the beam (see vertical arrows in Figs. 3.2 and 3.3). Further straining resulted in a decrease of the load. At the last recorded load point the beam was already badly deformed and the test was stopped.

The photographs in Figs. 3.4 and 3.5 show the final deformed shape and a local buckle in the compression flange, respectively.

**LOAD-DEFORMATION CURVES OF BEAMS**

The test just described represents a fairly typical beam history. There are, however, a variety of other possibilities. Some of these are illustrated in Fig. 3.6, where the curves give moment-versus-deflection relationships in the plane of loading. The solid curve OAB corresponds to the case where no
2.30 we shall set \( P = 0 \) (beam condition), and \( M_{by} = M_{ty} = 0 \) (forces acting only in the \( y-z \) plane; in-plane condition). The only forces acting on the beam are \( M_{b-x} \) and \( M_{r-x} \) as shown in Fig. 3.7, where the end moments are positive as defined in Fig. 2.32(b).

With \( M_b = -M_{b-x} + (z/L)(M_{r-x} + M_{b-x}) \) from Eq. (2.80), the differential equations of beam bending are [Eqs. (2.93) through (2.95)]

\[
B_w \ddot{v} = -M_b \tag{3.1}
\]

\[
B_w \ddot{u} + M_w \phi = 0 \tag{3.2}
\]

\[
C_w \phi'' - (C_r + \bar{K}) \phi' + M_w \phi - \left( \frac{M_{r-x} + M_{b-x}}{L} \right) u = 0 \tag{3.3}
\]

The first of these equations involves only the vertical deflection \( v \); the other two involve both the lateral deflection \( u \) of the shear center and the twist \( \phi \), but not \( v \). The first equation is independent of the other two, but the latter are interrelated through the occurrence of both \( u \) and \( \phi \) in each one. We have thus two independent sets of equations; each set will provide us with important information about the behavior of the beam.

**IN-PLANE CONDITION**

First we shall examine Eq. (3.1). If we differentiate it twice, we get \( B_w \ddot{v} = -M''_b \). But \( M_w = -M_{b-x} + (z/L)(M_{r-x} + M_{b-x}) \), and so \( M''_b = 0 \). With the notation \( B_w = EI_a \) [Eq. (2.90)], the differential equation becomes \( EI_a \ddot{v} = 0 \). This equation is the same as the differential equation of in-plane bending which we derived in Chapter 2 [Eq. (2.24) with \( \theta = 0 \)]. Thus the beam will deform in a plane through the shear center, and this plane is parallel to the \( y-z \) plane.

Let us look at the simple example shown in Fig. 3.8(a). In this beam two equal end moments \( M_b \) are bending the beam into single curvature deformation. With the sign convention of Fig. 3.7, \( M_{b-x} = -M_0 \), \( M_{r-x} = M_b \), and \( M_k = M_0 \). Thus, from Eq. (3.1), \( EI_a \ddot{v} = -M_b \). After a twofold integration we obtain

\[
v = -\frac{M_b z^2}{2EI_a} + C_1 z + C_2 \tag{3.4}
\]

where \( C_1 \) and \( C_2 \) are constants of integration which we determine from the boundary conditions \( v(0) = v(L) = 0 \) to be equal to \( C_1 = M_b L^2/2 \) and \( C_2 = 0 \). The final deflection equation is therefore

\[
v = \frac{M_b L^2}{2EI_a} \left( \frac{z}{L} \right) - \left( \frac{z}{L} \right)^2 \tag{3.5}
\]

From Eq. (3.5) we can determine the end slope \( \theta = v'(0) \); that is,

\[
\theta = \frac{M_b L}{2EI_a} \tag{3.6}
\]

This is a linear relationship between \( M_b \) and \( \theta \) for any given beam of length \( L \) and elastic stiffness \( EI_a \).

The relationship expressed in Eq. (3.6) becomes invalid when the maximum stress in the beam reaches \( \sigma_f \). From Eq. (2.1) \( \sigma = M_b y/L_x \); the maximum stress will be at the extreme fiber of the cross section \( y = d/2 \), where \( d \) is the beam depth). With the abbreviation \( 2L_x/d = S_x, \) the section modulus, and noting that \( M_b = M_0 \), we have

\[
\sigma_{max} = \frac{M_b}{S_x} \tag{3.7}
\]

We shall assume for the time being that the member contains no residual
stresses, and so we can set $\sigma_{\max} = \sigma_Y$; thus
\begin{equation}
(M_\sigma)_{\max} = S_x \sigma_Y
\end{equation}

We now introduce the term \textit{yield moment}
\begin{equation}
M_Y = S_x \sigma_Y
\end{equation}

Setting Eq. (3.9) into the slope equation [Eq. (3.6)], we get the slope corresponding to the initiation of yielding
\begin{equation}
\theta_Y = \frac{M_Y L}{2EI_x}
\end{equation}

If we use $\theta_Y$ from Eq. (3.10) to nondimensionalize Eq. (3.6) (dividing both sides by $\theta_Y$), we see that
\begin{equation}
\frac{\theta}{\theta_Y} = \frac{M_x}{M_Y}
\end{equation}

This relationship is shown as the solid straight line in Fig. 3.8(b).

\section*{LATERAL-TORSIONAL EQUATIONS}

Equation (3.11) is valid as long as $\theta/\theta_Y \leq 1.0$. In addition it is necessary that this relationship also comply with the conditions imposed by the remaining two differential equations [that is, Eqs. (3.2) and (3.3)]. Let us examine them now. The already familiar portions of the equations are $B_x = EI_x$ [Eq. (2.90)], $C_x = G K_x$ [Eq. (2.91)], and $C_T = EI_x$ [Eq. (2.92)]. The term $K$ was defined by Eq. (2.86) as $K = \int \sigma \tau dA$ where $\sigma$ is the stress anywhere in the cross section under consideration and $\tau$ is the distance between the point where $\sigma$ acts and the shear center. The stress is defined by Eq. (2.1), and from Fig. 2.31
\begin{equation}
a^2 = (x_0 - x)^2 + (y_0 - y)^2
\end{equation}

Therefore
\begin{equation}
K = \frac{M_x}{I_x} \left( x_0 \int A y dA - 2x_0 \int x y dA + \int (x - x_0)^2 y dA + \int (y - y_0)^2 x dA 
    - 2y_0 \int x^2 dA + \int y^2 dA \right)
\end{equation}

If we introduce the new cross-sectional property
\begin{equation}
\beta_x = \frac{\int A (x^2 + y^2) dA}{I_x} - 2y_0
\end{equation}

and note the relationships of Eqs. (2.2) and (2.3), we obtain
\begin{equation}
K = M_x \beta_x
\end{equation}

After differentiating Eqs. (3.2) and (3.3) and substituting the appropriate terms for $M_x$, $B_x$, $C_x$, $C_T$, and $K$, we obtain the following two differential equations for lateral-torsional buckling:\footnote{Equations (3.15) and (3.16) are identical with Eqs. 1 and 2 in Ref. 3.4; however they are not identical with the expressions given by Bleich (Eqs. 3.10, Ref. 1.34) for the same case of loading. Bleich neglected the $\beta_x M_x \theta''$ and the $M_x'' \theta$ term in Eq. (3.16). This omission has already been pointed out by Masur and Diller.}^1
\begin{equation}
EI_x \theta'' + M_x \phi'' + 2M_x \phi' = 0
\end{equation}

\begin{equation}
EI_x \phi'' - (G K_x + M_x \beta_x) \phi'' - M_x \phi' + M_x' \phi' = 0
\end{equation}

We should note that use has been made of the fact that $M_x' = (M_{xx} + M_{xx})/L$ and $M_x'' = 0$.

\section*{LATERAL-TORSIONAL BUCKLING OF A SIMPLE BEAM}

The application of Eqs. (3.15) and (3.16) will now be illustrated on the beam of Fig. 3.8(a). In addition to the conditions already stated, we shall specify that the cross section has \textit{double symmetry}, that is, $x_0 = y_0 = 0$, and that the ends are \textit{simplly supported}. This means that the end sections cannot deflect or twist, they are free to warp, and no end moment exists about the $y$ axis. The boundary conditions thus are $w(0) = w(L) = \phi(0) = \phi(L) = u''(0) = u''(L) = \phi''(0) = \phi''(L) = 0$.

For a section with double symmetry it can be shown that
\begin{equation}
\beta_x = \frac{1}{I_x} \int_{-a}^{a} \int_{-b}^{b} y(x^2 + y^2) dA = 0
\end{equation}

With $M_x = M_e$ and $M_x' = 0$, the differential equations [Eqs. (3.15) and (3.16)] now become equal to
\begin{equation}
EI_x \theta'' + M_e \phi'' = 0
\end{equation}

\begin{equation}
EI_x \phi'' - G K_x \phi'' + M_e \phi' = 0
\end{equation}

Next we integrate Eq. (3.18) twice, that is,
\begin{equation}
EI_x \theta'' + M_e \phi + C_t x + C_t = 0
\end{equation}

where $C_t$ and $C_t$ are constants of integration which are equal to zero from the boundary conditions. Thus from Eq. (3.20) we get
\begin{equation}
\phi'' = -\frac{M_e \phi}{EI_x}
\end{equation}

If we set Eq. (3.21) into the second differential equation [Eq. (3.19)], we obtain an equation which is only a function of $\phi$
\begin{equation}
EI_x \phi'' - G K_x \phi'' - \frac{M_e^2}{EI_x} \phi = 0
\end{equation}
This equation can be written as

\[ \phi'''' - \lambda_1 \phi''' - \lambda_2 \phi = 0 \]  

(3.23)

where

\[ \lambda_1 = \frac{GK_t}{EI_x} \quad \lambda_2 = \frac{M_t^2}{EJ_t L_w^2} \]  

(3.24)

The solution of Eq. (3.23) is\(^{1,14}\)

\[ \phi = C_1 \cosh \alpha_z z + C_2 \sinh \alpha_z z + C_3 \sin \alpha_z z + C_4 \cos \alpha_z z \]  

(3.25)

where \(C_1, C_2, C_3,\) and \(C_4\) are constants of integration and

\[ \alpha_1 = \sqrt{\lambda_1 + \sqrt{\lambda_1^2 + 4 \lambda_2}} \]  

(3.26)

\[ \alpha_2 = \sqrt{-\lambda_1 + \sqrt{\lambda_1^2 + 4 \lambda_2}} \]  

(3.27)

With the four boundary conditions \(\phi(0) = \phi(L) = \phi''(0) = \phi''(L) = 0\) we obtain the following four simultaneous equations for the unknown constants \(C_1, C_2, C_3,\) and \(C_4\) from Eq. (3.25):

\[
\begin{align*}
0 &= C_1(1) + C_2(0) + C_3(0) + C_4(1) \\
0 &= C_1(\alpha_1^2) + C_2(0) + C_3(-\alpha_2^2) + C_4(-\alpha_2^2) \\
0 &= C_1 \cosh \alpha_1 L + C_2 \sinh \alpha_1 L + C_3 \sin \alpha_2 L + C_4 \cos \alpha_2 L \\
0 &= C_1 \alpha_1^2 \cosh \alpha_1 L + C_2 \alpha_2^2 \sinh \alpha_2 L - C_3 \alpha_2^2 \sin \alpha_2 L - C_4 \alpha_2^2 \cos \alpha_2 L
\end{align*}
\]

These simultaneous equations are homogeneous, and they are satisfied only if \(C_1 = C_2 = C_3 = C_4 = 0\) (trivial solution, because then \(\phi = u = 0\)) or if the determinant of the coefficients of the unknowns is zero (nontrivial solution).\(^{1,3}\) We can formulate this latter condition as follows:

\[
\begin{vmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -\alpha_1^2 \\
\cosh \alpha_1 L & \sinh \alpha_1 L & \sin \alpha_1 L & \cos \alpha_1 L \\
\alpha_1^2 \cosh \alpha_1 L & \alpha_1^2 \sinh \alpha_1 L & -\alpha_2^2 \sin \alpha_2 L & -\alpha_2^2 \cos \alpha_2 L
\end{vmatrix} = 0
\]  

(3.28)

This determinant is called the characteristic determinant; by expanding this determinant and simplifying, we obtain the following characteristic equation or buckling condition:

\[ (\alpha_1^2 + \alpha_2^2)^3 \sinh \alpha_1 L \sin \alpha_2 L = 0 \]  

(3.29)

The terms \(\alpha_1\) and \(\alpha_2\) must fulfill this relationship in order to get a solution for which \(\phi\) and \(u\) are nonzero. The first bracket is the sum of two positive numbers, and so it is not equal to zero; \(\sinh \alpha_1 L\) is zero only if \(\alpha_1 = 0\), which is a trivial case. Thus Eq. (3.29) can only be true if

\[ \sin \alpha_2 L = 0 \]  

(3.30)

This will be so only for certain specific values of \(\alpha_2\); these are called the characteristic values or eigenvalues of the differential equations, and in this case they are

\[ \alpha_2 L = n \pi \]  

(3.31)

where \(n = 1, 2, 3, \ldots\) By substituting \(\lambda_1\) and \(\lambda_2\) [Eqs. (3.24)] into the equation for \(\alpha_2\) [Eq. (3.27)], we get after some rearrangement the critical or buckling moment

\[ (M_{kr})_c = \frac{\pi n^2}{L} \sqrt{\frac{EI_t G K_t}{1 + \frac{n^2 \pi^2 E L_w^2}{G K_t L^2}}} \]  

(3.32)

When \(M_c\) is equal to any one of the values \((M_{kr})_c\) in Eq. (3.32), it is possible that lateral-torsional deformations \(u\) and \(\phi\) exist, that is, the beam can buckle.

If we next substitute Eq. (3.30) into the original four homogeneous simultaneous equations, we find that \(C_1 = C_2 = C_3 = C_4 = 0\) and that we cannot determine anything else about \(C_0\) except that it is not equal to zero. Therefore the deflected shape is [from Eq. (3.25) with \(\alpha_2 = n \pi / L\)]

\[ \phi = C_1 \sin \frac{n \pi z}{L} \]  

(3.33)

Our analysis could not, however, tell us anything about the magnitude of the deflection since we do not know the value of \(C_0.\)\(^3\) We can also determine the lateral deflection \(u\) by substituting Eq. (3.33) into Eq. (3.21). After integration and using the boundary conditions \(u(0) = u(L) = 0\), we get

\[ u = C_1 M_c L^2 \sin \frac{(n \pi z) L}{n^2 \pi^2 E L_w^2} \]  

(3.34)

**COMMENTS ON ELASTIC BUCKLING**

We shall now examine the significance of the results obtained from the solution of the two differential equations. We found that the deflections \(u\) and \(\phi\) could assume nonzero values only if the loading (in this case the end moment \(M_c\)) was equal to certain specific values [eigenvalues, Eq. (3.31)]. At these critical loads the beam can be in a laterally undeformed position \((u = \phi = 0)\) since the trivial solution is valid for any load, but it can also be in equilibrium in a laterally deflected shape. Thus at a critical moment the

\(^3\) Discussions on the mathematical properties of buckling problems similar to the one given above are available in many references. The following list is but a brief sampling of these: Chaps. 2 and 6 in Ref. 1.30; Chaps. 2, 3, and 4 of Ref. 1.29; and Chaps. 5 in Ref. 2.14. We should also note here that the mathematical behavior of buckling problems is identical with that of vibration problems.
equilibrium can bifurcate, or as we defined it in Chapter 1, the beam buckles. Our analysis has given us the values for the critical moment at which buckling can occur [Eq. (3.32)] and the shape of the buckled member [Eqs. (3.33) and (3.34)]; that is all. In order to make use of this information intelligently, we must consider other factors which do not follow from the mathematical derivation presented.

To explain these factors we shall consider the simple model given in Fig. 3.9.\(^{111}\) A rigid bar is restrained at one end by a rotational spring of stiffness \(K\). This bar is subjected to a force \(P\) at its top and a moment \(M_0\) at its bottom. Equilibrium of moments about the pivotal point gives the relationship

\[
PL \sin \theta + M_0 - K\theta = 0
\]

or

\[
\frac{PL}{K} = \frac{\theta - \theta_0}{\sin \theta}
\]

where \(\theta_0 = M_0/K\). The curves relating \(\theta\) and \(PL/K\) for various values of \(\theta_0\) are shown as the solid-line curves in Fig. 3.10. For \(\theta_0 = 0, \theta = 0\) for any value of \(PL/K\). However, at \(PL/K = 1.0\) two branches originate, one with a positive \(\theta\) and one with a negative \(\theta\), for which \(PL/K\) increases monotonically as \(\theta\) becomes larger. This bar buckles where the equilibrium bifurcates, that is, \(P_0 = K/L\).\(^4\) The curves for nonzero values of \(\theta_0\) consist of two independent branches, one on the positive side of \(\theta\) and one on the negative side. As \(\theta_0\) approaches zero, the curves become asymptotic to the curve for \(\theta_0 = 0\).

We can think of \(\theta_0\) as an initial imperfection present in any structural system. We see that the bar remains nearly straight, with only a small initial imperfection, until \(PL/K = 1.0\) is approached; however, near this load large increases of deformation result from small increases in \(P\); the system becomes soft at this point. Such a situation should be avoided in a real structure.

Equation (3.36) gives information only about the equilibrium of the bar; it reveals nothing about the state of this equilibrium. We can test the stability of the system by disturbing it by a virtual rotation \(\theta^*\). If \(\theta^*\) can be applied without changing the forces, then the bar is at its limit of stability. The equilibrium equation [Eq. (3.36)] is changed only in the terms involving rotations, but the forces remain unchanged; thus the equilibrium of the disturbed system is, from Eq. (3.36),

\[
\frac{PL}{K} = \frac{\theta + \theta^* - \theta_0}{\sin (\theta + \theta^*)}
\]

After some trigonometric manipulations, and noting that \(\theta^*\) is small such that \(\sin \theta^* \approx \theta^*\) and \(\cos \theta^* \approx 1.0\), we find that

\[
\left(\frac{PL}{K} \sin \theta + \theta_0 - \theta\right) + \theta^* \left(\frac{PL}{K} \cos \theta - 1\right) = 0
\]

The first expression in parentheses is zero from Eq. (3.36), and since \(\theta^*\) is not zero,

\[
\frac{PL}{K} \cos \theta - 1 = 0
\]

gives the equation which delineates the boundary between stable and unstable equilibrium. The curve corresponding to Eq. (3.38) is shown as a
The dashed line in Fig. 3.10; the region below this curve is stable, and the region above it is unstable. Since disturbances are always likely to be present, the system will naturally assume stable deflection configurations.

The curves in Fig. 3.10 give the complete history of the simple system shown in Fig. 3.4. It will be appreciated that a similar analysis for even the simple beam of Fig. 3.8 would be vastly more complicated. The differential equations [Eqs. (3.1) through (3.3)] would not only have to contain terms accounting for initial crookedness, but they would also have to be formulated for large deflections. Such third-order analyses have been performed for columns: first by Euler, who called the deflected shape of the buckled column the *elastica* (Sec. 2.7, Ref. 1.29). Similar problems on the third-order analysis of beams were considered by Masur, who treated the effect of small moments about the weak axis ($M_{yy}$ and $M_{yz}$ in Fig. 2.30) in great detail by Pettersson, who, however, did not include the effects of large deformations.

The conclusions reached from these studies are essentially the same as those we arrived at from the simple model of Fig. 3.10: (1) the small unavoidable initial imperfections which are necessarily present in any member will not greatly influence deformations computed by assuming a perfectly straight system until the loads are near the buckling load of the perfect system, (2) near the buckling load the structure becomes soft, that is, small increases in load result in very large increases in deflection, (3) large deflections will always occur near the lowest buckling load, and therefore the higher modes have no real significance [that is, in Eq. (3.32) the only critical moment of interest is when $n = 1$], and (4) the load-deflection curve in the elastic region continues to rise, albeit almost imperceptibly for statically determinate beams and columns, and unloading can thus only be initiated by yielding which will invariably occur sooner or later.

The computation of the load-deflection curve for members with initial imperfections is tedious at best and is seldom justifiable in practice. Thus, even though we know that bifurcation-type buckling never really occurs, we use the critical load for an initially straight member in design as one real limit of strength in the elastic region. This is precisely what we did earlier for the beam of Fig. 3.8. Many laboratory tests and successful design practice have shown this to be a safe and reasonable approach.

In summary then, when $M_0$ is near $(M_0)_{cr}$ [Eq. (3.32)], relatively large values of $u$ and $\phi$ result, and we can consider $(M_0)_{cr}$ to be for all practical purposes the maximum moment which the beam can carry. Thus up to $M_0 = (M_0)_{cr}$, the deformations of the member are governed by Eq. (3.11), provided $(M_0)_{cr} \ll M_r$. At $M_0 = (M_0)_{cr}$, the beam will experience the start of lateral-torsional buckling, after which $M_0$ can increase only a negligible amount until unloading is triggered by yieldling after considerable deformation [see dashed curves in Fig. 3.8(b) and (c)]. Our mathematical analysis gives us only the value of the critical moment and the shape of the buckled member; no more. Information on the postbuckling behavior of the perfect member, or on the behavior of the real member can be obtained only at the cost of a great deal of extra work.

The deflection history of any cross section is shown in Fig. 3.11. Location 1 identifies the section when $M_0 = 0$, that is, in the unloaded state; location 2 shows the section when $M_0 = (M_0)_{cr}$, but buckling has not yet occurred; finally, location 3 gives the situation after buckling [$M_0$ still being equal to $(M_0)_{cr}$]. During buckling the section rotates about the point $C_{LB}$. This center of rotation is located a distance $y_{LB}$ below the shear center. Since $\tan \phi \equiv \phi = u/y_{LB}$, we can express this distance as

$$y_{LB} = \frac{u}{\phi} = \frac{M_0 L^3}{\pi^2 E I_y}$$

(3.39)

by substituting $\phi$ and $u$ from Eqs. (3.33) and (3.34). Setting $(M_0)_{cr}$ for $n = 1$

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**Fig. 3.11.** Positions of a cross section before and after buckling.
from Eq. (3.32) into Eq. (3.39), we find that
\[
y_{Lb} = \frac{L}{\pi^3 E I_b} \left[ \frac{L^3}{G K_T E I_b} \left( 1 + \frac{\pi^3 E I_b}{G K_T L^3} \right) \right]^{\frac{1}{2}}
\]  
(3.40)

After some more rearrangement we can also express Eq. (3.39) as
\[
y_{Lb} = \sqrt{\frac{I_b}{I_g}} \left( 1 + \frac{\pi^3 E I_b}{G K_T L^3} \right)
\]  
(3.41)

The second radical in Eq. (3.40) is nearly equal to unity for very long members, and the second radical is unity for very short members in Eq. (3.41). Thus \( y_{Lb} = (L/\pi)^3 G K_T / E I_b \) for long beams and \( y_{Lb} = \sqrt{I_b / I_g} \) for short beams. For wide-flange sections \( I_s = I_b (d - t)^3 / 4 \) (Table 2.4), and so the center of rotation is \( y_{Lb} = (d - t)/2 \), or at the center of the tension flange for short beams. Point \( C_{Lb} \) moves away from this point as the member becomes longer; \( y_{Lb} \) varies directly as the length for long beams.

Our discussion on the lateral-torsional buckling behavior of the beam in Fig. 3.8 applies equally well to any other type of a member, and we shall continue to make use of the concepts which were introduced here as we study the buckling strength of columns, beam-columns, and frames in later chapters.

**NUMERICAL EXAMPLE**

For the beam and loading of Fig. 3.8 the lowest critical moment is equal to [Eq. (3.32) with \( n = 1 \)]
\[
\left( \frac{M_c}{M_b} \right)_{cr} = \frac{\pi}{L} \sqrt{E I_b G K_T} \left[ 1 + \frac{\pi^3 E I_b}{G K_T L^3} \right]
\]  
(3.42)

The second term in the square root in this equation is nearly equal to unity for relatively long beams, and could be conservatively neglected; that is,
\[
\left( \frac{M_c}{M_b} \right)_{cr} > \frac{\pi}{L} \sqrt{E I_b G K_T}
\]  
(3.43)

We shall now compute the length \( L \) for which the critical moment is 0.6 \( M_b \) [arrow in Fig. 3.8(b)], for a 27WF94 ASTM A441 steel beam. The following properties are given: \( \sigma_y = 50 \text{ ksi, } E = 29,000 \text{ ksi, } G = 11,200 \text{ ksi, } S_2 = 242.8 \text{ in.}^4, I_y = 115.1 \text{ in.}^4, K_T = 4.39 \text{ in.}^4, \text{ and } I_x = 19,700 \text{ in.}^4. \) The yield moment \( M_y = S_2 \sigma_y = 242.8 \times 50 = 12,140 \text{ kip-in.}, \) and then \( \left( M_c / M_b \right)_{cr} = 0.6 M_y = 7,284 \text{ kip-in.} \). Substituting the known numerical values into Eq. (3.42) and solving for the unknown length, we get \( L = 276 \text{ in.} \) (23.0 ft). Thus a 27WF94 beam of a length of 23 ft would buckle laterally at \( M = 0.6 M_y. \)

We can manipulate Eq. (3.42) to obtain a formula which will permit investigation of the influence of the various parameters which govern the equation, using the wide-flange shape as an example. We first nondimensionalize the equation by \( M_y = S_2 \sigma_y, \) or
\[
\left( \frac{M}{M_y} \right)_{cr} = \frac{\pi^3 E I_b G K_T}{L^3 S_2 \sigma_y} \left[ 1 + \frac{\pi^3 E I_b}{G K_T L^3} \right]
\]  
(3.44)

After introducing the following identities:
\[
\epsilon_T = \frac{\sigma_y}{E}, \quad S_2 = 2I_y / d, \quad I_y = Ar^2, \quad I_x = Ar_x^2, \quad I_o = L(d - t)^3 / 4
\]  
(3.45)

and the abbreviation
\[
D_F = \frac{K_T}{Ad^4}
\]  
(3.46)

(where besides the already-defined symbols, \( r_s \) and \( r_n \) are the strong and weak axis radii of gyration, respectively, and the formula for \( I_x \) is from Table 2.4), we get the following expression for the nondimensional buckling moment:
\[
\left( \frac{M}{M_y} \right)_{cr} = \frac{\pi d r_s}{2 \epsilon_T} \left[ \sqrt{D_F G E / L} \right] \left[ 1 + \frac{\pi^3 E (1 - t / d)^3}{4GD_F L(r_s / d)^3} \right]
\]  
(3.47)

The following of these ratios are nearly constant for rolled steel-wide-flange shapes: \( d / r_s \approx 2.38 \) and \( (1 - t / d) \approx 0.95. \) These ratios vary less than 10 per cent from the average values given here. We can also note that the ratio \( G / E = 1 / 2(1 + v) \) (\( v \) is Poisson’s ratio, and it is equal to 0.3 for steel) is a constant equal to 0.385. If we substitute these values into Eq. (3.47), we obtain the following approximate formula for rolled-wide-flange shapes:
\[
\left( \frac{M}{M_y} \right)_{cr} = \frac{5.56 \epsilon_T D_F}{\epsilon_T (L/r_s)} \left[ 1 + \frac{5.78}{D_F (L/r_s)} \right]
\]  
(3.48)

The variables which control lateral-torsional buckling are the (1) slenderness ratio \( L/r_s, \) (2) the material coefficient \( \epsilon_T, \) and (3) the cross-sectional constant \( D_F. \) This latter coefficient varies from about \( 150 \times 10^{-4} \) for torsionally weak sections to about \( 7 \times 10^{-4} \) for very compact shapes. The majority of the shapes lie in the range between \( D_F = 150 \times 10^{-4} \) and \( D_F = 2,000 \times 10^{-4}. \)

The curves in Fig. 3.12 are a graphical representation of Eq. (3.48) for \( D_F \times 10^4 = 150, 1,000, \) and \( 2,000. \) From this figure we see that the critical moment is inversely proportional to the length and to the yield stress. Shapes usually used as beams fall into the range \( 150 < D_F \times 10^4 < 1,000. \) The dashed lines in this figure show the level above which the curves are not valid for three types of steel (ASTM A36, A441, and A514) because of yielding.

The curves in Fig. 3.12 illustrate how lateral-torsional buckling strength can be represented graphically. The shape of the curves—a rapid decrease in strength for short beams and a gradual decrease in strength for long beams—is typical of all buckling problems. These curves can be used to obtain a rapid estimate of the elastic critical moment of any wide-flange beam.

\(^7\) This dependence on \( D_F \) as the only cross-sectional parameter is also true in the inelastic range. \(^{3,8,89}\)
The first of these equations is identical with Eq. (3.18), and it can be transformed for simple boundary conditions to Eq. (3.21); similarly we can also express the second differential equation as before [Eq. (3.23)]

$$\phi^{iv} - \lambda_4 \phi^{iv} = 0$$

where \( \lambda_4 \) is identical with its value from Eq. (3.24), but

$$\lambda_4 = \frac{GK_x + M_0 \beta_x}{EI_x}$$  \hspace{1cm} (3.51)

From Eq. (3.23) the buckling condition for the boundary conditions

$$u(0) = u''(0) = \phi(0) = \phi''(0) = u(L) = u''(L) = \phi(L) = \phi''(L) = 0$$

(simple supports) becomes

$$\sin \alpha_s L = 0$$

where \( \alpha_s \) is again expressed by Eq. (3.27). The lowest critical moment is [from Eqs. (3.27) and (3.30)]

$$\alpha_s L = \pi$$  \hspace{1cm} (3.52)

and after some algebraic manipulation we obtain

$$(M_0)_{cr} = \frac{\pi^4 EI_x \beta_x}{2L^4} \left[ 1 \pm \sqrt{1 + \frac{4}{\beta_x^2} \left( \frac{GK_x L^3}{EI_x} - \frac{I_y}{I_x} \right)} \right]$$  \hspace{1cm} (3.53)

This equation reduces to Eq. (3.42) for \( \beta_x = 0 \). The two signs before the radical will give two critical moments, one positive and one negative. Thus the value of the critical moment depends on the direction of the end moment.

The value of \( \beta_x \) is computed by Eq. (3.13). Formulas for several common cross sections are given below.

$$\beta_x = 0$$  \hspace{1cm} (3.54)

for a channel

$$\beta_x = \frac{1}{L_x} \left[ (d' - \bar{y}) \left( \frac{b h_t}{12} + b t_s (d' - \bar{y})^2 \right) + \frac{w (d' - \bar{y})^3}{4} \right] - \frac{\left( \frac{b h_t}{12} + b t_s \bar{y}^2 + \frac{w \bar{y}^3}{4} \right)}{2\gamma_s}$$  \hspace{1cm} (3.55)

for a wide-flange shape of unequal flanges (Table 2.4)\(^{1,19}\)

$$\beta_x = \frac{1}{L_x} \left[ \frac{w}{4} (d - \bar{y}_w)^4 - (\bar{y}_w - t)^4 \right] - \frac{b \left( \bar{y}_w - t \right)}{\bar{y}_w}$$  \hspace{1cm} (3.56)

for a tee section (Fig. 3.13). For this section the shear center is at the intersection of the middle line of the flange and the stem, and so \( \gamma_s = -\bar{y}_w + t/2 \).

We shall illustrate the application of Eq. (3.53) by examining the lateral-torsional buckling strength of the three tee beams for which the properties are given in Table 3.1. The moment equation [Eq. (3.53)] can be transformed
into an equation for critical stress by dividing \((M_o)_{cr}\) by the section modulus \(S_x\):

\[
\sigma_{cr} = \frac{(M_o)_{cr}}{S_x} = \frac{A_1}{(L/r_y)^2}[1 \pm \sqrt{1 + A_2(L/r_y)^2}]
\]

where the coefficients \(A_1\) and \(A_2\) are functions of the material and cross-sectional properties, and they are given in Table 3.1 for the three sections. It should be noted that for tee shapes \(I_x = 0\), since \(\rho_y = 0\), and so \(\omega_y = 0\) everywhere on the section (see Sec. 2.5) and that the maximum stress will be at the lower end of the tee stem. Then resulting \(\sigma_{cr}\) versus \(L/r_y\) curves are shown in Fig. 3.14 for the three sections. The solid curves correspond to compression in the flange (radical in Eq. (3.57)), and the dashed curves are for tension in the flange (radical). For the positive radical \(\sigma_{cr} \to \infty\) as \(L/r_y \to 0\), but for the negative radical \(\sigma_{cr} = 0/0\) for \(L/r_y = 0\). However, with the application of L'Hospital's rule we get (two applications of the rule)

\[
\lim_{L/r_y \to 0} \sigma_{cr} = -\frac{A_1 A_2}{2}
\]

Table 3.1. Properties of Tee Beams

<table>
<thead>
<tr>
<th>Beam No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>ST 18WF150</td>
<td>ST 9WF25</td>
<td>T (A-N) 8 x 6 x 0.500</td>
</tr>
<tr>
<td>(E)</td>
<td>29,000 ksi</td>
<td>29,000 ksi</td>
<td>10,000 ksi</td>
</tr>
<tr>
<td>(G)</td>
<td>11,200 ksi</td>
<td>11,200 ksi</td>
<td>3,800 ksi</td>
</tr>
<tr>
<td>(\beta_x)</td>
<td>4.13 in.</td>
<td>2.14 in.</td>
<td>1.24 in.</td>
</tr>
<tr>
<td>(t)</td>
<td>1.68 in.</td>
<td>0.57 in.</td>
<td>0.86 in.</td>
</tr>
<tr>
<td>(d)</td>
<td>18.36 in.</td>
<td>9.00 in.</td>
<td>6.00 in.</td>
</tr>
<tr>
<td>(b)</td>
<td>16.66 in.</td>
<td>7.50 in.</td>
<td>8.00 in.</td>
</tr>
<tr>
<td>(w)</td>
<td>0.94 in.</td>
<td>0.36 in.</td>
<td>0.50 in.</td>
</tr>
<tr>
<td>(I_x)</td>
<td>1,227.7 in.(^4)</td>
<td>53.9 in.(^4)</td>
<td>22.93 in.(^4)</td>
</tr>
<tr>
<td>(K_x)</td>
<td>34.4 in.(^4)</td>
<td>0.67 in.(^4)</td>
<td>1.95 in.(^4)</td>
</tr>
<tr>
<td>(I_y)</td>
<td>612.6 in.(^4)</td>
<td>18.6 in.(^4)</td>
<td>36.76 in.(^4)</td>
</tr>
<tr>
<td>(\beta_y)</td>
<td>11.94 in.</td>
<td>6.19 in.</td>
<td>2.96 in.</td>
</tr>
<tr>
<td>(S_x)</td>
<td>85.9 in.(^3)</td>
<td>7.85 in.(^3)</td>
<td>4.82 in.(^3)</td>
</tr>
<tr>
<td>(r_y)</td>
<td>3.73 in.</td>
<td>1.59 in.</td>
<td>1.96 in.</td>
</tr>
<tr>
<td>(A_1)</td>
<td>87.58 \times 10^4</td>
<td>83.04 \times 10^4</td>
<td>29.00 \times 10^4</td>
</tr>
<tr>
<td>(A_2)</td>
<td>8.58 \times 10^{-4}</td>
<td>3.72 \times 10^{-4}</td>
<td>35.82 \times 10^{-4}</td>
</tr>
</tbody>
</table>

Fig. 3.14. Buckling curves for tee beams

Thus if the stem of the tee is in compression, a limiting critical stress is reached when the length approaches zero. This critical stress is equal to \(-376\) ksi, \(-154\) ksi, and \(-518\) ksi respectively for the 18-in. and the 9-in. steel tees and for the aluminum tee. The existence of such a limiting situation is somewhat different from the usual case in which the critical elastic stress approaches infinity for very short members. For very short tee beams in
which the stem is in compression it can be shown that twisting takes place about the shear center. This is the case of pure torsional buckling (see Chapter 4).\footnote{For a similar situation see also Sec. 42 in Ref. 1.30.}

**THE EFFECT OF THE BOUNDARY CONDITIONS**

We have now presented formulas for the lateral-torsional buckling strength of doubly symmetric [Eq. (3.42)] and asymmetric [Eq. (3.53)] simply supported beams. Simple boundaries are one extreme support condition; the other extreme condition is when the ends are laterally and torsionally fixed. In this case \( u = u' = \phi = \phi' = 0 \) at the boundary. The real situation is usually somewhere between these extremes.

In order to illustrate the effect of the boundary conditions, we shall now examine a beam of length \( L \) under uniform moment [Fig. 3.8(a)] with simple boundary conditions with respect to \( u \) and fixed ends with respect to \( \phi \). That is,

\[
\begin{align*}
  u(0) &= u'(0) = u(L) = u''(L) = \psi(0) = \psi'(0) = \psi(L) = \psi'(L) = 0
\end{align*}
\]

We can proceed as we did for the simply supported beam, that is, starting from Eqs. (3.18) and (3.19) and arriving at the solution of the differential equation [Eq. (3.31.7)] as

\[
\phi = C_1 \cosh \alpha_1 z + C_2 \sinh \alpha_1 z + C_3 \sin \alpha_2 z + C_4 \cos \alpha_2 z
\]

Substitution of the four boundary conditions for \( \phi \) results in the following buckling condition after evaluation of the resulting characteristic determinant:

\[
\cosh \alpha_1 L \cos \alpha_2 L - 1 + \left( \frac{\alpha_1^2 - \alpha_2^2}{2 \alpha_1 \alpha_2} \right) \sin \alpha_1 L \sin \alpha_2 L = 0
\]

(3.60)

In this equation \( \alpha_1 \) and \( \alpha_2 \) are defined by Eqs. (3.26) and (3.27), respectively. Instead of obtaining a simple relationship such as Eq. (3.30), we now have a rather complex transcendental equation from which it is more difficult to obtain a solution. After a calculation by trial and error we find that the critical moment satisfying Eq. (3.60) is equal to 0.8787\( M_r \) for a 27WF94 steel beam (\( E = 30,000 \) ksi; \( G = 11,500 \) ksi; \( \sigma_s = 33 \) ksi) of length \( L = 200r_y \). The corresponding critical moment for simply supported ends is 0.522\( M_r \). Thus the torsionally fixed end increases the critical moment considerably.

**APPROXIMATE SOLUTIONS**

Equations of the degrees of complexity of Eq. (3.60) are not uncommon for lateral-torsional buckling problems. In fact, most problems are not amenable to closed-form solutions at all. We thus have to resort to other ways of obtaining a solution. Such approximate and numerical procedures can be

roughly classified into the following three categories: (1) numerical integration, (2) finite difference methods, and (3) energy methods. Of these, the first two become really efficient only if electronic computing devices are used; the last is adapted to hand calculation.

The **numerical integration process** as used by American engineers was developed by Newmark,\(^{(3.11)}\) who adapted the Stodola-Vianello graphical procedure into an efficient numerical tool. The procedure has been applied by Austin, Yegian, and Tung\(^{(3.12)}\) for the solution of the lateral-torsional buckling strength of beams subjected to either a uniformly distributed load or a central concentrated load for various cases of end restraint. Both the lateral and the torsional boundary conditions were varied from the fully fixed to the fully unrestrained (simple) condition. Numerous charts and tables are provided in this work to permit easy computation of the buckling loads for these loading conditions.

The **finite difference procedure** consists in transforming the differential equations [such as Eqs. (3.15) and (3.16), for example] into difference equations. Some of the many authors treating the finite difference procedure are Salvatori and Baron,\(^{(3.13)}\) and Crandall.\(^{(3.14)}\) The second and the fourth derivatives of the twisting angle \( \phi \) in difference terminology for any point \( i \) on a beam subdivided into equal spaces \( gL \) (Fig. 3.15) are equal to

\[
\phi''_i = \frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{(gL)^2}
\]

(3.61)

\[
\phi^{iv}_i = \frac{\phi_{i-1} - 4\phi_i + 6\phi_{i-1} + \phi_{i+2}}{(gL)^4}
\]

(3.62)

We can thus write the differential equation

\[
\phi^{iv}_i - \lambda_\alpha \phi''_i - \lambda_\phi \phi = 0
\]

(3.63)

at the point \( i \) as

\[
\left( \frac{1}{gL} \right)^4 (\phi_{i-2} - 4\phi_{i-1} + 6\phi_i - 4\phi_{i+1} + \phi_{i+2}) - \frac{\lambda_\alpha}{(gL)^2} (\phi_{i-1} - 2\phi_i + \phi_{i+1}) - \lambda_\phi \phi_i = 0
\]

This equation is a linear and homogeneous equation in terms of the unknown deformation coefficients \( \phi_i \). With the aid of the boundary conditions for a simple support

\[
\phi_0 = \phi_n = 0, \quad \phi_{-1} = \phi_1, \quad \text{and} \quad \phi_{n+1} = -\phi_{n-1}
\]

(3.64)

![Fig. 3.15. Notation for finite difference equation](image-url)
we can in this case write \( n - 1 \) homogeneous equations for \( n - 1 \) unknown angles \( \phi \). The vanishing of the determinant of the coefficients of these unknowns gives us the desired buckling condition.

Setting up the finite difference equations is a simple matter, but the solution of the resulting determinant is performed efficiently only by a digital computer.\(^{[3.6,3.14]}\)

Energy methods have been used very successfully in applications ranging over a wide variety of engineering problems. They have been extensively used for the solution of elastic buckling problems, and their application is documented in many references, notably in Refs. 1.21, 1.22, 1.29, and 1.30. Before the high-speed large-capacity electronic computing device came into use, the energy methods held out the only hope for handling the increasingly complex problems posed by modern construction. As a result many methods were developed.\(^{[3.10]}\) We shall here briefly discuss only the method known as the Rayleigh-Ritz method.\(^{[1.21]}\)

This method is based on the principle that the total potential of a structural system must be a minimum if the system is to be in static equilibrium.\(^{[1.21]}\) The total potential \( \Pi \) for an elastic structure is composed of the strain energy \( U \) and the potential of the applied loads \( V_p \); that is,

\[
\Pi = U + V_p
\]

(3.65)

The equilibrium condition is expressed mathematically as\(^{[1.21]}\)

\[
\delta \Pi = 0
\]

(3.66)

This equation states that for equilibrium the first variation of the total potential must vanish. The total potential of an elastic system can always be expressed as a quadratic function of the deformations (assuming that the deflections are small), and we can use Eq. (3.66) in conjunction with the tools provided by the calculus of variations to develop the differential equations.\(^{[1.21,1.29]}\)

This, however, is not exactly our problem right now. Our problem is either having differential equations we cannot solve directly, or having solutions which result in expressions which are too unwieldy [see, for example, Eq. (3.60)]. Fortunately Eq. (3.66) is not very sensitive to variations of the deflected shape, and so we can expect reasonable results if we use an approximation of the deflected shape of the member.

THE RAYLEIGH-RITZ METHOD

In order to illustrate the energy method we shall now perform an approximate calculation according to the Rayleigh-Ritz method for the problem of the torsionally fixed ended beam we solved previously by the analytically exact method.

The total potential of a general thin-walled cross section is derived in Appendix 3A, and it is equal to

\[
\Pi = \frac{1}{2} \int_0^L \left[ EI_y (\psi')^2 + EI_z (\psi'')^2 + G K_y (\phi')^2 + 2 M_{ax} \phi' - 2 M_{ay} \phi' + M_{ax} \theta (\phi') \right] dz
\]

(3.67)

For our problem (see Fig. 3.16) \( x_0 = y_0 = \phi = 0 \) because of double symmetry of the cross section, and thus

\[
\Pi = \frac{1}{2} \int_0^L \left[ EI_y (\psi')^2 + EI_z (\psi'')^2 + G K_y (\phi')^2 + 2 M_{ax} \phi' - 2 M_{ay} \phi' \right] dz
\]

(3.68)

We are interested in finding that moment \( (M_x)_{cr} \) at which it will be possible to have both an unbuckled (deflection \( \psi \), only) and a buckled shape (deformations \( u, v, \) and \( \phi \)). The total potential of the buckled shape is given by Eq. (3.68); that of the unbuckled shape is \( \frac{1}{2} \int_0^L [EI_y (\psi')^2 + 2M_{ax} \phi'] \) since \( u = \phi = 0 \). The potential of buckling is the difference of the two, or

\[
\Pi_{cr} = \frac{1}{2} \int_0^L \left[ EI_y (u''')^2 + EI_z (\psi'')^2 + G K_y (\phi')^2 - 2 M_{ax} \phi' \right] dz
\]

(3.69)

We have stated before that the total potential is not very sensitive to the shape of the deformation. We can obtain an approximate value of it by assuming the following buckled shape:

\[
u = A \sin \frac{\pi z}{L}, \quad \phi = B \left(1 - \cos \frac{2\pi z}{L}\right)
\]

(3.70)

It can be easily shown that these deformed shapes fulfill the boundary conditions given in Fig. 3.16. If we substitute the deformations \( \psi \) and \( \phi \) into Eq. (3.69) and perform the integrations, we obtain after some algebraic manipulations the following formula for the approximate value of the total potential:

\[
\Pi = \frac{\pi^4 EI_y}{4L^4} A^2 - \frac{8 M_{ax} \pi^2 AB}{3L} + \left(\frac{4\pi^4 EI_y}{L^4} + \frac{G K_y \pi^2}{L}\right) B^2
\]

(3.71)

At the instant of buckling both the deformed shape and the unbuckled shape must be in equilibrium under the same loading. Accordingly, the total

\[
M_0
\]

BEAMS

SEC. 3.2

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Boundary conditions:

\[
v = v'' = u = u'' = \phi = \phi' = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = L
\]

Fig. 3.16. Beam example
potential of buckling must be a minimum, or in other words, the first variation of it must vanish [Eq. (3.66)]. There are various ways in which the expression $\delta \Pi = 0$ can be evaluated.\(^{[1,12]}\) We shall use the following approach: Eq. (3.71) represents a surface in a three-dimensional rectangular coordinate system with axes $\Pi$, $A$, and $B$. The condition of minimum potential is then that

$$\frac{\delta \Pi}{\delta A} = 0 \quad \text{and} \quad \frac{\delta \Pi}{\delta B} = 0$$

(3.72)

Upon performing the differentiations indicated by Eq. (3.72) on Eq. (3.71), we obtain the following two simultaneous equations:

$$\frac{E I}{L} \pi^3 A - \frac{16 M_0}{3} B = 0$$

(3.73)

$$-\frac{16 M_0}{3} A + \left( \frac{16 E I}{L^3} \pi^3 + 4 G K \pi \right) B = 0$$

(3.74)

The vanishing of the determinant of the coefficients of $A$ and $B$ gives the buckling condition, which, when solved for the value of $M_0$, gives

$$(M_0)^e = \frac{\pi}{0.85 L} \sqrt{\frac{E I G K}{1 + \frac{\pi^2 E I}{G K (0.85 L)^3}}}$$

(3.75)

Substitution of the numerical data for a 27WF94 beam gives us $(M_0)^e = 0.905 M_f$, which is in error by 3.3 per cent when compared with the answer from Eq. (3.60) $(0.878 M_f)$. We see that the relatively simple calculations with the Rayleigh-Ritz method gave good results. Better correlation yet could have been obtained by using

$$u = \sum_{i=1}^{n} a_i \sin \frac{i \pi x}{L} \quad \text{and} \quad \phi = \sum_{i=1}^{n} b_i \left( 1 - \cos \frac{2 i \pi x}{L} \right)$$

(3.76)

and setting $n = 2, 3, \text{or even more.}$

* In a more formal way we define

$$\delta \Pi = \Pi (A + \delta A, B + \delta B) - \Pi (A, B) = 0$$

where $\delta A$ and $\delta B$ are virtual deformations which are arbitrary except that they obey the boundary conditions. If we expand the above expression by McLaurin's expansion, we get

$$\delta \Pi = \left( \frac{\partial \Pi}{\partial A} \right) \delta A + \left( \frac{\partial \Pi}{\partial B} \right) \delta B + \text{second-order terms}$$

The second-order terms are identical with zero for the first variation.\(^{[1,12]}\) so

$$\delta \Pi = \left( \frac{\partial \Pi}{\partial A} \right) \delta A + \left( \frac{\partial \Pi}{\partial B} \right) \delta B = 0$$

Since this expression must hold for any disturbance $\delta A$ and $\delta B$, and since $\delta A$ and $\delta B$ are not zero,

$$\frac{\partial \Pi}{\partial A} = \frac{\partial \Pi}{\partial B} = 0$$

SEC. 3.2

INFLUENCE OF BOUNDARY CONDITIONS
AND NONUNIFORM BENDING

The energy methods have been used to develop many charts and tables to aid in the rapid computation of elastic buckling loads. As a result of this work it is possible to give some general formulae in which the effects of other than simple boundary conditions and other than equal end moments (Fig. 3.8) can be accounted for. Following is a formula for including different types of loading.\(^{[1,16]}\)

$$(M_e)^e = \frac{C_1 \pi}{L} \sqrt{\frac{E I G K}{1 + \frac{\pi^2 E I}{G K (0.5 L)^3}}}$$

(3.77)

where $C_1$ is a correction factor which accounts for the type of loading. Salvadori has used the Rayleigh-Ritz method\(^{[9,17]}\) to investigate the lateral-torsional buckling strength of wide-flange beam-columns under end moments $M_0$ and $k M_f$ (Fig. 3.17). As a result of this work it was concluded that the coefficient $C_1$ is not greatly influenced by cross-sectional parameters or the lateral-torsional end restraint conditions.\(^{[5,18]}\) The following relationship gives a safe but good approximation to $C_1$ for wide-flange beams under nonuniform moment.\(^{[5,10]}\)

$$C_1 = 1.75 - 0.05 \kappa + 0.3 \kappa^2 \leq 2.3$$

(3.78)

Values of $C_1$ are tabulated for a variety of other loading and support conditions in the guide of the Column Research Council\(^{[5,10]}\) and by Clark and Hill.\(^{[5,10]}\)

The effect of other than simple lateral-torsional boundary conditions can also be investigated by the Rayleigh-Ritz method as we showed above. Results of extensive calculations have been presented in Chapter V of Vlasov's book.\(^{[1,18]}\) The critical moment equation can be written as

$$(M_e)^e = \frac{\pi}{K_e L} \sqrt{\frac{E I G K}{1 + \frac{\pi^2 E I}{G K (K_e L)^3}}}$$

(3.79)

where the coefficients $K_e$ and $K_r$ are effective length factors which account for the boundary conditions of $u$ and $\phi$, respectively. Values for $K_e$ and $K_r$, as computed from the results given by Vlasov, are listed in Table 3.2 for a number of boundary conditions. A safe and simple rule to follow is to equate $K_e$ and/or $K_r$ to (1) 1.00 if both ends are simple, (2) 0.70 if one end is simple and the other is fixed, and (3) 0.50 if both ends are fixed. This procedure permits the calculation of the critical load if the boundary conditions are
mixed. If, for example, the lateral deflection $u$ is simply supported on both ends but warping is restrained by heavy coverplates, then use $K_y = 1.0$ and $K_z = 0.5$. If more precision is desired, we should use the values of $K_y$ and $K_z$ from Table 3.2. Tables and charts presented in Ref. 3.12 could also be used, as we mentioned before. For the 27WF94 beam with simple boundary conditions in $u$ and torsionally fixed ends (Fig. 3.16) we obtained $(M_0)_{cr} = 0.878 M_F$ from the solution of Eq. (3.60). The critical moment from Eq. (3.79) is equal to 0.890$M_F$, with $K_y = 0.883$ and $K_z = 0.492$ from Table 3.2; for $K_y = 1.0$ and $K_z = 0.5$, $(M_0)_{cr} = 0.78 M_F$, which is on the safe side.

**SUMMARY OF ELASTIC LATERAL-TORSIONAL BUCKLING OF BEAMS**

In the preceding sections we have presented an introduction to the problem on the lateral-torsional buckling of beams intentionally loaded to produce deformations in a plane parallel to one of the principal axes of the cross section. We have seen that an ideal critical condition exists, in which the beam, which has deformed only in the $y$ direction, suddenly can be in equilibrium in a deformation mode which involves lateral deflections $u$ and twist $\phi$. We have seen that this critical load represents a real and important limit of structural usefulness.

The moment causing the initiation of lateral-torsional buckling of a beam segment loaded by unequal end moments $M_y$ and $M_z$ (where we can always choose the moments in such a way that $-1.00 \leq \kappa \leq +1.00$) can be expressed by the following formula:\(^{(3.19)}\)

\[
(M_0)_{cr} = \frac{C_1 \pi^2 E I_y}{2 (K_y L)^2} \left\{ 1 + \sqrt{1 + \frac{4}{K_z^2} \left( \frac{I_y K_z^2}{I_y K_z^2} + \frac{G K_z (K_y L)^2}{\pi^2 E I_y} \right)} \right\} \tag{3.80}
\]

In this equation $C_1$ accounts for the variation of the moment [Eq. (3.78)], and $K_y$ and $K_z$ take care of the lateral-torsional boundary conditions. For $C_1 = 1.0$ and $\beta_y = 0$, Eq. (3.80) reduces to Eq. (3.79). The unusual appearance of $K_y$ and $K_z$ under the radical is obtained from Vlasov's E. V.13.4.\(^{(1.18)}\)

Equation (3.80) suffices for most cases likely to occur in design practice. Additional terms to account for loads not applied at the shear center, as well as charts and tables to facilitate the solution of Eq. (3.80), are given in the CRC guide\(^{(3,15)}\) and in Ref. 3.19. For cases not covered by this formula, reference should be made to the books by Bleich\(^{(1.30)}\) and Timoshenko and Gere\(^{(1.29)}\) and to the various works we have referred to previously. An extensive literature survey of lateral-torsional buckling solutions has been given by Lee\(^{(3,16)}\). Besides the usual listing of the references, this report gives a table which enables the quick location of the type of problem for which the solution is sought. Lee's literature survey is up to date through 1959. An extension of the problem to the lateral instability of simple frames is discussed in Ref. 3.21.

We shall illustrate the way in which Eq. (3.80) is applied on the problem

![Diagram of a channel beam](image)

Fig. 3.18. Example problem on a channel beam

Material: 2014-T6 aluminum alloy

<table>
<thead>
<tr>
<th>$L$</th>
<th>120 in.</th>
<th>$E = 10,000$ ksi</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_y$</td>
<td>9.63 in$^4$</td>
<td>$G = 3800$ ksi</td>
</tr>
<tr>
<td>$I_x$</td>
<td>36.15 in$^4$</td>
<td>$\sigma_y = 55$ ksi</td>
</tr>
<tr>
<td>$K_y$</td>
<td>1.17 in$^2$</td>
<td>$S_x = 41.97$ in$^3$</td>
</tr>
</tbody>
</table>

Factor of safety against yield: $\frac{S_x}{S_y} = 2.41$
given in Fig. (3.18). For a simply supported aluminum channel it is desired to find the load \( Q \) which can be supported while maintaining a factor of safety of 2.41. For the segment \( L \) of this member the moment ratio \( k = 0 \), and so from Eq. (3.78) we get \( C_1 = 1.75 \). Lateral support is provided under the load, and since both spans are identical, they will buckle at the same time; thus we can assume simple boundary conditions at the center. At the ends the lateral deflection \( u \) is fixed, but the end section is free to warp. For these conditions \( K_y = 0.626 \) and \( K_y = 1.0 \) from Table 3.2. Substituting now into Eq. (3.80) we get \( (M_k)_{cr} = 1,900 \) kip-in., as the buckling moment. Since \( M_o = QL/2 \), the load causing lateral-torsional buckling is \( Q_{cr} = 31.7 \) kips. The allowable load is then \( 31.7/2.41 = 13.2 \) kips.

### Biaxial Bending

In the previous portions of this chapter we assumed that intentional bending was applied only in one plane, that is, we set \( M_{xy} \) and \( M_{yz} \) equal to zero in Eqs. (2.93) through (2.95). We shall now consider the case in which bending occurs about both axes. We shall assume that equal end moments cause single curvature deflection (Fig. 2.30), that is, \( M_{xy} = -M_{yz} = -M_{ox} \) and \( M_{yz} = -M_{yx} = M_{oy} \). The three differential equations of bending and torsion now become

\[
E I_o \psi'' + M_{oy} \phi = -M_{oz}
\]

(3.81)

\[
E I_i \phi'' + M_{ox} \phi = M_{oy}
\]

(3.82)

\[
E I_o \phi''' - (GK_x + \bar{K}) \phi' + M_{ox} \phi' + M_{oy} \psi' = 0
\]

(3.83)

We can see that these equations are not independent of each other and that they are not homogeneous. We no longer have a buckling or an eigenvalue problem; we shall now be able to find a unique deflected shape.

Next we must determine \( \phi' \bar{K} \) in Eq. (3.83). The stress at any location in a cross section is [from Eq. (2.74)]

\[
\sigma = \frac{M_y}{I_y} \phi + \frac{M_x}{I_x} \psi + E \bar{\alpha}_y \phi''
\]

where

\[
M_y = M_x + \phi M_y \quad \text{and} \quad M_x = M_y - \phi M_x
\]

Thus

\[
\phi' \bar{K} = \int_A \left( \phi \frac{M_y}{I_y} + \phi' \frac{M_x}{I_x} + \phi' \frac{M_x}{I_y} + E \bar{\alpha}_y \phi'' \right) dA
\]

Since we are dealing with small deflections, we can neglect the second-order nonlinear terms \( \phi \phi' \phi' \phi'' \), and therefore

\[
\bar{K} = \frac{M_x}{I_x} \int_A yz \phi'' dA - M_y \int_A \phi'' dA
\]

(3.84)

The term \( \phi'' \) is defined by Eq. (3.12); after performing the integrations we get

\[
\bar{K} = M_x \beta_x - M_y \beta_y
\]

(3.85)

where \( \beta_x \) is defined by Eq. (3.13) and

\[
\beta_y = \frac{1}{I_x} \int_A x(x^2 + y^2) dA - 2 \lambda_x
\]

(3.86)

The third differential equation [Eq. (3.83)] can therefore be expressed as

\[
E I_o \phi''' - (GK_x + \bar{K}) \phi' + M_{ox} \phi' + M_{oy} \psi' = 0
\]

(3.87)

For a doubly symmetric cross section \( x_o = y_o = \beta_x = \beta_y = 0 \) this equation becomes equal to

\[
E I_o \phi''' - GK_x \phi' + M_{ox} \phi' + M_{oy} \psi' = 0
\]

(3.88)

If we differentiate Eq. (3.88) once and substitute \( \phi'' \) from Eqs. (3.81) and (3.82) respectively, then

\[
\phi'' = \lambda_x \phi - \lambda_y \phi
\]

(3.89)

where

\[
\lambda_x = \frac{GK_x}{E I_o}
\]

(3.90)

\[
\lambda_y = \frac{M_y}{E I_o} \left( 1 + \frac{I_x M_y^2}{I_y M_x^2} \right)
\]

(3.91)

\[
\lambda_o = \frac{M_{ox} M_{oy} (1 - 1/I_o I_y)}{E I_o}
\]

(3.92)

The solution to Eq. (3.92) is

\[
\phi = C_1 \sinh \alpha \gamma + C_2 \cos \alpha \gamma + C_3 \sin \alpha \gamma + C_4 \cos \alpha \gamma - \frac{\lambda_x}{\lambda_y}
\]

(3.93)

where \( \alpha_x \) and \( \alpha_y \) are given by Eqs. (3.26) and (3.27). From the boundary conditions we can next determine the coefficients \( C_1, C_2, C_3, \) and \( C_4 \). For simple boundaries \( \phi(0) = \phi''(0) = \phi(L) = \phi''(L) = 0 \), for example,

\[
C_1 = C_4 \left( \frac{\alpha_x}{\alpha_y} \right)^2 \left( 1 - \frac{\cosh \alpha_x L}{\sinh \alpha_x L} \right)
\]

(3.94)

\[
C_2 = C_4 \left( \frac{\alpha_x}{\alpha_y} \right)^2 \left( \frac{\sinh \alpha_x L}{\sin \alpha_y L} \right)
\]

(3.95)

\[
C_3 = C_4 \frac{1 - \cos \alpha_x L}{\sin \alpha_y L}
\]

(3.96)

\[
C_4 = \frac{\lambda_y}{\lambda_x (1 + \alpha_x/\alpha_y)}
\]

(3.97)

We shall next work out a numerical example to illustrate the effect of
secondary stresses introduced by the twisting of the member. The following data are given:

Section 27WF94

\[ E = 29,000 \text{ ksi}; \quad G = 11,200 \text{ ksi}; \quad M_{sx} = 5,905 \text{ kip-in.;} \]
\[ M_{sy} = -0.1 M_{sx}; \quad L = 200 \text{ in.} \]

Cross-sectional properties

\[ S_x = 242.8 \text{ in.}^3; \quad S_y = 23.0 \text{ in.}^3; \quad K_y = 4.35 \text{ in.}^3; \]
\[ I_x = 19,700 \text{ in.}^4; \quad I_y = 115.1 \text{ in.}^4; \quad I_z = 3,266.7 \text{ in.}^4; \]
\[ (\omega_{\phi})_{\text{max}} = 65.3 \text{ in.}^2 \]

If we perform a first-order analysis, the maximum stress is

\[ \sigma_{\text{max}} = \frac{M_{sx}}{S_x} - \frac{M_{sy}}{S_y} = \frac{5905}{242.8} - \frac{590.5}{23.0} = 50.0 \text{ ksi} \]

The second-order analysis will also involve the torsional stresses and the changes in the moments due to the twist \( \phi \)

\[ \sigma_{\text{max}} = \frac{M_{t}}{S_y} - \frac{M_{t}}{S_y} + E\alpha_{\phi} \phi'' = \frac{M_{sx} - M_{sy}}{S_x} + \phi M_{sy} - \phi M_{sx} + E\alpha_{\phi} \phi'' \]

From Eqs. (3.90) through (3.92) we compute

\[ \lambda_1 = 0.8528 \times 10^{-4}; \quad \lambda_2 = 1.8287 \times 10^{-4}; \quad \lambda_3 = 0.1764 \times 10^{-5} \]

and therefore, from Eqs. (3.26) and (3.27)

\[ \alpha_1 = 1.358 \times 10^{-3} \quad \text{and} \quad \alpha_3 = 0.996 \times 10^{-2} \]

The angle of twist is equal to [Eqs. (3.93) through (3.97)]

\[ \phi = -0.02955 \sin \alpha_x z + 0.03374 \cosh \alpha_x z + 0.09681 \sin \alpha_x z + 0.06272 \cos \alpha_x z - 0.09646 \]

The maximum values of \( \phi \) and \( \phi'' \) occur at the center of the beam \( (x = L/2) \) and are equal to

\[ \phi_{\text{max}} = 0.0352 \text{ radians}, \quad \phi''_{\text{max}} = -0.0853 \times 10^{-4} \text{ in.}^{-2} \]

With these values \( M_t = 5,905 - 0.0352(590.5) = 5,884 \text{ kip-in.;} \quad M_x = -590.5 - 0.0352(5905) = -798 \text{ kip-in.;} \quad \) and \( E(\omega_{\phi})_{\text{max}} \phi'' = 16.2 \text{ ksi.} \) The maximum stress is therefore

\[ \sigma_{\text{max}} = \frac{5,884}{242.8} + \frac{798}{23.0} + 16.2 = 75.1 \text{ ksi} \]

Thus by counting in the second-order effects we obtained a 50 per cent increase of stress. This increase can be cause for alarm if it is necessary that no yielding be present. However, the maximum stress occurs at only two of the four flange tips, and a slight amount of local yielding is not very detrimental to the overall performance of the beam.

The calculations for the determination of the stresses due to biaxial bending are lengthy and laborious. Peterson has solved a great many cases of loading for asymmetric wide-flange shapes and has presented the results in numerous charts. The Rayleigh-Ritz method can also aid in obtaining simpler solutions, and formulas and several examples are given by Dabrowski. Lee and Prawle have shown how such problems can be solved by an electric analog computer.

### LOCAL BUCKLING

In addition to the lateral-torsional buckling of the whole beam, in-plane behavior can also be modified by elastic buckling of the plate elements of which the member is composed. Local buckling is characterized by the development of wavelike deformations in the plate elements, and it will cause a subsequent deviation from the in-plane response, which is predicted by considering the whole undamaged cross section. In some cases local buckling may be the reason for immediate failure, that is, no increase of load beyond which causes its initiation is possible; or, more frequently, it results in a redistribution of the stresses in the plate, and thus it changes the character of the response of the member to further loading.

Problems related to elastic local buckling have been widely investigated and play an important role in the analysis and design of all types of metal structures. We shall not discuss in detail these problems here because of our emphasis on overall behavior and because of the very thorough coverage of the topic in the literature. The fundamental aspects of the problem are treated, among others, by Timoshenko (Chap. 9, Ref. 1.29) and Bleich (Chap. IX, Ref. 1.30). Applications to a variety of structural problems and further references are contained in the CRC guide (Ref. 3.10) and in Chap. 17 of Ref. 1.33.

The simplest case of local buckling is that of a rectangular plate of constant thickness loaded by a uniform stress \( \sigma_x \) along two edges (see Fig. 3.19). This plate will remain flat until \( \sigma_x \) becomes equal to a critical stress \( (\sigma_x)_{cr} \), provided \( \sigma_x \) is truly axial and the plate is initially perfectly flat. When the critical stress is reached, the plate will be able to deform out of its plane, that is, it will buckle. The mathematical problem is identical with that of the lateral buckling phenomenon, except that the differential equation is now a partial differential equation, and so the details of the solution are different and more complicated. These details will not be given here (see Refs. 1.29 and 1.30); the final critical stress can be expressed by the following formula:

\[ (\sigma_x)_{cr} = \frac{\pi^2 KE}{12(1 - \nu^2)} \left( \frac{t}{b} \right)^2 \]  

(3.98)

10 The total potential for biaxial bending is

\[ W = \frac{1}{2} \int \left[ EI_x (\phi'')^2 + EI_y (\phi'')^2 + EI_z (\phi'')^2 + G K (\phi')^2 + 2M_{x,\phi} \phi'' + 2M_{y,\phi} \phi'' + 2M_{z,\phi} \phi'' \right] \]
In this equation $E$ is the elastic modulus, $\nu$ is Poisson's ratio, $t$ and $b$ are the thickness and the width of the plate, respectively (see Fig. 3.19), and $K$ is a numerical factor which depends on the state of restraint at the unloaded edges. For example, $K = 4.00$ when both edges are fixed. When one edge is unsupported and the other is fixed or simply supported, $K = 1.277$ and $K = 0.425$, respectively. A half-flange plate of a wide-flange section, for example, is restrained by the web, and therefore one edge of it is somewhere between the fixed and the pinned condition. The recommended value of $K = 0.70$ is used in various specifications for this case (see Chap. 17 of Ref. 1.33). The corresponding value for the web of a beam in bending is $K = 23.9$. Many other cases of loading and restraint have been investigated and are available in the literature.

It is again emphasized that local buckling problems are very important and deserve a great deal of study, despite the short introductory treatment given here. We shall return to considerations of local buckling in connection with inelastic behavior in later parts of this chapter.

### 3.3. INELASTIC BEAMS

#### IMPORTANCE OF THE INELASTIC RANGE

The theory discussed in Sec. 3.2 is valid only as long as the material in the beams remains elastic. In metal beams, particularly steel beams, residual stresses are present owing to various manufacturing processes (Chapter 1). The stresses due to the loading are superimposed on the residual stresses, and so the start of yielding is accelerated or retarded, depending on the sign of the various stresses acting at the cross section. If, for example, we have a rolled wide-flange steel beam with a stress-strain curve of Fig. 1.5(a) and a residual stress distribution of Fig. 1.6, yielding due to bending will start at the two tips of the compression flange when $\sigma_y = M_y/S_y + \sigma_{re}$, or

$$\left(\frac{M_y}{M_y}^\text{yield}\right) = 1 - \frac{\sigma_{re}}{\sigma_y}$$

(3.99)

where $M_y$ is the yield moment defined by Eq. (3.9). For rolled shapes of A36 steel the value of $\sigma_{re}$ is approximately $0.3\sigma_y$, and thus yielding will commence when $M_y = 0.7M_y$. The elastic lateral-torsional buckling curves of Fig. 3.12, for example, are thus invalid for $1,000\sigma_y(M_y/M_y) > 0.87$. This means that even for the weakest beams ($D_y = 150 \times 10^{-6}$) our previous theory cannot predict what will happen for spans of less than about $140\sigma_y$. Most practical beams are braced on shorter spans, and thus their behavior must be determined by a theory which accounts for yielding.

We have essentially two types of problems to consider for inelastic beams, depending on the philosophy by which the member is designed: (1) In allowable stress design the beam is most efficiently utilized if the maximum moment times the load factor is near the yield moment $M_y$. In this method we consider the member to be elastic up to $M_y = S_y\sigma_y$; this assumption leads to a good agreement with reality as far as in-plane behavior is concerned. However, residual stresses can reduce the lateral-torsional buckling strength drastically, and so the assumed elastic member must be braced according to a theory which accounts for yielding. (2) In plastic design the maximum moments are equal to $M_y$ (Fig. 3.6), and certain portions of the beam must be able to maintain $M_y$ through considerable inelastic rotations. Thus we require, not only a capacity to carry moment, but also a capacity to deform without unloading. The tendency for local and lateral-torsional buckling is enhanced in the highly strained regions near plastic hinges, and we must therefore know how buckling can be effectively prevented from interfering with the optimum performance of the member.

The prediction of inelastic response is much more complicated than the prediction of elastic behavior. Because of the practical importance of the problem a great deal of theoretical and experimental research has been devoted to it, and as a consequence we can predict the behavior of beams with as much confidence in the inelastic range as we can in the elastic range. In the theoretical analysis of inelastic beams we can fortunately retain many of the assumptions used in elastic theory, especially the assumption of small deflections; we can also neglect the initial crookedness of the members.

#### IN-PLANE BEHAVIOR

We shall start our study of inelastic beams with their in-plane performance. In this case the member is loaded in a plane parallel to a principal axis, and deflection occurs in a plane parallel to the plane of loading.

The basic concepts of inelastic in-plane behavior will be explained by
considering a rectangular section [Fig. 3.20(a)] made of a material having an elastic, a plastic, and a strain-hardening portion in its stress-strain curve [Fig. 3.20(b)]. If a cross section of this beam is subjected to a bending moment $M$, the possible stress distribution is either fully elastic [Fig. 3.21(a)], partially elastic and partially plastic [Fig. 3.21(b)], or elastic, plastic, and strain-hardened [Fig. 3.21(c)]. In the following derivations we shall assume that (1) plane sections before bending remain plane after bending and (2) the deflections are small. These assumptions still lead to excellent agreement between theory and experiment even in the inelastic range (Refs 1.6 and 3.2).

The applied bending moment must be resisted by the internal moment, and so $M = \int_A \sigma y \, dA$. In the elastic region (Fig. 3.21)

$$M = \sigma \frac{bd^3}{6}$$  \hspace{1cm} (3.100)

From the stress diagram [Fig. 3.21(a)] we note that

$$\tan \Phi E \approx \frac{\sigma}{\Phi E} = \frac{\sigma}{d/2}$$  \hspace{1cm} (3.101)

where $\Phi$ is the curvature of the member. Setting $\sigma$ from Eq. (3.101) into Eq. (3.100), we find that

$$M = \frac{bd^3 E}{12}$$  \hspace{1cm} (3.102)

Fig. 3.20. A rectangular cross section and the stress-strain diagram

Fig. 3.21. Stress distributions of a rectangular cross section

We can nondimensionalize Eq. (3.102) by dividing it by $M_r = \frac{bd^3 \sigma_y}{6}$ (Eq. 3.9) and noting also that the curvature at the initiation of yielding is [from Eq. (3.101) with $\sigma = \sigma_y$]

$$\Phi_y = \frac{2\sigma_y}{dE}$$  \hspace{1cm} (3.103)

we finally get the following relationship between moment and curvature in the elastic range

$$\frac{M}{M_r} = \frac{\Phi}{\Phi_y} \quad (0 \leq \frac{\Phi}{\Phi_y} \leq 1.0)$$  \hspace{1cm} (3.104)

From Fig. 3.21(b) we find by a similar process that

$$M = \frac{bd^3 \sigma_y}{4} \left[1 - \frac{4}{3} \left(\frac{1}{2} - \gamma\right)\right]$$  \hspace{1cm} (3.105)
and from the stress diagram we see that

$$E\Phi = \frac{\sigma_y}{(d/2)^2 - \gamma d}$$  \hspace{1cm} (3.106)

If we now set $\gamma$ from Eq. (3.106) into Eq. (3.105) and nondimensionalize by $M_y$, we obtain the following relationship:

$$\frac{M}{M_y} = \frac{3}{2} - \frac{1}{2\left(\Phi/\Phi_y\right)^3}$$  \hspace{1cm} (3.107)

Equation (3.107) is correct as long as the strain is larger than $\epsilon_y$ and smaller than $\epsilon_{sys}$, or nondimensionally, if $1.0 \leq \Phi/\Phi_y \leq \epsilon_{sys}/\epsilon_y$.

Proceeding now to the strain-hardening case [Fig. 3.21(c)], we can show that

$$\frac{M}{M_y} = \frac{3}{2} - \frac{1}{2\left(\Phi/\Phi_y\right)^3} + \frac{E_{st}}{E} \left(\frac{\Phi}{\Phi_y} - \frac{3\epsilon_{sys}}{2\epsilon_y} + \frac{(\epsilon_{sys}/\epsilon_y)^3}{2\left(\Phi/\Phi_y\right)^3}\right)$$  \hspace{1cm} (3.108)

In the derivation of Eq. (3.108) we have used the following relationships from Fig. 3.21(c):

$$E\Phi = \frac{\sigma_y}{\epsilon_y}, \quad e = \frac{\Phi}{d/2}, \quad \eta d = \frac{e - \epsilon_{sys}}{\Phi}, \quad E_{st}\Phi = \frac{\sigma - \sigma_y}{\eta d}$$  \hspace{1cm} (3.109)

The $M-\Phi$ curve corresponding to Eqs. (3.104), (3.107), and (3.109) is plotted in Fig. 3.22 for $\epsilon_{sys}/\epsilon_y = 12$ and $E/E_{st} = 40$, which are typical values for structural carbon steel (Chap. 1 in Ref. 1.6).

![Fig. 3.22. Moment-curvature relationship for rectangular cross section](image)

The moment-curvature relationship is the basic tool with which we can start to analyze the in-plane behavior of inelastic beams. We recall that for small deflections $\Phi = -\nu''$ [Eq. (2.23)], where $\nu''$ is the second derivative of the deflection in the $y$ direction. Thus, for example, the differential equation of in-plane bending for the elastic-plastic case is [from Eq. (3.107)]

$$M = M_y \left[ \frac{3}{2} - \frac{(\Phi/\Phi_y)^3}{2(\nu'')^2} \right]$$  \hspace{1cm} (3.110)

Equations such as Eq. (3.110) can be solved with some effort analytically, but the procedure is complicated. We shall instead use the moment-area method and calculate the deformation history of a simply supported beam under a central concentrated load [Fig. 3.23(a)]. In order to simplify matters, we shall idealize the $M-\Phi$ relationship as shown by the dashed lines in Fig. 3.22. That is,

$$\frac{M}{M_y} = \frac{\Phi}{\Phi_y} \quad \text{for} \quad 0 \leq \frac{\Phi}{\Phi_y} \leq 1.50$$  \hspace{1cm} (3.111)

$$\frac{M}{M_y} = 1.50 \quad \text{for} \quad 1.5 \leq \frac{\Phi}{\Phi_y} \leq 12$$  \hspace{1cm} (3.112)

$$\frac{M}{M_y} = 1.5 + \frac{\Phi/\Phi_y - 12}{40} \quad \text{for} \quad 12 \leq \frac{\Phi}{\Phi_y}$$  \hspace{1cm} (3.113)

These equations represent the more precise analytical curve with reasonable accuracy, and they are much simpler to manipulate mathematically because they are straight lines.

According to the moment-area theorem, which is a statement of the equilibrium and is therefore valid also in the inelastic range, the end slope $\theta$ of the beam in Fig. 3.23(a) is equal to half the area under the curvature diagram. Two curvature diagrams are shown in Fig. 3.23. Each of these corresponds to the moment diagram (b), and one is for the elastic case (c) and one is for the strain-hardened case (d). As long as the maximum moment at the center of the beam $M = QL/4$ is less than $1.5M_y$, the curvature is $\Phi = p\Phi_y/4$ from Eq. (3.111). The symbol $p$ represents

$$p = \frac{QL}{M_y}$$  \hspace{1cm} (3.114)

When $QL/4$ exceeds $1.5M_y$, the $M-\Phi$ relationship of Eq. (3.113) holds everywhere in the beam where the moment is larger than $1.5M_y$. At the points where $M = 1.5M_y$, the curvature jumps from its value of $1.5\Phi_y$ where yielding starts to $12\Phi_y$, where strain hardening starts [Fig. 3.23(d)]. There is actually no region in which Eq. (3.112) applies, since $M = 1.5M_y$ only at a point and not over a region along the length of the beam, according to our simplified representation of the $M-\Phi$ curves.

From the moment area theorem we get the following expressions for $\theta$:
The nondimensional $p$-$\theta$ relationship is plotted as the lower dashed curve in Fig. 3.26. Up to $p = 6$ the beam is elastic; in the inelastic range ($p > 6$) the load-carrying capacity still increases, but small increases in $p$ are accompanied by large increases in end rotation.

We shall next contrast the behavior of a continuous beam with that of the simply supported beam of Fig. 3.23. The solution of statically indeterminate problems is not as straightforward as the solution of determinate problems. In order to determine the curvature diagram we must know the moments. These cannot be determined by the equilibrium condition alone, and we must utilize the conditions of compatibility.

One way of dealing with an inelastic statically indeterminate beam will now be illustrated by the three-span continuous beam shown in Fig. 3.24. The geometry and loading are given in Fig. 3.24(a), and the moment diagram is shown in Fig. 3.24(c). We do not as yet know the value of $M_o$ (the redundant moment), but we know that for a continuous deflected shape [Fig. 3.24(c)]

- For $0 \leq p \leq 6$
  \[ \theta = \frac{P}{16} \]

- For $p \gg 6$
  \[ \theta = 5p - 51 + \frac{513}{4p} \]

where

\[ \theta = \frac{\theta}{\Phi_r L} \]
3.24(b)] the slope at the right end of the side span must equal the slope at the left end of the center span.

From this condition the redundant moment is determined. The end slope \( \theta \) is equal to

\[
\theta = \frac{1}{L} \int_{0}^{L} \Phi \, dz
\]  

(3.118)

for the side spans, and

\[
\theta = \int_{0}^{L} \Phi \, dz
\]  

(3.119)

for the center span, according to the moment area theorem. We want to find out the relationship between the applied load \( Q \) and the slope \( \theta \).

At first the beam is elastic and the curvature is proportional to the moment [Eq. (3.114)]. The elastic curvature diagram is shown in Fig. 3.25(a). By integrating Eqs. (3.118) and (3.119) and setting the two slopes equal to each other, we find that in the elastic range

\[
M_0 = \frac{3QL}{40}
\]  

(3.120)

\[
\theta = \frac{QL^2 \Phi}{40M_Y}
\]  

(3.121)

In nondimensional form [see Eqs. (3.114) and (3.117)] these equations are

\[
m_0 = \frac{M_0}{M_Y} = \frac{3p}{40}
\]  

(3.122)

\[
\bar{\theta} = \frac{p}{40}
\]  

(3.123)

Elastic behavior is terminated when the maximum moment becomes equal to \( 1.5M_Y \). This occurs at the center of the beam when \( p = 60/7 = 8.57 \). Thus in the elastic range the load-deformation relationship is

\[
\bar{\theta} = 0.025p \quad \text{for} \quad 0 \leq p \leq 8.57
\]  

(3.124)

When the elastic limit is exceeded, the center of the beam will yield and the curvature diagram will be as shown in Fig. 3.25(b). The curvature is determined by Eq. (3.113) wherever the moment exceeds \( 1.5M_Y \). Performing the integration for the side span [Eq. (3.118)], we find that

\[
\theta = \frac{M_0L^2 \Phi}{3M_Y} \quad \text{or} \quad \bar{\theta} = \frac{m_0}{3}
\]  

(3.125)

The end slope for the center span, from Eq. (3.119), is

\[
\theta = \int_{0}^{L} \frac{\Phi L}{M_Y} \left(-M_0 + \frac{Qz}{2}\right) \, dz
\]

\[
+ \int_{0}^{L} \frac{40 \Phi}{M_Y} \left(-M_0 + \frac{Qz}{2} - 48 \Phi \right) \, dz
\]  

(3.126)

Fig. 3.25. Curvature diagrams of a continuous beam. (a) Elastic range, (b) strain-hardening under load, (c) strain-hardening at the supports and under the load.

After integration and nondimensionalization

\[
\bar{\theta} = \left(\frac{m_0 + 1.5}{p}\right) \left(39m_0 + 37.5\right) - 20m_0 - 24 + 2.5p
\]  

(3.127)

When \( p = 8.57, m_0 = 0.642 \), and \( \bar{\theta} = 0.214 \), giving a check on the lower limit of applicability of this equation. If we set \( \bar{\theta} \) from Eq. (3.125) equal to \( \bar{\theta} \) from
Eq. (3.127), we obtain a relationship between \( p \) and \( m_0 \)

\[
p^2 - p(8.13m_0 + 9.6) + (m_0 + 1.5)(15.6m_0 + 15) = 0 \tag{3.128}
\]

For any value of \( m_0 \) we can now compute \( p \) from this equation, and with Eq. (3.125) we also know the corresponding value of \( \bar{\theta} \). The lower limit of \( m_0 \) is 0.642, and the upper limit is when \( m_0 = 1.5 \) at the instant when yielding starts at the supports. The corresponding limits of \( p \) are 8.57 and 12.88, respectively, and the upper and lower limits of \( \bar{\theta} \) are 0.214 and 0.500, respectively. The computation of the \( p \) versus \( \bar{\theta} \) curve proceeds by specifying a value of \( m_0 \) between 0.642 and 1.5, computing \( p \) from Eq. (3.128) and \( \bar{\theta} \) from Eq. (3.125).

After \( p \) exceeds 12.88, yielding also takes place at the supports, and the corresponding curvature diagram is given in Fig. 3.25(c). Proceeding with the integration as in the previous case,

\[
\bar{\theta} = \frac{27}{m_0} - 24 + \frac{40m_0}{3} \tag{3.129}
\]

for the side span, and

\[
\bar{\theta} = \frac{288}{p} - 24 - 20m_0 + 2.5p \tag{3.130}
\]

for the center span. Equating the two values of \( \bar{\theta} \), we obtain an equation relating \( m_0 \) and \( p \). From this equation we compute \( p \) corresponding to any value of \( m_0 > 1.5 \), and \( \bar{\theta} \) is determined from either of Eqs. (3.129) or (3.130).

The curve relating \( Q \) and \( \bar{\theta} \) (nondimensionally as \( p \) and \( \bar{\theta} \), respectively) is given in Fig. 3.26. Even after yielding at the supports and the center, the curve still continues to rise.

The development of the \( p-\bar{\theta} \) curve in Fig. 3.26 is a time-consuming process, and it would hardly be used in routine calculations. We can achieve much simpler results by ignoring the strain-hardening region of the \( M-\Phi \) curve altogether, as is done in simple plastic analysis.\(^{1,4}\) If this is done, then the maximum moment which can be supported at the two peak points of the moment diagram of Fig. 3.25(b) is equal to \( M_p = 1.5M_F \). This can be expressed for the center of the continuous beam [Fig. 3.25(b)] as \( QL/4 - 1.5M_F = 1.5M_F \), from which the maximum load is \( Q = 12M_F/L \) or \( p_F = 12 \). From an elastic analysis of the continuous beam we can determine that \( M_0 = 3QL/40 \), and thus the moment at the center of the beam is \( 7QL/40 \). Yielding will commence first at the larger of these two, or \( 1.5M_F = 7QL/40 \) \((p = 8.57 \text{ as shown in Fig. 3.27})\). The \( p-\bar{\theta} \) curve corresponding to the \( M-\Phi \) curve without strain hardening is shown as a dashed curve in Fig. 3.26. It coincides with the more precise curve in the elastic region, and it closely approximates it until the maximum load according to simple plastic analysis is reached. Because of the neglect of strain hardening, an angular discontinuity exists at the supports and at the load point. These discontinuities represent plastic hinges.\(^{1,4}\)

An analysis which utilizes the elastic-plastic \( M-\Phi \) curve is called elastic-plastic analysis. It is the simplest model used for analyzing inelastic beams.

The discussion of the in-plane response of a rectangular beam in the inelastic range has shown that (1) the \( M-\Phi \) curve is the basic instrument from which we can start to develop the deformations of inelastic beams by integration, (2) continuous beams possess a large reserve of load-carrying capacity beyond the elastic limit (see Fig. 3.26), (3) the maximum load according to simple plastic theory is less than the actual capacity, and (4) the mechanism of yielding and strain hardening always result in a load-deflection curve which is rising. This latter point indicates that from in-plane behavior alone we cannot predict a drop in the load-deformation curve, and thus we must look elsewhere for finding the causes of unloading and thus of failure.

The development of the moment-curvature relationship for the rectangular
have been solved in this way (see, for example, Ref. 3.26). A simpler method, which, however, still utilizes strain hardening, is presented in Ref. 3.27.

For practical purposes it is often sufficient to use a simplified curve consisting of two straight lines (dashed curve in Fig. 3.27) representing the elastic part and the plastic part of the curve. This $M$-$\Phi$ curve is the basis of simple plastic design, and its use as a safe basis for predicting the in-plane load-deformation relationship has been demonstrated by many experiments on a variety of structures. On the other hand, it is necessary to consider the actual $M$-$\Phi$ curve if we want to predict the onset of lateral-torsional buckling in the inelastic range. Our discussion on in-plane behavior in the inelastic range was introductory and, only brief. For further study refer to the various texts on plastic design (Refs. 1.6, 1.7, 1.8, 1.41, 3.2, and 3.9).

**Inelastic Lateral-Torsional Buckling**

The danger of lateral-torsional buckling in the inelastic range is greater than in the elastic range because the stiffnesses $B_r$, $C_{\theta r}$, and $C_r$, which govern the critical load in the elastic case [Eqs. (3.2) and (3.3)], are reduced by yielding.

Lateral-torsional buckling occurs when it is possible for equilibriums exists under the same load both in a laterally undeformed position (deflections $v$ only) and in a laterally deformed position (deformations $u$ and $\phi$—see Fig. 3.11) which is infinitely close to the laterally undeformed position. We shall now apply this same philosophy to the case of lateraltorsional buckling in the inelastic range.

The resistance to lateral-torsional buckling depends on the resistance of the cross section to moments $M_\alpha$ and $M_\gamma$. In Chapter 2 we have found that $M_\alpha = -B_r u'' + C_{\theta r} \phi'' - E I \phi''''$ [Eqs. (2.27) and (2.65)].

The equations expressing the left side of these formulas [Eqs. (2.82) and (2.89)] were derived from equilibrium considerations alone, and thus they are valid in the elastic as well as the inelastic range. We need to worry only about the right-hand side terms.

Let us look at the yielded rectangular cross section of Fig. 3.28(a). At the instant before buckling $\phi = 0$, and the cross section is subjected only to a moment $M_\gamma$ which results in the stress distribution shown in Fig. 3.28(b) if the stress-strain diagram of the material is as shown in Fig. 3.29.

Just after lateral buckling the cross section will have rotated about $C_{\theta r}$ through an angle $\phi$ [Fig. 3.28(a)]. As a result the section will be subjected to moments $M_{\alpha}', M_{\gamma}'$, and $M_{\gamma}'$.

Let us disregard $M_{\gamma}'$ for the time being and look only at $M_{\alpha}'$ and $M_{\gamma}'$. From Eq. (2.82) $M_{\alpha}' = M_\alpha \pm \phi M_\gamma$ and $M_{\gamma}' = M_\gamma - \phi M_\gamma$. But we have no externally applied moment $M_\gamma$, and so $M_{\alpha}' = -\phi M_\gamma$ and $M_{\gamma}' = M_\gamma$. Our concern here is to derive the resistance of the already-yielded section to a small moment $M_{\gamma}'$. 

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**Fig. 3.27.** Moment-curvature relationship for wide-flange beams
path by which they arrived at the inelastic stage (unloading as shown by arrow $B$ in Fig. 3.29); in the second instance we have assumed that unloading is elastic (arrow $C$ in Fig. 3.29).

Let us now sum up our findings: at the instant of buckling the resistance to lateral-torsional buckling is determined by the elastic core. If we can find the stiffnesses $B_y$, $C_y$, and $C_w$, then we can use our previous theory [as expressed by Eqs. (3.2) and (3.3)] to determine the critical moments. We have two choices for computing these stiffnesses: They can be based on the condition existing just before buckling, or they can be based on the condition just after buckling. Since the forces induced by the buckling deformation will not significantly increase the extent of the areas already yielded but will significantly decrease this area, the latter stiffnesses are larger, and so we shall obtain a larger critical moment. The question now is: Which of these two critical moments is the correct one?

The duality introduced by the two concepts with regard to the stiffnesses resisting buckling was discovered in the last decade of the 19th century when Engesser presented two formulas for the buckling strength of axially loaded columns.\(^{(1.29,1.30,5.58)}\) One of these, the *tangent modulus* formula, was based on the assumption that unloading followed the same path as loading. The other formula, the *reduced modulus* formula, was based on the assumption of elastic unloading. The reduced modulus formula always leads to a larger value of the critical load. For 50 years or so, engineers considered the reduced modulus concept to be correct, but there was considerable controversy about this.

The paradox was settled by Shanley,\(^{(9.29)}\) who showed by a very simple theoretical model that in the inelastic range it was not necessarily correct to assume that the possibility of buckling existed when it could be proved that two neighboring deflected shapes exist under the same load. We shall deal in more detail with Shanley’s derivation in Chapter 4, but his conclusions, which are pertinent to any inelastic bifurcation-type buckling problem, will be listed here: (1) The tangent modulus load is always smaller than the reduced modulus load, (2) the tangent modulus load is the highest load at which no lateral deformations can exist for an initially perfect beam, (3) after the tangent modulus load is exceeded, lateral deformations may exist and an increased load can be carried until a maximum load is reached, (4) the maximum load is always less than the reduced modulus load, and (5) the reduced modulus load is an upper bound to inelastic buckling strength and also to the maximum load.

These conclusions are illustrated in Figs. 3.30 and 3.31. In Fig. 3.30 we have the relationship between the moment and the deformations $u$ or $\phi$; lateral deflection may initiate at the tangent modulus moment, but in order to obtain finite deformations, the moment must further increase. The tangent modulus moment is defined by the lateral buckling equations [Eqs. (3.2) and (3.3)], and
tically, and its determination depends on the stiffnesses which include the elastic unloading of previously yielded portions of the cross section. The maximum load is thus bounded by the tangent modulus and the reduced modulus moment. The three types of critical loads are shown in the \( M_r-L \) plot of Fig. 3.31. Up to \( M_0 = M_r(1 - \sigma_{rel}/\sigma_f) \) there is only one curve, the elastic lateral-torsional buckling curve, and in the region \( M_r(1 - \sigma_{rel}/\sigma_f) \leq M_0 \leq M_r \) there exist three theoretical limiting moments.

Obviously it would be most satisfactory if we could obtain the value for the moment corresponding to the maximum strength of the beam. The determination of this moment is, however, extremely complicated. The best we can do is to evaluate the bounds to the maximum strength from the reduced and the tangent modulus concept. Even this turns out to be a rather complicated procedure. Tangent and reduced modulus solutions were presented for rectangular beams by Wittrick and Neal, respectively; Horne presented a solution for the reduced modulus strength of wide-flange beams without residual stresses, and the tangent modulus solution for wide-flange shapes with a residual stress pattern equal to that shown in Fig. 1.6 has been given in Refs. 3.8, 3.25, and 3.35.

The computation of the tangent modulus moment is the simpler procedure. The value of the maximum moment is reduced because of practically unavoidable initial crookedness, and so the tangent modulus load represents in most cases a reasonable estimate of the maximum moment which an inelastic beam can support.

**THE TANGENT MODULUS BUCKLING LOAD FOR WIDE-FLANGE BEAMS WITH RESIDUAL STRESSES**

The tangent modulus solution for wide-flange shapes with residual stresses is the simplest of the three theoretical solutions. The effort to arrive at an answer is considerable, however, and we shall only describe the procedures in broad outline form here.

The steps in arriving at a solution are as follows:

1. For a given wide-flange shape and residual stress pattern we determine for a specified moment \( M_r[1 - \sigma_{rel}/\sigma_f] \leq M \leq M_r \) the extent of yielding. For example, the stress distribution in the two flanges and in the web due to such a moment \( M \) and for the residual stress pattern on Fig. 1.6 is shown in Fig. 3.32. Bending stresses from \( M_r \) are superimposed on the residual stresses. Yielding will occur wherever the sum of the two types of stresses becomes equal to the yield stress \( \sigma_f \). For the situation in Fig. 3.32 the compression flange will start to yield at the flange tips, and yielding will penetrate toward the center of the flange as the moment increases. The tension flange will start to yield in the center, and yielding will spread toward the flange tips.
and into the web. The shaded areas in Fig. 3.33 show the distribution of the yielded portions of the cross section corresponding to one particular value of the moment $M$. By considering the equilibrium conditions $\int_A \sigma \, dA = 0$ and $\int_A \sigma y \, dA = M$ and from the geometric relationships of the stress blocks (Fig. 3.32), we can develop formulas which relate the yield penetration parameters $\alpha$, $\beta$, and $\nu$ to the moment.

2. After the yielded portions of the wide-flange shape corresponding to $M$ have been determined, we can find the corresponding stiffness values $C_T$, $B_T$, $C_T$, and $K$. The stiffnesses $B_T$ and $C_T$ are the bending stiffness and the warping stiffness, respectively, of the elastic core. The parameter $K = \int_A \sigma a^2 \, dA$ can be obtained from the stress distribution (Fig. 3.32) and the member geometry, taking due account of the fact that the shear center now no longer coincides with the original centroid and including the residual stresses. The determination of the torsional stiffness $C_T$ is a matter of some controversy. According to proofs furnished among others by Neal, the value of $C_T$ for a perfectly straight member is not reduced by yielding, and so $C_T = GK_T$ [Eq. (2.91)]. Lay has argued that the discontinuous nature of yielding shows that a reduced value of $G$ is to be used in the yielded portions of the beam. Although this may be true, the end result is not greatly affected by either concept, as the inelastic beams are relatively short and St. Venant torsion has a small effect. The former concept (that is, $C_T = GK_T$) was used in the computations of Refs. 3.8, 3.25 and 3.35. The curves in Fig. 3.34 show the variations of the stiffness parameters in the yielded region.
3. With the stiffness coefficients $B_r$, $C_w$, $C_y$, and $\bar{K}$ known, we can proceed to solve the differential equations [Eqs. (3.2) and (3.3)]. If the moment varies along the beam, then a numerical procedure must be used, since now not only $M$ but also each coefficient varies along the $z$ axis of the member. For the case of uniform moment (Fig. 3.8) the differential equations can be solved in closed form, as was the case in the elastic region (Sec. 3.2). The corresponding buckling equation can be solved for the length, which is the only remaining unknown, as

$$L_{cr} = \frac{\pi}{M_s} \sqrt{\frac{B_r C_y (1 + \bar{K}/C_y)}{2} \left[ 1 + \sqrt{1 + \frac{4C_w M_0^2}{B_r C_y (1 + \bar{K}/C_y)}} \right]}$$  \hspace{1cm} (3.133)

The terms $B_r$, $\bar{K}$, and $C_w$ in Eq. (3.133) are all dependent on $M_0$.

The actual computation of $L_{cr}$ from Eq. (3.133) was performed by the use of families of auxiliary curves\(^{(13,8,135)}\) or by computer.\(^{(13,25)}\) A curve giving the relationship between $M_0/M_y$ and $L/r_y$ for an 8WF31 section ($\sigma_y = 33$ ksi, $\sigma_{cr} = 0.3\sigma_y$, and $E = 30,000$ ksi) is shown as the solid-line curve in Fig. 3.35. Also shown as a dashed curve is the relationship which results if residual stresses are neglected.

From this figure it is apparent that residual stresses have a considerable effect on the critical buckling moment, especially in the range of lengths in which the transition from the fully elastic to the partly yielded condition takes place. There the reduction in capacity may amount to as much as 30 per cent.\(^{(13,33)}\)

In the previous description we saw how the lateral-torsional buckling moment according to the tangent modulus theory can be determined for steel wide-flange beams. The method of solution is complicated, and, furthermore, it rests on a number of assumptions which are not fully valid. For example, the assumed residual stress pattern is in equilibrium with respect to axial force and bending (see Chapter 1), but not with respect to torsion. The shear modulus $G$ is controversial, and strain hardening and initial crookedness have been neglected. This latter effect may be more serious in the inelastic range than in the elastic range of buckling. However, the theory explains that yielding reduces the capacity to resist buckling, and it provides a gross and conservative approximation of a transition between the limit of elastic behavior and the full plastic moment. In view of an absence of a more refined theory, it is permissible to approximate the inelastic region of lateral torsional buckling by a straight-line transition curve between the length which corresponds to the termination of elastic buckling and the length where the beam is so short that the full plastic moment can be attained.\(^{(13,33)}\) Such a transition

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**Fig. 3.34.** Moment-stiffness curves for yielded wide-flange cross section

**Fig. 3.35.** Lateral buckling curve for a wide-flange beam
The type of performance is characterized by curve \textit{OAH} in Fig. 3.6. Under some conditions it is possible to reach or exceed the plastic moment and to maintain such a moment for a considerable range of deformation (see Fig. 3.2 and curves \textit{OAK} and \textit{OAC} in Fig. 3.6). The fact that such behavior is possible for a lateral bracing spacing which is still reasonably economical is the reason why plastic design of steel structures is not only possible but also economical.\(^{1,8}\) Final failure is due to a combined occurrence of local and lateral-torsional instability. In general, failure due to plastic instability is imminent when the average strains are near or equal to the strain-hardening strains (Fig. 3.20).

The questions which must be considered in plastic design are: (1) What is the optimum spacing for braces that will prevent inelastic instability from curtailing the moment and deformation capacity? (2) What are the maximum plate width-thickness ratios which will inhibit premature local buckling? (3) What is the maximum deformation which can be attained before the moment capacity begins to drop below \(M_p\)? These are vital questions in the plastic design of steel structures, and a great deal of research has been done to answer them. Some of this research is recorded in Refs. 1.24 and 3.1 and 3.37 through 3.48. Our discussion will first deal with the behavior of plastic wide-flange beams under uniform moment.

The beam model of Fig. 3.2 has been used in several studies on the required bracing spacing in plastic design,\(^{2,1,4,5,42}\) and we have already discussed its behavior in Sec. 3.1. In the plane of bending, which is the plane of the web, the two concentrated loads \(Q\) provide a uniform moment over the central third of the member. Lateral bracing is present at the ends and at the load points (Fig. 3.3). The central segment will be fully yielded when \(M_p\) is reached, and lateral and local buckling will eventually take place in this span. The two outside spans remain essentially elastic, and, being a great deal stiffer than the yielded central segment, they will act as a restraint against the lateral deflections of the buckled center span.

The in-plane performance of such a beam is illustrated by the load-deflection curve of Fig. 3.2, or by the moment-rotation curve of Fig. 3.37 (the rotation \(\theta\) being the slope at the supports). Both of these curves are typical of properly behaved beams: The plastic moment \(M_p\) is reached and maintained through a considerable deformation.\(^{13}\) The moment \(M_p\) is maintained by the beam for a rotation of about 8 times \(\phi_p\) before the plastic hinge behavior is terminated by unloading. Just before this unloading, local buckling was observed (see arrows in Fig. 3.37).

The cause of unloading in this beam was the formation of local buckles.

\(^{11}\) Another empirical approximation, which is based on consideration of a critical stress, will be presented in Sec. 3.4.

\(^{12}\) The same statement could also be made about local buckling if it occurs before the whole plate element is fully yielded.

\(^{13}\) The curve in Fig. 3.37 is a typical test curve from Ref. 3.42. The ordinate is the maximum moment \(QL = M\), nondimensionalized by \(M_p\); the abscissa is \(\theta\), nondimensionalized by \(\phi_p\), the hypothetical elastic slope at \(M = M_p\). This particular beam had an unbraced length of \(L = 30\ r_g\) (see inset in Fig. 3.2 for definition of \(L\)).
in the laterally deflected compression flange (see photographs in Figs. 3.4 and 3.5). This mechanism of failure is typical of wide-flange beams under uniform moment. Lateral deflection of the compression flange becomes noticeable as soon as $M_p$ is reached in the beam (at approximately $\theta = 1.5\theta_p$ in Fig. 3.37—see also Fig.3.3). At this time we can also see yield lines which extend across the whole width of the compression flange [Fig. 3.38(a)]. These yield lines are due to the axial shortening of this flange. As deformation in the plane of bending continues, the compression flange is shortened [3.38(b)]. Instead of taking up this shortening by axial straining, the compression flange will deflect laterally. No new axial yield lines are formed, but yield lines due to lateral bending develop in half of the flange [Fig. 3.38(a)]. A local buckle becomes evident in this doubly strained region when the average strain across the whole flange is equal to $\varepsilon_{ST}$, the strain at strain hardening.\(^{1,2,3}\)

We can appreciate that the whole process of deformation and final failure is very complicated. In order to be able to develop an analytical solution we shall assume that local and lateral buckling occur independently of each other and at the instant when the ideally straight compression flange is fully yielded, that is, when all the elastic regions in it have disappeared and the strain is everywhere $\varepsilon_{ST}$. Figure 3.39(a) shows this ideal flange, and Fig. 3.39(b) shows the actual stress-strain curve of steel; in the yield lines $\varepsilon = \varepsilon_{ST}$, and in the other parts of the flange $\varepsilon = \varepsilon_T$, whereas the stress remains at a constant value of $\sigma_T$.\(^{1,4,4}\)

The local buckling model is a plate of width $b$ and thickness $t$ (the compression flange subjected to an axial stress of $\sigma_T$ [Fig. 3.39(a)]. The critical stress at which this straight plate can buckle is\(^{2,4,5}\)

$$\sigma_{ct} = \sigma_T = G_{ST} \left( \frac{2t}{b} \right)^2$$

\[(3.134)\]
where $G_{st}$ is the shear modulus in strain hardening 

$$G_{st} = \frac{2G}{1 + \frac{E}{4G_{st}(1 + \nu)}} \quad (3.135)$$

In Eq. (3.135) $E_{st}$ is the strain-hardening modulus obtained from a tension coupon test of the material and $\nu$ is Poisson's ratio. For $E = 29,000$ ksi, $G = 11,200$ ksi, $E_{st} = 900$ ksi (typical of A36 steel), and $\nu = 0.3$, $G_{st} = 3,100$ ksi. Solving Eq. (3.134) for the critical width-thickness ratio $b/t$,

$$\frac{b}{t}_{cr} = \frac{110}{\sqrt{\sigma_y}} \quad (3.136)$$

The $b/t$ ratio of Eq. (3.136) is the maximum permissible ratio for the condition of $\sigma = \sigma_y$ and $\epsilon = \epsilon_{st}$. It neglects the effect of web restraint on local buckling. However, this effect increases the $b/t$ ratio only a small amount (about 3 per cent). 

Equation (3.136) is valid for steels with distinct strain hardening and for a strain-hardening modulus $E_{st} = 900$ ksi. Constructional steels with ASTM designations A7, A36, and A441 are typical materials to which Eq. (3.136) applies. A majority of the commercially rolled A36 and A441 steel wide-flange shapes have $b/t$ ratios less than the limiting value, and thus beams for these shapes can be expected to perform as required in plastic design, provided they are braced properly.

The optimum spacing of the lateral bracing can be determined by experiment or theoretically by considering the lateral-torsional buckling strength of the beam as a whole, or the buckling strength of a column consisting of the compression flange and half of the web.

The lateral-torsional buckling moment of an elastic simply supported beam under uniform moment [Fig. 38(a)] was derived in Sec. 3.2, and the critical relationship between $M_0$ and $L$ is given by Eq. (3.42). This equation can be rearranged as follows:

$$(M_{0})_{cr} = \frac{\pi^2 E}{E_L^2} \sqrt{I \sigma_y} \sqrt{1 + \frac{G K L^3}{\pi^2 E I_o}} \quad (3.137)$$

We shall now adapt this equation to the conditions of optimum bracing spacing (that is, local and lateral-torsional buckling occurring simultaneously).

During the interval in which the beam supports the plastic moment

14 Equation (3.136) gives $b/t = 18$ for $\sigma_y = 36$ ksi and $b/t = 15$ for $\sigma_y = 50$ ksi. The 1963 AISC specifications give $b/t = 17$ as the limit for $\sigma_y = 36$ ksi. This is conservative for uniform moment, but it also covers the case of beams under moment gradient for which 17 is the correct number.

15 Actually web local buckling should also be checked, but for beams this is not critical; web buckling, however, often controls for beam-columns and columns. Also, the presentation of the local buckling problem here is simplified, because the proper theoretical foundation has not been laid for a thorough study. Students are referred to Refs. 3.44, 3.45, and 3.49 for further study of this subject.

(Flat portion of $M$-$\theta$ curve in Fig. 3.37) the compression flange is subjected to a uniform stress $\sigma_r$ [Fig. 3.39(a)], and it becomes progressively more traversed by yield planes which become externally visible when the mill scale flakes off at the intersection of the plate surface and the yield plane (see Fig. 3.4 or 3.5). In the yield planes the material is at a strain equal to that corresponding to the initial strain hardening, and in the surrounding material which is still elastic, the strain is equal to $\epsilon_r$ [Fig. 3.39(b)]. Thus the stress-strain diagram as recorded by a tension test of a coupon (Fig. 3.20) represents an average condition. In the flat portion of this curve some of the material is either still elastic or already strain-hardened. In an initially straight compression specimen the yield lines will appear perpendicular to the longitudinal axis of the member [see Sec. 3.39(a)]. When the average strain reaches $\epsilon_{st}$, then the whole member is strain-hardened (that is, fully yielded), and we could expect to get an increase in stress above $\sigma_r$. The yield lines are then very closely spaced. We have shown that when this occurs, the compression flange will buckle locally. Thus we can at best attain the stress $\sigma_r$ and the strain $\epsilon_{st}$.

If at this time the flange is laterally disturbed, as in lateral-torsional buckling (Fig. 3.28), some fibers will strain according to $\sigma = E_{st}\epsilon$ (loading zones), and some fibers will strain according to $\sigma = E\epsilon$ (unloading zones). The buckling load according to the tangent modulus concept is determined from the assumption that only loading occurs, and therefore we can use $B = E_{st}$ in Eq. (3.137). Since we are concerned with buckling at the very start of strain hardening, the stress is still $\sigma_r$ and we can set $M_0 = M_r$ in this equation. The critical length $L$ will be relatively short, and therefore $G_{st}K_0L^3/E_{st}I_o << 1$, or that is, the second radical in Eq. (3.137) can be conservatively set equal to unity.

We now can rearrange the buckling equation [Eq. (3.137)] into the following form:

$$L_{cr} = \frac{\pi^2 E_{st} \sqrt{I \sigma_y}}{M_r} \quad (3.138)$$

With

$$I_o = \frac{(d - t)^3 I_s}{4}, \quad M_r = fS_o\sigma_y, \quad r_y^2 = \frac{l_y}{A}, \quad S_o = \frac{2E_A}{d}$$

this equation is

$$L_{cr} = \frac{\pi^2 E_{st}(d/r_y^3)(1 - t/d)}{4f\sigma_y} \quad (3.139)$$

For rolled wide-flange shapes we have used previously in Sec. 3.2 the following average cross-sectional properties: $f = 1.14$ (shape factor), $(1 - t/d) = 0.95$, and $d/r_y = 2.4$. With these values Eq. (3.139) becomes

$$L_{cr} = 3.44 \sqrt{\frac{E_{st}}{\epsilon_r}} \quad (3.140)$$
For A36 steel with $E_{yy} = 900$ ksi\(^{1,43}\) and $\sigma_y = 36$ ksi, this critical length is equal to $17r_y$. Theoretically a simply supported wide-flange beam will buckle locally and laterally when the spacing is equal to this value.

The derivation of Eq. (3.140) has neglected the restraint due to the elastic adjacent spans [Fig. 3.38(a)]. This restraint can be included by multiplying $L$ by an effective length factor $K$, which is equal to 0.54.\(^{1,43}\) We have also assumed that all the yield lines will be due to axial strains only. There are also yield lines present due to lateral bending [Fig. 3.38(a)], and by considering their effect it has been shown that the following equation gives a more realistic value for the critical length:\(^{1,3}\)

$$\frac{KL}{r_y} = 0.54 \frac{L}{r_y} = \frac{\pi}{\sqrt{\sigma_y [1 + 0.56E/E_{yy}]}}$$  (3.141)

According to this equation,

$$\frac{L}{r_y} = \frac{227}{\sqrt{\sigma_y}}$$  (3.142)

for $E = 29,500$ ksi and $E_{yy} = 900$ ksi. Values of $E_{yy}$ ranging from 700 to 900 ksi have been reported for A36 and A441 type steels, and so Eq. (3.142) represents a reasonable design equation which has also been verified by several test series.\(^{1,2,4,12}\) For $\sigma_y = 36$ ksi, $L_{cr} = 38r_y$ from this equation.

This value corresponds to $L_{cr} = 35r_y$ in the 1963 AISC specifications.\(^{1,3}\)

Our foregoing discussion has applied to beams under uniform moment with elastic adjacent spans. If the yielded region extends beyond one span of the lateral bracing, then we must use $K = 0.8$ in Eq. (3.142).\(^{1,43}\) Beams under moment gradient behave differently from beams under uniform moment because only a part of the braced segment will be yielded. Since the development of the rules for bracing spacing is quite lengthy, we shall omit it here (see Ref. 3.40) and state only the final results, which are simple: If the ratio of end moments in the braced span adjoining the hinge is between 1.0 and 0.5, use the bracing rules developed for uniform moment [Eq. (3.141) or (3.142)]. If the ratio of end moments is less than 0.5, the critical length is equal to:\(^{16}\)

$$\frac{L}{r_y} = \frac{0.7\pi}{\sqrt{\sigma_y}}$$  (3.143)

In beams under moment gradient, local buckling occurs before lateraltorsional buckling when a long enough segment of the beam has yielded so that one whole wavelength of a local buckle can form. In either case, that is, in beams under full or near uniform moment and beams under moment gradient, both local and lateraltorsional deformations occur, and these eventually overcome the strengthening effects of strain hardening until finally the capacity drops off. We should realize that these deformations are present in the form of initial imperfections from the start of loading, but they become magnified only when extensive yielding has occurred. Their separate treatment as two individual buckling problems was the best we could do to obtain analytical solutions. Future research will, hopefully, bring more realistic solutions.

We have covered here only the geometric requirements for behavior which are needed for plastic design. The topic of the available deformation capacities is left for a study of the works cited (for example, Refs. 3.3, 3.40, or 3.50).

**BEAM BEHAVIOR**

Our study in Secs. 3.2 and 3.3 has introduced a few important concepts about beam behavior. This study was not at all exhaustive, as many topics were left unmentioned. The treatment of local buckling, biaxial bending, and torsion was slight, and such important topics as the behavior of beams with variable section, box girders, plate girders, etc. were not covered at all. These are all separate and important studies, deserving more detailed attention. However, we attempted to point out three major ranges of beam behavior: the range in which full plastification is possible, the range in which the resistance to buckling is impaired by partial yielding, and the range in which the capacity is controlled by elastic buckling. These three types of behavior were illustrated by the very common and practical case of the prismatic beam loaded so that its primary response is in the plane symmetry.

The three ranges of moment capacity are dependent on the length of the unbraced segment, which controls lateral-torsional buckling, and the plate slenderness, which controls local buckling. These three ranges are illustrated schematically in Fig. 3.40. In the first range, where combined local and lateral instability of the fully plastified section limits the deformability but permits attainment of the full plastic moment, the beams are suitable for use in plastic design. In the other ranges neither full moment capacity nor adequate deformability exists, and those members are used in allowable stress design. In this range are the plate girders with stiffeners and thin webs, the box girders, the light-gage members, and the aluminum beams of common construction.

The limits between the three ranges are somewhat arbitrarily defined for convenience in design. The important fact to remember is that the transitions are gradual and that these are, for any type of beam, continuous relationships between the moment capacity and the geometry of the section. These relationships are entirely a function of instability. In addition to these relationships there are others—for example, those dictated by fatigue or brittle fracture—which may also need to be considered.

In the next section we shall consider ways in which our knowledge of beam behavior is adapted to design specifications.
3.4. DESIGN PRACTICE

USE OF BEAM THEORY IN DESIGN

In the previous two sections we discussed some features of the behavior of beams in the elastic, inelastic, and fully plastic ranges. We have concentrated our attention on problems of lateral-torsional buckling, and in this article we shall discuss the application of this information in design.

There are essentially two philosophies used in the design of beams: In the first and most widely used of these it is assumed that the internal forces, reactions, and the resulting stresses and deformations can be predicted by first-order elastic theory. The computed stresses are then limited to remain below values representing the limits at which the assumptions are no longer valid. Such limits are the attainment of the elastic limit or a critical buckling stress. This design philosophy is called allowable stress design." The second design philosophy for beams is plastic design, in which the maximum load is reached when a plastic mechanism forms.

The two design procedures make distinctly different demands on the structure with respect to instability phenomena. In allowable stress design the major portion of the beam remains elastic, although in some regions of

17 It is also frequently called elastic design; however, this is incorrect since at the limiting stress the material is not necessarily elastic anymore.

high residual stresses the beam may be partly yielded. Thus when the nominal yield stress is reached, the beam is still stiff. In plastic design, on the other hand, the beam must undergo considerable rotation while it is fully yielded. Thus a plastically designed beam must be braced more closely than a beam designed by the allowable stress method. This disadvantage is offset by the lighter section required to carry the same load.

In allowable stress design the allowable stress $\sigma_{bl}$ (subscript $b$ indicates bending, and subscript $a$ indicates allowable) is defined as

$$\sigma_b \leq \sigma_{bl} = \frac{\sigma_y}{F.S.} \quad \text{or} \quad \sigma_b \leq \sigma_{bl} = \frac{\sigma_y}{F.S.}$$

(3.144)

where $\sigma_y$ is the elastically computed bending stress, $\sigma_y$ is a buckling stress smaller than the yield stress $\sigma_y$, and F.S. is the factor of safety. Usually a check is made against both limits (yielding and buckling), and the lower of the two controls the design.

In plastic design we need to provide the necessary bracing so that hinges can develop. Unless special rotation calculations are made, we space the bracing for optimum rotation capacity [Eqs. (3.142) and (3.143)]. We have only one alternative in plastic design: to provide properly spaced bracing so that the beam can form a mechanism. In allowable stress design we have two choices: We can specify the bracing spacing and compute $\sigma_{bl}$, or we can use $\sigma_y$ as our limiting stress and compute the bracing necessary to insure that this will happen.

LATERAL-TORSIONAL BUCKLING STRESS
IN ALLOWABLE STRESS DESIGN

The elastic critical stress of a beam with a doubly symmetric cross section is equal to [from Eqs. (3.77) and (3.79)]

$$\sigma_{cr} = \frac{(M_c)_{cr}}{S_a} = \frac{C_c \pi}{K_c L S_a} \sqrt{E I G K_c} \left[ 1 + \frac{\pi^2 E I}{G K_c (K_c L_s)^2} \right]$$

(3.145)

The corresponding formula for a singly symmetric section is given as Eq. (3.80). We have shown in connection with Fig. 3.35 that the inelastic lateral-torsional buckling strength can be estimated by a straight-line relationship between $M_{cr}$ corresponding to the start of yielding, and $M_c$. Whereas this is a rather simple criterion to apply, it usually involves a graphical construction. In order to avoid this, a simple and conservative procedure has been advanced in Ref. 3.10 for dealing with the inelastic case. It is based on the assumption that the relationship between elastic and inelastic buckling is the same for both axially loaded columns and beams. The procedure consists of the following steps: The elastic lateral-torsional buckling stress ($\sigma_{cr}$) is for the given problem is computed by the means presented in Sec. 3.2, or by any other...
means. If \((\bar{\sigma}_r)_{\ell}\) is larger than \(\frac{1}{4}\sigma_r\), then it can be converted to an approximate inelastic stress by the following formula:

\[
(\sigma_{r\ell})_f = \sigma_r \left[ 1 - \frac{\sigma_r}{4(\bar{\sigma}_r)_{\ell}} \right]
\]  

(3.146)

The comparison between the straight-line approximation and this approximation is shown in Fig. 3.41 for the case of a 27WF94 beam. Whereas the actual strength will reach the value of \(M_p\) for a zero length beam, the procedure of Eq. (3.146) will always lead to \(M_p\). The comparison between analytically exact \(M_p\)-L curves and this approximation also shows the latter to be somewhat conservative.\(^{(3,32)}\)

In design specifications it is customary to use relatively simple rules for determining the critical stress. Usually the segment to be braced is assumed to have simple ends, uniform moment, and double symmetry. For this case we can write Eq. (3.145) in the following form:

\[
\sigma_{cr} = \sqrt{\frac{\pi^2EI_GK_r}{S_2L^2}} + \frac{\pi^2EI}{S_3L^2} = \sigma_r + \sigma_{wr}
\]  

(3.147)

The value of \(\sigma_{cr}\) is thus the hypotenuse of a right triangle where the other sides are \(\sigma_r\) and \(\sigma_{wr}\).\(^{(3,30)}\)

Simplified expressions for \(\sigma_r\) and \(\sigma_{wr}\) can be found for wide-flange shapes if the following relationships are used:

\[
A \cong 2bt + (d - 2t)w = 2A_r + A_{wr}
\]  

(3.148)

\(^{18}\) This relationship holds also for columns and beam-columns, and it is further discussed in Sec. 4.3, where enough background has been presented for the derivation.

---

Substituting Eqs. (3.148) through (3.152) into Eq. (3.147) we obtain

\[
\sigma_{r\ell} = \frac{\pi^2EGF}{9(Ld/\beta t)^3} \left[ \frac{\pi^2E}{(L/F)^3} \right] \left[ 1 + \frac{A_{wr}}{6A_r} \right]
\]  

(3.153)

where

\[
F = 1 + \frac{A_{wr}w^2}{2A_rL^2}
\]  

(3.154)

and

\[
\beta^3 = \frac{b^3}{12(1 + \frac{A_{wr}}{6A_r})}
\]  

(3.155)

The term \(F\) is nearly equal to unity for most rolled wide-flange shapes\(^{(18)}\) and will be neglected. The term \(\beta\) in Eq. (3.156) is the radius of gyration of the compression flange plus one-sixth of the web about the \(y\) axis. For steel \((E = 29,000\, \text{ksi}; G = 11,200\, \text{ksi})\) the lateral-torsional buckling stress is

\[
\sigma_{cr} = \sqrt{\frac{18,900}{(Ld/\beta t)^3} + \left[ \frac{\pi^2E}{(L/F)^3} \right]}
\]  

(3.156)

Using either \(\sigma_r\) or \(\sigma_{wr}\) will give a conservative critical stress. The first approximation \(\sigma_r\) is closer for long beams (St. Venant torsion only), and the second approximation \(\sigma_{wr}\) is better for short beams (warping torsion only). In the latter case inelastic buckling governs, and we must modify \(\sigma_r\) by Eq. (3.146), that is,

\[
(\sigma_{r\ell})_f = \sigma_r \left[ 1 - \frac{\sigma_r(L/F)^3}{4\pi^2E} \right]
\]  

(3.157)

The two stresses \(\sigma_r\) and \(\sigma_{wr}\) are the basis of the lateral buckling rules in the 1963 AISC specifications. According to this specification,

\[
\sigma_{br} = 12,000 \frac{Ld/\beta t}{1000}
\]  

(3.158)

\[
\sigma_{br} = 0.6\sigma r \left[ 1 - \frac{\sigma_r(L/F)^3}{4\pi^2E} \right]
\]  

(3.159)

\(^{19}\) This approximation is unconservative, since \(F < 1.0\). Neglect of this factor can lead to buckling during the erection of thin-walled plate girders \((F = 0.78\) for \(b = 20\) in.; \(t = 1\) in.; \(d = 120\) in.; \(w = 1\) in.).
The term $C_1$ accounts for moment gradient (Eq. 3.78). In any situation both equations must be checked, and both have a factor of safety of about 1.65 against $\sigma_y$ or $\sigma_{fy}$.

There are essentially four ways in which various specifications approach the problem of allowable stresses when lateral-torsional buckling must be considered. All four of these are illustrated in Fig. 3.42 for a 27WF94 steel member. The first of these uses Eq. (3.145) directly, and the inelastic stress due to Eq. (3.146) for $\sigma_y > \sigma_{fy}/2$. The curve corresponding to this method is shown as the solid-line curve in Fig. 3.42. In actual design the stress would be divided by an appropriate safety factor, say, 1.65. This type of approach is permitted as an alternate in the German DIN 4114.1354 The 1963 AISC specification is illustrated by one of the dashed lines in Fig. 3.42. There is no reduction in stress until the unbraced length is approximately 80$\text{r}_y$ for this particular member. For longer spans Eq. (3.159) governs, except for the region $105\text{r}_y \leq L \leq 155\text{r}_y$ where Eq. (3.160) governs, causing a slight bulge in the curve. The AISI gives three equations, each applicable for a given range of unbraced spans$^{3,36}$ (dot-dash curve in Fig. 3.42).20 A similar procedure is recommended by the ASCE specifications for aluminum alloys (Ref. 3.53). The equations for the curves plotted in Fig. 3.42 (AISI formulas) are as follows:

$$\sigma_{sa} = 0.606\sigma_y \quad \text{for} \quad 0 \leq \frac{L}{r_y} \leq \frac{408}{\sqrt{\sigma_y}}$$

(3.161)

![Graph of lateral-torsional buckling curves from specifications.](image)

$^{20}$ The AISI formulas were used on a rolled shape here for purposes of illustration. These equations are to be used only for light-gage cold-formed members.

$$\sigma_{sa} = 0.674\sigma_y \left[ 1 - \frac{\sigma_y (L/r_y)^2}{(166.3)(10^5)} \right] \quad \text{for} \quad \frac{408}{\sqrt{\sigma_y}} \leq \frac{L}{r_y} \leq \frac{909}{\sqrt{\sigma_y}}$$

(3.162)

$$\sigma_{sa} = \frac{280,000}{(L/r_y)^2} \quad \text{for} \quad \frac{909}{\sqrt{\sigma_y}} \leq \frac{L}{r_y}$$

(3.163)

Finally, the last method is illustrated by the dashed curve according to the AREA specification.1354 This curve represents essentially $\sigma_y$ [Eq. (3.163)] which is terminated where normally elastic buckling would take place. Thus only relatively short unbraced lengths are permitted by such a specification. The particular curve plotted in Fig. 3.43 is

$$\sigma_{sa} = 0.545\sigma_y \left[ 1 - 0.1386(10^{-4})\left(\frac{L}{r_y}\right)^2 \right] \quad \text{for} \quad 0 \leq \frac{L}{r_y} \leq 196$$

(3.164)

A similar rule is provided by the AASHO.1355 In the two specifications the formulas are actually given in terms of $L/b$ instead of $L/r_y$.21

The comparison of the types of specification curves with the maximum strength curve shows that the former have a reasonable factor of safety against lateral-torsional buckling. Each of the types of approaches is simple and has evolved through successful use in practice. They provide a quick and easy answer for the usual cases encountered in design. For unusual cases, or for careful analyses, we should of course use more precise methods.22

Further topics, such as plate girder design,23 box girders, etc. are left for individual study.

REFERENCES


$^{21}$ It should be noted that in Eqs. (3.161) through (3.163) the units of $\sigma_y$ are ksi.


### PROBLEMS

3.1. Show all steps in the derivations of Eqs. (3.15) and (3.16), starting with Eqs. (2.94) and (2.95).

3.2. Show that \( \beta_0 = 0 \) [Eq. (3.13)] for doubly symmetric sections.

3.3. Plot an \((M_{i})_{cr}/M_p\) versus \(L/r_y\) curve for an A441 steel 18WF50 beam. The loading and end conditions are shown in Fig. 3.8(a). Use Eq. (3.47) and compare with results from Eq. (3.48). Use computer for calculations.

3.4. Determine the critical moment of a wide-flange beam under end moment \((\kappa = 0, \phi = 0, \pi L = 0, \text{Fig. 3.17})\)
   (a) by Eq.(3.42).
   (b) By charts from Refs. 3.17 and 3.18.
   (c) By Rayleigh-Ritz method, assuming

\[
\begin{align*}
 u &= A \sin \frac{\pi x}{L} + B \sin \frac{2\pi x}{L} \quad \phi = C \sin \frac{\pi x}{L} + D \sin \frac{2\pi x}{L} \\
\text{(d) By Rayleigh-Ritz method, assuming} \\
& \quad u = A \sin \frac{\pi x}{L} + B \sin \frac{2\pi x}{L} \quad \phi = C \sin \frac{\pi x}{L} + D \sin \frac{2\pi x}{L} \\
\text{(e) By finite-difference method using a spacing of } L/3, L/4, \text{ and } L/10. \text{ Improve answer by extrapolation (Ref. 3.13).} \\
& \text{Compare answers and discuss the solutions.} \\
& \text{Given: } 27WF94 \text{ section} \\
& \quad L = 200 \, r_y \\
& \quad E = 30,000 \, \text{ksi} \\
& \quad G = 11,500 \, \text{ksi} \\
& \text{simply supported ends} \\
& \quad \Pi = \frac{1}{2} \int_0^L \left[ E I_x (u')^2 + E I_y (\phi')^2 + G K_x (\dot{\psi})^2 + 2 M_x \frac{\pi L}{L} u'' \right] \, dx \\
& \text{Solve (d) and (e) only if computer is available.}
\end{align*}
\]

3.5. A uniformly distributed load \( q \) acts through the centroid of a doubly symmetric cross section \((\kappa_0 = \gamma_0 = \beta_0 = 0)\)
   (a) Develop an expression for the total potential \( U = V_p \)
   (b) Develop an expression for \((M_{i})_{cr} = q_s L^3/8\) by the Rayleigh-Ritz method for

\[
\begin{align*}
 u &= A \sin \frac{\pi x}{L} + B \sin \frac{2\pi x}{L} \quad \phi = C \sin \frac{\pi x}{L} + D \sin \frac{2\pi x}{L} \\
\text{(c) Compare this expression with Eq. (3.42) and discuss the similarities and differences.}
\end{align*}
\]

3.6. Develop an expression for \((M_{i})_{cr}\) for channels.

P. 3.6

3.7. (a) Derive Eq.(3.60) and solve it for the beam of Fig. (3.16).
   (b) Solve this problem by finite difference method for spacings of \(L/10\) and \(L/12\). Improve answer by extrapolation (a computer is needed for this problem).

3.8. Derive Eqs. (3.55) and (3.56).

3.9. Solve the biaxial bending problem discussed at the end of Sec. 3.2 by the Rayleigh-Ritz method, assuming \( \phi = A \sin \pi z/L \). Note: Transform the total potential expression given at the end of Sec. 3.2 into a function of \( \phi \) only by noting that

\[
\begin{align*}
\psi'' &= \frac{1}{E I_y} (-M_{0e} - \phi M_{0y}) \\
u'' &= \frac{1}{E I_y} (M_{0y} - \phi M_{0x})
\end{align*}
\]
3.10. Work through in detail those portions of Sec. 3.3 which result in the curves of Fig. 3.26.


3.12. Review Refs. 3.3, 3.40, and 3.44 and compare them with the simplified treatment given to the lateral bracing and local buckling problem in Sec. 3.3 for beams in plastic design.

3.13. Ends are simply supported \( (\phi = \phi'' = u = u'' = v = v'' = 0) \) and \( u = \phi = 0 \) at the load points.

\[
L = 40 \text{ ft},
27WF94
A36 Steel
\]

P. 3.13

Determine the allowable value of \( Q \) by
(a) The 1963 AISC specifications.
(b) The 1966 AASHO specifications.
(c) Any method, using any degree of sophistication (text or Refs.). Factor of safety against lateral-torsional buckling must be 1.65.

3.14. A simply supported wide-flange beam is loaded by a vertical force \( Q \) through the centroid at the center. The bottom of the flange at the center is supported by an infinitely rigid pinned-end strut of length \( h \). What is the value of \( Q \) under which the steel beam can assume a deformed (buckled) configuration? What is the buckled mode shape?

\[
\varepsilon = 30,000 \text{ ksi}
\]
\[
\sigma = 11,800 \text{ ksi}
\]
\[
L = 300 \text{ in.}
\]
\[
h = 50 \text{ in.}
\]

P. 3.14

The Total Potential of a Thin-walled Beam

The strain energy per unit volume of an elastic body is defined as (Chapter 6, Ref. 2.1)

\[
U_0 = \frac{1}{2} \left( \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{xz} \gamma_{xz} \right)
\]  

(3A.1)

where \( \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz} \) are the normal and shear stresses acting on an element, and \( \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz} \) are the corresponding strains. For our problem, normal stresses exist only in the \( z \) direction \( (\sigma_z = \sigma, \sigma_x = \sigma_y = 0) \); see Fig. 2.1), and shear stresses occur only in the \( xy \) plane \( (\tau_{xy} = \tau, \tau_{yz} = \tau_{xz} = 0) \); see Fig. 2.4). Thus the total strain energy for a bar of length \( L \) and area \( \int A \) is equal to

\[
U = \frac{1}{2} \int_0^L \int_0^\varepsilon \left( \frac{\sigma^2}{E} + \frac{\tau^2}{G} \right) t \ ds \ dz
\]  

(3A.2)

In Eq. (3A.2) we used the relationship \( \varepsilon = \sigma/E \) and \( \gamma = \tau/G \) (Hooke's Law).
The normal stress at any cross section of the deformed bar is equal to [Eq. (2.74)]

\[
\sigma = \frac{P}{A} + \frac{M_2 y}{I_z} - \frac{M_3 x}{I_y} + E\omega_{0}\phi''
\]

For a beam \( P = 0 \), \( M_2 = -EI_\phi'' \), and \( M_3 = EI_\psi'' \) [Eqs. (2.27)].

Next we shall determine expressions for the strain energy due to each individual stress component by integrating Eq. (3A.2). The strain energy for \( \sigma = M_1 y/I_z = -E\psi'' \) is

\[
U_1 = \frac{1}{2} \int_0^L E(u'')^2 \int_0^y y'^2 t \, ds \, dz = \frac{1}{2} \int_0^L EI_\psi(u'')^2 \, dz
\]

(3A.3)

Similarly, \( U \) for \( -M_3 x/I_y \) is

\[
U_3 = \frac{1}{2} \int_0^L EI_\psi(\psi'')^2 \, dz
\]

(3A.4)

The strain energy for \( \sigma = E\omega_{0}\phi'' \) is

\[
U_4 = \frac{1}{2} \int_0^L E(\phi'')^2 \int_0^y \omega_0^2 t \, ds \, dz = \frac{1}{2} \int_0^L EI_\phi(\phi'')^2 \, dz
\]

(3A.5)

The shear stress is equal to the sum of the stresses from the shear forces \( V_x \) and \( V_y \), the warping shear stress, and the St. Venant torsional stress [Eq. (2.75)]. The stresses due to \( V_x \), \( V_y \), and warping are of a smaller order of magnitude than the St. Venant shear stress, and therefore they are usually neglected (as was done in the derivation of the differential equations of bending, Eqs. 2.93 through 2.95). Their effect is negligible unless the member is very short. The St. Venant shear stress at any location in the cross section is (Fig. 3A.1)

\[
\tau_{xy} = Gt\phi' \frac{2x}{t} = 2Gt\phi'\frac{x}{t}
\]

(3A.6)

The corresponding value of \( U \) is

\[
U_4 = \frac{1}{2} \int_0^L 4G(\phi'')^2 \int_0^y \frac{x^2}{t^2} d\tilde{z} \, ds \, dz = \frac{1}{2} \int_0^L G(\phi'')^2 \int_0^L \frac{x^2}{t^2} \, ds \, dz
\]

But \( \int_0^t \frac{t^3}{3} \, ds = K_t \), the St. Venant torsion constant [Eq. (2.35)], and so

\[
U_4 = \frac{1}{2} \int_0^L 4GK_t(\phi'')^2 \, dz
\]

(3A.7)

The total strain energy is now the sum of the four components

\[
U = \frac{1}{2} \int_0^L \left[ EI_\psi(u'')^2 + EI_\phi(\phi'')^2 + EI_\psi(\psi'')^2 + \frac{1}{2} GK_t(\phi'')^2 \right] \, dz
\]

(3A.8)
the following formula for the potential of the external forces:

\[ V_p = - \int_0^a \Delta L \frac{M_{ox} y}{I_x} \, ds \]  (3A.9)

The lengthening \( \Delta L \) of the strip \( ds \) consists of two components: (1) the strain due to the stresses acting along the length of the member, and (2) the shortening due to curvature [Fig. 3A.2(b)]. The first component is equal to

\[ \int_0^a e \, dz = \int_0^a \left( \frac{M_{iy}}{E I_y} - \frac{M_{ix}}{E I_x} + \omega_0 \phi'' \right) \, dz \]  (3A.10)

where the expression in the parentheses is the strain due to the forces \( M_t \), \( M_{ox} \), and \( M_{oy} \) at any cross section. The second component is computed from [Fig. 3A.2(b) and (c)] \( L + (\Delta L) = \int_0^a dL \), where \( dL = \sqrt{dz^2 + \left( \frac{du}{dz} \right)^2 + \left( \frac{dv}{dz} \right)^2} \) (\( u_0 \) and \( v_0 \) are the deflections of point Q, see Fig. 2.31). Since \( u_0 \) and \( v_0 \) are small,

\[ dL = dz \sqrt{1 + \left( \frac{du}{dz} \right)^2 + \left( \frac{dv}{dz} \right)^2} \approx dz \left[ 1 + \frac{1}{2} \left( \frac{du}{dz} \right)^2 + \frac{1}{2} \left( \frac{dv}{dz} \right)^2 \right] \]

From Eq. (2.77) we know that \( u_0 = u + \phi'(y_0 - y) \) and \( v_0 = v - \phi'(x_0 - x) \), and therefore

\[ dL = dz [1 + \frac{1}{2} (u' + \phi'(y_0 - y))^2 + \frac{1}{2} (v' - \phi'(x_0 - x))^2] \]

Noting that in Eq. (3.10) we can set \( M_t = -EI_y \phi'' \) and \( M_{ox} = EI_x \phi'' \) [Eqs. (2.27)], we obtain after some algebra the following expression for the elongation \( \Delta L \):

\[ \Delta L = \int_0^a e \, dz - (\Delta L)_0 = \int_0^a e \, dz - L + \int_0^a dL \]  (3A.11)

\[ \Delta L = \int_0^a \left\{ -y \phi'' - xu'' + \omega_0 \phi'' - \frac{1}{2} [u' + \phi'(y_0 - y)]^2 - \frac{1}{2} [v' - \phi'(x_0 - x)]^2 \right\} \, dz \]

If we substitute Eq. (3.11) into Eq. (3.9), perform the integrations, and note that \( \int_0^a x y \, ds = \int_0^a x \, ds \times \int_0^a y \, ds = \int_0^a y \, ds \times \int_0^a x \, ds = 0 \), \( \int_0^a y^2 \, ds = 2t \omega \), and

\[ \int_0^a x^2 \, ds = I_x \]

[Eq. (3.13)], then

\[ V_p = \frac{1}{2} \int_0^a [2M_{ox}y'' - 2M_{oo}u' + M_{ox} \beta_2 \phi'] \, ds \]  (3A.12)

The total potential \( \Pi = U + V_p \), or

\[ \Pi = \frac{1}{2} \int_0^a [EI_y (y'')^2 + EI_x (u'')^2 + EI_y (\phi'')^2 + G K_2 (\phi')^3 + 2M_{ox} (y'') - 2M_{oo} u' + M_{ox} \beta_2 \phi'] \, ds \]  (3A.13)

This equation is the total potential of a beam with a general thin-walled open cross section.

---

### Chapter Four

4.1. Introduction

**Columns**

Members subjected to a compressive axial load through the centroid of the member are called columns. Under some conditions of loading and framing many of the vertical members in multistory frames can be idealized as columns. It is customary also to design the compression elements of trusses in buildings and bridges as columns. Most of the members in such structures as transmission line towers, radio or television antennae, etc. are also columns. Columns thus represent an important constituent of the members which a structural designer encounters in his work.

**Historical Notes**

The attention of engineers and applied mathematicians was focused early on problems related to the instability of columns. Although the lateralel-torsional buckling problem of beams was not investigated until the end of
the 19th century, problems on column buckling were solved as early as 1752 by the mathematician Leonard Euler, who developed a column formula still in common use today.\(^{(1,40)}\) By the end of the 19th century many of the problems connected with elastic column buckling were well understood, and two working hypotheses about inelastic buckling were in existence.\(^{(2,28)}\) These two were the tangent modulus and the reduced modulus concepts which we discussed in Chapter 3. The controversy over these two concepts lasted until 1947, when Shanley finally brought them into a correct relationship to each other.\(^{(3,49)}\) In the 1940’s it was also realized that residual stresses play an important role in understanding inelastic column behavior.

As a consequence of more than two centuries of analytical and experimental study the behavior of columns is better understood than that of other types of members. Inasmuch as we could not hope to review all this work here, we shall cover only the highlights. The student may obtain more information from Refs. 1.29, 1.30, 1.34, 1.35, 1.36, 3.28, 4.1, and 4.2.

THE LOAD-DEFORMATION BEHAVIOR OF COLUMNS

There are two deformation parameters which are useful in describing the history of a column: (1) the axial shortening \(w\), and (2) the lateral deflection \(v\) (see lower right-hand corner in Fig. 4.1). Depending on the load-deformation history we have three kinds of columns: (1) short columns (or stub-columns), (2) columns of intermediate length, and (3) long columns. Schematic \(P-w\) and \(P-v\) curves for each of these columns is shown in Fig. 4.1(a) (b), and (c), respectively.

The stub column [Fig. 4.1(a)] acts essentially like a coupon (see Sec. 1.4), defining the behavior of the cross section; under load it will only shorten and will not deflect laterally. As we saw in Chapter 1, the \(P-w\) curve reflects the presence of residual stresses and the nonlinearity of the stress-strain curve. For a material with a stress-strain curve which becomes fully plastic at \(\sigma_y\), the stub-column curve will flatten out at the yield load \(P_y = A \sigma_y\).

Columns in frames and trusses are usually in the intermediate-length range [Fig. 4.1(b)]. For these columns lateral deflections begin at the tangent modulus load \(P_T\) after some portions of the cross section have already yielded. A relatively small increase of load is possible beyond \(P_T\) to \(P_x\), after which unloading of the load-deformation relationship begins.

Long columns buckle in the elastic range [Fig. 4.1(c)]. At the critical load it is possible to find two equilibrium configurations, one straight and one laterally deformed, and further lateral deflection occurs without an appreciable increase of load until yielding sets in at some part of the member, and as a result the load begins to drop off.

The load-deformation relationships described by the solid-line curves in Fig. 4.1 are for initially perfectly straight members. These curves must be modified owing to initial crookedness and the unavoidable eccentricities of the axial load. In the case of elastic buckling the load-deflection curve of initially crooked columns will approach \(P_x\) asymptotically as the lateral deflections increase [dashed part of \(P-v\) curve in Fig. 4.1(c)]. The presence of these imperfections reduces the maximum load which can be reached [Fig. 4.1(b)] for columns of intermediate length. In most instances it is sufficient to use the tangent modulus load as an index of column strength; however, in some cases the influence of even small initial imperfections needs to be considered for a safe analysis.

In the following discussion we shall first deal with long, that is, elastic
columns and then with problems in connection with columns of intermediate length (inelastic behavior).

4.2. ELASTIC COLUMNS

THE DIFFERENTIAL EQUATIONS OF ELASTIC BUCKLING

The practical maximum load which can be carried by an elastic column is the load under which the equilibrium bifurcates, or, in other words, the column buckles. One state of the equilibrium at buckling is the straight state, in which the column has undergone only axial shortening. The other state, which is the buckled configuration, may involve lateral deflections \( u \) or \( v \) in the \( x \) or \( y \) direction, respectively, or a twist \( \phi \) about the \( z \) axis in the case of a doubly symmetric cross section. For asymmetric sections the buckling mode will involve all three deformations \( u \), \( v \), and \( \phi \) [Fig. 2.31(a)].

The differential equations for prismatic members subjected to compressive axial forces at the ends have been derived in Chapter 2. For zero bending moments (that is, \( M_{xu} = M_{yu} = M_{xy} = M_{yx} = 0 \) in Fig. 2.30) these equations become [from Eqs. (2.93) through (2.95)]

\[
B_x v'' + P_v - P_x u' = 0 \quad (4.1)
\]

\[
B_y u'' + P_u + P_y v' = 0 \quad (4.2)
\]

\[
C_w \phi'''' - (C_{y} + \bar{K}) \phi' + P_{y} u' - P_{x} v' = 0 \quad (4.3)
\]

In the elastic range the coefficients \( B_x, B_y, C_w \), and \( C_{w} \) are defined as in Eqs. (2.90) through (2.92). The term \( \bar{K} \) is [from Eqs. (2.86) and (3.12)] equal to

\[
\bar{K} = \int_a^b a^2 dA = \int_a^b [(x_o - x)^2 + (y_o - y)^2] dA \quad (4.4)
\]

The stress \( \sigma \) is equal to

\[
\sigma = -\frac{P}{A} + \frac{M_x u}{I_x} - \frac{M_y v}{I_y} + E \omega \phi'' \quad (4.5)
\]

where \( P \) is compressive (Fig. 2.30), \( \sigma \) is positive when in tension, and

\[
M_x = P v - P_x u \phi \quad (4.6)
\]

\[
M_y = P u + P_y v \phi \quad (4.7)
\]

from Eqs. (2.80) through (2.82). In Eq. (4.3) \( \bar{K} \) occurs as a factor in the product \( \bar{K} \phi' \), and if we retain only the linear terms in \( u \), \( v \), and \( \phi \), as was done when the differential equations were derived in Chapter 2, then

\[
\bar{K} \phi' = -\left(\frac{P}{A}\right) \phi' \int_a^b [(x_o - x)^2 + (y_o - y)^2] dA \quad (4.8)
\]

After performing the integrations in Eq. (4.8) and noting that \( x \) and \( y \) are principal centroidal coordinates, we get the following expression for \( \bar{K} \):

\[
\bar{K} = -\frac{P R_o^2}{A} \quad (4.9)
\]

where

\[
r_o^2 = x_o^2 + y_o^2 + I_o^2 + I_x \quad (4.10)
\]

The differentiated versions of Eqs. (4.1), (4.2), and (4.3) can now be written as

\[
E I_x v'' + P_v - P_x u' = 0 \quad (4.11)
\]

\[
E I_y u'' + P_u + P_y v' = 0 \quad (4.12)
\]

\[
E I_x \phi'''' - (C_{y} + \bar{K}) \phi' + P_{y} u' - P_{x} v' = 0 \quad (4.13)
\]

SECTIONS WITH DOUBLE SYMMETRY

The three coupled differential equations above separate themselves in the case where \( x_o = y_o = 0 \) into three independent equations

\[
v'' + F_o^2 u'' = 0 \quad (4.14)
\]

\[
u'' + F_o^2 u'' = 0 \quad (4.15)
\]

\[
\phi'''' = 0 \quad (4.16)
\]

where

\[
F_o^2 = \frac{P}{E I_o}, \quad F_o^2 = \frac{P}{E I_o}, \quad F_o^2 = \frac{P R_o^2 - G K_t}{E I_o} \quad (4.17)
\]

The three equations are identical in form, and they have identical solutions. For example,

\[
v = C_1 \sin F_x x + C_2 \cos F_x x + C_3 x + C_4 \quad (4.18)
\]

Substitution of four boundary conditions results in four homogeneous linear simultaneous equations in the unknown constants \( C_i \) through \( C_4 \). The vanishing of the determinant of the coefficients of the constants furnishes us with the buckling condition, which in turn provides the critical load. The procedure is identical with that described in detail in connection with the lateraltorsional buckling of beams in Chapter 3. The determinant for simple boundary conditions \( [v(0) = v^{''}(0) = v(L) = v^{''}(L) = 0] \) is, for example,

\[
\begin{vmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & -F_o^2 & 0 \\
\sin F_x L & \cos F_x L & L & 1 \\
-F_o^2 \sin F_x L & -F_o^2 \cos F_x L & 0 & 0 \\
\end{vmatrix} = -LF_o^2 \sin F_x L = 0 \quad (4.19)
\]

from which the buckling condition becomes

\[
\sin F_x L = 0 \quad (4.20)
\]
Bifurcation can thus occur when

\[ F_n L = n \pi \quad (n = 1, 2, 3, \ldots) \quad (4.21) \]

or

\[ P_s = \frac{n \pi^3 EI_s}{L^3} \quad (4.22) \]

We have shown in Sec. 3.2 that only the lowest critical load is of interest, and so

\[ (P_s)_{cr} = \frac{\pi^3 EI_s}{L^3} \quad (4.23) \]

The critical loads for the other two buckling modes \((u\) and \(\phi\)) are equal to

\[ (P_u)_{cr} = \frac{\pi^3 EI_u}{L^3} \quad (4.24) \]

\[ (P_{\phi})_{cr} = \left[ \frac{\pi^3 EI_u}{L^3} + G K_r \right] \left[ \frac{1}{R_r^2} \right] \quad (4.25) \]

where

\[ R_r^2 = \frac{L_s + L_t}{A} \quad (4.26) \]

from Eq. (4.10).

A doubly symmetric prismatic column will thus have three independent buckling loads \((P_u)_{cr}, (P_{\phi})_{cr},\) and \((P_s)_{cr}\). In any situation the lowest of these governs, and therefore it is necessary that all three equations (4.23), (4.24), and (4.25) be checked. For rolled wide-flange profiles usually \((P_u)_{cr}\) is the smallest, but it is advisable to compute the torsional buckling load \((P_{\phi})_{cr}\) for cold formed light-gage steel columns and for aluminum columns.

**EXAMPLE PROBLEM ON THE BUCKLING OF WIDE-FLANGE COLUMNS**

We shall now illustrate the computational procedure by determining the elastic buckling strength of the wide-flange column shown in Fig. 4.2. The boundary conditions are

\[ u(0) = u''(0) = u(L) = u''(L) = \phi(0) = \phi''(0) = 0 \]

\[ u(L) = u''(L) = u'(L) = \phi(L) = \phi'(L) = 0 \]

that is, the member is simply supported about its strong axis and fixed about its weak axis, and warping is free at the top and prevented at the bottom. From Eq. (4.23) we have for the simply supported condition

\[ (P_s)_{cr} = \frac{\pi^3 EI_s}{L^3} \]

The deflection \(u\) is equal to [Eq. (4.18)]

\[ u = C_0 \sin F_s u + C_0 \cos F_s x + C_2 z + C_4 \quad (4.27) \]

Upon substitution of the boundary conditions we obtain the following characteristic determinant and buckling condition:

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
F_s & 0 & 1 & 0 \\
\sin F_s L & \cos F_s L & L & 1 \\
F_s \cos F_s L & -F_s \sin F_s L & 1 & 0
\end{bmatrix}
= \sin \frac{F_s L}{2} = 0 \quad (4.28)
\]
Equation (4.28) will be satisfied if
\[ \frac{F_s L}{Z} = n \pi \]  
(4.29)
and thus the lowest critical load \( n = 1 \) is equal to
\[ (P_e)_{cr} = \frac{4 \pi^2 EI_y}{L^2} \]  
(4.30)
The angle of twist is equal to [Eq. (4.18)]
\[ \phi = C_1 \sin F_s z + C_2 \cos F_s z + C_3 z + C_4 \]  
(4.31)
and after substitution of the boundary conditions for \( \phi \) and solving the characteristic determinant, we obtain
\[ \tan F_s L = F_s L \]  
(4.32)
as the buckling condition. This equation is solved by trial and error or by plotting one curve \( F_s = \tan F_s L \) and another curve \( F_s = F_s L \) and finding the lowest value of \( F_s L \) where these two curves intersect. This occurs for \( F_s L = 4.493 \). After substitution of \( F_s \) from Eq. (4.17) we obtain the following expression for the critical load \( (P_e)_{cr} \):
\[ (P_e)_{cr} = \frac{GK_y + 20.19 EI_y L^3}{r_s^2} \]  
(4.33)
We now nondimensionalize Eqs. (4.23), (4.30), and (4.33) by \( P_y = A \sigma_y \), noting that \( I_x = A r_s^2 \), \( I_y = A r_s^3 \), and \( I_z = (I_x + I_y) / A \). After some algebraic manipulation we obtain the following expressions:
\[ \left( \frac{P_e}{P_y} \right)_{cr} = \frac{\pi^2 E}{\sigma_y (L/r_s)^2} \]  
(4.34)
\[ \left( \frac{P_e}{P_y} \right)_{cr} = \frac{4 \pi^2 E}{\sigma_y (L/r_s)^2} = \frac{4 \pi^2 E (r_y/r_s)^3}{\sigma_y (L/r_s)^2} \]  
(4.35)
\[ \left( \frac{P_e}{P_y} \right)_{cr} = \left[ \frac{\pi^2 E G K_y}{\sigma_y (I_z + I_y)} \left( 1 + \frac{\pi^2 E I_y}{G K_y (0.6999 L/r_s)} \right) \right] \]  
(4.36)
The curves showing the relationship between \( (P_e)_{cr} \), \( (P_y)_{cr} \), \( (P_e)_{cr} \), and \( (P_y)_{cr} \), and the strong axis slenderness ratio \( L/r_s \) are given in Fig. 4.2 for a 24WF84 rolled beam section for which \( G/E = 0.385 \). For this shape \( r_y = 10.69 \) in., \( r_y = 19.17 \) in., \( I_x = 2825 \) in.\(^4 \), \( I_y = 95.7 \) in.\(^4 \), \( K_y = 3.00 \) in.\(^4 \), and \( I_z = 16,241 \) in.\(^4 \). For these geometric and material properties Eqs. (4.34) through (4.36) become equal to
\[ \left( \frac{P_e}{P_y} \right)_{cr} = \frac{1}{\sigma_y (L/r_s)^2} \left[ \frac{P_e}{P_y} \right]_{cr} \pi^2 E \sigma_y = 0.136 \]  
(4.37)
\[ \left( \frac{P_e}{P_y} \right)_{cr} = (0.401) (10^{-4}) \left[ 1 + \frac{2499}{(L/r_s)^2} \right] = \left[ 0.136 \right] \]  
(4.37)
The curves in Fig. 4.2 show that the strong axis buckling will not occur, because \( (P_e)_{cr} \) is always greater than \( (P_y)_{cr} \) or \( (P_e)_{cr} \). Weak axis buckling will govern for \( L > 32 r_s \); below this length torsional buckling will control. Since \( 32 r_s = 332 \) in., which is a relatively large length, torsional buckling will occur for most practical lengths for this particular column.

**BOUNDARY CONDITIONS**

The influence of boundary conditions other than simple supports can be incorporated into the critical-strength equations by making use of the effective length concept. We can express the equations for \( P_x \), \( P_y \), and \( P_z \) as
\[ (P_x)_{cr} = \frac{\pi^2 EI_x}{(K_x L)^2}, \quad (P_y)_{cr} = \frac{\pi^2 EI_y}{(K_y L)^2}, \quad (P_z)_{cr} = \frac{GK_y + \pi^2 EI_y (K_y L)^2}{r_s^2} \]  
(4.38)
The coefficients \( K_x \), \( K_y \), and \( K_z \) are coefficients by which the length \( L \) is multiplied so that the formulas for simple supports can be used. If both ends of a column are simply supported, then \( K = 1.0 \); if both ends are fixed, \( K = 0.5 \); and if one end is simple and one is fixed, then \( K = 0.699 \approx 0.70 \), as we have seen for the example of Fig. 4.2. Usually the end fixity is neither fixed nor simple, and thus the effective length factors lie between or outside the values given above. These factors can be computed from a knowledge of the end restraint conditions as shown in the following two examples:

**Example of Fig. 4.3.** The sketch in Fig. 4.3 shows a column which is pinned at the bottom and restrained by a beam at the top. We shall consider the effect of the restraint offered by this beam on the critical load for buckling in the plane of the framework where the buckled shape will consist only of the deflection \( \psi \).

The boundary conditions at the bottom of the frame (\( z = L \)) are \( \psi = \frac{P_y}{P_y} = \frac{20.19 EI_y}{L^2} \). For these geometric and material properties Eqs. (4.34) through (4.36) become equal to
\[ \left( \frac{P_y}{P_y} \right)_{cr} = \frac{1}{\sigma_y (L/r_s)^2} \left[ \frac{P_y}{P_y} \right]_{cr} \pi^2 E \sigma_y = 0.136 \]  
(4.37)
\[ \left( \frac{P_y}{P_y} \right)_{cr} = (0.401) (10^{-4}) \left[ 1 + \frac{2499}{(L/r_s)^2} \right] = \left[ 0.136 \right] \]  
(4.37)

![Fig. 4.3. Buckling strength curve for a restrained column](image-url)
In the case of a restrained column, the deflection is prevented at the top and bottom, so \( v'' = 0 \), and the moment at the top is \( M(0) = 0 \). Thus, the restraint condition is such that the column is prevented from deflection. During buckling, the column will deform as shown in Fig. 4.4. The rotation of the joint is the slope of the deformed curve, and it is equal to \( v'(0) \). The restraining beam will be subjected to a restraining moment \( M_r \), and the column will have a moment \( M(0) \) at this joint. Equilibrium requires that

\[
M_r + M(0) = 0 \quad (4.39)
\]

The value of \( M_r \) for the beam end can be computed by the moment-area method, and it is equal to

\[
M_r = \left( \frac{4EI_z}{L_a} \right) v'(0) \quad (4.40)
\]

The moment \( M(0) \) is equal to [Eq. (2.27)]

\[
M(0) = -v''(0) EI_c \quad (4.41)
\]

If we set Eqs. (4.40) and (4.41) into (4.39) we obtain the fourth required boundary condition

\[
\left( \frac{4EI_z}{L_a} \right) v'(0) - EI_c v''(0) = 0 \quad (4.42)
\]

Now we substitute the expression for \( v \) [Eq. (4.18)] and its derivatives into the four boundary conditions, and we obtain the characteristic determinant. The solution of this determinant furnishes the following buckling condition:

\[
\tan F_a L_c = \frac{4\gamma F_a L_c}{(F_a L_c)^2 + 4\gamma} \quad (4.43)
\]

where

\[
\gamma = \frac{L_c I_a}{L_a I_c} \quad (4.44)
\]

and

\[
(F_a L_c)^2 = \frac{P L_c}{EI_a} \quad (4.45)
\]

The limits of Eq. (4.43) occur when the restraining beam is infinitely stiff \( (I_b = \infty, \gamma = \infty) \) and when this beam is infinitely soft \( (I_b = 0, \gamma = 0) \). These two limits correspond to a fixed and a simply supported end at the top, respectively. We already know from our previous example that \( P_{cr} = 20.19EI/L_a^2 \) if one end of the column is fixed and \( P_{cr} = \pi^4 EI/L_a^2 \) if both ends are simply supported. Indeed, for \( \gamma = \infty \) Eq. (4.43) gives the relationship

\[
\tan F_a L_c = F_a L_c \quad (4.46)
\]

and for \( \gamma = 0 \)

\[
\sin F_a L_c = 0 \quad (4.47)
\]

These two equations correspond to the buckling conditions of the limiting cases, as can be seen by comparing them with Eqs. (4.32) and (4.20), respectively.

The relationship expressed by Eq. (4.43) between these limits is shown in Fig. 4.3. In this representation \( \gamma \) (the abscissa) is an index of frame geometry and \( P/P_x \) (the ordinate) corresponds to the buckling strength of the frame. The load \( P_x \) is the critical load for a simply supported column; it is defined by Eq. (4.23) as being equal to \( \pi^4 EI/L_a^2 \). The connection between the parameter \( F_a L_c \) in Eq. (4.43) and \( P_{cr}/P_x \) is defined by the following formula [see Eq. (4.17)]:

\[
\frac{P_{cr}}{P_x} = \frac{1}{\pi^4} (F_a L_c) \quad (4.48)
\]

From Fig. 4.3 we can see that a small increase of restraint (that is, \( \gamma \)) on the left part of the curve results in a relatively large increase in the buckling strength, whereas no great improvement in the buckling strength is achieved by adding a great deal of stiffness to the restraining beam. The effect of a finite amount of restraint can give full fixity for all practical purposes and that in design calculations it pays to include the effect of even a small amount of restraint.

For any given value of \( \gamma \) we can determine \( P_{cr} \) from Fig. 4.3 or by the formula [Eq. (4.38)]

\[
P_{cr} = \pi^4 EI_c (K_x L_c)^2 \quad (4.49)
\]

where \( K_x \) is the effective length factor of the column. This factor is equal to

\[
K_x = \frac{1}{\sqrt{P_{cr}/P_x}} \quad (4.50)
\]

It varies from \( K_x = 1.0 \) for a pinned end (\( \gamma = 0 \)) to \( K_x = 0.699 \) for a fixed end (\( \gamma = \infty \)), and for \( \gamma > 5 \) we can take \( K_x = 0.7 \) for practical purposes (full fixity at the top).

**Example of Fig. 4.5.** The sketch shows a column which is simply supported at the top (at \( z = L \)) and is restrained against twisting at the top in the horizontal plane by four beams of stiffness \( EI_b \) and length \( L_b \). The beams are attached to the end of the column by a collar which confines the deformation of the four restraining beams in a plane perpendicular to the \( z \) axis. This means that the top of the column can twist but it may not warp, or \( \psi'(0) = 0 \). The fourth boundary condition is provided by the restraint...
column must be equal to zero, or
\[ 4M_t + M_i(0) = \frac{12EI_b}{L_b} \phi(0) + M_i(0) = 0 \]  
(4.51)

The sense of the moment at the column top is negative with respect to the
z axis, and so \( M_i(0) \) is equal to [Eq. (2.65)]
\[ M_i(0) = -GK_T \phi'(0) + EL_1 \phi''(0) \]  
(4.52)

By substituting Eq. (4.52) into Eq. (4.51) we obtain the boundary condition
\[ \frac{12EI_b}{L_b} \phi(0) - GK_T \phi'(0) + EL_1 \phi''(0) = 0 \]  
(4.53)

If we now set our expression for \( \phi \) [Eq. (4.31)] into this and the other three
boundary conditions \([\phi'(0) = \phi(L) = \phi''(L) = 0] \), we can construct the
characteristic determinant, which when solved gives the following buckling
condition:
\[ \frac{1 - \tan F_T L}{(F_T L)^3} = \frac{I_b L_g}{12I_b L^3} \]  
(4.54)

The limits of Eq. (4.54) occur when \( I_b = 0 \) (no end restraint) and \( I_b = \infty \) (full fixity). The value of \( K_T L \) for these two limits is \( \pi/2 \) and 4.493, respectively.

From Eq. (4.17) we see that
\[ (P_c)_T = \frac{GK_T + (F_T L)^3 EI_a/L_3}{F_T^3} \]  
(4.55)

which can also be written as
\[ (P_c)_T = \frac{GK_T + \pi^2 EI_a(L_3)^3}{F_T^3} \]  
(4.56)

if \( K_T \), the effective length factor, is equal to
\[ K_T = \frac{\pi}{F_T L} \]  
(4.57)

The variation of \( K_T \) with \( L_a/L_3 \) is shown as the curve in Fig. 4.5. If \( I_b = \infty \), \( K_T = 0.699 \), corresponding to full fixity at the top. When no
restraint is present, this factor is equal to 2.00. Furthermore, we can see that for
practical purposes the top of the column is free to rotate if \( I_b L_a/12I_b L^3 > 5 \).

The two examples of Figs. 4.3 and 4.5 were solved to illustrate how the
critical loads of restrained axially loaded columns can be determined.
Columns in frames and trusses are usually restrained at their ends, and we could
find their elastic buckling strength by the same methods. Because of the
complex interaction between many members in such structures, we need,
however, more efficient procedures. These are discussed in Sec. 4.3.
The differential equations for asymmetric columns are Eqs. (4.11), (4.12), and (4.13). It can be shown, although this demonstration is omitted here, that these equations are fulfilled for simply supported boundary conditions if:

\[ v = C_1 \sin \frac{\pi z}{L}, \quad u = C_2 \sin \frac{\pi z}{L}, \quad \phi = C_3 \sin \frac{\pi z}{L} \]  

Substitution of the deflections and their derivatives into the differential equations gives the following three homogeneous simultaneous equations:

\[ C_1 \left( \frac{\pi^2 EI_z}{L^3} - P \right) + C_2 \left( P x_0 \right) = 0 \]
\[ C_2 \left( \frac{\pi^2 EI_z}{L^3} - P \right) + C_3 \left( -P y_0 \right) = 0 \]
\[ C_3 \left( \frac{\pi^2 EI_z}{L^3} + G K_x - P \phi \right) = 0 \]

(4.58)

The vanishing of the determinant formed by the coefficients of \( C_1, C_2, \) and \( C_3 \) gives the following buckling condition:

\[ (P - P_s)(P - P_p)(P - P_r) - P^2(P - P_p) \left( \frac{x_0^2}{L^2} \right) - P(P - P_z) \left( \frac{x_0^2}{L^2} \right) = 0 \]

(4.59)

where \( P_s, P_p, P_r \) are defined by Eqs. (4.23) through (4.25).

The lowest root of Eq. (4.59) is the critical buckling load of the simply supported column. This critical value of \( P \) is always less than either \( P_s, P_p, \) or \( P_r \), and must therefore be computed. We can prove this as follows: Let us call the left side of Eq. (4.59) \( F(P) \) and assume arbitrarily that \( P_s < P_p < P_r \). Then

for \( P = 0 \) \( F(P) = -P_s P_p P_r \) (negative)

for \( P = P_s \) \( F(P) = -P_s(P_s - P_r)x_0^2 (\frac{x_0^2}{L^2}) \) (positive)

for \( P = P_p \) \( F(P) = -P_p(P_p - P_s)x_0^2 (\frac{x_0^2}{L^2}) \) (negative)

for \( P = P_r \) \( F(P) = -P_r(P_r - P_s)x_0^2 (\frac{x_0^2}{L^2}) \) + \( (P_r - P_s)x_0^2 \) (negative)

and for \( P \to \infty \), \( F(P) \) is positive.

The shape of the resulting cubic curve is given in Fig. 4.7. According to this curve the lowest root must be smaller than either \( P_s, P_p, \) or \( P_r \). The same conclusion can be reached for any combination of the relative magnitudes of these three loads.

Sections with only one axis of symmetry occur frequently (channels, tee

sections, etc.). If we set, for example, \( x_0 = 0 \), the differential equations are

\[ EI_u'' - P u'' = 0 \]

\[ EI_u'' + Pu'' + Py_0 \phi'' = 0 \]

\[ EI_u + (P \phi - G K_x) + Py_0 \phi'' = 0 \]

(4.60)

(4.61)

(4.62)

The first of these is independent of the other two, and it provides one critical load

\[ P_s = \frac{\pi^4 EI_z}{L^3} \]

(4.63)

for the case of simply supported ends. The solution of Eqs. (4.61) and (4.62) for simple boundary conditions is given by Eq. (4.58) \( \phi \) and \( \phi \), and the critical load can be computed from the following buckling condition:

\[ \frac{\pi^4 EI_z}{L^3} \left( P_s - P_p \right) - (P_s x_0^2) = 0 \]

(4.64)

This is a quadratic equation, and therefore \( P \) can be also expressed directly as

\[ P = \left( P_s + P_p \right) \left[ 1 - \sqrt{\frac{4P_s P_p J}{(P_s + P_p) J}} \right] \]

(4.65)

where

\[ J = 1 - \frac{y_0^2}{x_0^2} = \frac{1}{1 + A \phi_0 (I_s + I_p)} \]

(4.66)

In any situation both \( P_s \) from Eq. (4.63) and \( P \) from Eq. (4.65) must be computed; the lowest value of \( P \) controls the design.

We shall take for an example a simply supported column of a hat-shaped cross section which is formed out of a thin sheet of uniform thickness \( t \). The cross section and its properties, as well as formulas for \( P_s, P_p, \) and \( P_r \) are

\[ F(P) = F(P') = \frac{P}{P'} \]

\[ P = F(P) x_0^2 \]

\[ F(P) = F(P') = \frac{P}{P'} \]

\[ P = F(P) x_0^2 \]
Cross sectional properties:

\[ A = 6bh \]
\[ I_x = \frac{22}{3} b^3 t = 7.333 b^3 t \]
\[ I_y = \frac{7b^4 t}{6} = 1.167 b^4 t \]
\[ I_w = \frac{3.5}{22} b^4 t = 0.591 b^4 t \]
\[ K = 2b^3 t \]
\[ J_b = 2.066 b^2 \]
\[ J = 0.679 \]

Buckling properties:

\[ \frac{P_x}{P_y} = 12.06 \left( \frac{E}{\sigma_y} \right) \left( \frac{L}{R_0} \right)^2 \]
\[ \frac{P_y}{P_y} = 1.919 \left( \frac{E}{\sigma_y} \right) \left( \frac{L}{R_0} \right)^2 \]
\[ \frac{P_x}{P_y} = 0.160 \left( \frac{G}{\sigma_y} \right) + 0.466 \left( \frac{E}{\sigma_y} \right) \left( \frac{L}{R_0} \right)^2 \]

Fig. 4.9. Column curves for an aluminum alloy hat-shaped column

Fig. 4.10. Effect of material properties on column strength

given in Fig. 4.8. The curves showing the variation between the slenderness parameter \( L/b \) and the four loads \( P_x, P_y, P_n, \) and \( P_{tr} \) are shown in Fig. 4.9 for the case of \( b/t = 10 \) and for an aluminum alloy column. The critical load from Eq. (4.65) governs for lengths less than \( 52.6b \); for longer columns, buckling along the \( y \) axis governs.

The curves in Fig. 4.10 afford a study of the effect of the variation of the material properties \( \sigma_y, E, \) and \( G. \) The curves show the buckling strength envelopes for the cross section of Fig. 4.8 for A36 steel (\( \sigma_y = 36 \) ksi), A441 steel (\( \sigma_y = 50 \) ksi), and for an aluminum alloy (\( \sigma_y = 37 \) ksi). For the two types of steel having the same value of \( G \) and \( E, \) it is seen that the higher-strength steel has a lower relative buckling strength. The aluminum section with approximately the same yield stress as the A36 steel is considerably weaker because of its smaller value of \( G \) and \( E. \)

So far we have talked only about simply supported asymmetric columns. The differential equations could be solved also for other boundary conditions, and in simple cases this would not be too difficult. A relatively easy approximate solution can be attained by using the effective length factors \( K_x, K_y, \) and \( K_z \) for the individual loads \( P_x, P_y, \) and \( P_n, \) and then solving for the critical loads from Eq. (4.59) or (4.65), whichever is appropriate for the problem under consideration.
4.3. ELASTIC BUCKLING OF FRAMES

In the analysis of the strength of plane frames (see Chapter 6) it is necessary to know the elastic buckling load of the frame in the plane of the frame. The knowledge of this load in itself is not enough to assess the strength of the frame, but it is one important factor which must be considered, as we shall see in Chapter 6.

The sketch in Fig. 4.11 shows a simple frame with axial loads acting on the column tops. If \( P \) is less than a critical value, the frame is undeformed except for the shortening of the columns. When \( P = P_c \), a buckled shape is possible (dashed lines in Fig. 4.11). We are interested in the value of this critical load. It can be obtained in the same manner by which we found the critical load of the structure in Fig. 4.3, that is, by substituting the boundary conditions into the deflection equation and setting the resulting buckling determinant equal to zero. For frames with many members such a procedure is, however, too complex, and an approximate procedure must be found. We shall derive one possible approximate procedure after the introduction of the slope-deflection equation method of solving frame-buckling problems.

**SLOPE-DEFLECTION EQUATIONS**

The slope-deflection equations of classical indeterminate structural analysis relate the end rotations \( \theta_a \) and \( \theta_b \) and the bar rotation \( \rho \) to the end moments \( M_a \) and \( M_b \) [Fig. 4.12(d)]. For a prismatic bar these equations are equal to:

\[
M_a = \frac{EI_x}{L} [2\theta_a + \theta_b - 6\rho], \quad M_b = \frac{EI_x}{L} [2\theta_a + 4\theta_b - 6\rho]
\] (4.67)

Equations (4.67) are valid only if \( P = 0 \). In the presence of \( P \) we need new equations, and these are derived as follows:

The differential equation governing the deformations of the beam-column in Fig. 4.12(a) is [from Eq. (2.93), after a twofold integration],

\[
EI_x \psi'' + P \psi'' = 0
\] (4.68)

if the member has double symmetry. This equation is identical with Eq. (4.14) and its solution is given therefore by Eq. (4.18). The unknown coefficients \( C_1 \) through \( C_4 \) are found from the boundary conditions \( \psi(0) = \psi(L) = \psi''(0) = -M_a/EI_x \), and the deflection is

\[
v = \frac{M_a}{P} \left( \cos F_{z} - \frac{\sin F_{z}}{\tan \theta_a + \frac{z}{F_{x}L} - 1} \right)
\] (4.69)

![Fig. 4.12. Deformations of a beam-column](image)
The end rotations are obtained by differentiation, that is,

\[ \psi'(0) = \theta_A = \frac{M_A}{P} \left( \frac{1}{L} - \frac{F_v}{\tan \frac{F_v}{L}} \right) \]  
(4.70)

\[ \psi'(L) = \theta_B = \frac{M_B}{P} \left( \frac{1}{L} - \frac{F_v}{\sin \frac{F_v}{L}} \right) \]  
(4.71)

In a similar way we can show that the end rotations for the case of Fig. 4.12(b) are

\[ \theta_A = \frac{M_B}{P} \left( \frac{1}{L} - \frac{F_v}{\sin \frac{F_v}{L}} \right) \]  
(4.72)

\[ \theta_B = \frac{M_B}{P} \left( \frac{1}{L} - \frac{F_v}{\tan \frac{F_v}{L}} \right) \]  
(4.73)

We can superimpose these two sets of rotations to obtain the end slopes for the bar of Fig. 4.12(c), that is,

\[ \theta_A = \frac{L(cM_A - sM_B)}{EI_a}, \quad \theta_B = \frac{L(-sM_A + cM_B)}{EI_a} \]  
(4.74)

where

\[ c = \frac{1 - F_vL/\tan \frac{F_v}{L}}{E(I_aF_v)}, \quad s = \frac{(F_vL/\sin \frac{F_v}{L}) - 1}{(F_vL)^2} \]  
(4.75)

If in addition we also have a bar rotation \( \rho \) [Fig. 4.12(d)], then

\[ \theta_A = \frac{L(cM_A - sM_B)}{EI_a} + \rho, \quad \theta_B = \frac{L(-sM_A + cM_B)}{EI_a} + \rho \]  
(4.76)

To get Eqs. (4.76) into the form of Eqs. (4.67), we solve them for \( M_A \) and \( M_B \), that is,

\[ M_A = \frac{EI_a}{L} [C\theta_A + S\theta_B - \rho(C + S)] \]  
(4.77)

\[ M_B = \frac{EI_a}{L} [S\theta_A + C\theta_B - \rho(C + S)] \]  
(4.78)

where

\[ C = \frac{c}{c^2 - s^2}; \quad S = \frac{s}{c^2 - s^2} \]  
(4.79)

Equations (4.77) and (4.78) are the slope-deflection equations modified for axial load. When \( P = 0 \), it can be shown that \( C = 4 \) and \( S = 2 \). With these values Eqs. (4.77) and (4.78) are identical with Eqs. (4.67). In these equations the rotations and the moments are positive when clockwise; therefore the forces shown in Fig. 4.12(d) are positive.

We shall next solve the problem of Fig. 4.3 by the slope-deflection equations. The sketches in Fig. 4.13 show the buckled shape of the frame, with a positive joint rotation \( \theta_B \) (clockwise) and positive moments. The slope-deflection equations for the ends of the two members are

\[ M_{AB} = \frac{EI_a}{L_o} [C_{AB}\theta_A + S_{AB}\theta_B - \rho_{AB}(C_{AB} + S_{AB})] \]

\[ M_{BA} = \frac{EI_a}{L_o} [S_{AB}\theta_A + C_{AB}\theta_B - \rho_{AB}(C_{AB} + S_{AB})] \]

\[ M_{BC} = \frac{EI_B}{L_o} [C_{BC}\theta_B + S_{BC}\theta_C - \rho_{BC}(C_{BC} + S_{BC})] \]

\[ M_{CB} = \frac{EI_B}{L_o} [S_{BC}\theta_B + C_{BC}\theta_C - \rho_{BC}(C_{BC} + S_{BC})] \]

These equations can be considerably simplified by noting that joint B does not translate, and thus \( \rho_{AB} = \rho_{BC} = 0 \). There is no axial force in bar BC, hence \( C_{BC} = 4 \) and \( S_{BC} = 2 \). We also note that \( \theta_C = 0 \) and \( M_{AB} = 0 \). With the abbreviation \( S_{AB} = S \) and \( C_{AB} = C \), the slope-deflection equations become

\[ 0 = C\theta_A + S\theta_B, \quad M_{BA} = \frac{EI_a(S\theta_A + C\theta_B)}{L_o} \]

\[ M_{AB} = \frac{4EI_a\theta_B}{L_o}, \quad M_{CB} = \frac{2EI_B\theta_B}{L_o} \]

Equilibrium requires that \( M_{BA} + M_{BC} = 0 \); thus

\[ \frac{EI_a(S\theta_A + C\theta_B)}{L_o} + \frac{4EI_a\theta_B}{L_o} = 0 \]

If we substitute \( \theta_A = -S\theta_B/C \) into this equation and note that \( \gamma = L_oI_B/L_oI_C \) [Eq. (4.44)], then

\[ \theta_B \left( -\frac{S^2}{C} + C + 4\gamma \right) = 0 \]  
(4.80)

But \( \theta_B \neq 0 \) in the buckled shape, and so the critical condition is

\[ -\frac{S^2}{C} + C + 4\gamma = 0 \]  
(4.81)
It can be shown by using Eqs. (4.75) and (4.79) that Eq. (4.81) is identical with Eq. (4.43), and so the problem is solved.

The advantage of the slope deflection equation method over the previous procedure is that values of $S$ and $C$ corresponding to practical values of $F_c/L$ are available in tabulated form (Refs. 1.30, 4.3, and in Appendix 4A).

**A Simple Frame Buckling Problem**

The slope deflection method will now be illustrated in solving the problem of a single-story, single-bay rectangular frame (Fig. 4.14). This frame may buckle in a mode in which the top of the frame does not translate [Fig. 4.15(a)] or in a mode where the frame can sway [Fig. 4.16(a)]. We denote these cases as the nonway and the sway mode of buckling, respectively. We turn our attention first to the nonway condition (Fig. 4.15).

In Fig. 4.15(a) we see the positive end rotations $\theta$ and the positive end moments $M$ of each bar. These deformations and moments exist just after buckling when $P$ is its critical value. There are no bar rotations $\rho$, and the axial force in the horizontal bar is zero (hence $C_{bc} = 4$ and $S_{bc} = 2$). The axial force in each column is $P$, and thus $C_{ab} = C_{cd}$ and $S_{ab} = S_{cd}$. The values of the coefficients $C$ and $S$ in the slope deflection equations are computed on the basis of the force distribution just before buckling when $P = P_c$. An instant after buckling the axial force distribution in the members will change. This change is depend-

![Fig. 4.14. Dimensions of a rigid frame](image1)

**Fig. 4.14.** Dimensions of a rigid frame

![Fig. 4.15. Nonway buckling of the rigid frame of Fig. 4.14](image2)

**Fig. 4.15.** Nonway buckling of the rigid frame of Fig. 4.14

ent on the deformations. However, we are only interested in the happenings at the instant of buckling, so these deformations can be infinitesimally small.

The slope deflection equation for joint $A$ [Fig. 4.15(a)] is, from Eq. (4.77),

$$M_{aa} = 0 = (EI_c/L_c)(C_{ab} \theta_a + S_{ab} \theta_b),$$

$$\theta_a = -S_{ab} \theta_b / C$$  \hspace{1cm} (4.82)

Similarly, $\theta_b = -S_{bc} \theta_c/C$. At the upper end of column $AB$

$$M_{ab} = EI_c \frac{C_{ab} (S_{ab} \theta_a + C_{ab} \theta_b)}{L_c} = EI_c \left( \frac{C_{ab} - S_{ab}^2}{C} \right) \theta_b$$  \hspace{1cm} (4.83)

By the same procedure we obtain

$$M_{cd} = EI_c \left( \frac{C_{cd} - S_{cd}^2}{C} \right) \theta_c$$  \hspace{1cm} (4.84)

For the ends of the beam (member $BC$) we get

$$M_{bc} = EI_a (4 \theta_c + 2 \theta_b) / L_b, \quad M_{db} = EI_a (2 \theta_b + 4 \theta_c) / L_b$$  \hspace{1cm} (4.85)

The joint equilibrium conditions are

$$M_{ba} + M_{bc} = 0 \quad \text{and} \quad M_{cb} + M_{cd} = 0$$  \hspace{1cm} (4.86)

Substitution of Eqs. (4.83), (4.84), and (4.85) into Eqs. (4.86) results in the following two simultaneous equations with unknowns $\theta_a$ and $\theta_c$:

$$\left( \frac{C_{ab} - S_{ab}^2}{C} + 4 \gamma \right) \theta_a + (2 \gamma) \theta_c = 0$$  \hspace{1cm} (4.87)

$$\left( 2 \gamma \right) \theta_a + \left( \frac{C_{cd} - S_{cd}^2}{C} + 4 \gamma \right) \theta_c = 0$$  \hspace{1cm} (4.88)
where \( \gamma = L_{o}L_{o}/L_{o}L_{o} [\text{Eq. (4.44)}] \). The vanishing of the determinant of the coefficients of the unknowns gives the following buckling condition:

\[
\left( \frac{C^2 - S^2}{C} \right)^2 + 2\gamma = 0
\]  

(4.89)

This equation relates the critical buckling load \( P_{cr} \) with the dimensional properties of the frame \( (L_{o}, L_{o}, E, I_{o}, \text{ and } I_{o}) \). It is best solved by assuming a value of \( P_{cr}L_{o} = PL_{o}/E I_{o} \), looking up the corresponding values of \( C \) and \( S \) in the tables (see Appendix 4A) and finally solving for \( \gamma \). The corresponding curve is shown as the upper curve in Fig. 4.17(a). The limits of this curve are \( \pi^2 \lesssim P_{cr}L_{o}/E I_{o} \lesssim 20.19 \) for \( 0 \lesssim \gamma \lesssim \infty \). These limits correspond to the case of a pinned-ended column \( (\gamma = 0) \) and a column with one end pinned and one end fixed \( (\gamma = \infty) \), respectively. The limiting critical loads can also be computed by setting \( \gamma = 0 \) and \( \gamma = \infty \) into Eq. (4.87) and finding \( P_{cr} = \pi^2 E_{o}L_{o}^2 \) and \( P_{cr} = 2P_{cr}L_{o}^2/\pi^2 \).

The results of the calculation are more conveniently expressed by using effective lengths. That is, \( P_{cr} = \pi^2 E_{o}L_{o}/K_{s}L_{o}^2 \), where \( K_{s} = \sqrt{P_{cr}L_{o}/E I_{o}} [\text{Eq. (4.49)}] \). The \( K_{s} \) versus \( \gamma \) curve is shown as the lower curve in Fig. 4.17(b). For any given frame we can now determine \( K_{s} \) from this curve. The effective length factor varies from 1.0 to 0.699 for \( 0 \leq \gamma \leq \infty \), as expected.

The final information to be obtained from the calculations is the shape of the buckled frame. From Eq. (4.87) we find that \( \theta_{o} = -(1/2\gamma)[(C^2 - S^2)/C + 4\gamma] \theta_{o} \), and from the buckling condition \( 2\gamma = -(C^2 - S^2)/C \). Thus \( \theta_{o} = -\theta_{o} \) and from Eq. (4.82), \( \theta_{a} = -S\theta_{b}/C \) and \( \theta_{b} = S\theta_{b}/C \). All joint rotations are now expressed as functions of \( \theta_{o} \). The value of \( \theta_{o} \) cannot be determined from our analysis; however, we can define the shape of the buckled frame. The two possible buckled shapes are given in Figs. 4.15(b) and (c). It does not matter into which of the two modes the frame buckles, since the critical load is the same.

The critical load of the sway mode of buckling can be computed in a similar manner (Fig. 4.16). Here again \( C_{o} = 4, S_{o} = 2, C_{ab} = C_{cd}, \) and \( S_{ab} = S_{cb} \). There is no bar rotation \( \rho_{bc} \), and we can assume that \( \rho_{ab} = \rho_{cd} = \rho \). This can be done because the axial force in bar \( BC \) is zero.

Since \( M \) at the bases is zero, that is, \( M_{ab} = 0 = E_{o}I_{o}[C\theta_{a} + S\theta_{b} - \rho(C + S)] \), the base rotations are

\[
\theta_{a} = -S\theta_{b}/C + (C + S)\rho \quad \text{and} \quad \theta_{b} = -S\theta_{b}/C + (C + S)\rho
\]  

(4.90)

The four slope deflection equations are

\[
M_{ab} = E_{o}(C^2 - S^2)(\theta_{b} - \rho)/C_{o}, \quad M_{cd} = E_{o}(C^2 - S^2)(\theta_{c} - \rho)/C_{o}
\]  

\[
M_{bc} = E_{o}(4\theta_{b} + 2\theta_{c})/L_{b}, \quad M_{bc} = E_{o}(2\theta_{b} + 4\theta_{c})/L_{b}
\]  

(4.91)

The equilibrium conditions at joints \( B \) and \( C \) are expressed by Eqs. (4.86), giving two equations for the three unknowns \( \theta_{b}, \theta_{c}, \) and \( \rho \). An additional equation is provided by the condition that during buckling the total base shear must vanish (that is, sway buckling occurs without addition of a horizontal load to cause deflection), or [Fig. 4.16(b)]

\[
H_{a} + H_{b} = 0
\]  

(4.92)

where \( H_{a} \) and \( H_{b} \) are determined by taking moments about \( B \) and \( C \), respectively,

\[
P_{o}L_{o} + H_{b}L_{o} + M_{ba} = 0 \quad \text{and} \quad P_{o}L_{o} + H_{b}L_{o} + M_{cd} = 0
\]

By adding these two equations we get

\[
2P_{o}L_{o} + L_{o}(H_{a} + H_{b}) + M_{ba} + M_{cd} = 0
\]

Since \( H_{a} + H_{b} = 0 \) [Eq. (4.93)], the additional equilibrium condition becomes equal to

\[
2P_{o}L_{o} + M_{ba} + M_{cd} = 0
\]  

(4.94)

Substitution of the moments \( M_{ba}, M_{bc}, M_{cb}, \) and \( M_{cd} \) [Eqs. (4.91) and (4.92)] into the equilibrium equations [Eqs. (4.86) and (4.94)] results in the following three simultaneous homogeneous linear equations:

\[
\left( \frac{C^2 - S^2}{C} + 4\gamma \right) \theta_{a} + (2\gamma)\theta_{b} + \left( \frac{C^2 - S^2}{C} \right) \rho = 0
\]

\[
(2\gamma)\theta_{a} + \left( \frac{C^2 - S^2}{C} + 4\gamma \right) \theta_{b} + \left( \frac{C^2 - S^2}{C} \right) \rho = 0
\]

\[
(1\theta_{a} + (1)\theta_{b} + \left[ \frac{2PL_{o}^2}{E_{o}}(C^2 - S^2) - 2 \right] \rho = 0
\]

The vanishing of the determinant of the coefficient gives the following buckling condition:

\[
6\gamma = \frac{(PL_{o}^2/E_{o})(C^2 - S^2)/C}{[(C^2 - S^2)/C - PL_{o}^2/E_{o}]
\]  

(4.95)

This equation can be solved in the same manner as the buckling condition for the nonsway case [Eq. (4.89)], and the corresponding curves are given in Fig. 4.17. The buckled shape is shown in Fig. 4.16(c). The limits of the critical load for \( 0 \leq \gamma \leq \infty \) are \( 0 \leq PL_{o}/E_{o} \leq \pi^2/4 = 2.47 \), while \( 0 \leq K_{s} \leq 2.0 \).

We can make certain conclusions by comparing the various curves presented in Fig. 4.17 (these conclusions are also true qualitatively for more complex frames): (1) The buckling loads for the sway case are considerably smaller than for the nonsway case, and therefore the former governs. (2) A considerable increase in buckling strength can therefore be achieved if the top of the frame can be prevented from translating. This can be done by diagonal bracing or shear walls. (3) Changes in strength are pronounced for small changes in beam stiffness for small values of \( \gamma (\gamma < 1) \); however, for larger values of \( \gamma \) no great changes in strength (or, great reductions in \( K_{s} \)) result for relatively large increases of beam stiffness.
The slope-deflection method can be used in setting up the equilibrium equations for frames of any degree of complexity. For example, in a frame for which sway is inhibited by diagonal bracing [Fig. 4.18(a)] we can write one equation for the moment equilibrium of each joint. We obtain as many equations as there are joints and unknown joint rotations. The buckling
condition is then found by setting the determinant of the coefficients of the unknown rotations equal to zero.

If the frame can sway, then the unknowns are the joint rotations and the story deflections $p_c L_c$ [Fig. 4.18(b)]. In addition to the moment equilibrium equations we can set up a shear equilibrium equation for each story, giving us, therefore, as many equations as there are unknowns. The shear equilibrium equation is derived in the same manner as Eq. (4.94), and it is equal to

$$L_c p_c \sum_{i=1}^{i=n} P_i + \sum_{i=1}^{i=n} M_{ix} + \sum_{i=1}^{i=n} M_{ib} = 0$$  (4.96)

In Eq. (4.96) $n$ is the number of columns in the story, $P_i$ is the axial load of each column, and $M_{ix}$ and $M_{ib}$ are the column end moments at the top and the bottom of each column, respectively.

This method of calculating the buckling loads of frames breaks down for structures with many members because of the complex transcendental equations which result. For this reason various approximations have been proposed. The simplest one is suggested in the CRC guide and is used as the basis for the 1963 AISC specifications. This approximation determines the critical load of each column in a frame by using a subassemblage consisting of the column (member $AB$ in Fig. 4.18) and the four restraining beams (members $AC$, $AD$, $BE$, and $BF$ in Fig. 4.18) which frame into the column. The method assumes that each column in the frame buckles simultaneously. Reasonably good predictions of the critical loads can be achieved for regular frames.

We shall first examine the nonsway case [Fig. 4.18(a)], where the joints are prevented from sidesway. It is assumed that the rotations at the far ends of the restraining beams are equal in magnitude but opposite in sense to the joint rotations at the column ends. Thus $\theta_c = \theta_d = -\theta_A$ and $\theta_b = \theta_x = -\theta_x$. From the slope-deflection equations

$$M_{ac} = \frac{2EI_{BL} \theta_A}{L_{BL}}, \quad M_{ad} = \frac{2EI_{BR} \theta_A}{L_{BR}}$$  (4.97)

$$M_{bc} = \frac{E I_{BL} \theta_B}{L_{BL}}, \quad M_{bd} = \frac{2EI_{BR} \theta_B}{L_{BR}}$$  (4.98)

$$M_{ba} = \frac{EI_c (C \theta_A + S \theta_B)}{L_c}, \quad M_{ba} = \frac{EI_c (S \theta_A + C \theta_B)}{L_c}$$  (4.99)

It is further assumed that the restraining moments $M_{bc}, M_{bd}$ are distributed between column $AB$ and the column above in the ratio of the I/I values of the two columns. The joint equilibrium equations thus become equal to

$$M_{ab} + \frac{L_c}{L_c + I_C/I_{BC}} M_{ad} + \frac{M_{ab}}{L_c/L_c + I_C/I_{BC}} = 0$$  (4.100)

$$M_{ba} + \frac{L_c}{L_c + I_C/I_{BC}} M_{ba} + \frac{M_{ba}}{L_c/L_c + I_C/I_{BC}} = 0$$  (4.101)

By substituting the moments from Eqs. (4.97) through (4.99) into these equilibrium equations and setting the determinant of the coefficients of the two unknowns $\theta_A$ and $\theta_B$ equal to zero, we obtain the following buckling condition:

$$C^2 - S^2 + 2C \left( \frac{1}{G_r} + \frac{1}{G_b} \right) + \frac{4}{G_r G_b} = 0$$  (4.102)

In this equation

$$G_r = \frac{I_c/L_c + I_{BC}/I_{BC}}{I_{BL}/I_{BL} + I_{BR}/I_{BR}}, \quad G_b = \frac{L_c/L_c + I_{BC}/I_{BC}}{I_{BL}/I_{BL} + I_{BR}/I_{BR}}$$  (4.103)

Equation (4.102) can be further reduced into its form given in the CRC guide by noting the relationships between $C, S, c, s,$ and $PL_b/EI_c$ [Eqs. (4.75) and (4.79)]

$$\frac{4\pi^2 G_r G_b}{K_s^2} \left[ 1 - \frac{\pi}{K_s \tan (\pi/K_s)} \right] + 2K_s \tan (\pi/K_s) = 0$$  (4.104)

where $K_s = \pi/\sqrt{PL_b/EI_c}$ is the effective length factor. This equation relates the stiffness ratio $G_r$ and $G_b$ with $K_s$ which is used for the formula $P_r = \pi^2 EI_c/(K_s L_c)^2$ for each column. The relationship is conveniently represented in nomographic form in Fig. 4.19(a). A straight line drawn between the two known values of $G_r$ and $G_b$ gives the corresponding value of $K_s$.

The sway buckling condition is shown in Fig. 4.18(b). Here it is assumed that $\theta_c = \theta_d = -\theta_A$ and $\theta_b = \theta_x = -\theta_x$. The moment equations are

$$M_{ac} = \frac{6EI_{BL} \theta_A}{L_{BL}}, \quad M_{ad} = \frac{6EI_{BR} \theta_A}{L_{BR}}$$  (4.105)

$$M_{bc} = \frac{6EI_{BL} \theta_A}{L_{BL}}, \quad M_{bd} = \frac{6EI_{BR} \theta_B}{L_{BR}}$$  (4.106)

$$M_{ba} = \frac{EI_c (C \theta_A + S \theta_B)}{L_c} - (C + S)p, \quad M_{ba} = \frac{EI_c (S \theta_A + C \theta_B)}{L_c} - (C + S)p$$  (4.107)

The joint equilibrium equations are the same as Eqs. (4.100) and (4.101); the story shear equilibrium equation is

$$M_{ab} + M_{ba} + P \rho I_c = 0$$

Proceeding as before, we obtain the buckling condition in the following form (Eq. 2.22b in Ref. 3.10):

$$\frac{\pi^2 G_r G_b K_s^2 - 36}{6(G_r + G_b)} = \frac{\pi}{K_s \tan (\pi/K_s)}$$  (4.108)

The corresponding nomograph is given in Fig. 4.19(b).

**SUMMARY OF FRAME BUCKLING**

In this section we showed how the elastic buckling load of plane frames can be determined by the slope-deflection method, and we derived approximate equations for the effective length of columns in regular plane frames.
The elastic buckling load of a plane frame is an important parameter used in the approximate determination of the maximum strength of frames, as we shall learn in Chapter 6.

In our discussion here we have treated but one method of solution (slope deflection) and one approximate model (CRC). The material presented represents only a small fraction of that available in the literature. Other successful methods of solution are the three-moment equation method of Bleich and Lundsquist's convergence criterion, based on the method of moment distribution, and the energy method. More extensive multistory frame models than the CRC model have been proposed by Bleich and Goldberg. Auxiliary charts, tables, and approximate formulas for aiding the determination of the buckling strength of frames are available in many works, for example, in Refs. 1.21, 1.30, 3.51, 4.3, 4.8, and 4.9. A survey of the literature has been prepared by Lu who not only lists the pertinent references but also provides a convenient table for quickly locating references to the problem for which information is desired.

The determination of critical elastic buckling loads of complex frames can be best achieved by computer. The approximate methods of Bleich and Goldberg, for example, are adaptable to computer solution. It is also not difficult to write the slope-deflection equations and the equilibrium equation for any rigid frame in terms of the stability functions $S$ and $C$. For a given load level, the stability functions have a definite value, and a check could be made to determine whether the determinant of the coefficients of the deformations is zero or not. We could start the analysis at a load level at which we are sure that the frame is stable and then increase this load and test the determinant at each increment. The critical load would be between the two levels where the determinant changes to a negative value. Such analyses for plane and space frames are presented in Refs. 4.10 through 4.15. The elastic critical load is also determined as the by-product of an elastic second-order frame analysis. We shall return to the significance of the critical load in connection with our study of frame behavior in the last chapter of this book.

4.4. INELASTIC COLUMNS

THE TANGENT AND REDUCED MODULUS CONCEPTS

In Sec. 4.2 we showed that a simply supported symmetric elastic column will buckle when the axial load becomes equal to the lowest of the critical loads defined by Eqs. (4.23), (4.24), and (4.25). This critical load is equal to

$$P_c = \frac{\pi^2 EI}{L^2}$$
for buckling about the x axis. This equation can also be expressed in terms of a critical stress as

\[
\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L/r_s)^2} \tag{4.109}
\]

The meaning of the elastic critical load is that the column will begin to deflect laterally when \( P_{cr} \) is reached. In the elastic range no change in load is required to obtain this deflection, and the column can be perfectly straight and slightly deflected under the same load \( P_{cr} \). We call this phenomenon the bifurcation of the equilibrium.

This same reasoning was expanded into the inelastic range in 1889 by Engesser, who postulated that a homogeneous column made of a material having a stress-strain curve as shown in Fig. 4.20(a) will experience bifurcation of the equilibrium in the range above the proportional limit when the average stress \( P/A \) is equal to

\[
\sigma_{cr} = \frac{\pi^2 E_i}{(L/r_s)^2} \tag{4.110}
\]

In this equation \( E_i \) is the slope \( d\sigma/d\varepsilon \) of the stress-strain curve at the stress \( \sigma_{cr} \). The axial load corresponding to this stress is called the tangent modulus load \( P_T \) and is equal to

\[
P_T = \frac{\pi^2 E_i L}{L} \tag{4.111}
\]

In Eq. (4.110) the stress can now no longer be computed directly as \( E_i \) is also a function of the stress, and so the computational process is reversed, that is,

\[
\left( \frac{L}{r_s} \right)_{\sigma_{cr}} = \frac{\pi}{\sqrt{E_i}} \tag{4.112}
\]

Since it is usually not convenient to express the \( \sigma-\varepsilon \) relationship analytically, a column curve (\( \sigma_{cr} \) versus \( L/r \) curve) can be constructed graphically as shown in Fig. 4.20. From an experimentally determined \( \sigma-\varepsilon \) curve [Fig. 4.20(a)]

\[\text{Fig. 4.20. Construction of the tangent modulus column curve}\]

\[\text{For a historical review of the controversy about the tangent modulus and the reduced modulus concept see Chap. 1 in Ref. 1.30, or Refs. 1.40 and 3.28.}\]
on the assumptions that (1) the stress-strain curve of the material is known, (2) the displacements are small, (3) plane sections before bending remain plane after bending, and (4) no change of load is associated with bifurcation.3

The sketch in Fig. 4.22(b) shows a cross section which is symmetric about the y axis and which is part of a column that is assumed to buckle at a stress

\[ \sigma = P y / A \] about its x axis. During buckling an infinitesimal curvature \( d\Phi \) is introduced; the resulting strain distribution is shown in Fig. 4.22(c). The portion of the cross section to the right of the neutral axis of the bending strands will be further compressed, and to the left the existing strain due to \( P \) will be reduced. In the loading portion the strain is \( d\epsilon \), and the corresponding stress \( d\sigma \) [Fig. 4.22(d)] is equal to

\[ d\sigma = E E \epsilon \] (4.113)

In the unloading portion

\[ d\sigma = E E \epsilon \] (4.114)

that is, the stress-strain relationship is governed by the elastic modulus \( E \).

From the geometry of the strain distribution we find [Fig. 4.22(c)] that

\[ d\epsilon = [y + (y - y_1)] d\Phi \] and \[ d\sigma = [y - (y - y_1)] d\Phi \] (4.115)

where \( y \) and \( y_1 \) are defined in Fig. 4.22.

The curvature \( d\Phi \) is related to the deflection of the column \( v \) by the formula \( d\Phi = -v'' \) [Eq. (2.23)], and therefore the stresses are

\[ d\sigma = -v'' E (y - y_1) \] and \[ d\sigma = -v''(y - y_1 + y) \] (4.116)

According to our assumption, the change in \( P \) due to buckling is zero, and so

\[ dP = 0 = \int_{y-y_1}^{y-y_1} d\sigma dA - \int_{-(y-y)}^{y-y_1} d\sigma dA \] (4.117)

If we substitute Eqs. (4.116) into Eqs. (4.117), we find that

\[ E S_1 - E S_2 = 0 \] (4.118)

where

\[ S_1 = \int_{y-y_1}^{y-y_1} (y - y_1 + y) dA \] (4.119)

\[ S_2 = \int_{-(y-y)}^{y-y_1} (y - y_1 + y) dA \] (4.120)

are the statitical moments of the areas to the left and to the right of the neutral axis (N.A. in Fig. 4.22) about this axis, respectively. The expression in Eq. (4.118) permits the determination of the location of the neutral axis of the bending stresses.

Equilibrium of moments due to the bending stresses [Fig. 4.22(d)] about the neutral axis is expressed as follows:

\[ M = P v = \int_{y-y_1}^{y-y_1} d\sigma (y - y_1 + y) dA + \int_{-(y-y)}^{y-y_1} d\sigma (y - y_1 + y) dA \] (4.121)

From Eq. (4.121) we can obtain after substitution of \( d\sigma \) and \( d\sigma \) from Eqs. 4.116 the following formula:

\[ P v = -v'' (E I_1 + E I_2) \] (4.122)
where
\[ I_1 = \int_{y_{min}}^{y_{max}} (y - \bar{y} + y_1)^2 \, dA \]  
(4.123)
\[ I_2 = \int_{y_{min}}^{y_{max}} (\bar{y} - y_1 + y)^2 \, dA \]  
(4.124)
are the moments of inertia about the neutral axis of the area to the left and to the right of this axis, respectively. Rearrangement and differentiation of Eq. (4.124) give us
\[ v'' + F_v v = v'' + F_v v'' = 0 \]  
(4.125)
where
\[ F_v = \frac{P}{E I_x} \]  
(4.126)
and
\[ \bar{E} = E \left( \frac{I_1}{I_2} \right) + E_i \left( \frac{I_2}{I_1} \right) \]  
(4.127)
This latter term is the reduced modulus and is a function of both the material properties E and E_i and the cross-sectional geometry.

The differential equation is of the same form as Eq. (4.14), and we can compute the buckling load from it to be
\[ P_R = \frac{\pi^2 \bar{E} I_x}{L^2} \]  
(4.128)
The load P_R is the reduced modulus load. Since \( \bar{E} > E_{R} \), P_R will always be larger than \( P_T \).

**Shanley's Contribution**

For almost fifty years engineers were faced with a dilemma: They were convinced that the reduced modulus concept was correct, but the test results had an uncomfortable tendency to lie near the tangent modulus load. For this reason the tangent modulus concept was used in column design, and the discrepancy was ascribed to initial crookedness and end eccentricities of the load which could not be avoided when testing columns.

In order to see if this was so, Shanley conducted very careful tests on small aluminum columns. He found that lateral deflection started very near the theoretical tangent modulus load but that additional load could be carried until unloading set in. At no time could Shanley's columns support as much load as that predicted by the reduced modulus theory.

Shanley then used the model shown in Fig. 4.23 to explain what is happening. A column made up of two inextensible bars connected in the center of the column by a deformable cell comprise the model. Upon buckling, all deformations take place in the cell, which consists of two flanges of area \( A/2 \); these two flanges are connected by a web of zero area. One flange consists of a material with an elastic modulus \( E_1 \), and the other of a material with a modulus \( E_2 \). The length of the column L is much larger than the dimensions d of the cell.

In the buckled shape [Fig. 4.23(a)] the following relationships exist between the end slope \( \theta_o \), the lateral deflection \( v_o \) (where \( v_o \ll d \)), and the strains \( e_1 \) and \( e_2 \) of the flanges [Fig. 4.23(b)]
\[ v_o = \frac{\theta_0 L}{2} \quad \text{and} \quad \theta_0 = \frac{1}{2d} (e_1 + e_2) \]  
(4.129)
By combining these two equations we can eliminate \( \theta_0 \), and thus
\[ v_o = \frac{L}{4d} (e_1 + e_2) \]  
(4.130)

The external moment at the midheight of the column is
\[ M_e = P v_o = \frac{PL}{4d} (e_1 + e_2) \]  
(4.131)
The forces in the two flanges due to buckling are
\[ P_1 = \frac{E_1 e_1 A}{2d} \quad \text{and} \quad P_2 = \frac{E_2 e_1 A}{2d} \] (4.132)

The internal moment is then
\[ M_1 = \frac{d P_1}{2} + \frac{d P_2}{2} = \frac{A}{4} (E_1 e_1 + E_2 e_2) \] (4.133)

With \( M_1 = M_2 \) we get an expression for the axial load \( P \), or
\[ P = \frac{Ad}{L} \left( \frac{E_1 e_1 + E_2 e_2}{e_1 + e_2} \right) \] (4.134)

In case the cell is elastic \( E_1 = E_2 = E_3 \), and so
\[ P_s = \frac{AEd}{L} \] (4.135)

For the tangent modulus concept \( E_1 = E_2 = E_3 \), and so
\[ P_s = \frac{AEd}{L} \] (4.136)

When we consider the elastic unloading of the "tension" flange, then \( E_1 = E_1 \) and \( E_3 = E_3 \), and thus
\[ P = \frac{Ad}{L} \left( \frac{E_1 e_1 + E_3 e_3}{e_1 + e_3} \right) \] (4.137)

Upon substitution of \( e_1 \) from Eq. (4.130) and \( P_s \) from Eq. (4.136) and using the abbreviation
\[ \tau = \frac{E_1}{E} \] (4.138)

we find that
\[ P = P_s \left[ 1 + \frac{Le_2}{4dv_0} \left( \frac{1}{\tau} - 1 \right) \right] \] (4.139)

There will be forces \( P_1 \) and \( P_2 \) acting on the two flanges if the member is deformed, and the difference of these two loads will be the amount by which \( P \) is increased beyond the tangent modulus load, that is,
\[ P = P_s + (P_1 - P_2) \] (4.140)

By employing Eqs. (4.130), (4.132), and (4.138) we can show that
\[ P_1 - P_2 = \frac{AEd}{2d} \left[ \frac{4v_0}{2d} \left( \frac{1}{\tau} + 1 \right) e_2 \right] \] (4.141)

and that then
\[ P = P_s \left[ 1 + \frac{2v_0}{d} - \frac{Le_2}{2d} \left( 1 + \frac{1}{\tau} \right) \right] \] (4.142)

By eliminating \( e_2 \) from Eqs. (4.139) and (4.142) we finally get
\[ P = P_s \left[ 1 + \frac{1}{(d/2v_0) + (1 + \tau)(1 - \tau)} \right] \] (4.143)

If we say, for example, that \( \tau = 0.5 \) and that \( \tau \) remains constant after \( P_\tau \) is exceeded, then Eq. (4.143) is equal to
\[ \frac{P}{P_\tau} = 1 + \frac{1}{(d/2v_0) + 3} \] (4.144)

and the curve relating \( P/P_\tau \) and \( v_0/d \) is the solid line curve in Fig. 4.24. This curve approaches \( P = 1.333P_\tau \) asymptotically as \( v_0/d \to \infty \). In reality \( \tau \) will vary with the strain, and as \( P \) is increased, \( \tau \) will become progressively smaller and a \( P \) versus \( v_0 \) curve as shown by the dashed line in Fig. 4.24 results. This curve will have a maximum point somewhere between \( P_s \) and \( 1.333P_\tau \).

The reduced modulus concept gives the load at which deformation occurs without a change in load. Thus in Eq. (4.141) \( P_1 - P_2 = 0 \), and therefore
\[ e_2 = \frac{4v_0d}{L} \left[ \frac{1}{(1 + (1/\tau))} \right] \] (4.145)

The reduced modulus load is obtained if we set \( e_2 \) from Eq. (4.145) into Eq. (4.139), or
\[ P_s = P_\tau \left( 1 + \frac{1 - \tau}{1 + \tau} \right) \] (4.146)

which becomes for \( \tau = 0.5 \) equal to \( 1.333P_\tau \). Indeed, the actual load will be equal to \( P_s \) if we set \( v_0/d = \infty \) in Eq. (4.143). Thus \( P_s \) is the absolute maximum load which can be reached. But it will never be attained because the
value of \( \tau \) is reduced, and so the maximum load on this column will be somewhere between the tangent modulus and the reduced modulus load.

Equation (4.143) defines the relationship between the axial load \( P \) and the resulting deflection of the mid-height of the column \( v_o \) for any constant value \( \tau \) (Fig. 4.24). When \( v_o = 0 \), this equation gives \( P = P_T \), the tangent modulus load. For any value of \( v_o \) the axial load is larger than \( P_T \), or conversely, if \( P > P_T \), a deflection \( v_o \) is necessary for maintaining equilibrium. The maximum value of \( P = P_{P_T} \) is reached when the deflection \( v_o \) becomes infinitely large. This load represents the maximum possible theoretical load. Because bending after \( P_T \) is accompanied by further straining of the compression flange of the cell, \( \tau \) is reduced from its initial value and the \( P-v_o \) curve therefore reaches a peak below the reduced modulus load (dashed curve, Fig. 4.24).

Although the Shanley model (Fig. 4.23) has no resemblance to an actual column, the conclusions from its analysis apply also to such real members. These conclusions are (1) the tangent modulus concept gives the maximum load up to which an initially straight column will remain straight, (2) the actual maximum load exceeds the tangent modulus load, but it cannot be as large as the reduced modulus load, (3) any load above \( P_T \) will cause the column to be laterally deflected and (4) in the load range \( P_T \leq P \leq P_{P_T} \) there is always strain reversal present.

Shanley resolved the paradox which had existed for half a century, and he defined the concepts on which a rational inelastic column theory could be based. Following Shanley, many investigators refined and expanded these concepts. Among these were Duberg and Wilder,\(^{6,17}\) Johnston,\(^{3,18,4,19}\) Hoff,\(^{6,10}\) and Augusti,\(^{6,10,4,20}\) to mention only a few. In addition to the basic theoretical work of these authors, Shanley's theory also permitted the development of a rational column theory for steel columns, of which we shall learn more in a subsequent article of this chapter. This research has further proved Shanley’s model, and it has brought out the following additional facts about column behavior: (1) Theoretically it is possible to have an infinite number of “critical” loads (that is, loads at which a column may deflect) between the tangent modulus load \( P_T \) and the elastic buckling load (see Fig. 4.25). Upon deflection the gradient of the load-deflection curve is positive when \( P_T \leq P \leq P_{P_T} \), zero when \( P = P_{P_T} \), and negative (that is, deflection is accompanied by unloading) when \( P_{P_T} \leq P \leq P_{P_T} \). (2) In reality there are no initially straight columns. If the behavior of a straight column is regarded as the limiting case of an initially crooked column as the crookedness vanishes (Fig. 4.25), then the only significant critical load among the many theoretical possibilities is the tangent modulus load. (3) Methods were developed for the determination of the initial gradient of the \( P-v_o \) curve\(^{4,18}\) and the complete \( P-v_o \) curve up to the maximum load.\(^{3,18,4,17}\) The evaluation of \( P_{P_T} \) is based on complicated calculations based on equilibrium considerations similar to those we used in developing Shanley’s equations. In the cases investigated it was found that the maximum load was usually less than 5 per cent higher than \( P_T \). (4) Because of the closeness of the two loads it was recommended that design formulas for metal columns should be based on the tangent modulus concept. In case such an approach is not desired, we can always determine both \( P_T \) and \( P_{P_T} \) to obtain a lower and an upper bound to the maximum column strength. Only if the difference between the two is too large would it be practical to compute \( P_{P_T} \).

**EXAMPLE ON THE TANGENT MODULUS AND REDUCED MODULUS CONCEPTS**

To illustrate the tangent modulus and the reduced modulus concepts we shall now determine these two loads for a simply supported column of a rectangular cross section [Fig. 4.26(a)] when buckling is about the \( x \) axis. Let the stress-strain relationship be given by the formula

\[
e = \frac{\sigma}{E} + \left( \frac{3\sigma_T}{7E} \right) \left( \frac{\sigma}{\sigma_T} \right)^{19}
\]  

(4.147)
This is a special case of a generalized formula suggested by Ramberg and Osgood for materials with a nonlinear stress-strain curve.\(^{(4.12)}\) We can nondimensionalize this equation by \(e_y = \sigma_y/E\), and then

\[
\frac{e}{e_y} = \frac{\sigma}{\sigma_y} + \frac{3}{7} \left( \frac{\sigma}{\sigma_y} \right)^{10}
\]  
(4.148)

The curve representing this relationship is shown in the top right-hand corner of Fig. 4.27. Up to about 0.7\(\sigma_y\) the curve is nearly linear; beyond this point the curve bends sharply, but it continues to rise even after \(\sigma_y\) is reached. Such a curve is typical of the behavior of aluminum alloys.

The tangent modulus \(E_t\) is obtained by differentiating Eq. (4.147), or

\[
\frac{de}{d\sigma} = \frac{1}{E} \left[ 1 + \frac{30(\sigma/\sigma_y)^{10}}{7} \right]
\]  
(4.149)

\[
\tau = \frac{E_t}{E} = \frac{(d\sigma/dx)}{E} = \frac{1}{1 + (30/7)(\sigma/\sigma_y)^{10}}
\]  
(4.150)

The tangent modulus load is equal to [Eq. (4.111)] \(P_t = \pi^2 E_t I_y/L^3\). In a nondimensional form

\[
\frac{P_t}{P_y} = \frac{(\pi^2 E/\sigma_y)}{(L/\gamma)^3}
\]  
(4.151)

With the abbreviation

\[
\lambda_x = \frac{(L/\gamma)\sqrt{\sigma_y/E}}{\pi}
\]  
(4.152)

we can express Eq. (4.151) as

\[
(\lambda_x)_t = \sqrt{\frac{\tau}{P/P_y}}
\]  
(4.153)

This equation can be solved for the critical nondimensional tangent modulus slenderness ratio \(\lambda_x\) by specifying values of \(P/P_y\) and using Eq. (4.150) to compute \(\tau\). The resulting curve is shown as the lower of the two curves in Fig. 4.27.

The reduced modulus load is [from Eq. (4.125)] equal to \(P_n = \pi^2 \bar{E} I_y/L^3\) or, in nondimensional form

\[
(\lambda_x)_n = \frac{\sqrt{\bar{E}/E}}{\sqrt{P/P_y}}
\]  
(4.154)

The reduced modulus \(\bar{E}\) is computed by Eq. (4.127) as

\[
\frac{\bar{E}}{E} = \frac{I_1}{I_e} + \tau \left( \frac{I_2}{I_e} \right)
\]  
(4.155)

where \(I_1\) and \(I_2\) are the moments of inertia of the inelastic and the elastic zones about the neutral axis. The location of this axis is determined from Eq. (4.118) as

\[
\frac{E_t}{E} = \tau = \frac{S_i}{S_y}
\]  
(4.156)

where \(S_i\) and \(S_y\) are the statical moments of the yielded and the elastic zones about the neutral axis of the bending stresses. For the case of a rectangle [Fig. 4.26(b)],

\[
S_i = \frac{b}{2} (d - \bar{y})^2 \quad \text{and} \quad S_y = \frac{b\bar{y}^2}{2}
\]  
(4.157)

The distance \(\bar{y}\) which locates the neutral axis can be determined by substituting Eqs. (4.157) into Eq. (4.156) and solving for \(\bar{y}\); that is,

\[
\bar{y} = d \left( \frac{1 - \sqrt{\tau}}{1 - \tau} \right)
\]  
(4.158)
From Fig. 4.26(b) we also find that
\[ I_z = \frac{bd^3}{12}, \quad I_y = \frac{b(d - \bar{y})^3}{3}, \quad \text{and} \quad I_x = \frac{b^2h^3}{6} \]  \hspace{1cm} (4.159)

With Eqs. (4.159), (4.158), and (4.155) we finally get from Eq. (4.154) the reduced modulus relationship
\[ (\lambda_{x})_{r} = 2 \sqrt{\frac{\tau(1 - \sqrt{\tau})^3 + (\sqrt{\tau} - \tau)^3}{(\mu_1\mu_2)(1 - \tau)^3}} \]  \hspace{1cm} (4.160)

The resulting curve is shown as the upper curve in Fig. 4.27.

The curves in Fig. 4.27 show that indeed the reduced modulus load is above the tangent modulus load. The maximum strength of the column lies between the narrow band bounded by the two curves.

The dashed curve in Fig. 4.27 gives the column curve corresponding to a simplified straight-line elastic-plastic stress-strain curve. We see that such a simplification would in this case result in an overestimation of the critical loads in some regions and an underestimation in others.

By methods similar to that illustrated above we can construct column curves for any cross section if the stress-strain law is known to us. This method is widely used for the design of columns of lightweight alloys, and it represents a very satisfactory solution with regard to both simplicity and safety if the tangent modulus load is used as the maximum useful load. In the case of steel columns a very serious paradox is introduced if the concepts described above are used without modification.

**Buckling Strength of Steel Columns**

Constructional steels have stress-strain curves which are very nearly elastic-plastic before strain hardening sets in (Fig. 1.5). By the strict application of the tangent modulus concept the critical stress below \( \sigma_y \) is then governed by the elastic formula, and the column curve would take the form shown in Fig. 4.28. A great variety of tests on steel columns have, however, very convincingly shown that the column strength predicted by the reasoning above is usually higher than the actual strength if the columns are of intermediate length (0.3 < \( \lambda < 1.4 \), approximately). The test points in Fig. 4.28 illustrate this. These test results were taken from a test program performed at the Fritz Engineering Laboratory of Lehigh University.\(^{1,20}\) The specimens were 8WF31 columns of A7 steel. Other tests, notably those on welded H shapes, show an even larger discrepancy.\(^{1,14}\)

The traditional explanation of the apparent scatter of the test points has been that initial crookedness and unavoidable eccentricity of the load are the causes for the difference between theory and practice. However, the tests in Fig. 4.28 were performed carefully on well-centered and relatively straight specimens, and the measured amount of initial crookedness could by no means account for the low test results.

The idea that residual stresses can be held accountable for the lower strength of steel columns of intermediate length has been advanced as early as 1888,\(^{1,7}\) but it was not until 1951 that it was convincingly shown in a paper by Osgood\(^{1,20}\) that this was indeed so. At this time extensive research was already in progress at the Fritz Engineering Laboratory, where under the guidance of Dr. B. G. Johnston and later Dr. L. S. Beedle the distribution and the magnitude of residual stresses, and their effect on column strength, was extensively investigated. Rolled, riveted, and welded columns of various shapes and types of steel were tested and their column strength was compared with theoretical values computed from the measured material and residual stress properties.\(^4\)

This research has shown that residual stresses account for a large share of the deviation of the column strength from the ideal curve predicted from the stress-strain relationship (Fig. 4.28). A new interpretation of the tangent modulus concept which includes the effects of premature yielding due to residual stresses was introduced (Ref. 4.23). With this modification we can predict column strength very satisfactorily. Residual stresses do not, however, account for all effects, as can be seen in Fig. 4.28, where the as-rolled shapes (those containing residual stresses from uneven cooling after hot rolling)\(^4\)

\(^4\) We shall not attempt to summarize this work here. The student is referred for further study to Refs. 1.34, 1.35, 1.36, 3.10, and Refs. 4.24 through 4.29 for but a sampling of this work.
portions of the cross section will be yielded [cross-hatched area in Fig. 4.30(a)], and the stress-distribution will be as shown in Fig. 4.30(b). From this figure we can obtain the following relationship between the applied stress $\sigma$ and the parameter $\alpha$, which defines the extent of the elastic core, by using the equilibrium condition $P = \int \sigma \, dA$:

$$\frac{P}{P_r} = \frac{\sigma}{\sigma_r} - \left( \frac{\sigma}{\sigma_r} - \frac{1}{2} \right) \left( \frac{1}{2} - \alpha \right)$$  \hspace{1cm} (4.161)

From the geometry of similar triangles in Fig. 4.30 we can also show that

$$\frac{\sigma}{\sigma_r} = \frac{3}{2} - 2\alpha$$  \hspace{1cm} (4.162)

By combining Eqs. (4.161) and (4.162) we see that

$$\alpha = \sqrt{\frac{1}{2} \left( 1 - \frac{P}{P_r} \right)}$$  \hspace{1cm} (4.163)

We shall now establish the equation for $\alpha$ from another direction. If an infinitesimally small increase of stress $d\sigma$ is added to the column cross section [Fig. 4.30(c)], then the increase in $P$ is equal to

$$dP = (d\sigma)(d)(2ab) = A_x d\sigma$$  \hspace{1cm} (4.164)

where $A_x$ is the area of the elastic core. The average stress increase was defined in Sec. 1.4 as

$$d\sigma_A = \frac{dP}{A}$$  \hspace{1cm} (4.165)

and the change in strain is

$$d\varepsilon = \frac{d\sigma}{E} = \frac{dP}{A_x E}$$  \hspace{1cm} (4.166)

When a short length of this column is compressed in a testing machine, and we plot the value of $P/A$ and $\varepsilon$, we obtain a stub-column stress-strain curve. We have already discussed such curves in Sec. 1.4, and we found that the shape of this curve is influenced by the presence of residual stresses. The slope of the stub column $\sigma$-$\varepsilon$ curve at any location is

$$\frac{d\sigma_A}{d\varepsilon} = \frac{dP/A}{dP/A_x E} = \frac{EA_x}{A}$$  \hspace{1cm} (4.167)

We now can think of this slope of the stub-column stress-strain curve as the tangent modulus of the cross section $E_t$, and so

$$\frac{E_t}{E} = \tau = \frac{A_x}{A}$$  \hspace{1cm} (4.168)

Thus in the yielded region the ratio of the tangent modulus to the elastic modulus is equal to the ratio of the area of the elastic core to the area of the whole cross section. This is in contrast to the tangent modulus defined pre-
vously for the material; the tangent modulus of the cross section includes the effect of residual stresses, and it can be obtained analytically from the known residual stresses, or it can be obtained experimentally by compressing a short stub-column and plotting the average stress $P/A$ versus strain $e$ curve. This is a routine test and has been internationally standardized. The stub-column $e$ curve plays the same role that the actual $e$ curve did for a homogeneous material in the determination of the inelastic buckling loads (Fig. 4.20).

In the case of the rectangular cross section of Fig. 4.30(a),

$$
\tau = \frac{A_b}{A} = \frac{2axbd}{bd} = 2a 
$$

(4.169)

and so from Eq. (4.163),

$$
\tau = \sqrt{2(1 - P/P_r)} 
$$

(4.170)

We are now ready to develop expressions for the buckling strength of the rectangular column. In the elastic range ($0 < P/P_r < 0.5$) this is equal to

$$
P_x = \frac{\pi^2 E l_x}{L^2} \quad \text{and} \quad P_y = \frac{\pi^2 E l_y}{L^2} 
$$

(4.171)

With the abbreviations

$$
\lambda_x = \frac{(L/r_x)\sqrt{\sigma_y/E}}{\pi} \quad \text{and} \quad \lambda_y = \frac{(L/r_y)\sqrt{\sigma_y/E}}{\pi} 
$$

(4.172)

we can nondimensionalize Eqs. (4.171) to

$$
(\lambda_x)_{re} = (\lambda_y)_{re} = \frac{1}{\sqrt{P/P_r}} 
$$

(4.173)

The tangent modulus buckling strength is defined by the elastic core since $E = 0$ in the yielded zone; that is,

$$
P_x = \frac{\pi^2 E l_x}{L^2} 
$$

(4.174)

In nondimensional form this equation is

$$
\lambda_x = \sqrt{\frac{I_x}{I_y}P/P_r} 
$$

(4.175)

For buckling about the $y$ axis

$$
I_{x} = \frac{d}{12}(ab)^3 \quad \text{and} \quad I_y = \frac{db^3}{12} 
$$

(4.176)

and for buckling about the weak axis

$$
I_{xw} = \frac{2a}{12}(d)^3\quad \text{and} \quad I_{yw} = \frac{bd^3}{12} 
$$

(4.177)

Setting Eqs. (4.169) into Eqs. (4.176) and (4.177), and substituting these in turn into Eq. (4.175), we obtain the following equations for the tangent mod-ulus load:

$$
(\lambda_x)_{re} = \sqrt{\frac{\sigma_y}{P/P_r}} = \sqrt{\left(\frac{2(1 - P/P_r)}{P/P_r}\right)^{\frac{1}{2}}} 
$$

(4.178)

$$
(\lambda_y)_{re} = \sqrt{\frac{\sigma_y}{P/P_r}} = \sqrt{\left(\frac{2(1 - P/P_r)}{P/P_r}\right)^{\frac{1}{2}}} 
$$

The curves showing the $P/P_r$ versus $(\lambda_x)_{re}$ and $(\lambda_y)_{re}$ relationship are shown as the solid-line curves in Fig. 4.33. We see that for weak axis bending the tangent modulus is nearly a straight line between $P/P_r = 1.0$, where the whole cross section is yielded, and $P/P_r = 0.5$, where yielding commences. The curve for buckling about the strong axis is considerably above the weak axis curve.

The reason for this is seen from Eqs. (4.176) and (4.177): the moment of inertia of the elastic core about the $y$ axis varies as $x^2$, whereas this property varies as $a^2$ about the $x$ axis. Since $a < 0.10$, $x^2 < a^2$, and so $P_{xr}/P_r > P_{yr}/P_r$. As the buckling strength of wide-flange shapes is governed principally by the stiffness of the flanges, and the residual stress distribution in these flanges is similar to that in our rectangular column (compare Figs. 4.29(b) and 1.6), the nondimensional buckling strength of wide-flange shapes should be larger for strong axis buckling than for weak axis buckling. That is this is indeed so has been verified by many experiments on as-rolled wide-flange steel columns (see for example Refs. 1.34, 1.35, and 1.36).

The reduced modulus strength depends on the area which is elastic immediately after buckling, and this area consists of the elastic core before buckling plus the area of the unloaded zones. Two types of unloading exist if buckling is about the $y$ axis, as shown in Fig. 4.31: in one case the neutral axis of the bending strain is in the originally elastic zone [Fig. 4.31(a)], and in the other case the neutral axis is in the yielded zone [Fig. 4.31(b)].

The location of the neutral axis, as well the effective area, is determined from the condition that the change in $P$ at the reduced modulus load is equal to zero. By inspection of the stress change we find that $x_1 = (b/2)(\alpha + \alpha)$ for the case of Fig. 4.31(a); for the situation in Fig. 4.31(b),

$$
dP = 0 = E(d\Phi)\frac{d}{2} - E(d\Phi)\left(\frac{b}{2} + \alpha x_1\right)^2 
$$

and upon solution for $x_1$, we obtain

$$
x_1 = 2\alpha \left(\sqrt{1 + \frac{1}{2\alpha^2}} - 1\right) b 
$$

(4.179)

The boundary between the two cases occurs when $x_1 = b/2 - \alpha b$, or, from Eq. (4.179), $\alpha = \frac{1}{\alpha}$; this corresponds to $P/P_r = \frac{3}{4}$ [Eq. (4.163)]. The effective moment of inertia is computed as the moment of inertia of the
Fig. 4.31. Reduced modulus concept for buckling about the \( y \) axis of a rectangular column

Effective area about the neutral axis; that is, for Fig. 4.31(a),

\[
I_{y_{\text{eff}}} = \left( \frac{2xd}{3} \right) \left( \frac{12}{d} \right) = \frac{(1 - 2\alpha)^2}{8} = \frac{(1 + \gamma)^2}{8} \tag{4.180}
\]

and Fig. 4.31(b)

\[
I_{y_{\text{eff}}} = \frac{12}{b} \left[ \frac{x_{d}d}{3} + \frac{d(2\alpha b)^2}{12} + 2\alpha bd \left( \frac{b}{2} - x_{d} \right)^2 \right] \tag{4.181}
\]

which can be written as [using Eqs. (4.179) and (4.169)]

\[
I_{y_{\text{eff}}} = \tau^2 \left[ 1 + 4 \left( \frac{\sqrt{1 + \frac{1}{\tau}} - 1}{\tau} \right)^2 + 3\tau \left[ 1 - 2\tau \left( \frac{\sqrt{1 + \frac{1}{\tau}} - 1}{\tau} \right) \right] \right] \tag{4.182}
\]

By a similar process we find that for buckling about the \( x \) axis (Fig. 4.32)

\[
y_{i} = \frac{d}{2} \left( \frac{2\alpha}{1 - 2\alpha} \right) \left[ -1 + \sqrt{1 + \frac{(1 - 2\alpha)}{\alpha}} \right] \tag{4.183}
\]

and

\[
I_{x_{\text{eff}}} = 4 \left\{ \tau^3 - \tau^2(-1 + \sqrt{1 + 2(1 - \tau)\gamma})^2 + \tau(-1 + \sqrt{1 + 2(1 - \tau)\gamma}) \right\} \tag{4.184}
\]

The reduced modulus buckling load is then

\[
\lambda_{R} = \frac{\sqrt{I_{x_{\text{eff}}}/I_{y_{\text{eff}}}}}{P/P_{y}} \tag{4.185}
\]
THE TANGENT MODULUS STRENGTH OF PRACTICAL STEEL SECTIONS

The preceding example showed how the tangent modulus and reduced modulus column curves can be obtained for a simple rectangular section. The procedure for the more complicated practical sections is identical, except that the algebraic manipulations become more cumbersome, especially if the residual stress distribution is more complex and the actual instead of the simplified elastic-plastic stress-strain curve is used. However, if the material and the cross-sectional properties are known, we can develop the column curve.

It is often more convenient to perform a stub-column test than to section the material to find out its residual stress distribution or to make the time-consuming computations required to determine the residual stresses analytically. We have shown the simple relationships which exist between \( E \) of the stub-column stress-strain curve and the tangent modulus load for rectangular columns [Eqs. (4.178)]. The same relationships hold approximately also for rolled wide-flange shapes. For example,

\[
P_x = \frac{\pi^2 E I_x \tau}{L^2} \quad \text{and} \quad P_y = \frac{\pi^2 E I_y \tau}{L^2}
\]

(4.189)

for strong and weak axis buckling of rolled and welded wide-flange columns which have the residual stress distribution of Fig. 1.6. The corresponding curves for an 8WF31 column \( \sigma = 33 \text{ ksi}, \sigma_{rd} = 0.3\sigma \) are shown in Fig. 4.34 as the solid-line curves. A similarly simple relationship holds for solid round bars with symmetric residual stresses for which

\[
P = \frac{\pi^2 E I \tau}{L^2}
\]

(4.190)

The experimental work performed on steel columns has shown that except for solid round bars having initial crookedness and residual stresses and welded box-shaped columns, the tangent modulus strength provides an excellent measure of actual column strength, and the Column Research Council has suggested that design formulas for steel should be based on the tangent modulus concept. This practice has been used for many years very successfully for aluminum columns. We must realize that the size and the shape of the member, the yield stress of the steel, and the manufacturing procedure all introduce different kinds of residual stress patterns, and thus different tangent modulus curves result. It matters very much whether the compressive residual stress zones are far from the axis of bending or close to it, since the loss of the same amount of area far from the axis produces a much greater reduction in the effective moment of inertia than is caused by the loss of area due to yielding near this axis. Research information is now available to handle almost any situation (see the previously cited literature), and it is possible to utilize this information directly in design. Specifications could contain either tables or curves for rolled wide-flange shapes, welded H shapes, tubular members, etc. to permit a very rapid design of columns.

INELASTIC TORSIONAL BUCKLING OF COLUMNS

The discussion on inelastic buckling was up to now only concerned with in-plane buckling of symmetric sections. The possibility of torsional buckling of such sections, and the lateral-torsional buckling of asymmetric sections, may need to be considered also. The differential equations of torsional buckling for symmetric sections (that is, \( x = y = 0 \)) is [Eq. (4.3)]

\[
C_w\phi'''' + (C_r + \bar{K})\phi' = 0
\]

If \( C_w, C_r, \) and \( \bar{K} \) remain constant along the length of the member, then the buckling condition for simple end conditions can be shown to be

\[
C_r + \bar{K} + \frac{\pi^2 C_w}{L^4} = 0
\]

(4.191)

The solution for the torsional buckling load from Eq. (4.191) proceeds in a way very similar to the determination of the lateral-torsional buckling moment for beams in the inelastic range (see Sec. 3.3). For a given cross section, material, and residual stress distribution, a load is assumed which is
higher than the load required for the initiation of yielding. Next, the elastic core is determined from the applied stress and the residual stress pattern. Such a partially yielded cross section is shown in Fig. 4.35 for a wide-flange shape with high compressive residual stresses in the flange tips. The St. Venant torsional stiffness may be assumed as \( K_T \) for the whole cross section (see our discussion in Sec. 3.3 on this assumption), and the warping stiffness is determined for the elastic core. For the section in Fig. 4.35, for example,

\[
C_T = \frac{2EI_0(d - t)^3}{4EI_0(d - t)} = 2EI_0(d - t)^3
\]  

(4.192)

The coefficient \( \bar{K} \) is determined by integration of the expression [Eq. (4.4)]

\[
\bar{K} = \int_4 a^2 \, dA
\]  

(4.193)

where \( \sigma \) is the stress at any point due to the compressive force \( P \) and integration is over the whole cross section. The residual stress pattern must be also torsionally in equilibrium so that \( \bar{K} \) for the residual stresses is zero.\(^{1,22}\) The resulting residual stress pattern is therefore more complex than that shown in Fig. 1.6. When \( C_T, C_T, \) and \( \bar{K} \) for the assumed load are computed, the length of the column is computed from Eq. (4.191).

---

**Fig. 4.35. The yielded wide-flange column cross section**

---

**Fig. 4.36. Inelastic flexural and torsional buckling curves for rolled and welded wide-flange columns**

---

The fundamental concepts of inelastic torsional buckling of wide-flange columns is treated in Ref. 4.32. The inelastic torsional buckling of steel columns with different types of steel, wide-flange and tee cross section, and different distributions of residual stress was studied in detail by Nishino,\(^{1,33}\) who developed a numerical procedure for the computer solution for such problems. He found that the torsional buckling curve for the cases investigated is nearly a straight line between \( P/P_T = 1.0 \) and the start of yielding for rolled sections. For wide-flange shapes made of three plates by welding the residual stress distribution is quite different than that for rolled shapes, and therefore both the flexural and the torsional column curve in the inelastic range are different. However, in each case investigated by Nishino it was found that for wide-flange shapes flexural buckling was the controlling factor. The curves in Fig. 4.36 illustrate this for one particular problem. Investigation of the far more serious problem of the torsional buckling of inelastic light-gage asymmetric columns is carried out at present (1967) at Cornell University (Ref. 4.34 is a preliminary report on this research).

**THE MAXIMUM STRENGTH OF COLUMNS**

The maximum strength of "perfect" columns in the inelastic range lies between the tangent modulus and the reduced modulus load. Of these three loads the tangent modulus load is the easiest one to compute, and also a large number of tests under a variety of test conditions show that this load gives a reasonable estimate of the strength of the test column. For this reason we usually base the design loads on the tangent modulus load. In most instances this is a very good procedure, but unfortunately there are some exceptions, for example, when the tangent modulus concept sensibly underestimates or overestimates the actual strength of the column.

Let us first see where the tangent modulus concept gives a good estimate of actual column strength. The work of Duberg and Wilder\(^{4.17}\) and Johnston\(^{4.38}\) showed that the maximum strength is only a few per cent higher than the tangent modulus load for ideal H columns and rectangular columns made of a homogeneous material—such as aluminum. Hill, Brungabler, and Clark have shown with many tests that the strength of aluminum welded columns is also very satisfactorily predicted by the tangent modulus concept.\(^{4.36,4.40}\) The column research at Lehigh University has indicated that the strength of rolled wide-flange columns is their tangent modulus load.\(^{5.4}\) In this case, however, the tangent modulus load is determined from the residual stress distribution. Thus for extruded and welded aluminum columns and steel rolled wide-flange columns the design can be based satisfactorily on the tangent modulus load. In these cases the theoretical maximum strength is only slightly above this load, and the usual unavoidable variations in material and sectional properties, the eccentricity of the load, and the initial crookedness of the test specimen do not sensibly influence the results.
The maximum strength can be considerably higher than the tangent modulus load for perfectly straight steel welded H and box shapes.\(^{8,31,4,37}\)

The theoretical maximum load for a hybrid column (\(\sigma_Y = 100 \text{ ksi steel in the flanges and } \sigma_Y = 36 \text{ ksi steel in the web}) of \(L = 20r\) is, for example, 20 per cent above \(P_f\).\(^{8,47}\) In view of the fact that the computation of the maximum load is so much more involved than the determination of \(P_f\), we might overlook this reserve of strength and base the design on the tangent modulus load. This unfortunately may cause trouble, because the usual small initial crookedness of the test columns can have a great effect for these sections. One welded built-up steel column was about 25 per cent weaker than its strength predicted by the tangent modulus theory.

Initially crooked columns are actually beam-columns (Fig. 4.25) which deflect from the beginning of loading. Asymmetrical residual stresses caused by cold bending act in a similar way.\(^{6,34}\) In many columns both initial crookedness and asymmetrical residual stress must be considered to evaluate the maximum strength,\(^{6,34}\) and the results may fall considerably short of the tangent modulus load. The design of welded steel columns and solid round columns may then have to be based on the actual maximum strength for expected unavoidable initial deflections. Design formulas based on ultimate strength have been proposed for solid round columns.\(^{5,35}\) The research on welded box and H shapes is not yet complete (1967), and so no final design recommendations have been made.

The computation of the maximum strength of columns, including symmetrical and asymmetrical residual stresses and initial crookedness, can be done in the same way in which we treat beam-columns (Sec. 5.3). Simplified procedures, based on assuming that the column deflects like a portion of a sine wave, have also proved to be successful in predicting the maximum strength.\(^{6,34,6,36,4,47}\) Maximum strength curves are shown for two cases in Fig. 4.37 from the study reported in Ref. 4.24. The upper solid curve indicates the case of no residual stress, and the lower solid curve was computed for the case in which the residual stress pattern was as shown by the sketch.\(^{6,34,6,36,4,46}\) Such a residual stress pattern would result if a bar, which emerges curved from its last heat-treatment operation, is cold-straightened to a required straightness tolerance. Both maximum strength curves apply for \(\sigma_Y = 100 \text{ ksi (ASTM A514 steel) and an initial crookedness of}

\[
\frac{\psi_0}{R} = \frac{L/r}{360} \quad (4.194)
\]

where \(\psi_0\) is the initial maximum deflection and \(R\) is the column radius. This crookedness is equal to a standard mill tolerance of \(\frac{1}{4} \text{ in. in 10 ft of length.}

The CRC basic column curve (to be discussed in Sec. 4.5) and the elastic buckling curve are also shown for comparison. As the length approaches \(120r\), the maximum strength curves converge from below to the elastic curve. The allowable crookedness expressed by Eq. (4.194) is seen to cause a con-

---

**Fig. 4.37. Column curves for solid round bars**

Considerable reduction of strength in the range where most practical columns occur (20r to 70r). Curves such as those shown in Fig. 4.37 were used as the basis for design equations in Ref. (4.26). Related studies on aluminum and steel H columns were performed by computer, and the results are reported in Ref. 4.38.

**4.5. THE DESIGN OF COLUMNS**

**STATUS OF KNOWLEDGE ON COLUMN BEHAVIOR**

Research on the behavior of columns has been conducted vigorously for more than 300 years, and as a result more is known about columns than about any other structural member. Future work must utilize statistical and probabilistic techniques on the vast amount of information developed in laboratories and computers so as to establish the correct relationships between the reality of a column in an actual structure and the idealized situation in the laboratory and on paper.

We are now able to predict with very good accuracy the tangent modulus, the reduced modulus, and the maximum load of any inelastic column if we know the material properties, the cross-sectional properties, the residual...
stresses, and the initial deformations. The methods used to obtain these limiting loads are research tools, and as such they are not generally adaptable to design office use. However, enough is known, for example, about aluminum columns, rolled steel wide-flange columns, and solid round columns that definite design recommendations have been made. These recommendations have been included or will soon find their way into design specifications. Those designers who would want to design columns with more refinement than is possible with the specifications can easily use the wealth of information available in the many publications on this subject.

**COLUMN DESIGN**

Materials and design situations vary a great deal, and therefore various design specifications call for different approaches to the column design problem. In the following paragraphs we shall discuss some of these approaches and examine their merits.

*Direct use of research information.* This knowledge may be used to advantage if it is available. This is the "clearest" design approach, but there is by no means enough information available to warrant general use of it. It is also very time consuming.

*Tangent modulus equation.* The design of aluminum columns is based on the straightforward application of the tangent modulus concept [Eq. (4.110)], and design curves relating the allowable stress

$$\sigma_{ac} = \frac{\sigma_y}{F.S.} = \frac{\pi^2 E_i}{F.S. (L/r)^3}$$

(4.195)

and the slenderness ratio are provided in the specifications.

*Empirical column formulas.* These formulas provide a transition between the elastic buckling curve and the yield load and are the oldest and most traditional forms of column design. Such formulas are based on experiments, and they usually have a limited range of application. Some forms of these equations are

*Rankine-Gordon formula*

$$\sigma_{ac} = \frac{\sigma_y}{F.S.} \left[1 + K_e (L/r)^3\right]$$

(4.196)

*Straight-line formula (Tetmaier)*

$$\sigma_{ac} = \frac{\sigma_y - K_e (L/r)}{F.S.}$$

(4.197)

*Parabolic formula (Johnson parabola)*

$$\sigma_{ac} = \frac{\sigma_y - K_e (L/r)^3}{F.S.}$$

(4.198)

In these equations $\sigma_{ac}$ is the allowable stress, F.S. is the factor of safety, and $K_e$, $K_t$, and $K_{r}$ are numerical constants.

Examples for each of these formulas are

AISC specification, 1949 (secondary members)

$$\sigma_{ac} = \frac{18}{1 + [L/r]^2/18000}$$

(4.199)

Chicago building code

$$\sigma_{ac} = 16 - 0.07 \frac{L}{r}$$

(4.200)

AASHO specification (columns with riveted ends)

$$\sigma_{ac} = 15 - \frac{(L/r)^2}{4000}$$

(4.201)

These column equations apply for A7 steel ($\sigma_y = 33$ ksi), and $\sigma_{ac}$ is in units of ksi.

The CRC basic column strength formula, which was used in Chapter 3 and which was alluded to several times in this chapter, is also an empirical curve in the form of a parabola. In its most general form it is equal to

$$\sigma_{cr} = \sigma_y \left[1 - \frac{\sigma_{cr}}{\sigma_y} \left(\frac{1 - \sigma_{cr}}{\sigma_y} \left(K_e \frac{L}{r}\right)^2\right)\right]$$

(4.202)

where $\sigma_{cr}$ is the maximum compressive residual stress and $KL/r$ is the effective slenderness ratio. In the CRC guide this formula, with $\sigma_{cr} = 0.5\sigma_y$, was suggested as a compromise to having two separate equations for the strong and weak axis buckling strength of rolled steel wide-flange columns

$$\sigma_{cr} = \sigma_y \left[1 - \frac{\sigma_{cr} (KL/r)}{4E} \right]$$

(4.203)

and it is shown in Fig. 4.34. Over most of the inelastic range we can see that the CRC curve lies between the strong-axis and weak-axis solution. It provides a reasonably good estimate of the strength of rolled wide-flange shapes, but it is not conservative for welded steel columns (Fig. 9.22, Ref. 1.34) and for initially crooked solid round columns (Fig. 4.37).

The CRC equation [in the form of Eq. (4.203)] is the basis for column design in the 1963 AISC specifications for steel building construction and the 1962 AISI specifications for light-gage cold-formed steel construction. In these specifications $\sigma_{ac} = \sigma_{cr}/F.S.$, where $\sigma_{cr}$ is determined from Eq. (4.203). In the AISI specifications F.S. is a constant, whereas in the AISC specifications it is a variable quantity which depends on the slenderness ratio.

The CRC equation can also be used to estimate the buckling strength of columns which fail by torsional or lateral-torsional buckling, as was already discussed in connection with beam behavior in Chapter 3. The procedure of calculation is as follows: (1) Compute $P_{cr}$ by assuming an elastic column, using the methods and formulas of Sec. 4.2, (2) set $\sigma_{cr} = P_{cr}/A$, (3) compute $\sigma_{y}$ using Eq. (4.203), and (4) use $\sigma_{y}$ to replace $\sigma_{cr}$ in Eq. (4.203).
(3) if \( \sigma_{crE} \leq \sigma_E/2 \), the elastic solution is valid, (4) if \( \sigma_{crE} > \sigma_E/2 \), we compute an equivalent flexural buckling slenderness ratio

\[
\left( \frac{L}{r} \right)_{\text{equivalent}} = \pi \sqrt{\frac{E}{\sigma_{crE}}} \tag{4.204}
\]

and (5) compute the inelastic stress from the CRC formula for this slenderness ratio; that is,

\[
(\sigma_{cr})_I = \sigma_E \left[ 1 - \frac{\sigma_E (L/r)^2_{\text{equivalent}}}{4 \pi E} \right] = \sigma_E \left( 1 - \frac{1}{4 \sigma_{crE}} \right) \tag{4.205}
\]

This equation affords a simple way to estimate the critical torsional buckling stress in the inelastic range. Although it is not exact, it provides a satisfactory solution to an otherwise messy problem. One other alternate to the torsional buckling problem in the inelastic range would be to use a straight-line approximation between the end of the elastic range and \( P = P_r \).

The secant formula approach assigns the cause of the deviation of the inelastic column strength from the elastic column strength to initial crookedness, Fig. 4.38(a), or eccentricities of load application [Fig. 4.38(b)]. The loading on the column of Fig. 4.38(b) is identical with that on the column in Fig. 4.38(c), and for this latter case the maximum moment occurs at the center of the member. From Chapter 5 [Eqs. (5.21) and (5.22), \( \kappa = 1.0 \)] it is equal to

\[
M_{\text{max}} = P e_i \left[ \frac{\sqrt{2} - 2 \cos \frac{F_e L}{sin F_e L}}{\sin F_e L} \right] = P e_i \sec \left( \frac{L}{2} \sqrt{\frac{P}{E}} \right) \tag{4.206}
\]

Fig. 4.38. Initial curvature and unavoidable eccentricity of load

If the maximum allowable stress is \( \sigma_f/F_{S,E} \), and elastic theory is used, then

\[
\frac{\sigma_E}{F_{S,E}} = \frac{P}{A(F_{S,E})} + \frac{M_{\text{max}}}{S(F_{S,E})} \tag{4.207}
\]

We can now set Eq. (4.206) into (4.207), and if the average stress \( P/A \) is also the allowable stress, then we can solve Eq. (4.207) for this stress; that is,

\[
\sigma_{ar} = \frac{P}{A} = \frac{\sigma_E}{F_{S,E} \left[ 1 + (e_i A/S) \sec (L/2r) \sqrt{\sigma_{ar}/E} \right]} \tag{4.208}
\]

The term \( e_i A/S \) is the "initial eccentricity ratio," and it takes on the value of 0.25 in the AASHO and the AREA specifications. A similar formula is the Perry-Robertson formula used in the British specifications.

A variation of the initial eccentricity approach is used in the German specification DIN 4114 in which the allowable stress is computed from the maximum inelastic strength of an eccentrically loaded tee column with an initial eccentricity of \( e_i = r/20 + L/500 \). The allowable stress is expressed as

\[
\sigma_{ar} = \frac{\sigma_f}{\omega(F_{S,E})} \tag{4.209}
\]

where \( \omega \) is a tabulated numerical factor dependent on the type of steel and slenderness ratio.

Column design is not a direct process. First a section size and material must be assumed. The capacity of the assumed column is then checked against whatever column formula or critical load applies. If this check is not satisfactory, a new section is assumed and the process is repeated until the final section safely and economically fulfills the requirements. A great deal of help in the selection of columns is provided by the load tables given, for example, in the AISC manual. The most useful tool for column designers is the CRC guide.

**SUMMARY**

In this chapter we have examined methods for determining the buckling loads and maximum loads of metal columns. Our discussion was brief and incomplete, not only because of space limitations, but also because of the ready availability of much additional information from the literature. We did not cover such topics as (1) columns with variable cross section and variable axial force (see Chap. 2, Ref. 1.29 and Chap. VI, Ref. 1.30, for example), (2) columns with open webs, that is, laced, batten-braced, or perforated columns (see Chap. 2, Ref. 1.29; Chap. VI, Ref. 1.30; and Chap. 3, Ref. 3.10), (3) local buckling of column plate elements (see Chap. 3, Ref. 3.10, and Chap. IX, Ref. 1.30) and many other topics. Hopefully this brief introduction to columns has whetted the appetite of the reader for further study and research.
REFERENCES


PROBLEMS

4.1. Determine \((P_x)_{tr}, (P_y)_{tr}, (P_z)_{tr}\) [Eqs. (4.38)] and \(P_{tr}\) [from Eq. (4.59) or Eq. (4.65)] as a function of the column height \(L\) and plot the \(P-L\) curves (e.g., Fig. 4.9).

If no computer is available, solve only for the value of \(L\) given below.

(a) 14 WF142 and 27 WF94 steel shapes. \(\sigma_{tr} = 33\) ksi; \(E = 30,000\) ksi; \(G = 11,500\) ksi. Plot \(P/A\sigma_{tr}\) versus \(L/r_s\) curves, or compute \(P-s\) for \(L/r_s = 50\).

End conditions: \(u = v = \phi = u'' = v'' = \phi'' = 0\)

(b) 9 by 4 by 1 in. angle shape.

\(E\) and \(G\) as in (a) above; \(\sigma_{tr} = 36\) ksi.

Plot \(P-L\) curves or compute \(P-s\) for \(L = 50\) in.

End conditions as in (a) above.

(c) ST 9WF25 shape.

\(E\) and \(G\) as in (a) above; \(\sigma_{tr} = 100\) ksi.

Plot \(P-L\) curves or compute \(P-s\) for \(L = 120\) in.

End conditions: \(u = u' = v = v'' = \phi = \phi' = 0\).

(d) 6 by 3 by \(\frac{1}{8}\) in. zee shape.

\(\sigma_{tr} = 37\) ksi; \(E = 10,300\) ksi; \(G = 3,850\) ksi.

Plot \(P/A\sigma_{tr}\) versus \(L/r_{max}\) curves, or compute \(P-s\) for \(L/r_{max} = 40\).

End conditions as in (a) above.

Wherever \(P/A > \sigma_{tr}/2\), use CRC reduction method for inelastic columns. [Eq. (4.205)].

4.2.

Show that the buckling condition is

\[
\frac{F_L L_C}{F_c L_C} - \frac{F_L L_B}{6\lambda} + \frac{L_C}{L_B - L_C} = 0
\]

where

\[
F_L L_B = \frac{PL_b}{E I_C}; \quad \lambda = \frac{I_B L_C}{I_C L_B}
\]

4.3.

(a) Determine the buckling condition for (1) nonsway buckling and (2) sway buckling by the slope-deflection method.

(b) Plot curves relating \(PL_b/EI_C\) versus \(\lambda = L_B L_C/L_B L_C\) for both cases of buckling.

(c) Check end conditions when \(\lambda = 0\) and \(\lambda = \infty\).

(d) Determine the same curves as in (b) by the CRC nomographs (Fig. 4.19).

4.4. A rectangular rigid frame is braced by two diagonal braces which can only transmit tension. Determine the least required bracing area to just force a nonsway buckling mode [Fig. 4.15 and Eq. (4.89)]. Assume elastic behavior.
4.5. Determine the effective length factor in the equation $P_{cr} = \frac{\pi^2 EI}{(KL)^2}$ for the following simple frames:

(a)

(b)

(c)

P. 4.5

P. 4.6(a)

4.6. Develop expressions relating $P/P_Y$ and $\lambda = L/r(1/\pi) \sqrt{\sigma_Y/E}$ for the full range $0 \leq \lambda \leq \infty$ and plot the $P/P_Y - \lambda$ curves for the following cases. All column ends are simply supported.

(a) The yield point of the flange is twice the yield point of the web, $P_Y = 2\sigma_Y$ by definition. In the inelastic range consider both the tangent modulus and the reduced modulus solution about both the $x$ and $y$ axes.

(b) Consider buckling about both the $x$ and $y$ axes, and in the inelastic range compute only the tangent modulus load.

(c) 8WF31 column of ideal elastic-plastic material and with the residual stress pattern of Fig. 1.6 ($\sigma_{rs} = 0.3\sigma_Y$; $\sigma_T = 36$ ksi). Determine the $P/P_Y - \lambda$ curves for buckling about both the $x$ and the $y$ axes, and in the inelastic range consider both the tangent modulus and the reduced modulus solutions. Compare results with CRC curve and with Fig. 4.34.
## Stability Functions

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### Stability Functions

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### Chapter Five

#### 5.1. Introduction

**Beam-Columns**

Members which support loads causing both bending and axial compression are beam-columns. Such members are encountered frequently in the design of framed structures where, typically, the vertical members are both bent and compressed [Fig. 5.1(a)]. Some of the frequently found beam-column loading conditions in frames are shown in Fig. 5.1(b). The axial loads \( P \) derive from the loads above the particular member and from the loads on the adjacent beams framing into its end. The bending moments at the ends of the beam-column represent the resistance of this member to the bending moments acting at the ends of the members which frame into it.

Beam-columns in frames are usually subjected to end forces only. It is possible to find, however, beam-columns which support transverse forces in
addition to end forces [Fig. 5.1(c)]. Such loadings are encountered for compression cords in trusses, for example.

The analysis of beam-columns involves a combination of the features we discussed previously for beams and columns. In fact we could consider all members as beam-columns, beams representing the limiting case in which the axial force becomes zero and columns the case in which the bending moments vanish.

Fig. 5.1. Beam-columns and beam-column loadings. (a) Beam-columns in a frame, (b) typical beam-columns in frames, (c) typical beam-columns with transverse loads.

Fig. 5.2. End forces, dimensions, and sign conventions. (a) Beam-columns with end forces, (b) beam-column sign conventions.

We shall start the discussion of beam-column behavior in this chapter by considering a member of uniform section which is loaded by a compressive axial force $P$ and end moments which bend the member in its plane of symmetry. Such a member is shown in Fig. 5.2(a). The end moments $M_{En}$ and $M_{Fn}$ are shown positive as counterclockwise, this convention being consistent with that adopted in Eqs. (2.93) through (2.95). In the following derivations it will be more convenient to use the numerically larger of the two end moments as a reference, and so the slightly modified convention of Fig. 5.2(b) will be adopted. Here the larger end moment is $M_0$ and the origin of the $z$ coordinate is at the end where $M_0$ acts. The moment at the other end is $\kappa M_0$, where $\kappa$ represents the factor of proportionality between the larger and the smaller end moments. Thus $-1 \leq \kappa \leq +1$; when $\kappa$ is positive the moment at $z = L$ is positive in a clockwise sense.

LOAD-DEFORMATION BEHAVIOR OF BEAM-COLUMNS

The behavior of the beam-column shown in Fig. 5.2(b) can be studied by relating the applied end moment $M_0$ to the end rotation $\theta_0$ during the loading history of the member. Such behavior is illustrated by the experimentally obtained $M_0-\theta_0$ curve shown in Fig. 5.3.\(^{61,42}\) In the experiment a steel wide-flange beam-column was first subjected to a concentric force $P = 0.49 P_0$.\(^{62}\)
The bending moment inside the span of the beam-column is composed of the primary moment $M_0[1 - (v/L)]$ due to the applied end moment $M_0$ and the secondary moment $Pv$, where $v$ is the deflection of the member. As some portions of the member became inelastic, the deflection $v$ increases, and the $Pv$ moment requires a proportionately larger share of the moment capacity of the cross section. Eventually the beam-column is unable to support an increase of $M_0$, and thereafter the value of $M_0$ decreases as the deflection increases.

The in-plane performance as represented by the curve in Fig. 5.3 and by the solid curve in Fig. 5.4 corresponds to optimum conditions. Downward deviations from this optimum are caused by lateral-torsional deformations and/or local buckling. Several possibilities are illustrated in Fig. 5.4: Curve $A$ shows lateral-torsional buckling in the elastic range, maximum moment $(M_0)_{let}$; curve $B$ shows inelastic lateral-torsional buckling, maximum moment $(M_0)_{let}$ and bifurcation moment $(M_0)_{bif}$; curve $C$ shows local buckling.

The critical load (bifurcation) in the elastic range is equal to the maximum moment; in the inelastic range, however, bifurcation occurs before the maximum moment is reached (see curve $B$).

In the following discussion we shall consider these and various other aspects of the beam-column problem. We shall start with elastic behavior.

**Fig. 5.3.** Experimental-moment-end-rotation curve for a beam-column

While $P$ was kept constant, an end moment $M_0$ was applied about the major axis of the member at one end ($v = 0$). The moment was applied through rotating the end of the beam-column by monotonically increasing rotations $\theta_0$. The measured relationship between $M_0$ and $\theta_0$ is the curve in Fig. 5.3. Lateral bracing along the length of the member was provided to curb deformations out of the plane of bending. The $M_0-\theta_0$ curve in Fig. 5.3 represents deformation behavior which takes place in one plane only (in-plane behavior), and thus the disturbing effects of lateral deformation are not present.

The behavior of beam-columns is different from the behavior of columns and beams. The axial load which is applied is smaller than the maximum force $P$ which can be supported as a column, and thus there exists some reserve of capacity to carry bending moment. On the other hand, the moment which can be carried is less than the full plastic moment $M_p$ which could be supported if $P$ were zero. Furthermore, the maximum moment which is reached is not maintained through an indefinite rotation. We recall that for beams the reduction in moment is accompanied by effects such as lateraltorsional or local buckling. In the experiment these effects were deliberately avoided by lateral bracing, and no local buckling was observed until well beyond the peak of the $M_0-\theta_0$ curve. Thus the unloading in the curve of Fig. 5.3 is entirely due to causes which are occurring in the plane of bending.

**Fig. 5.4.** Effect of lateral-torsional and local buckling
5.2. THE ELASTIC RESPONSE OF BEAM-COLUMNS

THE DIFFERENTIAL EQUATIONS OF ELASTIC BEAM-COLUMNS

The differential equations governing the elastic behavior of the beam-column of Fig. 5.2(b) are given by Eqs. (2.93) through (2.95). These equations can be written as follows for $M_y = M_z = 0$, $M_{xz} = -M_z$, and $M_{yz} = \kappa M_y$:

$$B_y\psi'' + Pu - Px\phi = M_y\left[-1 + \frac{z}{L}(1 - \kappa)\right]$$

(5.1)

$$B_z\psi'' + Pu + M_z\phi\left[1 - \frac{z}{L}(1 - \kappa)\right] + Py\phi = 0$$

(5.2)

$$C_y\phi''' - (C_y + \tilde{K})\phi' + M_y\psi\left[1 - \frac{z}{L}(1 - \kappa)\right] + Py\Phi' - Px\alpha' + M_z\psi = 0$$

(5.3)

These three equations are not independent of each other, and the first one is not homogeneous. They do not, therefore, describe a bifurcation, as deformations $u$, $v$, and $\phi$ will be present from the start of loading.

In order to obtain deflections only in a plane parallel to a principal axis we must have at least one axis of symmetry. Furthermore, the applied bending moments $M_y$ and $M_z$ must act in the plane of symmetry which will be assumed to be the $y-z$ plane. This is achieved by setting $x_0 = 0$ in Eqs. (5.1) through (5.3), that is,

$$B_y\psi'' + Pu = M_y\left[-1 + \frac{z}{L}(1 - \kappa)\right]$$

(5.4)

$$C_y\phi''' - (C_y + \tilde{K})\phi' + M_y\psi\left[1 - \frac{z}{L}(1 - \kappa)\right] + Py\Phi' + M_z\psi = 0$$

(5.5)

Equation 5.2 is unchanged since it does not contain $x_0$.

We shall now differentiate Eqs. (5.2), (5.4), and (5.5) and replace the stiffness coefficients $B_y$, $B_z$, $C_y$, and $C_y$ by their elastic values $EI_y$, $EI_z$, $GK_z$, and $EI_o$, respectively. This gives

$$EI_y\Phi'' + Pu'' = 0$$

(5.6)

$$EI_y\psi'' - M_y\left[1 - \frac{z}{L}(1 - \kappa)\right]\Phi'' - 2M_z(1 - \kappa)\psi' + Py\Phi'' = 0$$

(5.7)

$$EI_y\phi''' - (GK_z + \tilde{K})\phi'' - \tilde{K}'\phi' + M_y\left[1 - \frac{z}{L}(1 - \kappa)\right]u'' + Pyu'' = 0$$

(5.8)

The value of $\tilde{K} = \int \sigma\alpha^2 dA$ [Eq. (2.86)] is computed for a stress $\sigma = -(P/A) + (M_z\gamma/I_z)$. The component due to $-P/A$ is equal to $-P\gamma$ [from Eq. (4.9)], and the component due to $M_z\gamma/I_z$ is $M_z\beta_z$ [from Eq. (3.14)]. Thus

$$\tilde{K} = -P\gamma + M_z\beta_z$$

(5.9)

where $\gamma$ and $\beta_z$ are defined by Eqs. (4.10) and (3.13), respectively. In the development of Eq. (5.9) certain terms involving products of small quantities were neglected [see the development of Eq. (3.85) in Chapter 3]. The moment $M_z$ in Eq. (5.9) is the bending moment anywhere within the span of the beam-column, and it is equal to [from Eq. (2.80)]

$$M_z = M_y\left[1 - \frac{z}{L}(1 - \kappa)\right]$$

(5.10)

Thus the third differential equation (Eq. 5.8) can now be written as

$$EI_y\phi''' - [GK_z - P\gamma + M_z\beta_z]\phi'' + M_y\beta_z\phi'(1 - \kappa)\gamma + M_z\beta_z\phi' + Py\Phi'' = 0$$

(5.11)

Equation (5.6) is independent of the other two differential equations [Eqs. (5.7) and (5.11)], and it defines the deformations in the plane of bending ($y$-$z$), which is also the plane of symmetry. The remaining two differential equations define the critical condition when lateral-torsional buckling commences.

THE IN-PLANE STRENGTH OF BEAM-COLUMNS

The deformation of the beam-column in the $y$-$z$ plane is defined by Eq. (5.6), which can be rearranged as follows:

$$\psi'' + P\gamma\phi'' = 0$$

(5.12)

where

$$\psi'' = \frac{P}{EI_y}$$

(5.13)

The solution of Eq. (5.12) is [Eq. (4.18)]

$$\psi = C_1 \sin F_z x + C_2 \cos F_z x + C_3 z + C_4$$

(5.14)

The boundary conditions are [Fig. 5.2(b)] $\psi(0) = \psi(L) = 0$, $\psi''(0) = -M_z/EI_y$, and $\psi''(L) = -\kappa M_y/EI_y$. The last two boundary conditions are a result of the end moments $M_y$ and $\kappa M_z$. From these boundary conditions we

1 In a more refined analysis we should also include in this equation the term $+P\gamma$ [Eq. (2.80)]. Unlike the deformations $u$ and $\phi$, which occur only after the onset of lateraltorsional buckling, the deflection $\psi$ can contribute a significant portion to the bending moment. However, if we did this, we could considerably complicate Eq. (5.11) by terms such as the products $u\phi'$ and $u'\phi$. Since similar terms were already neglected in the derivations of Eqs. (2.93) through (2.95), the introduction of this refinement here is not warranted.
can find the constants of integration in Eq. (5.14), and the final expression 
for the deflection $v$ is

$$v = \frac{M_0}{E} \left[ \frac{\kappa - \cos F_L z}{\sin F_L L} \sin F_L z + \cos F_L z + \frac{z}{L}(1 - \kappa) - 1 \right]$$  \hspace{1cm} (5.15)

The bending moment at any point $z$ is computed as

$$M = -EI \frac{dv}{dz} = M_0 \left[ \frac{\kappa - \cos F_L z}{\sin F_L L} \sin F_L z + \cos F_L z \right]$$  \hspace{1cm} (5.16)

In order to be able to determine the maximum elastic stress in the member at 
a given load we need to know the maximum value of $M$. The maximum 
moment will occur at a location $z$ which is defined by\footnote{A graphical procedure is available also by which moment diagrams of elastic beam-columns can be rapidly developed.\footnote{\textsuperscript{5.4}}}

$$\frac{dM}{dz} = 0 = M_0 F_L \left[ \frac{\kappa - \cos F_L z}{\sin F_L L} \cos F_L z - \sin F_L z \right]$$  \hspace{1cm} (5.17)

After some rearrangement of Eq. (5.17) we find that

$$\tan F_L z = \frac{\kappa - \cos F_L z}{\sin F_L z}$$  \hspace{1cm} (5.18)

With Eq. (5.18) substituted into Eq. (5.16), the maximum moment is equal to

$$M_{\text{max}} = M_0 (\tan F_L z \sin F_L z + \cos F_L z)$$  \hspace{1cm} (5.19)

which can be simplified to

$$M_{\text{max}} = \frac{M_0}{\cos F_L z}$$  \hspace{1cm} (5.20)

From Eq. (5.18) and the trigonometric relationships given in Fig. 5.5 we find that

$$M_{\text{max}} = M_0 \varphi$$  \hspace{1cm} (5.21)

where\footnote{\textsuperscript{5.4}}

$$\varphi = \frac{1}{\cos F_L z} = \frac{\sqrt{1 + \kappa^2 - 2\kappa \cos F_L L}}{\sin F_L L}$$  \hspace{1cm} (5.22)

is a factor by which $M_0$ is modified to obtain the maximum moment.

The solution of Eq. (5.18) may result in values of $z$ which are negative. 
This means that the relationship expressed by Eq. (5.16) has no maximum 
within the region $0 \leq z \leq L$. The possible cases are illustrated in Fig. 5.6 
where a moment diagram is shown for the case in which $M_{\text{max}}$ is within 
(solid curve) and without (dashed curve) the span. In the latter case the 
maximum moment within the region $0 \leq z \leq L$ is at $z = 0$ and it is equal to 
$M_0$, that is, $\varphi = 1.0$. The boundary between the two types of situations 
occurs when $z = 0$. This happens when $\kappa = \cos F_L L$ [Eq. (5.18)]. Thus $\varphi$ 
from Eq. (5.22) is to be used when $\kappa \geq \cos F_L L$ and $\varphi = 1.0$ when $\kappa \leq \cos F_L L$.

Another type of a limiting situation arises when the axial load $P$ equals 
the critical load [Eq. (4.23)]

$$P_c = \frac{\pi^4 E \alpha}{L^4}$$

No moment at all can be carried in addition to $P_c$, that is, from Eq. (5.21)

$$M_0 = \frac{M_{\text{max}}}{\varphi} = 0$$  \hspace{1cm} (5.23)

This can occur only if $\varphi = \infty$; from Eq. (5.22) we see that this takes place 
when $\sin F_L L = 0$, and thus

$$(F_L L)_{\text{max}} = \pi$$  \hspace{1cm} (5.24)

The limit of applicability of the equations for elastic beam-columns is the 
point at which the maximum stress in the member just reaches $\sigma_r$, or 
the proportional limit of the material. If we take $\sigma_r$ as the limiting stress, 
then

$$\sigma_{\text{max}} = \frac{P}{A} + \frac{M_{\text{max}}}{S_e}$$  \hspace{1cm} (5.25)

After rearrangement and introduction of the nondimensional relationships 
$P/P_r = P/\sigma_r A$ and $M_0/M_r = M_0/S_e \sigma_r$, we obtain the following interaction equation\footnote{\textsuperscript{5.4}} (that is, an equation giving the interrelationship between the axial force and the end moment)

$$\frac{P}{P_r} + \varphi \left( \frac{M_0}{M_r} \right) = 1.0$$  \hspace{1cm} (5.26)

This interaction equation permits the calculation of one load parameter 
when the other one is given for any given beam-column when the maximum 
stress is $\sigma_r$. It applies for any value of $0 \leq F_L L \leq \pi$ (that is, $0 \leq P \leq P_c$). 
The coefficient $\varphi$ is equal to its value from Eq. (5.22) if $\kappa \geq \cos F_L L$ and $\varphi =$
1.0 when \( \kappa \leq \frac{P_0 \pi}{3L} \). Curves relating \( \varphi \) and \( F_xL \) are given in Fig. 5.7 for \( \kappa = 1.0, 0.5, 0, -0.5, \) and \(-1.0 \) (solid curves).

With the interaction equation [Eq. (5.26)] we are now able to study the effects of the variables which influence the elastic strength of beam-columns loaded by terminal moments. We are here defining the elastic limit as the criterion of strength (Fig. 5.4). We have not as yet presented any information on the end moments causing lateral-torsional buckling, nor have we said anything about the real maximum strength of the beam-column. This maximum moment can only reach after yielding (Fig. 5.4).

The parameters affecting the elastic strength are the nondimensional ratios \( P/P_r, M_0/M_r, \kappa, \) and \( F_xL \). This latter term is equal to

\[
F_xL = L\sqrt{\frac{P}{P_r}} = \frac{L}{r_e}\sqrt{\sigma_f(P/P_r)}
\]

(5.27)

Where \( \sigma_f = \sigma_r/E \). We can thus represent the elastic in-plane strength of beam-columns by the following four nondimensional parameters: \( P/P_r, \)

\[
M_0/M_r, \kappa, \text{ and } (L/r_e)\sigma_f.
\]

The interaction equation [Eq. (5.26)] is usually represented in the form of

\[
(P/P_r) - (M_0/M_r) \text{ interaction curves (Figs. 5.8 and 5.9). These curves are valid for all singly or doubly symmetric prismatic beam-columns which are loaded by an axial force and by end moments acting in the plane of symmetry.}

The influence of the variation of the slenderness parameter \( (L/r_e)\sigma_f \) is illustrated in Fig. 5.8 for a beam-column bent into single curvature by equal end moments \( \kappa = 1.0 \). Three curves are given: \( (L/r_e)\sigma_f = 0, 1.99, \) and 3.98 (corresponding to \( r_e = 0, 60, \) and 120, respectively, when \( E = 30,000 \) ksi and \( \sigma_r = 33 \) ksi). All three curves converge to \( M_0 = M_r \) when \( P = 0 \). At this limit the member is a beam, and \( M_0 \) will reach \( M_r \) at yielding. When \( M_0 = 0 \), we have a column which can carry \( P = P_r \) when \( P_r > P_r \) and \( P = P_r \) when \( P < P_r \). As the slenderness ratio increases, beam-column strength decreases. The curve for \( (L/r_e)\sigma_f = 3.98 \) is considerably lower than the curve for the case of a zero length member. This is to be expected since slender columns deflect more, and thus a larger share of the total moment is due to \( P_v \).

The effect of the variation of the end moment ratio \( \kappa \) is illustrated in Fig. 5.9 for a constant ratio \( (L/r_e)\sigma_f = 4.00 \) (solid curves). Single curvature bending with equal end moments \( \kappa = 1.0 \) is seen to give the lowest strength.
Double curvature bending with $\kappa = -1.0$ is the strongest. For this case the maximum moment occurs at the ends of the member, regardless of its length; that is, $\varphi = 1.0$ in Eq. (5.26). This situation applies only as long as $P < P_E$. When $P = P_E$, which is at $P = 0.617\ P_F$ for $(L/r_a)_{\varphi} = 4.00$, the curve is cut off by the horizontal line $P = P_E$, as shown in Fig. 5.9.

OTHER LOADING CASES
AND AN APPROXIMATION

The elastic in-plane strength of any singly or doubly symmetric prismatic beam-column loaded by terminal moments in the plane of symmetry is defined by the interaction equation, Eq. (5.26). This interaction equation can be shown to apply also to other situations, for example, to cases of nonprismatic members and members with concentrated and distributed loads applied in the plane of symmetry between the ends. Solutions to many of the latter problems are presented in Ref. 1.29. The principle of superposition can be applied if it is kept in mind that the axial force must be included in each subsolution. For example, the problem of Fig. 5.10(a) can be solved by superimposing the solution of the member with end moments only [Fig.

\footnote{Equations and curves for $\varphi$ for tapered beam-columns are given in Ref. 5.5.}

5.10(b)] on the solution of the member with the distributed load only [Fig. 5.10(c)]. In each of the two cases $P$ must be included since moment is increased in both cases by $P_E$.

The differential equation for in-plane behavior of a beam-column with distributed load $q$ [Fig. 5.10(c)] is equal to

$$EI\frac{d^4v}{dx^4} + Pu'' = q$$

The solution of this differential equation for simple boundary conditions $\psi(0) = \psi(L) = \psi''(0) = \psi''(L) = 0$ is

$$v = \frac{q}{PE_F} \left(\frac{1 - \cos F_z L}{\sin F_z L}\right) \sin F_z x + \frac{F_z L^2}{2} \left(\frac{z}{L}\right)^2 - 1$$

(5.29)

After superimposing $v$ from Eq. (5.29) on the deflection $v$ from Eq. (5.15), we can show by a similar procedure to that used for terminally loaded beam-columns that the maximum moment for the beam-column of Fig. 5.10(a) is equal to

$$M_{max} = \varphi M_o$$

(5.30)

where

$$\varphi = 1.0 \quad \text{for} \quad A \leq \cos F_z L$$

(5.31)

and

$$\varphi = \left[\frac{1 + B}{1 + \frac{A^2 - 2A \cos F_z L}{\sin F_z L} - B}\right] \quad \text{for} \quad A \geq \cos F_z L$$

(5.32)

In Eqs. (5.31) and (5.32)

$$B = \frac{qE_Fz}{M_o F_p}$$

(5.33)

and

$$A = \kappa + B$$

(5.34)

With Eq. (5.30) we see that the interaction equation for this beam-column can
This expression can be derived in a similar manner to the expression for the potential energy of beams. It contains like terms \( \int_0^L \cos^2 (ix\pi/L) \, dz = L/2 \) and unlike terms which are zero, and thus

\[
V_p = -\frac{P \pi^2}{4L} \sum_{n=1}^{\infty} n^4 a_n^2
\]

The potential energy of the end moments \( M_0 \) for \( \kappa = 1.0 \)

\[
V_\kappa = -M_0 \omega'(0) + M_0 \omega'(L) = \frac{M_0 \pi}{L} \sum_{n=1}^{\infty} a_n [1 - (-1)^n]
\]

The potential energy is a minimum when, for any \( n = i \),

\[
\frac{\partial (U + V)}{\partial a_i} = 0 = \frac{EI_i \pi^4}{4L^3} (2a_i d^3) - \frac{P \pi^2}{4L} \sum_{n=1}^{\infty} a_n n^2 [1 - (-1)^n]
\]

from which

\[
a_i = -\frac{2M_0 L^4 [1 - (-1)^n]}{\pi^4 E I_x} \left[ a_i d^3 + \frac{P}{P_x} \sum_{n=1}^{\infty} n^2 [1 - (-1)^n] \right]
\]

and thus

\[
v = \frac{2M_0 L^4}{\pi^4 E I_x} \sum_{n=1}^{\infty} \frac{n^2 [1 - (-1)^n] \sin (n\pi z/L)}{n^2 - 4 \pi^2 / P_x}
\]

The maximum moment occurs at \( z = L/2 \), and it is equal to

\[
M_{\text{max}} = M_0 + P v
\]

If we set \( n = 1 \) and neglect the term \([4/\pi^2 - 1](P/P_x)\) as small (conservatively), then we obtain Eq. (5.38).

Various approximate formulas have been suggested for \( C_m \). The following two were proposed by Massonnet \( (8, 10) \) and Austin \( (5, 11) \) respectively:

\[
C_m^* = \sqrt{0.5 \pi^2 + 0.4 \pi^2} + 0.3
\]

\[
C_m^* = 0.6 + 0.4 \pi^2 > 0.4
\]

Neither of these equations gives a very good approximation for \( C_m \) [Eq. (5.37)] because \( C_m \) does not depend on \( \kappa \) alone but also on \( F_L \). However, when these values are used in the interaction equation [Eq. (5.35)], the end result (see dashed curves in Fig. 5.9) is nevertheless quite reasonable, especially in the range \(-0.5 \leq \kappa \leq 1.0\).

For beam-columns with end moments we can therefore use the following interaction equation to define the elastic limit:

\[
\frac{P}{F_T} + \frac{C_m^* M_0}{M_T (1 - P/P_x)} = 1.0
\]

where \( C_m^* \) from Eq. (5.39) or (5.40) is used. The approximate curves in Fig. 5.9 were computed with \( C_m^* \) from Eq. (5.40). The approximate interaction
equation no longer accounts for the case in which the maximum moment occurs at the end of the member, and so the additional condition
\[
\frac{P}{P_r} + \frac{M_0}{M_r} = 1.0
\] (5.42)
must also be satisfied. In Fig. 5.9 we see that the dashed curves [from Eq. (5.41)] are cut off at the intersection with this line.

If transverse forces are present between the ends of the member, it is possible to use the following approximate formula for finding the maximum moment: (5.10, 5.8)
\[
M_{\text{max}} = \frac{\bar{M}_{\text{max}}}{1 - \left(\frac{P}{P_r}\right)}
\] (5.43)
In Eq. (5.43) \(M_{\text{max}}\) is the maximum moment in the span if \(P = 0\) and \(\bar{M}_{\text{max}}\) is the maximum deflection for \(P = 0\). For the beam-column with the distributed load [Fig. 5.10(c)], for example,
\[
M_{\text{max}} = \frac{qL^2}{8} + \frac{5PqL^4}{384EJ(1 - \frac{P}{P_r})}
\] (5.44)

ELASTIC LATERAL-TORSIONAL BUCKLING

Another limit of elastic beam-column strength to be considered is lateral-torsional buckling. Lateral-torsional buckling is governed by the last two of the three basic differential equations [Eqs. (5.7) and (5.11)]. The phenomenon is identical with the lateral-torsional buckling of beams (Sec. 3.2) and columns (Sec. 4.2), and in the elastic range the critical combination of loads producing it is the maximum practical load which can be sustained by the member.

The solution of Eqs. (5.7) and (5.11) for singly symmetric sections and for unequal end moments is best attempted by numerical or approximate methods (finite difference, numerical integration, or energy methods). For doubly symmetric sections \(y_u = \beta_u = 0\), and the differential equations of lateral-torsional buckling then become
\[
EL_u'' + Pu'' + M_0\left[1 - \frac{Z}{L}(1 - \kappa)\right]u'' = 0
\] (5.45)
\[
EL\phi''' - (GK_T - P\phi)'' + M_0\left[1 - \frac{Z}{L}(1 - \kappa)\right]\phi'' = 0
\] (5.46)
The solution of these differential equations is again best attempted by numerical or energy methods. Extensive charts and tables for wide-flange beam-columns were presented by Salvadori (1931, 1938) who used the Rayleigh-Ritz method in his computations.

A simple solution of Eqs. (5.7) and (5.11) is possible for the case of uniform moment (\(\kappa = 1.0\)) where the differential equations are
\[
EL_u'' + Pu'' + (M_0 + PY_u)u'' = 0
\] (5.47)
\[
EL\phi''' - (GK_T - P\phi)'' + (M_0 + PY_u)\phi'' = 0
\] (5.48)
These equations have the following solution for simply supported boundary conditions [that is, \(u(0) = u''(0) = u(L) = u''(L) = \phi(0) = \phi''(0) = \phi(L) = \phi''(L) = 0\)]
\[
u = C_1 \sin \frac{\pi Z}{L} \quad \text{and} \quad \phi = C_3 \sin \frac{\pi Z}{L}
\] (5.49)
Substitution of Eqs. (5.49) and their derivatives into Eqs. (5.47) and (5.48) results in the following two homogeneous simultaneous equations:
\[
\left[\left(\frac{\pi^2EL}{L^3} - P\right)C_1 - (M_0 + PY_u)C_3\right] \frac{\pi^2Z}{L} \sin \frac{\pi Z}{L} = 0
\]
\[
\left[-(M_0 + PY_u)C_1 + \left(\frac{\pi^2EL}{L^3} + GK_T - P\phi + M_0\beta_u\right)C_3\right] \frac{\pi^2Z}{L} \sin \frac{\pi Z}{L} = 0
\]
The term \((\pi^2/L^3) \sin (\pi Z/L) \neq 0\), and thus the determinant of the coefficients of \(C_1\) and \(C_3\) becomes equal to zero
\[
\left|\begin{array}{cc}
(P - P_u) & -(M_0 + PY_u) \\
-(M_0 + PY_u) & (\phi + P\phi) \\
\end{array}\right| = 0
\]
from which the buckling condition becomes
\[
(P_r - P)(\phi + P\phi - P\phi + M_0\beta_u) = (M_0 + PY_u)^2
\] (5.50)
In Eq. (5.50) \(P_r\) and \(P_u\) are the critical axial loads [Eqs. (4.24) and (4.25)]
\[
P_r = \frac{\pi^2EL}{L^3} \quad \text{and} \quad P_u = \frac{\pi^2EL}{L^3} + GK_T
\] (5.51)
For any given beam-column Eq. (5.50) can be solved for the critical combination of \(P\) and \(M_0\).

For doubly symmetric sections \(\beta_u = \gamma_u = 0\), and therefore Eq. (5.50) is simplified to
\[
(M_0)_{cr} = \sqrt{(I_u + I_u)A(K(P_r - P)(P_r - P))}
\] (5.52)
or in nondimensional form
\[
\left(\frac{M_0}{M_r}\right)_{cr} = \sqrt{\frac{d(1 + I_u/I_u)}{4\pi^2}} \left(\frac{P_r}{P_r - P}\right) \left(\frac{P_r - P}{P_r}\right)
\] (5.53)
where \(d\) and \(r_u\) are the depth and the radius of gyration about the \(x\) axis, respectively.

The effect of lateral-torsional buckling for an SWF31 rolled steel shape is shown in Fig. 5.11 for \((L/r_u)\sqrt{\varepsilon_r} = 1.99\) and 3.98. The two curves for the elastic in-plane strength are reproduced in this figure from Fig. 5.8. The lateral-torsional buckling curves were computed from Eq. (5.53), which is equal to the following expressions for our particular problem here:
\[
\left(\frac{M_0}{M_r}\right)_{cr} = 1.337 \sqrt{\left(\frac{0.836 - P}{P_r}\right)\left(\frac{1.629 - P}{P_r}\right)} \quad \text{for} \quad \frac{L}{r_u} \sqrt{\varepsilon_r} = 1.99
\]
\[
\left(\frac{M_0}{M_r}\right)_{cr} = 1.337 \sqrt{\left(\frac{0.209 - P}{P_r}\right)\left(\frac{1.066 - P}{P_r}\right)} \quad \text{for} \quad \frac{L}{r_u} \sqrt{\varepsilon_r} = 3.98
\]
A conservative check of the bracing spacing can be made by the simple method illustrated in Fig. 5.13. In Fig. 5.13(a) is shown the loading in the \( y-z \) plane for a given beam-column, and Fig. 5.13(b) shows the moment diagram. The maximum moment in the span is \( \varphi M_o \), where \( \varphi \) can be found from Fig. 5.7 or Eq. (5.22). The tentative location of the lateral bracing, which

\[
\frac{P}{P_f} = 0.3, \quad \epsilon_y = 0.0016, \quad \frac{L}{r_s} = 2.62
\]
must be able to prevent twisting and deflection in the \( x \) direction at the points where it is applied, is next assumed. If \( M_x \) for the length \( L_{eEF} \) [Fig. 5.13(c)] from Eq. (5.33) is larger than or equal to \( \varphi M_y \), then the bracing spacing is satisfactory; if not, then a closer spacing needs to be tried.

**THE INFLUENCE OF THE BOUNDARY CONDITIONS**

Our previous discussion on lateral-torsional buckling applies only if the ends are laterally and torsionally simply supported. It is possible to include the effect of other boundary conditions by making use of the effective length concept. Vlasov\(^{11,18}\) has shown that Eq. (5.50) can be more generally formulated as follows:

\[
(P_x - P)(P_x^2 - P_x^2 + M_yB_x) = K_{13}(M_x + Py)^2
\]

(5.54)

where

\[
Py = \pi^2 E_0 I_y
\]

and

\[
P_x = \pi^2 E_0 I_y (K_x L)^2 + G K_x
\]

(5.55)

and \( K_{13} \) is a numerical factor depending on the boundary conditions. Values of \( K_{13} \) are tabulated for several common end conditions in Table 3.2. Conservatively this factor can be made equal to unity. The effective length factors \( K_y \) and \( K_x \) are computed as was done for various examples in Sec. 4.2, or they can be estimated from the known limits of \( K_y = 1 \) if both ends are pinned, \( K_y = 0.7 \) if one end is fixed and the other is pinned, \( K_y = 0.5 \) if both ends are fixed, and \( K_y = 2.0 \) if one end is fixed and the other is free. In any situation \( K_y \) and \( K_x \) need not be the same, that is, Eq. (5.54) is also valid if the lateral and the torsional boundary conditions are not identical.

For an example we shall consider the problem in Fig. 5.11 where \( L / \sqrt{E_y / \tau_x} = 3.98 \) and where the lateral boundary conditions are \( u(0) = u'(0) = u(L) = u'(L) = 0 \) and the torsional boundary conditions are \( \phi(0) = \phi'(0) = \phi(L) = \phi'(L) = 0 \). For this case \( K_y = 0.5, K_x = 0.7 \), and \( K_{13} = 0.875 \) (Table 3.2), and the resulting lateral-torsional interaction equation is

\[
M_x / M_y = 2.16 \sqrt{(0.836 - \frac{P}{P_T})(1.251 - \frac{P}{P_T})}
\]

The resulting curve is not plotted in Fig. 5.11, but it can be easily ascertained that it will in all cases be above the in-plane strength curve.

**SUMMARY OF ELASTIC BEAM-COLUMNS**

The elastic in-plane strength is defined as the combination of loads which cause \( \sigma_{max} = \sigma_y \) in the beam-column. We showed that for singly and doubly symmetric beam-columns the elastic strength can be represented by a non-dimensional interaction equation [Eq. (5.26)]. In addition to the elastic strength we must also determine whether elastic lateral-torsional buckling controls. We have given the differential equations which must be solved to determine the critical moment, and we have developed solutions for the case of \( \varphi = 1.0 \).

Where we have asymmetric sections or where end moments are applied about both principal axes (Fig. 2.30), we have biaxial bending, and deformations \( u, v, \) and \( \phi \) exist from the start of loading. We must then solve the general differential equations of bending [Eqs. (2.93) through (2.95)]. Such problems have been solved by analytically exact methods by Culver.\(^{6,53}\) Lee and Prawel\(^{6,54}\) have presented solutions obtained by an analog computer, and Thürlimann\(^{6,109}\) and Dabrowski\(^{6,52}\) have given approximate solutions.

**5.3. THE MAXIMUM STRENGTH OF BEAM-COLUMNS**

**THE IMPORTANCE OF THE INELASTIC RANGE**

Elastic theory (discussed in Sec. 5.2) gives a rational, internally consistent, and relatively simple means for analyzing and designing metal beam-columns. In this theory the limits of usefulness are either \( \sigma_{max} = \sigma_y \) or the lateral-torsional buckling load. There are, however, several shortcomings to this approach: (1) The maximum strength may be considerably higher (30 to 40 per cent in some cases) than the elastic limit load,\(^{6,54}\) (2) there is no satisfactory way of accounting for the limiting case when \( M_x = 0 \), that is, when the member is a column and there exists a transition between elastic buckling and full yielding, as shown in Fig. 4.34, for example, and (3) the elastic limit has no real meaning in relation to the maximum strength of the frame of which the beam-column is part.\(^{6,141}\)

A great deal of research has been performed on the problem of the inelastic behavior of beam-columns since the beginning of this century. We shall only be able to cover a very small part of this work here. Two problems will be considered: (1) the maximum strength of beam-columns and (2) the moment-rotation characteristic of inelastic beam-columns.

**IN-PLANE BEHAVIOR**

In-plane behavior in the inelastic range will be discussed first. Typical in-plane \( M_x-\theta_x \) curves are shown in Figs. 5.3 and 5.4 (solid curves). After an initially linear response up to the elastic limit, the \( M_x-\theta_x \) relationship becomes nonlinear. A peak is reached, and thereafter the moment capacity decreases. In steel beam-columns the elastic limit is reached when the sum of the stresses due to the applied loads and the residual stresses becomes equal to \( \sigma_y \). The
interaction equation defining this limit is, from Eqs. (5.25) and (5.26),
\[
\frac{P}{F_r} + \frac{M_a}{M_r} = 1 - \frac{\sigma_r}{\sigma_y}
\]
(5.56)

The \( M_a,\theta_a \) curves have a peak because eventually \( M_a + P \theta \to M_{pc} \), where \( M_a \) is the primary moment and \( M_{pc} \) is the plastic moment of the cross section. Since \( P \) and \( M_{pc} \) are constant, \( M_a \) must decrease as the deflection increases.

In our present discussion we are interested in the magnitude of the loads at the peak. We shall again use the conditions of Fig. 5.2(b) as the basis of further calculations. The moment at any location \( x \) is
\[
M_a = M_r \left[ 1 - \frac{x}{L} (1 - \kappa) \right] + P \theta
\]
(5.57)

This external moment is resisted by an internal moment which is a function of the curvature \( \Phi \). In the elastic region \( (M_a)_e = \Phi EL \); in the inelastic range this function is considerably more complicated; that is, \( (M_a)_m = f(\Phi, P, \text{material, and cross-sectional properties}) \). The curvature \( \Phi \) is equal to \(-\theta^a\) if we retain the assumptions of small deflections and plane sections and if shear deformations are neglected. These are reasonable assumptions in the inelastic range also, as shown by the fact that even here rotations are small and \( \theta^a \ll 1 \). The equilibrium equation, \( M_{ext} = (M_a)_m \) is thus
\[
M_r \left[ 1 - \frac{x}{L} (1 - \kappa) \right] + P \theta = f(\theta^a, P, \text{material, section})
\]
(5.58)

The deflections and end rotations of the beam-column are obtained by integrating Eq. (5.57). Because of the complicated form of the \( M-P-\Phi \) relationship, this is best performed by numerical techniques.

THE MOMENT-THRUST-CURVATURE RELATIONSHIP

The \( M-P-\Phi \) relationship on the right side of Eq. (5.57) will depend on the sequence in which \( P \) and \( M \) are applied to the cross section. For any given \( P \) and \( M \) the resulting inelastic curvature \( \Phi \) will be different if, for example, \( P \) and \( M \) increase proportionally or if \( P \) is applied first and then \( M \). There is thus no unique relationship that exists among the three quantities. Usually \( M-P-\Phi \) curves are obtained by applying \( P \) first and holding it constant while the corresponding relationship between \( M \) and \( \Phi \) is established. This method has been shown to be quite satisfactory (Ref. 5.11 and Chap. 14 of Ref. 3.9). Even though most beam-columns are not loaded in this sequence, the resulting error is only small.\(^{\text{10,11}}\)

We shall start with the simple problem of a rectangular section [Fig. 5.14(a)] made of an ideal elastic-plastic material [Fig. 5.14(b)]. At a cross section the member is first subjected to a compressive axial force \( P \), which is held constant, and to a bending moment, which will be varied from zero to its maximum possible value when the whole cross section has become plastified. There are three possible stress distributions [Fig. 5.14(c)]. The \( M-\Phi \) relationship for \( P = \text{constant} \) is obtained from equilibrium \( \int_A \sigma \, dA = P \) and \( \int_A \sigma \, dA = M \) and the geometry of the stress blocks, as was done for the same cross section in Sec. 3.3 for the case of \( P = 0 \) (Fig. 3.21). The resulting nondimensional \( M-P-\Phi \) equations are
\[
\frac{M}{M_r} = \frac{\Phi}{\Phi_r} \quad \text{for} \quad 0 \leq \frac{\Phi}{\Phi_r} \leq \left(1 - \frac{P}{P_r}\right)
\]
(5.58)

\[
\frac{M}{M_r} = \frac{3}{2} \left(1 - \frac{P}{P_r}\right) - \frac{2(1 - \frac{P}{P_r})^{3/2}}{(\Phi/\Phi_r)^{3/2}} \quad \text{for} \quad \left(1 - \frac{P}{P_r}\right) \leq \frac{\Phi}{\Phi_r} \leq \left(1 - \frac{P}{P_r}\right)^{1/2}
\]
(5.59)

\[
\frac{M}{M_r} = \frac{3(1 - \frac{P}{P_r})^2}{2} - \frac{1}{2(\Phi/\Phi_r)} \quad \text{for} \quad \left(1 - \frac{P}{P_r}\right)^{1/2} \leq \frac{\Phi}{\Phi_r} \leq \infty
\]
(5.60)

where
\[
P_r = b d \sigma_y, \quad M_r = \frac{b d^2 \sigma_y}{6}, \quad \Phi_r = \frac{2 \sigma_y}{Ed}
\]
(5.61)
The curves corresponding to Eqs. (5.58) through (5.60) are given in Fig. 5.15 for various values of $P/P_r$. As $\Phi \to \infty$, the curves become asymptotic to $M = \frac{3}{4}[1 - (P/P_r)^2]M_r$. This moment corresponds to the full plastification of the cross section. The plastic moment is reduced from its value $M_r$ when $P = 0$ to

$$M_{pc} = M_r\left[1 - \left(\frac{P}{P_r}\right)^2\right]$$  \hspace{1cm} (5.62)

where $M_r = \frac{3}{4}M_Y = bd^3\sigma_y/4$.

The $M$-$P$-$\Phi$ curves for other shapes and materials can be derived in a manner similar to that shown above. Timoshenko and Gere\(^{(1,4)}\) present a general method whereby $M$-$P$-$\Phi$ curves can be determined graphically for any cross section and $\sigma$-$\epsilon$ curve. A semigraphical procedure for both strong and weak axis bending of wide-flange shapes with residual stresses is given in Ref. 1.37, where formulas and curves as well as experimental verifications are also presented. Corresponding information for tubular shapes is found in Ref. 5.12. Formulas for the strong axis bending of wide-flange shapes with the residual stress pattern of Fig. 1.6 are tabulated in Ref. 3.25. The $M$-$P$-$\Phi$ curves for an 8WF31 shape are shown in Fig. 5.16 for an assumed perfectly elastic-plastic $\sigma$-$\epsilon$ curve [Fig. 5.14(b)]. The maximum compressive residual stress is 0.3 $\sigma_r$, and its distribution is given in Fig. 1.6. The nondimensional curves for other rolled A36 steel wide-flange shapes, which have a similar residual stress distribution and magnitude, differ only slightly from the curves for the 8WF31 shape, and thus the curves in Fig. 5.16 can be conveniently used for all wide-flange shapes without introducing significant errors.\(^{(1,37)}\) The maximum residual stress for high-strength steels is usually less than 0.3 $\sigma_r$, and therefore the use of the curves in Fig. 5.16 is conservative.\(^{(3,42)}\) Other $M$-$P$-$\Phi$ curves would need to be computed for welded shapes since their residual stress distribution is quite different from that shown in Fig. 1.6.\(^{(1,34)}\)

Wide-flange $M$-$P$-$\Phi$ curves exhibit plastic hinge formation at relatively lower curvatures than the rectangular shape (compare Figs. 5.15 and 5.16). Formulas for computing $M_{pc}$ are listed in Chap. 7 of Ref. 1.41. However, the following two empirical equations give satisfactory results, as can be seen from Fig. 5.17:

**Fig. 5.15.** Moment-thrust-curvature relationships for rectangular shape

**Fig. 5.16.** Moment-thrust-curvature relationships for wide-flange shape
procedures. For the rectangular shape there have been several analytically exact solutions. We shall first examine such a solution and then discuss approximate and numerical solutions.

## THE ANALYTICALLY EXACT METHOD

An exact solution was given by Jezeck\(^{5,11}\) for the rectangular eccentrically loaded beam-column shown in Fig. 5.18. For this section and for an elastic-plastic \(\sigma-\epsilon\) curve \[Fig. 5.14(b)\] the \(M-P-\Phi\) relations of Eqs. \(5.58\) through \(5.60\) are valid. The bending moment at any location \(z\) within the span is, from Fig. 5.18(a),

\[
M = P(e + v) \tag{5.65}
\]

By equating this external moment to the internal moment defined by Eqs. \(5.58\) through \(5.60\), we obtain the following differential equations for the three zones indicated in Fig. 5.18(b):

### Elastic range

\[
P(e + v) = -EI\ddot{\nu} \tag{5.66}
\]

Yielding in compression only

\[
P(e + v) = M_r \left[ 3 \left( 1 - \frac{P}{P_r} \right) - \frac{2(1 - P/P_r)^{3/5}}{(-E \epsilon_0/2\sigma_y)^{1/5}} \right] \tag{5.67}
\]

Yielding in tension and compression

\[
P(e + v) = M_r \left[ 3 \left( 1 - \frac{P}{P_r} \right) - \frac{2\sigma_y^2}{2EI\epsilon_0} \right] \tag{5.68}
\]

In these equations we replaced \(\Phi\) by \(-\nu''\). Equations \(5.67\) and \(5.68\) can be

---

**Fig. 5.17.** Plastic moment modified by axial force

\[
\frac{M_{pc}}{M_{pc}} = 1.18 \left( 1 - \frac{P}{P_r} \right) \tag{5.63}
\]

\[
\frac{M_{pc}}{M_{pc}} = 1.19 \left[ 1 - \left( \frac{P}{P_r} \right)^2 \right] \tag{5.64}
\]

Under the assumption of an ideal elastic-plastic \(\sigma-\epsilon\) curve \[Fig. 5.14(b)\] the values of \(M_{pc}\) from Eqs. \(5.63\) and \(5.64\) represent the largest moments a wide-flange cross section will be able to sustain for a particular axial load.

### METHODS OF SOLUTION

**FOR INELASTIC BEAM-COLUMNS**

The \(M-P-\Phi\) curves are the basic information from which the integration of the beam-column equilibrium equation \[Eq. (5.57)\] can proceed. Since the usual \(M-P-\Phi\) curves are too complicated in analytical form, Eq. \(5.57\) is best integrated by using one of the several available numerical integration techniques.
rearranged and solved with the aid of tables of elliptical integrals. By the matching of deflections and slopes at the points where two regions meet, the constants of integration can be found and the $P-u$ curve can be computed. The peak point on this curve is the desired maximum value of $P$. Similar problems were treated by this method by Baker, Horne and Heyman,\textsuperscript{1\text{*}3\text{*}} Oxford,\textsuperscript{4\text{*}1\text{t}7} and Hauk and Lee.\textsuperscript{6\text{*}1\text{h}}

**APPROXIMATE METHODS**

The integration of Eq. (5.57) by exact analysis is time consuming even for a rectangular shape. For more complicated shapes it is best not to attempt such a solution.

Many approximate solutions have been developed for beam-columns with various loadings. Some of these use the exact $M-P-\Phi$ curves and assume a suitable deflected shape (for example, Refs. 5.19 through 5.22), and others base the solution on integrable analytical approximations of the $M-P-\Phi$ relationship.\textsuperscript{5\text{*}2\text{h}}

We shall illustrate an approximate procedure by solving the rectangular beam-column of Fig. 5.18. The pertinent $M-P-\Phi$ relations are given by Eqs. (5.58) through (5.60). The solution is based on the reasoning that when the maximum load is reached, that is, the load-deflection curve has attained its peak, the external moment at the most strained section increases at the same rate as the resisting moment supplied by the internal stress system at this point.\textsuperscript{5\text{*}1\text{a}} If the increase in external moment is greater than the moment which the internal stress system can supply, then equilibrium can be maintained only by a drop in the external moment and thus a drop in the external loads, and therefore the gradient of the load-deflection curve becomes negative. If, on the other hand, the internal moment resistance is greater, the gradient is positive, and the peak has not yet been reached. The peak point is thus characterized by the equality of the changes in the external and the internal moments at the most strained section.

For the eccentrically loaded member of Fig. 5.18 the most strained section is at the mid-height (at $z = L/2$) where the deflection is $v_0$ and the external moment is $P(e + v_0)$. We shall denote the changes in moment with respect to changes in the curvature at this critical section. Since the curvature is proportional to the deflection $v_0$, we can define the condition of the member at the attainment of the maximum value of the applied load as

$$\frac{dM}{dv_0} = \frac{d[P(e + v_0)]}{dv_0} = \frac{dM_{\text{ext}}}{dv_0}$$

In order to complete the solution, we shall need to assume a convenient deflected shape for the beam-column, thus introducing an approximation. We shall assume that the deflected shape of this member is

$$v = v_0 \sin \frac{\pi z}{L}$$

This deflected shape fulfills the condition that $v(0) = v(L) = 0$, and the curve is symmetric with respect to the mid-height deflection $v_0$. By introducing this deflected shape, we can circumvent the problem of solving the differential equations [Eqs. (5.67) and (5.68)].

The curvature at the mid-height is

$$\Phi_0 = -v''(\frac{L}{2}) = v_0 \frac{\pi^2}{L^2} = v_0 \left(\frac{\pi}{L}\right)^2$$

In nondimensional form

$$\frac{\Phi_0}{\Phi_r} = 6\frac{v_0}{P_x} \frac{Q}{P_r}$$

where $\Phi_r$ is defined by the last of Eqs. (5.61) and

$$\frac{P_x}{\sigma_r L/r^2} = \frac{\pi^2 d/L^2}{12r}$$

The nondimensional external moment at $z = L/2$ is

$$\frac{M}{M_r} = \frac{P(e + v_0)}{M_r} = \frac{P e}{M_r} + 6\frac{v_0}{P_f} = \frac{M_0}{M_r} + 6\frac{v_0}{P_f}$$

where $Pe = M_0$ and $P_f = A\sigma_r = b d \sigma_r$.

Equilibrium at this most yielded section requires that $M_{\text{ext}} = M_{\text{int}}$. There will be two types of yielding at this section: (1) yielding in compression only and (2) yielding in compression and tension [Fig. 5.14(c)]. The external moment is described by Eq. (5.74), and the internal moments are given by Eqs. (5.59) and (5.60) respectively, for the two cases of yielding. The resulting equilibrium equations are then

$$\frac{M_0}{M_r} + 6\frac{v_0}{P_f} = 3\left(1 - \frac{P}{P_r}\right) - \left[\frac{(1 - P/[P_r])^{1/2}}{6v_0/d}\right]$$

In the equations of the internal moment we replaced $\Phi_r$ by its expression from Eq. (5.72).

The condition at the maximum point on the $M_e-v_0$ curve is given by Eq. (5.69), which is expressed nondimensionally as

$$\frac{d(M_{\text{ext}}/M_r)}{d(v_0/d)} - \frac{d(M_{\text{int}}/M_r)}{d(v_0/d)} = \frac{d(M_{\text{ext}} - M_{\text{int}}/M_r)}{d(v_0/d)}$$

For the case of compressive yielding only, for example [Eq. (5.75)],

$$\frac{d}{d(v_0/d)} \left[\frac{M_0}{M_r} + 6\frac{v_0}{P_f} \left(1 - \frac{P}{P_r}\right) + \frac{2(1 - P/[P_r])^{1/2}}{6v_0/d}\right] = 0$$

If we multiply Eq. (5.78) by $(v_0/d)^{1/2}$ and perform the differentiation, we get after some algebra

$$\frac{v_0}{d} = \frac{1 - P/[P_r]}{6P/P_r} - \frac{M_0/M_r}{18P/P_r}$$

(5.79)
This is the value of the mid-height deflection for yielding in compression only at the maximum value of \( M_0 \). By setting \( v_0/d \) from Eq. (5.79) into the equilibrium equation [Eq. (5.75)], we finally obtain an interaction equation which relates the pertinent geometric and loading parameters at the maximum load

\[
\frac{M_0}{M_y} = 3 \left( 1 - \frac{P}{P_T} \right) \left[ 1 - \left( \frac{P}{P_T} \right)^2 \right] \quad (5.80)
\]

If we perform a similar sequence of operations for the case of yielding in both tension and compression, then we obtain the following center deflection at the maximum load:

\[
v_0 = \frac{(3/2)[1 - (P/P_T)^2] - M_0/M_y}{9(P/P_T)} \quad (5.81)
\]

and the following interaction equation:

\[
\frac{M_0}{M_y} = \frac{3}{2} \left[ 1 - \left( \frac{P}{P_T} \right)^2 - \left( \frac{P}{P_T} \right)^{3/2} \right] \quad (5.82)
\]

The range of applicability of the two interaction equations [Eqs. (5.80) and (5.82)] is defined by the limits in which the two relevant \( M-P-\Phi \) relations apply [Eqs. (5.59) and (5.60)]. From Eqs. (5.72), (5.79), and (5.80) we find that for the case of yielding in compression only

\[
\Phi = \Phi_T = \left( 1 - \frac{P}{P_T} \right) \left( \frac{P_x}{P_T} \right)^{3/2} \quad (5.83)
\]

The curvature defined by Eq. (5.83) applies over the range \( 1 - P/P_x \leq \Phi/\Phi_T \leq 1/1(1 - P/P_x) \) [Eq. (5.59)]. One limit of Eq. (5.83), and thus also the interaction equation Eq. (5.80), is when \( \Phi/\Phi_T = 1 - P/P_x \) or when \( P/P_x = 1 \). The other limit is when \( \Phi/\Phi_T = 1/1(1 - P/P_x) \) or when [from Eq. (5.83)] \( P_x = 1 - P/P_T \). Thus Eq. (5.80) is valid for the range \( 1 - P/P_x \leq \Phi/\Phi_T \leq 1 \). Similarly we can show that the other interaction equation [Eq. (5.82)] is valid for the range \( 0 \leq \Phi/\Phi_T \leq (1 - P/P_x) \). The two interaction equations are then

\[
\frac{M_0}{M_y} = 3 \left( 1 - \frac{P}{P_T} \right) \left[ 1 - \left( \frac{P}{P_T} \right)^{3/2} \right] \quad \text{for} \quad \left( 1 - \frac{P}{P_T} \right)^2 \leq \frac{P}{P_x} \leq 1 \quad (5.84)
\]

\[
\frac{M_0}{M_y} = \frac{3}{2} \left[ 1 - (P/P_T)^2 - (P/P_T)^{3/2} \right] \quad \text{for} \quad 0 \leq \frac{P}{P_x} \leq \left( 1 - \frac{P}{P_T} \right)^3 \quad (5.85)
\]

These two equations relate the end moment \( M_0 = P e \), the axial load \( P \) and the ratio \( (L/r) \sqrt{\varepsilon}, \) which is the variable part of the term \( P_e/P_v, [Eq. (5.73)] \), at the instant when a given rectangular beam-column loaded as shown in Fig. 5.18(a) supports its maximum end moment \( M_0 \) under a given axial load \( P \). The three nondimensional parameters which define the maximum condi-

**Fig. 5.19.** Maximum-strength interaction curves for rectangular beam-columns (\( e = 1.0 \))

The two types of curves in Fig. 5.19 show that rectangular steel beam-columns possess a great deal of reserve of strength beyond the first attainment of the yield stress, especially in the range \( (L/r) \sqrt{\varepsilon} < 2 \), in which most practical beam-columns fall.

The interaction curves in Fig. 5.19 give also a good approximation for steel wide-flange beam-columns bent about the weak axis if no significant residual stresses are present.

The maximum strength of other beam-columns can also be solved in a
similar manner to that illustrated above. The derivatives, which were here obtained from analytic expressions, can be also obtained graphically from actual \( M P \Phi \) curves. A variety of problems have been solved in this manner by Ketter.\(^{25,45}\)

The approximate method described here is useful only for the determination of the maximum strength, and it usually gives reliable results for cases of loading where the location of the most strained region is readily apparent—such as the mid-height of the member discussed above. By using more refined deflected shapes, the usefulness of the approximate method can be extended.\(^{19,24}\) The approximate methods based on an assumed deflected shape are not, however, suitable for finding the complete load-deformation response of beam-columns.

**MAXIMUM STRENGTH**

**BY NUMERICAL INTEGRATION**

The maximum in-plane strength of beam-columns under any in-plane loading condition can be also determined by numerical integration, as is illustrated in Fig. 5.20 for a beam-column with an end moment ratio of \( \kappa = \frac{1}{2} \).

The steps of the solution for given values of \( P \) and \( L \) are:

1. Obtain the \( M P \Phi \) curves for the given type of material, cross section, and residual stress distribution (Fig. 5.15 or 5.16, for example).
2. Specify a value of \( M_0 \) which is above that which would cause yielding.
3. Assume a reasonable deflected shape—as a first try the elastic deflection [Eq. (5.15)] may be used.
4. From the end moments and the deflected shape compute the bending moment \( M(\zeta) + P \epsilon \) at desired even intervals along the length of the member. In most cases satisfactory results are obtained when the spacing is \( \frac{1}{4} \) to \( \frac{1}{2} \) of the length.
5. Obtain the curvature corresponding to the total moment at each location from the \( M P \Phi \) curves.
6. Determine a new deflected shape by numerically or graphically integrating the curvature diagram. The numerical integration can be conveniently performed by Newmark's method.\(^{1,11}\)
7. Repeat steps 4, 5, and 6 with the new deflected shape until (a) the deflected shape at the start is equal to the deflected shape at the end of the cycle of integration (that is, the deflection converges to its true shape), or (b) it becomes obvious that no convergence can be obtained (in which case the assumed value of \( M_0 \) in step 2 is larger than the maximum which can be carried by the member). In either case, a satisfactory answer is obtained after three or four cycles.\(^{1,18}\)
8. Repeat steps 2 through 7 with a new value of \( M_0 \).

Finally a curve \( M_s \) can be drawn, for which the highest value of \( M_s \)

\[ P \]
\[ M_0 \]
\[ nM_0 = \frac{M_0}{2} \]
\[ L \]
\[ Loading \]
\[ Assumed deflection \]
\[ M(\zeta) \]
\[ M_0 \]
\[ M_0 \]
\[ Secondary moments \]
\[ P \epsilon \]
\[ Moment diagram \]

\[ Curvature diagram \]

\[ Fig. 5.20. Construction of the curvature diagram \]

can be made as close to, but yet still somewhat below, the actual value of \( (M_0)_{\text{max}} \), as desired.

The end result of one set of such calculations is the maximum moment \( M_s \) corresponding to a given \( P \) and \( L \) for the chosen cross section and material. The method of solution is illustrated by the example given in Table 5.1, where the maximum moment is determined for a beam-column with only one end moment \( (\kappa = 0) \) and with an axial load of \( 0.8P \). Four cycles of integration lead to a convergence of the deflected shape.

If this numerical integration procedure is repeated for many values of \( P \) and \( L \), we can finally construct interaction curves. Such interaction curves are given in Refs. 1.38 and 5.4 for beam-columns with \( \kappa = 1.0, 0.5, 0, -0.5, \) and \( -1.0 \). This same information is given in tabular form in Ref. 5.25. One such set of interaction curves for \( \kappa = 0 \) is reproduced from Ref. 1.38 in Fig. 5.21. These curves were computed for an 8WF31 shape of A7 steel (\( E = 30,000 \) ksi, \( \sigma_Y = 33 \) ksi) having the residual stress pattern of Fig. 1.6 (\( \sigma_{n0} = 0.3\sigma_Y \)). As we noted before, the nondimensional \( M P \Phi \) curves for this shape closely and conservatively approximate the \( M P \Phi \) relationship of all other rolled steel wide-flange shapes. Thus the interaction curves in Fig. 5.21 (and
those in Refs. 1.38, 5.54, and 5.55 can be used. For steel with yield stress other than 33 ksi, the curves can be used for steel with yield stress other than 33 ksi. For example, for an adjusted yield stress of 30 ksi, we use Eq. 5.1 with an adjusted yield stress ratio of 0.9, $\bar{M}/M_y = 0.233 M_y$.

### Table 5.1

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<td>Moment due to $M_y = 181$ kip-in.</td>
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<td>(guessed) $P \times v$, kip-in.</td>
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<td>0.112</td>
<td>0.131</td>
<td>0.129</td>
<td>0.112</td>
<td>0.081</td>
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<td>(k) 0</td>
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<td>0.113</td>
<td>0.132</td>
<td>0.130</td>
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<td>0.082</td>
<td>0.043</td>
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Starting deflection, in.
Final deflection, in.
Adjusting for a variation in the yield stress can be avoided if the interaction curves are further nondimensionalized, as shown in Fig. 5.22. Here the curves relate $M_0/M_P$ with the slenderness ratio factor $(L/r_e)\sqrt{\sigma_t}$ for constant values of $P/P_P$. The curves in this figure are for $\kappa = 1.0$, and for rolled wide-flange shapes.

Information shown in the interaction curves can be used directly in structural analysis, or it can be used as the basis for checking empirical design equations. They represent an easy way of relating the basic variables of the in-plane maximum strength of beam-columns. They can be computed as needed for a variety of shapes, residual stress patterns, and loading conditions by computer.

**VERIFICATION BY EXPERIMENT**

The interaction curves developed for the 8WF31 rolled shape with the residual stress pattern of Fig. 1.6 are a great deal more useful than we would expect. We have shown that they can be applied for steel rolled wide-flange beam-columns where bending is about the strong axis, where the residual stress is as shown in Fig. 1.6 ($\sigma_{re} = 0.3\sigma_t$), and where $\sigma_t$ can take on any value. In actual members these conditions are not really fulfilled. Residual stress distribution and magnitude can vary considerably for different wide-flange shapes. The residual stress pattern in Fig. 1.6 is already an idealization of the real situation even for 8WF31 shapes. There are also residual stresses in rolled members due to cold straightening after cooling, and the yield stress in the web is usually higher than in the flanges.

Thus there are a great number of parameters which can be different for different rolled shapes. If these parameters are known, it is possible to include their effects into the in-plane strength determination of beam-columns by the method we discussed previously. However, we shall now show that the interaction curves developed for the 8WF31 shape are quite reasonable for other situations also.

We shall first show this by experimental means. The ratio of the experimental maximum load to the theoretical maximum load based on the 8WF31 shape with the residual stress pattern of Fig. 1.6 is shown in Fig. 5.23 for 83 beam-column tests. These experiments were performed over a period of almost 40 years by a number of investigators in several laboratories. (Refs. 5.2, 5.26 through 5.30). The value of $\sigma_t$ varied from 33 to 55 ksi, and all except the hat-shaped members of Ref. 5.26 were rolled wide-flange or I shapes.

In all cases in-plane behavior was predominant up to the maximum load, and thus the theory of in-plane interaction curves is valid.

The histograms in Fig. 5.23 show that the distribution of the ratio of the experimental to the theoretical strength is approximately normal and that the average ratio is 1.005. The tests between 0.95 and 1.05 comprise 64 per cent of all tests, and those between 0.90 and 1.10 comprise 89 per cent.
EFFECT OF CROSS-SECTIONAL SHAPE

Although not enough information is yet available to compare the maximum in-plane strength of all types of shapes, we can make some comparisons and observe some trends. In-plane interaction curves are given in Fig. 5.24 for three slenderness ratios and four sections. These are for strong and weak axis bending of wide-flange shapes, solid circular sections, and circular tubes. The curves for a constant slenderness ratio seem to fall into a relatively narrow band, and the curve for the strong axis bending of the rolled wide-flange shape represents an average condition. For these shapes at least one could use with confidence this average interaction curve. As mentioned previously, welded steel shapes should be further investigated before we can state any general conclusions about their strength.

The preceding discussion has shown that in-plane interaction curves for

![Graph showing interaction curves for different sections and slenderness ratios.]

The interaction curves given in Figs. 5.21 and 5.22 (with further curves and tabulations in Refs. 1.38, 5.4, and 5.25) give thus a rather reasonable representation of in-plane beam-column strength for these experiments. Even though the interaction curves do not precisely give the maximum strength of all beam-columns, they represent a fairly good average for rolled steel shapes. For these shapes the residual stress in the flange tips is compressive. The interaction curves for the rolled 8WF31 shape should not, however, be used for welded shapes because of the greatly different residual stress pattern. Further studies on such shapes should be performed.
the strong axis bending of the 8WF31 shape are satisfactory for the prediction of the strength of the strong axis bending strength of all rolled wide-flange beam-columns, and that these curves are also useful for the weak axis bending strength of wide-flange shapes and for solid round bars and circular tubes if not too great a precision is desired. These interaction curves are thus of great practical importance.

**APPROXIMATE INTERACTION EQUATIONS**

The use of interaction curves or tabulated values\(^{(1,2,5,6,16)}\) can be avoided by suitable empirical formulas obtained from or checked by the more exact information. The in-plane interaction curves for \(\kappa = 0\) and \(\kappa = 1.0\) (Figs. 5.21 and 5.22) are approximated to a reasonable degree of accuracy by the following formula:

\[
P \frac{P}{P_e} + \frac{C^*_e M_0}{M_0 (1 - P/P_e)} = 1.0
\]

where \(P_e\) is the maximum axial load which can be supported by the member if the bending moment is zero, \(C^*_e\) is a coefficient which accounts for the effect of unequal end moments (\(\kappa \neq 1.0\)), \(M_0\) and \(P\) are the maximum end moment and axial force, respectively (Fig. 5.4), \(M_p\) is the full plastic moment for \(P = 0\), and \(P_e\) is the elastic buckling load of the member in the plane of the applied moments. The value of \(C^*_e\) can be approximated by Eq. (5.39) or by Eq. (5.40). The interaction equation [Eq. (5.86)] is very similar to the elastic limit integration equation [Eq. (5.41)], with the exception that \(P_e\) is either the elastic or the inelastic column strength. This equation provides a simple empirical transition between one extreme where \(M_0 = 0\) and \(P = P_e\), and the other extreme where \(P = 0\) and \(M_0 = M_p\), provided that \(C^*_e = 1.0\). If \(C^*_e < 1.0\) (that is, \(\kappa \neq 1.0\) and \(\kappa < 1.0\)), Eq. (5.86) will give \(M_0 > M_p\) when \(P = 0\), and thus a cut-off point exists where Eq. (5.86) no longer applies. This cut-off point occurs when a plastic hinge forms at one end of the beam-column. For wide-flange members bent about the strong axis the plastic hinge condition is defined by Eq. (5.63), that is,

\[
\frac{M_0}{M_p} = 1.18 \left(1 - \frac{P}{P_e}\right) \leq 1.0
\]

The relationships defined by Eqs. (5.86) and (5.87) are shown in Fig. 5.25, where the interaction curves are plotted on a \(P/P_e\) versus \(M_0/M_p\) coordinate system for one particular value of the slenderness ratio. The lower curve (for \(\kappa = 1.0\)) represents the most severe loading conditions, and it is always concave downward, connecting \(P = P_e\) and \(M_0 = M_p\). For \(\kappa < 1.0\), the curve of Eq. (5.86) pivots upward from its origin at \(P = P_e\), and it does not pass through \(M_p\). Beyond its point of intersection with Eq. (5.87) it is no longer valid. The envelope then defines the estimated maximum strength.

In any situation we must check both Eqs. (5.86) and (5.87), and the one giving the lower strength governs.

The accuracy of the empirical interaction equations [Eq. (5.86) and (5.87)] has been checked by experiment (Refs. 5.2, 5.6, and 5.7) and against the interaction curves determined by numerical integration.\(^{(1,6)}\) They give a good representation of the actual maximum in-plane strength of beam-column in the range \(0 \leq \kappa \leq 1.0\). The correlation becomes less satisfactory as \(\kappa\) approaches \(-1\), but it is still good enough for the empirical interaction equations to be used with confidence as design equations.\(^{(1,6)}\) Several design specifications use them as a basis for beam-column design.\(^{(1,6,9,12,13)}\)

**5.4. COLUMN DEFLECTION CURVES**

**IN-PLANE DEFLECTIONS**

The numerical integration procedure described in Sec. 5.3 determines the maximum moment for a given axial load. This maximum moment corresponds to the peak of the \(M_e\) curve where relatively large changes of the end rotations result for very small changes in moment (Fig. 5.4). This method, as well as the other approximate procedures, is thus ill-suited for determining
deformations. The whole load-deflection relationship in the plane of the applied moments can be obtained by using the concept of the Column Deflection Curves (CDC-s). These curves define equilibrium deformed shapes of members under axial load and bending. In the following discussion we shall define the properties of CDC-s and examine how they can be computed and used.

LITERATURE ON COLUMN DEFLECTION CURVES

CDC-s were first used by von Karman in 1910 in the determination of the maximum strength of rectangular steel beam-columns subjected to axial loads with small eccentricities.\textsuperscript{[5,33]} Chwalla generalized von Karman’s work to other loading conditions (1934)\textsuperscript{[5,24]} and cross sections\textsuperscript{[5,35]} and he demonstrated the use of CDC-s for statically indeterminate problems.\textsuperscript{[5,36]} Chwalla’s basic ideas have since been expanded in various directions by Ellis,\textsuperscript{[5,37]} Neal and Mansell,\textsuperscript{[5,38]} Horne,\textsuperscript{[5,39]} Bijlaard,\textsuperscript{[5,40]} Lee and Hauk,\textsuperscript{[5,41]} Ojalvo,\textsuperscript{[5,40]} and Lay.\textsuperscript{[5,11]}

BASIC CDC RELATIONSHIPS

The in-plane equilibrium equation of a beam-column has been already defined by Eq. (5.57) for the condition of loading and geometry of the member in Fig. 5.2(b). A more useful formulation of this relationship results if the end forces $M_0$, $\kappa M_0$, and $P$ are replaced by a single equivalent force $F$.\textsuperscript{[5,11]} In Fig. 5.26 is shown a beam-column $AB$ with the forces $P$, $M_0$, and $\kappa M_0$ acting at its end, and the equivalent force $F$ which is inclined to the $z$ axis of the member $AB$ by the angle $\gamma$. The line of action of $F$ intersects the $z$ axis at $z = L + N$. Summation of moments about point $A$ gives

$$F(L + N) \sin \gamma - M_0 = 0 \quad (5.88)$$

Summation of moments about point $C$ and summation of the horizontal forces give, respectively,

$$M_0 - \frac{M_0}{L}(1 - \kappa)(L + N) = 0 \quad (5.89)$$

$$P - F \cos \gamma = 0 \quad (5.90)$$

From Eqs. (5.88) through (5.90) we obtain the following relationships which define the location and magnitude of $F$ as

$$N = L \left(\frac{\kappa}{1 - \kappa}\right)$$

$$F = \frac{P}{\cos \gamma}$$

$$\tan \gamma = \frac{M_0(1 - \kappa)}{PL} \quad (5.91)$$

The external moment at any location on the $z$ axis is now $M_{ext} = F \bar{\psi}$, where $\bar{\psi}$ is the distance between the line of action of $F$ and the deflected shape of the beam-column $AB$. The internal moment is a function of the curvature, the axial force, and the cross-sectional and material properties of the member (that is, the $M-P-\Phi$ relationship as discussed in Sec. 5.3). We shall denote the internal moment as $M_{int} = f(\Phi)$, and thus the equilibrium relationship $M_{ext} = M_{int}$ can be expressed as

$$F\bar{\psi} = f(\Phi) = f(-\bar{\psi}') \quad (5.92)$$

The deflected shape of member $AB$ can be determined from the equilibrium condition by integration if we know the moment-thrust-curvature relationship for the cross section. The integration could start at point $A$ (Fig. 5.26) and proceed to point $B$. If the integration is continued beyond the end $B$ of the member until $\bar{\psi} = 0$ and if integration proceeds also to the left of point $A$ again until $\bar{\psi} = 0$, we obtain a symmetric curve as shown by the dashed line in Fig. 5.27. The equilibrium deflected shape of the original column $AB$ is a part of this new curve which is a half wavelength of a Column Deflection Curve.\textsuperscript{[5,11], [5,40]} The length of this half wavelength is $L_{CDC}$ and the slope at its end is $\theta_0$.

Relative to Eq. (5.57) the formulation of the equilibrium by Eq. (5.92) represents only a shift in coordinates, and thus it does not simplify the problem. However, just as portion $AB$ of the CDC in Fig. 5.27 represents the deflected shape of the actual beam-column $AB$ in Fig. 5.26, so some other portion of the CDC is the deflected shape of another beam-column. Thus any one CDC provides information about many problems.

The CDC-s are subject to the same assumptions which underlie the $M-P-\Phi$ curves from which they are obtained by integration. If we in addition assume that the deflections are relatively small, then $\tan \gamma \equiv \sin \gamma \equiv \gamma$ and $\cos \gamma \equiv 1.0$. As a consequence of this, $F \equiv P$ [Eq. (5.91)]. This assumption is very satisfactory except when the axial load is small. Lay suggests that the

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig5_26.png}
\caption{Resolution of beam-column end forces into single equivalent force}
\end{figure}

\textsuperscript{6} Von Karman’s work is reviewed in Sec. 3.2 of Ref. 1.29 and in Sec. 10 of Ref. 1.30.
Fig. 5.27. The beam-column as part of a CDC

subsequent applications of the CDC-s, when based on the small deflection assumption, should not be used if $P < 0.12P_Y$ for a steel with $\sigma_Y = 36$ ksi.\textsuperscript{117}

In the elastic range the bending moment in member $AB$ of Fig. 5.26 with respect to the $y-z$ coordinate system is equal to [Eq. (5.16)]

$$M = M_0 \left[ \frac{\kappa \cos F_F L}{\sin L} \right] \sin F_F z + \cos F_F z$$

Since the deflected shape of $AB$ is also a part of the CDC, Eq. (5.16) applies for the whole CDC. The point of zero moment, that is, the intersection of the CDC with the $z$ axis (Fig. 5.27) occurs at a distance $z = 0$. At this location $F \theta = P \bar{\theta} = M = 0$. Setting $M = 0$ in Eq. (5.16), we obtain for $z = \xi L$

$$\tan F_F \xi L = \frac{-\sin F_F L}{\kappa - \cos F_F L}$$

which defines the inflection points of the CDC wave with respect to the $z$ axis.

The ordinate $\tilde{\theta}$ of the CDC is equal to (from Fig. 5.28, which is an inverted and rotated view of Fig. 5.27)

$$\tilde{\theta} = \frac{v}{\cos \gamma} + (L + N - z) \sin \gamma$$

Noting that $\cos \gamma \cong 1.0$, $N = L \kappa (1 - \kappa)$, and $\sin \gamma \cong \gamma \cong M_0 (1 - \kappa)/PL$ [Eq. (5.91)] and substituting $v$ from Eq. (5.15), we find that

$$\tilde{\theta} = \frac{M_0}{P} \left[ \frac{\kappa \cos F_F L}{\sin F_F L} \right] \sin F_F z + \cos F_F z$$

Fig. 5.28. CDC geometry

A comparison of Eqs. (5.95) and (5.16) shows that the ordinate of the CDC, $\tilde{\theta}$, is equal to the moment in the beam-column divided by the axial load $P$. Since $P$ is assumed constant, the CDC is also the moment diagram of the beam-column.

Finally, in the elastic range $f(-\tilde{v})$ in Eq. (5.92) equals $-EI_\theta\tilde{\theta}$, and so the differential equation of a CDC is

$$EI_\theta \tilde{\theta}'' + F \tilde{\theta} = 0$$

which can be written as

$$\tilde{v}'' + F_F \tilde{v}'' = 0$$

if $F_F = F/EI_\theta \cong P/EI_\theta$. This equation is the same as the differential equation of buckling in the $y-z$ plane [Eq. (4.14)]. Thus in the elastic range the CDC takes the shape of the buckled form of an axially loaded column of length $L_{\text{CDC}}$ (Fig. 5.27). The deflected shape of a CDC is thus a sine wave, that is,

$$\tilde{\theta} = \tilde{\theta}_0 \sin F_F \bar{z}$$

where $\bar{z}$ is defined as the longitudinal coordinate of the CDC, with an origin at an inflection point (Fig. 5.28) and $\tilde{\theta}_0$ is the ordinate of the CDC at $\bar{z} = L_{\text{CDC}}/2$. The length of the CDC is

$$L_{\text{CDC}} = \frac{\pi F_F}{P_0} \cong \pi \sqrt{\frac{EI_\theta}{P}}$$

in the elastic range.

If within any CDC yielding takes place, then the shape of the CDC is affected and the length of the wave is reduced.\textsuperscript{12-46} In the inelastic range it is most convenient to determine the shape of the CDC by numerical integration.

Let us now summarize what we learned about CDC-s. Column Deflection Curves are periodic symmetric curves which define the equilibrium of a member subjected to a constant axial force $P$. The deformations consistent with equilibrium occur in the plane of symmetry of the bar, and thus the bar is
assumed to have at least one axis of symmetry. There are an infinite number of CDC-s for any given value of $P$. These CDC-s can be differentiated from each other by the different deflections $\theta$ at the center of the curve or by the end slope $\theta$. One full wavelength of several CDC-s is shown in Fig. 5.29. As long as the stresses $P/A + P \frac{\partial \theta}{\partial S}$ are below the proportional limit of the material, the CDC-s are elastic and have a half wavelength of $L_{CDC} = \pi \sqrt{E \frac{I_z}{P}}$. The wavelength is reduced due to yielding if $\theta \gg M_{YIELD}/P$. The maximum ordinate $\theta$ occurs when a plastic hinge forms at the center of the CDC when $\theta = M_{PQ}/P$, where $M_{PQ}$ is the plastic moment under the combined effect of the axial force and the bending moment. CDC-s of shorter wavelength are possible, however an angular discontinuity exists at the center due to the rotation of the plastic hinge.⁸

Column Deflection Curves can be thought of as deformed shapes of an axially loaded column. In a more real sense they represent equilibrium shapes which contain the deformed shapes and the moment diagrams of beam-columns, as we showed in connection with Figs. 5.26 through 5.28. This relationship between CDC and beam-column is valid only within the theory of small deflections, at least in the framework in which it was presented here.

In Fig. 5.30 we can see how CDC-s and various beam-columns can be correlated. The vertically shaded portions of the area under the CDC represents the moment-diagram of the beam-column, and the diagonally shaded area is the deflected shape. The practical use of CDC-s lies in the fact that by proper graphical arrangement of the coordinates of the CDC we can obtain the complete in-plane load-deflection curve of any beam-column. Furthermore, it is possible to solve statically indeterminate problems with their aid.

**COMPUTATION OF CDC-s**

The pertinent information required to define a CDC with a given value of $P$ and an end slope $\theta$ for a given cross section consists of the wavelength and values of the deflection $\theta$ and the slope $\theta$ at any point within the CDC wave. Analytical expressions can only be obtained in the elastic range and for very idealized cross sections in the inelastic range.⁹¹ In general it is more convenient to use a numerical integration procedure which gives values of $\theta$ and $\theta$ at discrete points along the CDC. Before the numerical integration process can start we must have available the pertinent moment-thrust-curvature relations for the material and the cross section.

The numerical integration process is explained with the aid of Fig. 5.31 which shows a portion of a CDC near the end $z = 0$. We start by specifying a value of $P$ and $\theta$. Next we choose convenient subdivisions of $\rho$, along the length of the axis through which $P$ passes. These length elements need not be equal, but it is convenient if they are. Usually a length $r \leqslant \rho \leqslant 4r$ is chosen, where $r$ is the radius of gyration in the plane of bending. Excellent results have been obtained by using $\rho = 4r$ for wide-flange shapes bent about their $x$ axis (except where $M$ approaches $M_{PQ}$). We assume that between the ends of the segments $\rho$, the deflected shape is a circular segment and that the deflections are small. Within the segment $\rho$, the radius of curvature is then $1/\rho$ (Fig. 5.31). It can be shown that if terms involving products

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⁸ A further discussion of CDC-s beyond hinge formation is given in Ref. 5.11.
The quarter wavelength is found when $\theta = 0$. This last point can be computed to within any desired degree of accuracy by making $\rho_i$ near $\theta = 0$ as small as needed to obtain $\theta = 0$. If the quarter wavelength is computed, the remaining portions of the CDC are also known because of symmetry. The half wavelength is equal to

$$L_{CDC} = 2 \sum_{i=1}^{n} \rho_i$$  \hspace{1cm} (5.106)

The value of $\Phi_i$ is the curvature from the $M$-$P$-$\Phi$ curve corresponding to the mean value of the moment in the segment $\rho_i$. This average moment is computed by the following formulas (see Fig. 5.31):

$$M_{m1} = \frac{P_0 \rho_{1}}{2}$$  \hspace{1cm} (5.107)

$$M_{m2} = P_0 \rho_{1} + \frac{P_0 \rho_{2}}{2}$$  \hspace{1cm} (5.108)

$$M_{m1} = P_0 \rho_{n-1} + \frac{P_0 \rho_{n}}{2}$$  \hspace{1cm} (5.109)

The operations described above are extremely simple, and they can be conveniently performed by desk calculator or digital computer. Usually it is desirable to work with nondimensional quantities. The slope $\theta_i$, the moment $M_i = P_0 \rho_i$, and the average moment $M_{m1}$, can be expressed in nondimensional form for doubly symmetric wide-flange shapes which are bent about their strong axis as follows:

$$\theta_i^* = \frac{\theta_{i-1}^* + 2 \left( \frac{r_s}{d} \right)^2 \rho_i^* \frac{\Phi_i}{\Phi_y}}{2}$$  \hspace{1cm} (5.110)

$$\frac{M_i}{M_y} = \frac{P_0 \rho_i}{M_y} \left[ \left( \frac{P_0 \rho_i}{M_y} \right) \left[ \frac{M_{m1}}{M_y} + \frac{\rho_i^* \Phi_i}{\Phi_y} - \frac{r_s}{d} \right] \right]$$  \hspace{1cm} (5.111)

$$\frac{M_{m1}}{M_y} = \frac{M_{m1}}{M_y} + \left( \frac{P_0 \rho_i}{M_y} \right) \rho_i^* \frac{\Phi_i}{\Phi_y}$$  \hspace{1cm} (5.112)

In these equations

$$\theta_i^* = \frac{\theta_i (d/r_s) \sqrt{\epsilon_y}}{\epsilon_y}$$  \hspace{1cm} (5.113)

$$\rho_i^* = \frac{\rho_i \sqrt{\epsilon_y}}{r_s}$$  \hspace{1cm} (5.114)

$\epsilon_y = \sigma_y/E$ is the yield strain, $M_y = S_0 \sigma_y$, $P_y = A \sigma_y$, $\Phi_y = 2 \pi r_s^2$, $r_y$ is the strong axis radius of gyration, and $d$ is the depth of the wide-flange shape. In order to find the pertinent values of the CDC at a location $x = \sum \rho_i$, we compute $\theta_i^*$ and $M_i/M_y$ from Eqs. (5.110) and (5.111) starting with $\rho_i^*$ and $\theta_i^*$. For any given type of steel $\epsilon_y$ is constant, and so it need not enter the calculations. Furthermore, the ratio $d/r_s$ is nearly constant ($d/r_s \approx 2.4$) for all rolled wide-flange shapes. We also noted in Sec. 5.3 that the $M$-$P$-$\Phi$
curves of Fig. 5.16 represent an average situation for all wide-flange shapes. Thus for any given value of $P/P_r$ and $\theta^*_P$, we can compute a CDC which is nearly correct for all rolled wide-flange shapes. For all other shapes, residual stress distributions, and materials these CDC-s do not apply and new calculations need to be performed.

The computational procedure outlined above becomes invalid as soon as $M = M^{pc}$ at the center of the CDC, since an angular discontinuity occurs after the formation of the plastic hinge. It is not possible to give a precise value for $\theta^*_P$ at the limit at which the plastic hinge forms. This condition will show up in the CDC calculations by the inability of the computer to produce a situation where $\theta = 0$. A rough approximation for the prediction of this limiting value is given by the formula

\[(\theta^*_P)_{\text{plastic hinge}} \cong 4.35 \left(1 - \frac{P}{P_r}\right) \sqrt{\frac{2}{P/P_r}} - 1\]  

(5.115)

The procedure outlined above will give values of $\theta^*_P$ and $M_r/M_r$ at any location within the CDC for values of $\theta^*_P$ less than its value from Eq. (5.115).

**USES OF CDC-S**

We have shown how CDC-s can be computed in a simple and in as precise a manner as desired by numerical integration. We shall now see how this information can be utilized in the solution of beam-column and frame problems. We can essentially do the following with the CDC-s: We can determine the end moment versus end rotation characteristic of a beam-column in the plane of the moments having any axial load and any end moment ratio $\kappa$. The procedure is graphical, and we assume that we have a sufficient number of CDC-s available for the given axial load $P$ and the given material and the cross section.

The concept will be explained with the example of a beam-column under constant axial load $P$ and equal end moments $M_o$, as shown in the center of Fig. 5.32. We want the curve relating $M_o$ and the center deflection $v_o$ as $v_o$ varies from zero to its final value and as $P$ remains constant. To the left of the beam-column in Fig. 5.32 we have shown a family of CDC-s for the same material, cross section, and axial load as the beam-column itself, arranged so that the center of the CDC corresponds to the center of the member. The various CDC-s are distinguished from each other by different end slopes of the CDC-s. To the right of the member we have shown the moment diagram for a particular value of $M_o$, and the corresponding CDC. The relationship between the CDC and the member is that at the ends $P\delta_a = M_o$, and in the center $\delta_a = v_o$. The coordinates $M_o$ and $v_o$ represent one point on a $M_o - v_o$ curve. If we scale off values of $\delta_a$ and $v_o$ from each CDC, where $\delta_a$ is obtained from the CDC-s at the level of the ends of the beam-column, we can construct the $M_o - v_o$ curve as shown in Fig. 5.33. This curve gives not only the ascending portion of the relationship, but also the descending portion. The descending part of the curve is due to the fact that as the CDC-s become shorter, $v_o$ at the level of the ends of the beam-column becomes smaller (see left side of Fig. 5.32).

The $M_o - v_o$ curve portrays the equilibrium configuration of the beam-column in Fig. 5.32. The maximum moment on this curve represents the information we obtained in Sec. 5.3 by a different process of integration.

The same CDC-s which were used for finding the $M_o - v_o$ curve could have also been used to determine the $M_o - \theta$ curve for this beam-column, since we
also know the slope of the CDC at every point along its length from the numerical integration [Eqs. (5.107) and (5.109)]. Also, the same CDC-s provide $M = v_0$ or $M = \theta$ curves for beam-columns of any length, since $L$ can be varied in the construction scheme shown in Fig. 5.32. Furthermore, if the CDC-s are oriented differently, as indicated in Fig. 5.30, the $M = \theta$ curves for values of $\kappa$ other than 1.0 can also be drawn.

The CDC information, that is, the moment and the slope at any location along the curve, is best presented in the following graphical form, as suggested by Ojalvo. The curve at the top of Fig. 5.34 is a full wavelength of a CDC for a given value of $\theta_0$ and $P$. From the numerical integration we know the deflection $\bar{z}$ (and thus $M = P \bar{z}$) and the slope $\theta$ at any point $\bar{z}$ from the end or at any point $\bar{z}$ from the center of the CDC. The three interrelated items of information, that is, $M$, $\bar{z}$, and $\theta$ can be arranged as shown in the center and lower portion of Fig. 5.34. One of these curves relates $M$ to $\theta$, whereas the other one relates $\bar{z}$ and $\theta$. In Fig. 5.34 the corresponding points on each of the three curves are marked by capital letters.

Families of $M = \theta$ and $\bar{z} = \theta$ curves are called CDC nomographs, and one set of such curves is given in Fig. 5.35 for the case of $P = 0.2 P_f, \sigma_f = 33$ ksi, and for strong axis bending of an 8WF31 member. Only half of the nomographs are shown, as the information in the other two quadrants is symmetric on top and antisymmetric on the bottom (see Fig. 5.34).

In Fig. 5.35 is shown also an illustration of how the nomograms can be used to obtain the $M = \theta$ curve for a particular beam-column. The member is under equal end moments and is of length $80 r_s$. The axial force is $0.2 P_f$. The value of $\bar{z} = 40 r_s$, since the end of the member will be a distance $40 r_s$ from the peak of the CDC. The $M = \theta$ curve is obtained as follows: (1) Draw a horizontal line in the $\bar{z}$ versus $\theta$ plot at $\bar{z}/r_s = 40$ (dashed line). Each intersection of this line with a curve with a constant $\theta_0$ corresponds to a point on the upper curve for the same $\theta_0$. (2) Draw vertical lines from one system of curves ($\bar{z} = \theta$) to the other system ($M = \theta$). (3) Connect the point of intersection to obtain the $M = \theta$ curve for the beam-column under consideration. The resulting curve is the heavy curve in the top half of the nomographs.

The same nomograph could be used to find the $M = \theta$ curves for beam-
columns of any other length. By rearranging the CDC information in other forms, similar curves could be obtained for any other end moment combination \( \kappa \).^{(5.44)}

The nomographs in Fig. 5.35 are strictly valid for the condition shown in the figure. However, the same curves closely approximate all rolled wide-flange shapes since the \( M-P-\Phi \) curves, from which they were obtained by integration, are nearly the same for these shapes. Thus one set of nomographs for one constant value of \( P/P_Y \) is sufficient for rolled steel wide-flange shapes. Similar nomographs, for various values of \( P/P_Y \), and for \( \kappa = 1.0 \) and \( \kappa = 0 \) are published in Ref. 5.43. In the same reference are also given families of \( M_\phi-\theta \) curves for both strong and weak axis bending for a wide range of values of \( P/P_Y \) and \( L/r \). Similar information is also available in Refs. 5.44, 5.45, and 5.46. Several \( M_\phi-\theta \) curves are given in Fig. 5.36 for \( P = 0.4P_Y \) (strong axis bending, rolled wide-flange shape) and for several values of \( L/r \). In this case the ordinate has been nondimensionalized by \( M_{Pc} \), the plastic moment under \( P = 0.4P_Y \).

The curves illustrate several facts about in-plane \( M_\phi-\theta \) curves: (1) The maximum strength decreases as the length increases. (2) The cut-off point for the curves (dashed line) indicates inception of local buckling. Beyond the cut-off point the predictions, which are based on in-plane performance only, no longer hold. It is not desirable in design to deform the beam-columns beyond this limit.\(^9\)

We have shown here how it is possible to obtain end moment versus end rotation curves for beam-columns having any value of \( L \) and \( P \). Many such curves are available for rolled steel wide-flange shapes in the literature. If it is desired to generate such information for any new shape or material, we must perform the following steps: (1) Develop \( M-P-\Phi \) curves, (2) develop CDC-s by numerical integration, (3) construct nomographs, and (4) construct \( M_\phi-\theta \) curves. The use of a computer can greatly facilitate this task.\(^{(5.45, 5.44)}\)

**RESTRAINED COLUMNS**

The final end result of the CDC manipulations is the availability of \( M_\phi-\theta \) curves for beam-columns for any value of \( \kappa \) and \( P/P_Y \).\(^{(5.44)}\) One use of this information is, of course, that we can obtain the maximum moments corresponding to each curve and thus construct maximum strength interaction curves. In addition, the concept of the CDC-s makes it possible to solve statically indeterminate problems in the inelastic range.

In many instances it is not possible to isolate a beam-column and its

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\(^9\) The determination of the point of local buckling is discussed in Ref. 5.1 or 5.11.
loads. Beam-columns are usually a part of a framed structure. The loads are transmitted to the beam-column from the members which frame its ends. These framing members do not only transmit loads, but they also restrain the rotation at the ends of the beam-columns. As a result the end forces on the beam-column are dependent on the end restraints.

It can be shown that the beam-columns in a rigid frame can be idealized as shown in Fig. 5.37 if the ends of the beam-column do not translate with respect to each other. The member is subjected to an axial force \( P \), applied end moments \((M_o)_P\) and \((M_e)_P\), and the restraint due to the adjacent members is represented by springs of known or assumed stiffness. At each joint the compatibility condition requires that the spring system and the end of the beam-column rotates an equal amount. Equilibrium requires that the sum of the moments on the joints be equal to zero. A more complicated situation is shown in Fig. 5.38, where the ends of the member translate with respect to each other and the system is subjected also to a horizontal force \( H \) and translation is restrained by a spring.

The procedure for the solution of a restrained column problem is illustrated by the simple example of Fig. 5.39. The structure consists of a vertical member which has two horizontal members framing into it at right angles. These beams are rigidly connected to the beam-column. We assume that the vertical member is first subjected to an axial force \( P \) which remains constant. An external moment \( M_o \) is then applied at each joint. We wish to know the maximum value of this moment. Compatibility requires that the slope at the end of the beam is equal to the slope at the end of the beam-column at each joint [Fig. 5.39(a)]; equilibrium requires that \( M_c + M_b = M_o \) [Fig. 5.39(b)].

The end moment versus end rotation curves for the beam and the beam-column are shown in Fig. 5.39(c). The \( M-\theta \) curve for the beam-column is obtained from a CDC nomograph, as discussed before. The corresponding curve for the beam is assumed to deform like a plastic hinge after \( M_b \) reaches the plastic moment of the section. The \( M_c-\theta \) curve of the joint is constructed by adding \( M_b \) to \( M_c \) at each rotation \( \theta \), thus fulfilling both conditions of compatibility and equilibrium. We can see from this simple schematic construction that the maximum moment which the joint,

**Fig. 5.39. Moment-rotation behavior of a restrained column**

and thus this particular structure, can support is less than the sum of the maximum moment capacities of the individual components of the structure. We can also note that \((M_o)_{max}\) occurs beyond the peak of the \( M-\theta \) curve of the beam-column. Thus the unloading portion of the characteristic behavior of this member is also utilized, and so the reduction of the moment capacity
of an individual member does not necessarily mean that the strength of the whole structure is exhausted.

The simple example of Fig. 5.39 has illustrated how the \( M_{cr} \theta \) curves of beam-columns, obtained from CDC curves, can be utilized in the determination of the strength of beam-columns. Similar, though more complicated, procedures have been used to determine the maximum strength of restrained columns with unequal end moments and end restraints and with side-ways (Fig. 5.38) in Refs. 5.41, 5.43, and 5.45 through 5.50 in connection with studies of the design of planar multistory frames. It is possible to determine the maximum strength of any restrained column in this manner if sufficient \( M_{cr} \theta \) curves or CDC nomographs are available. This information is available for beam-columns made of rolled steel wide-flange shapes, and it can be generated by computer for any other shape.

FURTHER COMMENTS ON BEAM-COLUMN BEHAVIOR

CDC nomographs (Fig. 5.35) and \( M_{cr} \theta \) curves can be further generalized by presenting the data in the terms of the nondimensional parameters described by Eqs. (5.110) through (5.114). For doubly symmetric shapes, such as the wide-flange shape, the coordinates of the nomographs (Fig. 5.34) could be \( M/M_E \) (or \( M/M_E \) or \( M/M_{E,cr} \)), \( (\varepsilon_1/r_1)\sqrt{\varepsilon_f} \), and \( \theta(d/r_2)(1/\sqrt{\varepsilon_f}) \). With this representation, for example, the CDC information for all rolled wide-flange shapes and for all types of steel could be given by a set of curve families relating \( M, \varepsilon_1, \) and \( \theta \) for constant values of \( P/P_f \). Such curves are available in Ref. 5.44.

The in-plane load deformation curves predicted by the CDC concept for individual beam-columns with \( \varepsilon = 1.0 \) and \( \varepsilon = 0 \) and for restrained columns of the type shown in Fig. 5.39 have been verified by experiment, and excellent correlation has been noted.\(^{(5.1, 5.20, 5.30, 5.53)}\) These experiments were performed on full-sized rolled wide-flange members. The curves in Fig. 5.40 show the correlation between experiment and theory for one of the restrained column tests reported in Ref. 5.30.

There are, as yet, some notable limitations of CDC theory. Load-deformation curves based on it are valid only as long as the deformations are in the plane of the moments, which is also a plane of symmetry of the cross section. Limited use has been made of the concept for inelastic rectangular biaxially loaded beam-columns,\(^{(5.52)}\) but this effort has not been able to account yet for the effect of warping deformations.

Therefore, the results of the CDC concept as presented here become invalid as soon as out-of-plane phenomena take over. One of these, local buckling, has already been discussed in connection with the \( M_{cr} \theta \) curves of Fig. 5.36. These curves are terminated where the strain in the compression flange at the center of the column becomes equal to \( \varepsilon_{cp} \), the strain-hardening

![Fig. 5.40. Experimental and theoretical \( M_{cr} \theta \) curves for restrained columns](image)

strain of the material, where it is assumed that local buckling will take place.\(^{(5.1, 5.11)}\) Local buckling usually occurs well beyond the peak of the \( M_{cr} \theta \) curves for wide-flange beam-columns as long as the flange width-thickness ratio is smaller than that defined by Eq. (3.136).

Another factor terminating in-plane performance is lateral-torsional buckling. As shown in Fig. 5.4, lateral-torsional buckling may occur before the peak of the curve is reached, and it reduces the deformation capacity of the beam-column. It has not yet been possible (1967) to determine the post-buckling load-deformation curve, but we can compute the critical moment at which lateral-torsional buckling sets in. The effect of lateral-torsional buckling is drastically illustrated in Fig. 5.41, where the moment-rotation curves for two almost identical test members are compared. One test member was laterally braced, and the other was not. Lateral-torsional buckling reduced both the strength and the capacity to deform for the beam-column which had no lateral bracing between the ends.

The determination of the critical moment corresponding to the initiation of lateral-torsional buckling \((M_{cr})_L\), in Fig. 5.4] in the inelastic range proceeds similarly to the method discussed for beams in Chapter 3. The procedure involves the solution of Eqs. (5.2) and (5.3) by finite difference equations.\(^{(3.8, 3.20, 5.53)}\) The cross-sectional properties \( B_p, C_{pl}, \), \( K_p \), and \( y_p \) depend on the values of \( P \) and \( M \) acting at the cross section. Since the moment \( M \) also depends on the deflection, the final solution involves a series of iterative steps.\(^{(5.20)}\)

The calculations described above are extremely laborious even with a digital computer, and not a great many results are available. These results
are presented in Ref. 5.53, and the curves in Fig. 5.42 show typical interaction curves for an 8WF31 member under an axial load of \( P = 0.3 P_Y \). The upper curve represents the maximum in-plane strength, and thus it denotes the best we can expect from this beam-column. The inelastic lateral-torsional buckling curve was determined by the method we discussed above, and the elastic curve is defined by Eq. (5.52). From Fig. (5.42) we see that in the inelastic range the effect of lateral-torsional buckling is quite severe for this cross section. The reason for this is that yielding starts at the tips of the compression flange due to residual stresses, and a large amount of bending and warping stiffness is lost for small amounts of yielding.

It is obvious that computations could not be repeated as many times as needed to produce enough information for determining design charts. A sufficient number of calculations were made, however, to be able to make the following conclusion: A reasonable procedure for estimating the value of the critical moment in the inelastic range is to calculate the elastic critical moment and to reduce the resulting critical stress by the CRC basic column formula. We shall illustrate the reduction procedure by the example of a simply supported wide-flange beam-column subjected to equal end moments \( M_0 \) about the strong axis (\( \kappa = 1.0 \)). The critical elastic moment is equal to, from Eq. (5.53),

\[
\left( \frac{M_0}{M_{cr}} \right)_{S} = \sqrt{\frac{(d/x_s)(1 + L_s/L_s)}{4 \left( \frac{P_{x}}{P_{y}} - \frac{P}{P_{y}} \right) \left( \frac{P_{x}}{P_{y}} - \frac{P}{P_{y}} \right)}}
\]

(5.116)

Just prior to lateral-torsional buckling the deformation of the member is in the plane of bending only, and the maximum moment occurs at \( z = L/2 \) for this loading condition. It is equal to \( M_{max} = M_0 \rho \), where \( \rho \) is defined by Eq. (5.22). The stress is maximum at the same location along the \( z \) axis, and it is equal to

\[
\frac{(\sigma_{x})_{max}}{\sigma_{y}} = \frac{P}{P_{y}} + \left( \frac{M_{cr}}{M_{cr}} \right)_{S} \rho
\]

at the instant of buckling. We shall assume now that this elastic stress is reduced by the CRC basic column formula [Eq. (3.146)] to an equivalent inelastic stress, or

\[
\frac{(\sigma_{y})_{max}}{\sigma_{y}} = 1 - \frac{1}{4 \left( (\sigma_{x})_{max}/\sigma_{y} \right)}
\]

The corresponding reduced inelastic critical moment is then

\[
\left( \frac{M_{cr}}{M_{cr}} \right)_{S} = \frac{(\sigma_{x})_{max}/\sigma_{y} - P/P_{y}}{\rho}
\]

(5.117)

where the moment is computed from Eq. (5.116) with the assumption that the member is still elastic, but it is stressed to a smaller, reduced stress \( \sigma_{y} \). This procedure is empirical, but comparisons with the analytically exact tangent modulus solution show reasonable correlation. An interaction curve from the CRC reduction procedure is compared with the exact solution in Fig. 5.42 for an 8WF31 beam-column. In this case the empirical procedure shows a conservative prediction of the buckling moment.

Other loading conditions can be solved in a similar manner. We first compute the critical moment (or load) from an elastic buckling solution. Next the maximum elastic stress corresponding to the critical moment is
determined, assuming in-plane behavior. This stress is then modified, and the corresponding loading is computed from elastic theory, again assuming in-plane behavior. This final computed value of the critical load is then the estimate of the inelastic lateral-torsional buckling load.

Another empirical method for computing the critical load has also been checked against the analytically exact solutions and again, reasonable correlation was found. This method employs the interaction equation [Eq. (5.86)] in a modified form, that is,

\[ P_{cr} + \frac{M_c C_m}{(M_c)_{cr}(1 - P/P_e)} = 1.0 \] (5.118)

where \( P_{cr} \) is the smallest buckling load which the member can sustain in the \( x \) or the \( y \) plane if bending is absent, \( (M_c)_{cr} \) is the critical moment in the absence of axial load, \( P_e \) is the elastic critical load in the plane of the moment, and \( C_m \) is a factor used for adjusting for the different end moment ratios \( \kappa \) [Eq. (5.39) or (5.40)]. The values of \( P_{cr} \) and \( (M_c)_{cr} \) can be determined by any of the methods discussed in Chapters 3 and 4, with due account given to inelastic reductions where this is applicable. A comparison of the interaction curve obtained from Eq. (5.118) with the analytically exact solution is shown in Fig. 5.42 for an 8WF31 member. The prediction is quite conservative. In this case \( P_{cr} \) and \( P_e \) were determined as follows:

\[ \frac{P_{cr}}{P_e} = \frac{\pi^2}{(L/r_a)^3} \quad \text{for} \quad \frac{P_0}{P_e} \leq 0.5 \] (5.119)

\[ \frac{P_{cr}}{P_e} = 1 - \frac{\pi^2}{4\pi^2} \quad \text{for} \quad \frac{P_0}{P_e} \geq 0.5 \] (5.120)

\[ \frac{P}{P_e} = \frac{\pi^2}{(L/r_a)^3} \] (5.121)

\( (M_c)_{cr} \) was found from Eq. (3.44), using the reduction procedure [Eq. (3.146)] where \( (M_c)_{cr} > 0.5M_y \).

5.5. DESIGN OF BEAM-COLUMNS

PHILOSOPHIES OF DESIGN

In the preceding portions of this chapter we have presented information which permits the determination of the maximum in-plane strength of beam-columns if the dimensions and the material of the member and the forces and restraints acting at its ends are known. We have also discussed how the critical loads associated with the start of lateral-torsional buckling can be estimated. The methods presumed that everything about the beam-column except one variable parameter (axial load, end moment or length) is known. In all cases it was assumed that the cross section of the member is given. In design, unfortunately, the cross section is the unknown parameter, and we are in search of that section which is able to support the loads placed upon the beam-column.

In design we resolve this problem by assuming a cross section (from past experience or from some rough preliminary calculations) and then determine whether the beam-column can actually perform as required. After the first check it is usually necessary to adjust the size of the cross section and try again until the conditions imposed on the problem are satisfied. Beam-column design is thus a trial-and-error procedure.

There are three basic philosophies of designing beam-columns: in one, the limit of usefulness is the attainment of the yield stress, in the second the limiting condition is the attainment of the maximum inelastic strength, and in the third the limit is the maximum strength of a frame.

THE SECANT FORMULA APPROACH

This design method is based on the maximum elastic strength of the beam-column, and it is the method underlying the AASHO and the AREA specifications. The design formula applying for the case of a beam-column with equal end eccentricities [Fig. 5.18(a)] can be developed as follows:

The end moment for this member is \( M_y = P e \) and the maximum moment occurs at the midpoint \( (x = L/2) \) and is equal to [from Eqs. (5.21) and (5.22) for \( \kappa = 1.0 \)].

\[ M_{max} = M_y \varphi = Pe \sec \frac{F_{l}L}{2} \] (5.122)

The maximum stress is

\[ \sigma_{max} = \sigma_\varphi = \frac{P}{A} + \frac{M_{max}}{S} = \frac{P}{A} \left( 1 + \frac{eA}{S} \sec \frac{F_{l}L}{2} \right) \] (5.123)

In the limiting case of \( e = 0 \), \( P = A \sigma_\varphi \) and no allowance is made for the reduction in load due to buckling. To alleviate this shortcoming, an initial eccentricity of \( eA/S = 0.25 \) is introduced by both the AREA and the AASHO specifications. Rearranging Eq. (5.123) and introducing \( e_a \), we obtain the following formula:

\[ \frac{P}{A} = 1 + [0.25 + eA/S] \sec \left( \frac{L/2r}{\sqrt{P/E}} \right) \] (5.124)

For a given cross section and material, \( P \) in Eq. (5.124) will just cause the initiation of yielding. For a working stress formula the factor of safety, F.S., is introduced as shown in Eq. (5.125).

\[ \sigma_a = \frac{P_{w}}{A} = \frac{\sigma_y}{(F.S.)} = \frac{\sigma_y/F.S.}{1 + (0.25 + eA/S) \sec \left( \frac{L/2r}{\sqrt{\sigma_y(F.S.)/E}} \right)} \] (5.125)

The method can be also used as an alternate in the German Buckling Specifications (Ref. 3.51).
This formula is of the same form as the AREA equation, except for differences of notation. The AASHO formula is more general, as it permits the inclusion of unequal end eccentricities. It is considerably more complicated than Eq. (5.125), and it is not reproduced here. It can also be derived from Eq. (5.26).

The disadvantage of the elastic strength approach lies in the fact that the resulting maximum loads for different beam-columns have no constant relationship to the actual maximum strength. The introduction of an artificial initial eccentricity and the necessity of making an additional check for the lateral-torsional buckling strength are also undesirable features.

**INTERACTION EQUATIONS**

The most widely used method of beam-column design utilizes some form of an interaction equation which approximates the maximum strength of the member. Interaction equations are the basis of the AISC(1.20) and AISI(3.20) specifications for steel, the ASCE specifications for aluminum alloys, as well as the German (DIN 4114)(3.31) and the British (BS 449) specifications, to mention only a few examples. The AISC interaction formulas are typical and will be used as an illustration. The empirical maximum strength interaction equation, including the allowance for lateral-torsional buckling, is defined by Eq. (5.118) as

\[
\frac{P}{P_e} + \frac{M_c M_m^*}{(M_c)_e(1 - P/P_e)} = 1.0
\]

This equation gives the maximum strength if the maximum moment occurs within the span of the member. An additional equation must be checked for the condition of full plastification at the column end [Eq. (5.87)], that is,

\[
\frac{M_0}{M_P} = 1.18 \left(1 - \frac{P}{P_e} \right) \leq 1.0
\]

If in these equations \( P \) and \( M_0 \) are working loads, and if we set

\[
P = \sigma_e M_0 \quad S = \sigma_s
\]

\[
\frac{P}{A(F.S)_e} = \sigma_{es} \quad \frac{M_0}{S(F.S)_e} = \sigma_{bs}
\]

we obtain the following interaction equation:

\[
\frac{\sigma_e}{\sigma_{es}} + \frac{\sigma_e C_s^*}{\sigma_{bs}(1 - \sigma_e/\sigma_s)} = 1.0
\]

We should note that in the AISC specifications (F.S)_e is different from (F.S)_s (the factor of safety in compression and bending, respectively). If we use the familiar AISC notation \( \sigma_e = f_{es}, \sigma_s = f_{ss}, \sigma_{es} = F_s, \sigma_{bs} = F_s \)

\[
\frac{f_s}{F_s} + \frac{f_s C_s^*}{F_s(1 - f_s/F_s)} \leq 1.0
\]

The second equation [Eq. (5.87)] is conservatively expressed as

\[
\frac{f_s}{0.6\sigma_s} + \frac{f_s}{F_s} \leq 1.0
\]

In any given design both equations must be checked.

The interaction equation approach is very versatile and can be conveniently used. It has the disadvantage that it is empirical, and therefore it may err at times either on the conservative or the unconservative side. Until the time that reliable charts are available for all conceivable design situations, this method is likely to remain very popular with specification writers. We shall return to this formula in Chapter 6, where we shall examine its components in connection with the design of beam-columns in frames.

For steel wide-flange shapes it is possible to utilize the available interaction curves for the determination of the in-plane strength of beam-columns. In the plastic design provisions of the AISC specifications (Ref. 1.20, Part 2), equations and tables are given which were derived from the curves by curve fitting.

The design methods described above are discussed in more detail, with appropriate examples, in Ref. 5.8.

**SUMMARY**

In this chapter a selection was given from the vast material which is available on the behavior of metal beam-columns. We did not cover problems in connection with beam-columns of variable section, nor did we treat biaxial bending (Refs. 3.22, 3.23, 3.31, and 5.9). It is expected that the beam-column design methods discussed in Ref. 5.46, where the effect of the restraining members is included in the assessment of the strength of beam-columns, will be used to a greater extent in the future in the design of multi-story frames. This method will be further discussed in Chapter 6.

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**PROBLEMS**

5.1. (a) Derive Eq. (5.32).

(b) Define the various parameters which describe this beam-column and determine the range of these parameters.

(c) Flow-chart and program a computer solution.

(d) Compute interaction curves and plot them for representative parameter groups.

(e) Develop an approximate interaction equation and check it with the analytically exact elastic limit solution obtained above.

5.2.

(a) Develop expression for deflection and moment by assuming elastic in-plane behavior.

(b) Develop an elastic limit interaction equation [Eq. (5.26)].

(c) Develop a simpler approximate interaction equation, based on the ideas presented in connection with Eq. (5.43).

(d) Plot and compare the exact and the approximate interaction curves for \((L/l) = 0, 1.0, 2.0, \text{and} 4.0\). Use the computer for the calculations.

5.3. Derive, discuss, and recommend a simpler treatment of the beam-column formulas and charts given in the current AASHO Specifications for Highway Bridges (Appendix C in the 1963 edition).

5.4.

```
\[ M_0 \quad \text{ST9WF25} \quad M_0 \]
```

\( L = 21.6 \text{ in.} \) \( \sigma_T = 100 \text{ ksi} \)
\( P = 100 \text{ k} \) \( E = 30,000 \text{ ksi} \)
\( g = 11,500 \text{ ksi} \)

End conditions are simple with respect to lateral and torsional deformations. What is the maximum \( M_0 \) which can be supported by this member before either (1) yielding or (2) elastic lateral-torsional buckling occurs.
5.5. Lateral-torsional boundary conditions: \( u = u'' = \phi = \phi'' = 0 \) at both ends.

(a) What is the value of \( M_0 \) to cause yielding?
(b) Is it necessary to provide intermittent lateral bracing to avoid lateral-torsional buckling before this \( M_0 \) is reached?

5.6. Determine the maximum value of \( P \) which can be supported by this subassembly. \( M_0 \) is constant, \( M_0 = 20 \) units, and \( P \) varies from zero to its maximum value.

5.7. Loads are factored (ultimate).
\[ \sigma_y = 50 \text{ ksi} \]
The plane of deformation is the plane of the web.

(a) What is the maximum value of \( M_0 \) which can be supported by this subassembly? Use charts in Ref. 5.44. Assume that beam will remain elastic until its \( M_P \) is reached; after that it rotates without a change in moment.

(b) Review contents of Ref. 5.44 and Chaps. 9, 10, and 17 in Ref. 5.47. (Further study on CDCs and subassemblies can be made by considering Refs. 5.40, 5.43, 5.46, 5.48, 5.49, 5.50, and 5.51.)

5.8. Two simply supported very slender rectangular concrete beams are subjected to equal end moments. One of the beams is prestressed, and the other is reinforced by steel bars. Is the elastic critical moment causing lateral-torsional buckling equal for the two beams, or is it different? Explain reasons for your answer. Is there any literature on this subject?

5.9. A 20-ft long column is subjected to an axial force of 150 k and end moments of 3,000 in.-kips on one end and 500 in.-kips on the other end. The moments cause double curvature deformation, and they act in the same plane. Lateral support out of the plane of bending is assumed. The loads above are working loads. Make a study of the most economic design, using A36, A441, or A514 steel and rolled wide-flange, welded H or welded box columns. No particular code applies, but the factor of safety against ultimate load must be demonstrated to be at least 1.65. Use the following fictitious cost figures:

Material:
- $0.13 per lb for A36 steel
- $0.14 per lb for A441 steel
- $0.18 per lb for A514 steel
Shipping: $0.10 per lb
Welding: $0.50 per ft of weld

This problem leaves a wide choice of solutions, and attempts at computerized optimum solution can be made.

5.10. Explain with schematic sketches how the CDC nomographs (e.g., Fig. 5.35) can be used to obtain the complete $M_2-\theta$ curves for beam-columns with unequal end moments.

Frames

6.1. THE LOAD-DEFORMATION RELATIONSHIP OF PLANAR FRAMES

In the previous three chapters we discussed the behavior of individual members: beams, columns, and beam-columns. These members do not occur in isolation, but many of them joined together make up a structural frame. This frame is the skeleton which supports the loads which the structure is called upon to support.

The purpose of frame analysis is to determine the limits of structural usefulness of a given frame and to compare the predicted performance with the required performance. Such an analysis is part of the design process, wherein adjustments and new analyses are made until the predicted performance matches as closely as possible the design requirements.

In this chapter we shall examine methods of determining the maximum load capacity and the deformation response of frames. This topic is a vast one and we shall only be able to cover a small portion of it. Emphasis will
be placed on basic behavior and the discussion will center around very simple examples. Computational efficiency and elegance will be sacrificed, where necessary, to provide a basic understanding of this complex subject.

**BASIC CONCEPTS OF FRAME ANALYSIS**

In the analysis of a frame we must know the geometry of the frame itself, the material and section properties of its members, as well as the character of the loads. From the previous chapters we know the response of the members to forces acting at their ends and to loads acting on them. These load-deformation relationships of the individual members give one basic ingredient of frame analysis. They relate the geometrical deformations of each member to the forces producing them. The other two ingredients of frame analysis are the principles of equilibrium and compatibility. Equilibrium requires that the internal and external forces balance everywhere in the frame, and compatibility ensures that the various geometric restraints at the ends of the member are not violated.

The three ingredients: equilibrium, compatibility, and the load-deformation relationship of the members, are the bases for all methods of frame analysis.

**LOAD-DEFORMATION RESPONSE OF FRAMES**

Frame behavior is characterized by the relationship between the loads, as they vary during the loading history, and the resulting deformations. The load-deformation relationship becomes unique if the load set is static and proportional and if the deflections increase monotonically from zero (see discussion in Sec. 1.1 and Fig. 1.1). This uniqueness of the load-deformation curve is, in addition, based on the conditions that the frame is elastic in its initial unloaded state, and that once yielding has started at any location in the frame, the strains at this location continue to increase (that is, no strain reversal). These various conditions are by no means met by any real frame and its loading. However, the stipulations of static and proportional loads and no strain reversal permit a comparative study of frame behavior, and it is strongly suspected that these assumptions lead to safe results as regards the maximum strength.

A typical load-deflection curve is shown in Fig. 6.1. It resembles the load-deformation curves of the individual members [see Figs. 3.2, 4.1(b), and 5.3]. The relationship is nonlinear from the start because of second-order geometric effects (that is, the forces produce deformations which in turn influence the forces). After the elastic limit is reached, the slope of the curve is further reduced owing to material nonlinearities (that is, yielding), and finally the slope becomes zero at the maximum load \( P_M \).

A curve such as Fig. 6.1 gives the value of the maximum load which can be carried by the frame, as well as the magnitude of the deformation corresponding to any load intensity. Furthermore, at least the ascending branch of the curve is in stable equilibrium (see discussion in Sec. 1.3). In design the curve can be used to check if (1) the ratio \( P_M/P_w \) (where \( P_w \) is the actual or working load) is sufficiently near a specified load-factor (as determined by judgment or prescribed by a code or a specification), and (2) the deflection at working load \( v_w \) is less than or equal to a specified minimum value. In the design operation we try to match these requirements with the structure behavior, each time adjusting the structure until the requirements are met.

Ideally it would be desirable to construct a load-deflection curve for each structure. We then could obtain the various items of information in which we are interested. Unfortunately we are only able to construct load-deflection curves for very simple structures. For more complex frames we need to introduce assumptions which will permit us eventually to obtain bounds for the value of \( P_M \). We shall discuss these procedures in the next section.

**6.2. FRAME ANALYSIS**

**INTRODUCTION**

The purpose of this article is to describe (1) how the load-deflection curve of frames may be constructed in as exact a manner as possible and (2) to describe approximate methods whereby the load-deflection curve, and particularly \( P_M \), can be estimated. We shall assume that the loads and the geometric and the material properties of the frame are known. In order to keep the discussion as simple as possible, it will also be assumed that the loads and the resulting deformations act in the same plane (planar frames with in-plane response).

**EXACT ANALYSIS**

The principles of frame analysis will be illustrated on the simple structure of Fig. 6.2. The frame is a single-story single-bay symmetric rectangular frame made up of two 30WF108 beam-columns and a 30WF132 beam. The height and the width of the frame are both 59 ft. The loads on the frame con-
Fig. 6.2. Dimensions of an example frame

The corresponding moment rotation curves for the beam and for the same values of \( q \) are given in Fig. 6.4. These curves were obtained by numerically integrating the \( M-\theta \) curve for rolled wide-flange members (Fig. 3.27). Integration is started at the center of the beam where the slope is zero with an assumed moment \( M_c \) and it is continued, using small intervals of integration along the length of the beam until the integrated length is \( L/2 \). Integration gives the slope and the moment at the end of the beam. The end moment is maximum when \( \theta = 0 \) (fixed-end moment) and minimum when a plastic hinge has formed, as shown by the flattening out of the \( M-\theta \) curves in Fig. 6.4.

The end moment \( M_{ex} \) is given by the expression

\[
M_{ex} = P \theta_{ex}
\]

and the corresponding rotation \( \theta_{ex} \) is given by

\[
\theta_{ex} = \frac{M_{ex}}{EI}
\]

where \( P \) is the applied load, \( \theta_{ex} \) is the corresponding rotation, \( M_{ex} \) is the applied moment, \( EI \) is the flexural rigidity of the beam.

The information in Figs. 6.3 and 6.4 represents the load-deformation relationship of the members. These two sets of curves will now be used to satisfy the conditions of equilibrium and compatibility. They are fulfilled if \( M_{ex} = M_{ex} \) and \( \theta_{ex} = \theta_{ex} \)(Fig. 6.5). For any given value of \( q \) both conditions are met at the point (or points) of intersection of the \( M-\theta \) curves for the beam and the beam-column (Fig. 6.6). In this figure the curves from Figs. 6.3 and 6.4 are superimposed (dashed lines), and the loci of their intersection points are connected to give the curve \( M_{ex} \theta_{ex} \) (solid curve). By taking the values of \( \theta_{ex} \) at the intersection points from Fig. 6.6 and plotting them against the corresponding values of \( q \), we can construct the load-deformation curve (\( q \) versus \( \theta_{ex} \)) in Fig. 6.7. The maximum value of \( q \) is \( q_{ex} = 4.11 \) kips/ft.
Fig. 6.4. End-moment-end-rotation curves of the beam in the frame of Fig. 6.2.

Fig. 6.5. Forces and deformations of the frame of Fig. 6.2

Symmetry: $\theta_{BD} = -\theta_{DB}$

Equilibrium: $M_{BA} = M_{DB}$

Compatibility: $\theta_{BD} = \theta_{DB}$

Fig. 6.6. Equilibrium and compatibility at the joints of the frame of Fig. 6.2

$q$, (kips/ft)

Linear behavior

$F = \frac{2.5qL}{2}$

$F = \frac{\theta_{B}L}{59}$

$30$ WF $132$

$30$ W $106$

Fig. 6.7. Load-deflection curve for the frame of Fig. 6.2

The analysis of the simple frame of Fig. 6.2 illustrated the application of the three concepts—equilibrium, compatibility, and member $M$-$\theta$ relations—to the solution of the frame load-deformation relationship. The principles involved are elementary, but we can easily see that such a solution for more complex frames can become quite involved, and therefore we must look for
simplifying assumptions to obtain approximate solutions. Other simple frame problems have been solved in the literature (see Refs. 5.17 and 6.3 through 6.7). The frames considered in these references are all single-story single-bay rectangular frames. The load-deformation curves obtained all exhibit the same character as the curves shown here in Figs. 6.1 and 6.7.

**EXAMPLE OF APPROXIMATE FRAME ANALYSIS**

Various approximate procedures for estimating the maximum load \( P_m \) will be illustrated now on the portal frame shown in Fig. 6.8. The frame has fixed bases, and two equal vertical loads \( P \) act on the beam-column top. A horizontal load \( \alpha P \) acts at the roof level, where \( \alpha \) is a factor of proportionality relating the vertical and horizontal loads. The beam has a moment of inertia \( I_b \) twice that of the beam-columns. The material is A441 steel (\( \sigma_y = 50 \) ksi). The cross-sectional dimensions are: shape factor of all members \( f = Z/S = 1.10 \), ratio of the radius of gyration to the depth of the beam-column \( r_b/d_b = 0.430 \), slenderness ratio of the beam-columns \( L_c/r_b = 60 \). The factor of proportionality is \( \alpha = 0.1 \), and we consider only in-plane behavior.

The deformed shape of this frame is shown in Fig. 6.9. The panel deflections \( \delta_b \) and \( \delta_c \) and the joint rotation \( \theta_b \) and \( \theta_c \) are shown positive. The end moments on each member (positive clockwise) as well as the base reactions are shown in this figure.

**Fig. 6.8. Properties of an example frame**

\[
\begin{align*}
P & \quad 2I = I_b \\
\alpha P & \\
\sigma_y & = 50 \text{ ksi} \\
E & = 30,000 \text{ ksi} \\
f & = Z/S = 1.10 \\
\frac{Z_c}{I_c} & = 0.430 \\
\alpha & = 0.1 \\
\frac{L_b}{r_b} & = 100 \\
\frac{L_c}{r_c} & = 60 \\
\end{align*}
\]

**Note:** Moments and deformations shown in assumed positive direction

**Fig. 6.9. Forces and deformations of the frame of Fig. 6.8**

Before proceeding with the analysis we shall assume that (1) the axial force in the beam is relatively small and (2) the axial forces in the two beam-columns are nearly identical. We shall examine the validity of these assumptions later. As a result of these assumptions we can neglect the geometry changes due to axial shortening. That is, points \( B \) and \( C \) remain on the same level and \( \delta_b = \delta_c \). The bar rotations are thus \( \rho_{ab} = \rho_{cd} = \rho = \frac{\delta_b}{L} \) and \( \rho_{bc} = 0 \).

**Equilibrium equations.** From Fig. 6.9 we can derive the following equilibrium relationships for the forces:

**Horizontal equilibrium**

\[
H_A + H_B = \alpha P \quad (6.1)
\]

**Vertical equilibrium**

\[
R_A + R_B = 2P \quad (6.2)
\]

**Sum of moments about \( A \)**

\[
\alpha P(L) + P(\rho L) + P(2L + \rho L) - R_B(2L) + M_{ab} + M_{bc} = 0 \quad (6.3)
\]

From Eq. (6.3) we find that

\[
R_B = P \left( 1 + \rho + \frac{\alpha}{2} \right) + \frac{M_{ab} + M_{bc}}{2L} \quad (6.4)
\]

and from Eqs. (6.4) and (6.2)

\[
R_A = P \left( 1 - \rho - \frac{\alpha}{2} \right) - \frac{M_{ab} + M_{bc}}{2L} \quad (6.5)
\]

**Joint equilibrium**

\[
M_{ba} + M_{bc} = 0 \quad (6.6)
\]

\[
M_{cb} + M_{cd} = 0 \quad (6.7)
\]
Shear equilibrium

Member AB  \[ R_A(pL) + H_A(L) + M_{AB} + M_{AB} = 0 \] (6.8)

Member CD  \[ R_D(pL) + H_D(L) + M_{DC} + M_{CD} = 0 \] (6.9)

Adding Eqs. (6.8) and (6.9) and substituting from Eqs. (6.1) and (6.2), we get

\[ PL(2\theta + \alpha) + M_{AB} + M_{BA} + M_{DC} + M_{CD} = 0 \] (6.10)

First-order elastic analysis. This behavior of this frame is defined if we know the beam-column bar rotation \( \rho \), the joint rotations \( \theta_b \) and \( \theta_c \), and the moments, shears, and axial forces acting at the ends of each of the three members. Analysis is simplest if we neglect the effect of deformation on the equilibrium equations [that is, \( \rho = 0 \) in Eq. (6.10)] and if we neglect the influence of the axial force on the stiffness of the members. We shall term this type of analysis first-order elastic analysis. This implies also that we assume elastic behavior. The end moments, the end rotations, and the bar rotations are related through the slope-deflection equations [Eqs. (4.77) and (4.78); see also Fig. 4.12 for the sign convention]. For the frame of Fig. 6.9 these equations become equal to the following formulas, noting that \( \theta_A = \theta_D = 0 \) because of the fixed end condition:

\[ M_{AB} = \frac{EI}{L} [S_{AB} \theta_b - \rho(C_{AB} + S_{AB})] \] (6.11)

\[ M_{BA} = \frac{EI}{L} [C_{AB} \theta_b - \rho(C_{AB} + S_{AB})] \] (6.12)

\[ M_{DC} = \frac{2EI}{2L} [C_{Dc} \theta_c + S_{DC} \theta_c] \] (6.13)

\[ M_{CB} = \frac{2EI}{2L} [S_{BC} \theta_c + C_{BC} \theta_c] \] (6.14)

\[ M_{CD} = \frac{EI}{L} [C_{CD} \theta_c - \rho(C_{CD} + S_{CD})] \] (6.15)

\[ M_{DC} = \frac{EI}{L} [S_{CD} \theta_c - \rho(C_{CD} + S_{CD})] \] (6.16)

The coefficients \( C \) and \( S \) [Eqs. (4.79) and (4.75)] are functions of the axial load acting in each member. In the first-order elastic analysis we assume that the axial force effect on the \( M-\theta \) relations will be neglected, and therefore \( C_{AB} = C_{BC} = C_{CD} = 4 \) and \( S_{AB} = S_{BC} = S_{CD} = 2 \).

We have three unknown deformations: \( \rho \), \( \theta_b \), and \( \theta_c \), and thus we need three independent equations of equilibrium. We shall use Eqs. (6.6) (6.7), and (6.10) in the following analysis. If we substitute the moments from Eqs. (6.11) through (6.16) into the three equilibrium equations (noting \( C = 4 \) and \( S = 2 \)), we obtain the following three simultaneous equations for the three unknowns:

\[ 8\theta_b + 2\theta_c - 6\rho = 0 \] (6.17)

\[ 2\theta_b + 8\theta_c - 6\rho = 0 \] (6.18)

\[ 6\theta_b + 6\theta_c - 24\rho = -\frac{PL^2\alpha}{EI} \] (6.19)

We should note that in the equilibrium equation [Eq. (6.10)] we must set \( \rho = 0 \) (first-order analysis), but that in the slope-deflection equations [Eqs. (6.11) through (6.16)] \( \rho \) is included. From the above equations we find that

\[ \theta_b = \theta_c = \frac{PL^2\alpha}{28EI} \] (6.20)

\[ \rho = \frac{5PL^2\alpha}{84EI} \] (6.21)

The corresponding moments from Eqs. (6.11) through (6.16) are equal to

\[ M_{AB} = M_{DC} = -\frac{2PL\alpha}{7} \] (6.22)

\[ M_{BA} = M_{CD} = -M_{BC} = -M_{CB} = -\frac{3PL\alpha}{14} \] (6.23)

The load-deformation relationship which will be used to describe the behavior of this frame is the \( P-\rho \) curve. For the first-order analysis, from Eq. (6.21), we obtain, after introducing \( \alpha = 0.1 \) and

\[ \frac{PL^2}{EI} = \left( \frac{P}{\frac{A}{\frac{E}{L^2}}} \right) \left( \frac{L^2}{E} \right) = \left( \frac{P}{\frac{E}{L^2}} \right) \left( \frac{L^2}{E} \right) \]

\[ = \left( \frac{(50)(60)^2}{30,000} \frac{P}{\frac{E}{L^2}} \right) = 6 \left( \frac{P}{\frac{E}{L^2}} \right) \] (6.24)

the following relationship:

\[ \rho = \frac{PL}{28} \] (6.25)

The resulting straight line is shown in Fig. 6.10.

Second-order elastic analysis. In this analysis we shall include the effects of the deformations in the equilibrium equations, notably in Eq. (6.10) where it will now be necessary to include the rotation \( \rho \). The determination of the coefficients \( C \) and \( S \) in the slope-deflection equation will also involve the axial force in each member. We shall assume that the axial force in the beam is small, so that \( C_{BC} = 4 \) and \( S_{BC} = 2 \). It is further assumed that the axial forces in the two beam-columns are not significantly different from each other, so that it will be possible to use the same values of \( C \) and \( S \) in both members.

We shall return and examine the validity of these two assumptions later. Thus we shall use \( C_{AB} = C_{CB} = C \) and \( S_{AB} = S_{CB} = S \).
The coefficients $C$ and $S$ are related to $PL^3/EI$ through Eqs. (4.79) and (4.75). The curve relating $P$ and $\rho$ from Eq. (6.30) is plotted in Fig. 6.10 for $\alpha = 0.1$ and $PL^3/EI = 0(P/P_r)$. For any constant value of $P$ the rotation from the first-order elastic analysis [Eq. (6.25)] is always smaller than $P$ from the second-order analysis [Eq. (6.30)]. As $P$ increases, $\rho$ from this analysis increases at an increasing rate until the rotation approaches infinity when $P = 1.236P_r$. This occurs when the denominator in Eq. (6.30) becomes zero. The load $P = 1.236P_r$ is the critical elastic buckling load $P_e$ of the frame (see also Sec. 4.3). At this load the frame, which is loaded by only the vertical loads (Fig. 4.11), can exist in both a straight and a laterally deflected position. The load $P_e$ is the largest load $P$ which can be supported by this frame under the assumption of unlimited elastic behavior.

The elastic buckling load $P_e$ can also be determined by setting the determinant of the coefficients of the unknown deformations in Eqs. (6.26) through (6.28) equal to zero, that is,

$$
\begin{vmatrix}
(C + 4) & 2 & -(C + S) \\
2 & (C + 4) & -(C + S) \\
(C + S) & (C + S) & \frac{2PL^3}{EI} - 4(C + S)
\end{vmatrix}
= 0
$$

Upon performing the operations indicated by this determinant, we obtain, after some algebraic manipulation, the following buckling condition:

$$
\frac{PL^3}{EI} = \frac{(C + S)(C - S + 12)}{C + 6}
$$

which can be solved for $P_e$ by trial and error or graphically. From it we indeed obtain $P_e = 1.236P_r$.

Thus the elastic buckling load is identical with the load at which the deformation $\rho$ (and also $\theta_b$ and $\theta_e$) of the frame of Fig. 6.8 is horizontal load $\alpha P$ becomes infinitely large. This load represents the highest possible load which can be supported by this frame.

**Rigid plastic analysis.** The frame in Fig. 6.8 does not possess the ability to remain always elastic. Eventually yielding will set in, and therefore the behavior described by the elastic analysis is no longer valid. We shall use the load predicted by rigid plastic theory as a first estimate of the actual load which can be supported. According to this theory the frame remains undeformed until the bending moment is equal to a plastic moment at sufficient locations in the frame so that a plastic mechanism has formed. This means that the moment-curvature relationship is composed of a vertical line (Fig. 6.11) until the plastic moment $M_{pl}$ is reached. That is, no deformation takes place as long as $M < M_{pl}$. Thereafter the curvature increases indefinitely at a constant moment (horizontal lines in Fig. 6.11). This representation of the moment-curvature relationship is an idealization of the $M-\Phi$ curves discussed for beam-columns in Sec. 5.3 (such curves are...
beam since the beam contains only a small axial force and \( M_p \) of the beam is larger than \( M_r \) of the beam-column (Fig. 6.8).

In order to simplify the problem we shall again assume, for the present, that the axial force in the two beam-columns is equal to \( P \), so that

\[
M_{AB} = M_{BA} = M_{CD} = M_{DC} = -M_{PC} = -1.18 M_p \left( 1 - \frac{P}{P_r} \right)
\]

(6.37)

where \( P \) is the applied load on the frame. By substituting these end moments into the equilibrium equation [Eq. (6.10)], we obtain the following relationship:

\[
PL(2\rho + \alpha) - 4M_{PC} = 0
\]

(6.38)

We can solve Eq. (6.38) for the load \( P \) as follows:

\[
\frac{P}{P_r} = \frac{1}{1 + (A\sigma_i L/4.72M_p)(2\rho + \alpha)}
\]

(6.39)

For \( \alpha = 0.1 \) and

\[
\frac{A\sigma_i L}{M_p} = \frac{A\sigma_i L}{Z\sigma_v} = \frac{AL}{L} = \frac{M_p}{E} = \left( \frac{L}{r} \right) = \left( \frac{1}{2T} \right)
\]

(6.40)

this equation becomes

\[
\frac{P}{P_r} = \frac{1}{2.342 + 26.84\rho}
\]

The curve from this equation is plotted in Fig. 6.10 as the one marked second-order rigid plastic. It is a second-order curve because it involves the rotation \( \rho \) in the equilibrium equation [Eq. (6.10)]. When \( \rho = 0 \), at the start of mechanism motion, we find that \( P_{r1} = P_r/2.342 = 0.426P_r \). This is the first-order rigid plastic maximum load we would obtain from simple plastic analysis. As soon as any rotation takes place, the capacity to carry load is reduced in accordance with Eq. (6.40), and we get the second-order curve shown in Fig. 6.10.

We have now two limiting loads: the elastic buckling load, \( P_b = 1.236P_r \) and the first-order rigid plastic load \( P_{r1} = 0.426P_r \). Both of these are higher than the actual load which can be carried because the true \( P_d \) curve is bounded by the second-order elastic and rigid plastic curves. We can obtain a better, but still an upper bound, estimate of the maximum load by using \( P \) at the point of intersection of the two curves (Fig. 6.10), that is \( P_{sl} = 0.356P_r \). This value was obtained graphically from Fig. 6.10.

**Examination of the assumptions.** The previous discussion has shown that the frame of Fig. 6.8 can at most support a load \( P = 0.356P_r \). In order to obtain this answer, we have assumed that the axial load in the beam is zero
and that the axial loads in the two beam-columns are nearly equal. We shall now examine the validity of these assumptions.

The axial force in the beam is less than $\alpha P$ (Fig. 6.8). Since $P$ is no larger than 0.356$P_T$, and the beam and the beam-column areas are equal (Fig. 6.8), the axial force in the beam is less than 0.036$P_T$. For this small axial load $C = 3.95$ and $S = 2.01$, as compared with the assumed $C = 4$ and $S = 2$. These are small differences, and they have little effect on the end results. Thus our assumption of neglecting the axial force in the beam was justified in this case.

The axial force in the beam-columns is dependent on the deformations and the end moments [Eqs. (6.4) and (6.5)]. We shall compute them first by assuming that the axial force in each beam-column is equal to $P = 0.356P_T$ and then use an iterative procedure to find the actual values. The corresponding values of the various components of the deflections are: $PL^4/EI = 2.14$, $C = 3.71$, $S = 2.08$ (Appendix 4A), $\rho = 0.0171$ radians [Eq. (6.29)], $M_{AB} = M_{DC} = -0.0208 PL$, and finally $R_p = 1.046P$ and $R_s = 0.954P$ [Eqs. (6.4) and (6.5)]. These values of the axial force are $\pm 4.6$ per cent different from $P$. Such a small difference will not influence the second-order elastic and rigid plastic calculations significantly, and therefore it was reasonable to use the assumption that the two axial forces are equal in computing $C$ and $S$ in this case.

An examination of Fig. 6.10 shows an additional simplification which may be introduced for this problem. If we substitute into the $P-\rho$ relationship defined by Eq. (6.30)(second-order elastic analysis) $C = 4$ and $S = 2$, that is, if we neglect the effect of the reduction in stiffness due to axial force, we obtain

$$\rho = \frac{5\alpha (PL^4/EI)}{84 - 10(PL^4/EI)} \quad (6.41)$$

This is a considerably simpler expression than Eq. (6.30), and the resulting curve is seen to be little different from the more exact relationship for $\alpha = 0.1$ and $PL^4/EI = 6(P/P_T)$ of this problem if $P$ is less than the load $P_{st} = 0.356P_T$. Thus for all further calculations we may set $C = 4$ and $S = 2$ even in the second-order analysis in this problem.

**Lower bound of the frame strength.** From our previous calculations we know that the frame of Fig. 6.8 can support at most $P = 0.356P_T$. A lower-bound solution results if we compute the load at which the first plastic hinge forms. The elastic second-order moments are given by Eqs. (6.31) and (6.32). If we set $\alpha = 0.1$ $PL^4/EI = 6(P/P_T)$, $C = 4$, and $S = 2$ into these equations, we find that

$$M_{AB} = M_{DC} = -\frac{0.4(P/P_T)P_T L}{14 - 10(P/P_T)}$$

$$M_{BA} = M_{CD} = -\frac{0.3(P/P_T)P_T L}{14 - 10(P/P_T)}$$

From the relationship

$$\frac{-0.4(P/P_T)P_T L}{14 - 10(P/P_T)} = -1.18M_P \left(1 - \frac{P}{P_T}\right) \quad (6.43)$$

we obtain, by noting that $M_P/P_T L = 2f(r)/d [1/(L/r)] = 0.0158$, $P_{st} = 0.332P_T$. When $P = P_{st}$, hinges form at the base of the frame. This load represents a lower bound, and the frame will at least carry $P_{st}$. From our analysis we now know that the maximum load of the frame of Fig. 6.8 is bounded by $P_{st}$ and $P_{st}$, or $0.332P_T < P_M < 0.356P_T$ (representing a 7 per cent difference).

**Elastic-plastic analysis.** In the previous analysis of plastic behavior we have used the rigid-plastic theory ($M$-$\Phi$ curve of Fig. 6.11). An analysis based on the step-by-step determination of the load-deformation relationship as successively more and more hinges develop is called *elastic-plastic analysis*. In this analysis elastic (not rigid) behavior is assumed up to the hinge formation where $M = M_{PC}$, and the $M$-$\Phi$ relationship of Fig. 6.13 applies. According to this assumption the segments of the frame between hinges remain elastic.

We have already shown that plastic hinges develop first at the base of the frame shown in Fig. 6.8. After these hinges have formed, the frame will respond to additional loads as if real hinges existed at these locations. The corresponding additional deformations are $\delta\theta_A, \delta\theta_B$, and $\delta\Phi$ (Fig. 6.14). The frame is assumed elastic everywhere except at the hinges, and thus we can use the elastic slope-deflection equations. We have previously noted that at the load levels of importance in this problem we can conveniently use $C = 4$ and $S = 2$ in these equations. We shall also assume that the deformations of the frame are symmetric, so that $\delta\theta_A = \delta\theta_B, \delta\theta_A = \delta\theta_B, \delta M_{BA} = \delta M_{AB}$.

![Fig. 6.13. Elastic-plastic $M$-$\Phi$ relationship](image-url)
from Eq. (6.41) (or from Fig. 6.10 from the second-order elastic curve). Thus \( \rho = 0.0155 + \delta \rho \). The moments \( M_{AB} \) and \( M_{BC} \) are equal to the plastic moment \( M_{PC} = 1.18M_{e}(1 - P/P_{Y}) \). The other moments are

\[
M_{BA} = M_{EA} = (M_{BA})TA + \delta M_{BA}
\]

(6.51)

where \( (M_{BA})TA \) is equal to the value from Eq. (6.32) (second-order elastic analysis) for \( P = P_{TA} = 0.332P_{Y} \). Upon substitution into Eq. (6.32) we obtain \( (M_{BA})TA = -0.00934P_{Y}L \). The moment \( \delta M_{BA} \) is the change due to the increased load above \( P_{TA} \) and it is equal to [from Eq. (6.48)]

\[
\delta M_{BA} = -2EI\frac{\delta \rho}{L} = -2 \left( \frac{EI}{P_{Y}L} \right) (\rho - \rho_{TA})P_{Y}L = \frac{(\rho - 0.0155)P_{Y}L}{3}
\]

(6.52)

The equilibrium equation is thus, from Eq. (6.49),

\[
P_{Y} \left( \frac{2\rho + \alpha}{2} \right) - 2(1.18) \left( \frac{M_{e}}{P_{Y}L} \right) \left( 1 - \frac{P}{P_{Y}} \right) - 2 \left( 0.00934 + \frac{\rho - 0.0155}{3} \right) = 0
\]

(6.53)

This equation can be simplified into the following form by noting that \( \alpha = 0.1 \) and \( M_{e}/P_{Y}L = 0.0158 \):

\[
P_{Y} = \frac{0.667\rho + 0.0455}{2\rho + 0.1372}
\]

(6.54)

Equation (6.54) relates the frame load \( P \) to the corresponding rotation in the range where the plastic hinges have developed at the bases of the beam-columns, but the mechanism has not yet formed. It can be shown that Eq. (6.54) is almost insensitive to practical variations of \( \rho \), that is,

\[
P_{Y} = \frac{0.667\rho + 0.0455}{2\rho + 0.1372} = \frac{455}{1 + 14.62\rho} \approx 0.332
\]

Thus the curve describing the behavior of the frame after the formation of the plastic hinges at the bases is nearly a horizontal straight line (Fig. 6.15). The curve of Eq. (6.54) is invalid after the hinges at the top of the beam-columns have formed, that is, after it intersects the curve from rigid plastic analysis [Eq. (6.40) and Fig. 6.15].

The load-deformation behavior of the frame of Fig. 6.8 is thus bounded by the three curves representing second-order elastic behavior [Eq. (6.41)], second-order elastic behavior but with the first hinges included [Eq. (6.54)], and the second-order rigid-plastic behavior [Eq. (6.40)]. The region bounded by these curves is shown crosshatched in Fig. 6.15. In this particular problem the maximum load from these curves is equal to the load corresponding to \( P_{TA} \) where the hinges first develop at the base of the frame.

According to this latest and most elaborate analysis the best estimate of the maximum load which can be supported by the frame of Fig. 6.8 is equal to \( P_{TA} = P_{PS} = 0.332P_{Y} \). The actual maximum load should be close to this value.
SEC. 6.2  FRAMES

FRAME REFERENCE LOADS

In the previous example we showed how various estimates of the actual maximum load of the simple frame of Fig. 6.8 could be obtained. The procedures can be extended in general to any plane frame. The various loads which were obtained are reference loads, each having a particular significance with respect to the actual maximum load \( P_{r} \). The possible reference loads are classified in Figs. 6.16 and 6.17. In these figures the heavy solid curve marked true behavior is the actual load-deformation relationship for the frame, obtained by an experiment.

Four additional curves are shown in Fig. 6.16: The first- and second-order elastic and the first- and second-order rigid-plastic curves. The various reference loads in this figure are (1) \( P_{r} \), the elastic buckling load, (2) \( P_{r1} \), the rigid-plastic first-order collapse load, (3) \( P_{r2} \) and \( P_{r3} \), the loads at the intersection of the rigid-plastic curve with the first- and second-order elastic curves, respectively, and (4) \( P_{r4} \) and \( P_{r5} \), the loads at which hinges begin to develop on the first- and second-order elastic curves.

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In Fig. 6.17 we have the first- and second-order elastic-plastic curves with the following additional reference loads: (1) $P_{cr}$, the point of intersection between the second-order rigid-plastic and the first-order elastic-plastic curve, (2) $P_{c2}$, the maximum point on the second-order elastic-plastic curve, and (3) $P_{cr}$, the point of intersection between the second-order elastic-plastic and rigid-plastic curves.

In the following discussion we shall further consider the significance of these various reference loads.

**FIRST-ORDER ELASTIC ANALYSIS**

In this analysis we assume fully elastic response, and we formulate the equilibrium equations on the undeformed structure, and thus the response of the structure is linear (Fig. 6.16).

First-order elastic analysis is the simplest of all possible analyses. Engineers are most familiar with it since it has been traditionally the major content of courses in structural analysis in civil engineering curricula and it is most used in professional practice. Since the beginning of the 20th century, a multitude of methods have been developed for analytically exact procedures (slope-deflection, virtual work, column analogy, moment distribution, for example) and for procedures which result in more or less approximate solutions (portal method, cantilever method, etc.). In recent years procedures based on matrix algebra for the force or displacement methods have been advanced (see Refs. 1.1 through 1.5 and 1.9 through 1.11, for example). These latter procedures are particularly useful in conjunction with digital computers, and they will most likely supplant the other methods as computer use becomes more and more universal.

It is usually customary to neglect the influence of axial force and shear force on the stiffness of the members, although such a formulation is possible. These effects are, however, of a smaller magnitude than the effects of secondary deformations which play so important a role in the second-order analysis.

Besides furnishing the reference load $P_{cr}$ at the point of intersection with the second-order rigid-plastic curve (Fig. 6.16), which is an upper bound to $P_{cr}$, the first-order elastic solution also gives the load $P_{c2}$ at the point where elastic behavior terminates. It also gives the elastic deflections corresponding to these loads.

The load $P_{c2}$ can be defined in several ways. The most commonly used definition is that $P_{c2}$ is the load at which the elastic stress in the most stressed fiber in the frame is equal to the yield stress $\sigma_y$. This definition may be modified for the limiting stress $\sigma_y - \sigma_y$, to account for the presence of residual stresses $\sigma_y$ or for a critical stress $\sigma_c$, as the limiting stress to account for elastic lateral-torsional or local buckling. Another way of defining $P_{c2}$ is the load at which the first plastic hinge forms in the frame. The first definition leads to the limiting load on which most specifications are based, and the second definition is used in the so-called ultimate strength design for concrete structures. The difference in load level between the two definitions is equal to $f - 1$, where $f$ is the shape factor. Since the shape factor for wide-flange shapes bent about their major axis is only about 1.14, this difference is not great.

Regardless of the definition of $P_{c2}$, it is a lower bound of $P_{cr}$ for structures having low axial loads (lightly loaded rigid frames for which $P_a \gg P_{c2}$, and where the axial load is less than 0.15$P_{c2}$). For structures with heavier axial loads it is not necessarily a lower bound, and here caution should be exercised in placing too much reliability on $P_{c2}$ because it neglects possibly dominant second-order effects.

**SECOND-ORDER ELASTIC ANALYSIS**

In this analysis the frame is still assumed to be elastic, but the equations of equilibrium are formulated on the deformed structure. The moments in the members are magnified by the product of the axial force and the deflection, and as a result the deformations increase at a faster rate than the load. Eventually these deformations become quite large until they become, theoretically at least, equal to infinity as the load approaches the frame buckling load $P_b$ (Fig. 6.16). \(^\dagger\)

A second-order elastic analysis is performed by assuming small deflections (that is, the curvature $\Phi = -\psi''$ where $\psi''$ is the second derivative of the deflection). The reduction in the member stiffness due to the axial forces (for example $C$ and $S$ in the slope-deflection equations) is included. Except in very simple structures, such as that of Fig. 6.8, we do not know the magnitude of these axial forces at the beginning of the analysis. However, the axial forces can be estimated, or obtained from a first-order analysis in a first cycle of the calculations. Since the coefficients $C$ and $S$ (or any other such functions which may enter the analysis) are not very sensitive to minor changes in $P$ over most of their range, good results are obtained by only a few cycles of the calculations. A further simplification is possible if $P L^3/EI < 1.0$, because then we can conveniently use $C = 4$ and $S = 2$ without introducing a significant error. \(^\dagger\) This fact was demonstrated in connection with the example of Fig. 6.8.

The results of a second-order elastic analysis are a load-deflection curve which now includes geometric nonlinearities and a value for the elastic critical load $P_a$ (Fig. 6.16). This load is an important reference load because it rep-

\(^\dagger\) This is true if the effects of axial shortening are neglected (Ref. 6.8).

\(^\dagger\) This was illustrated in the example of Fig. 6.8, where no second cycle of analysis was necessary to account for the differences in axial load between the two columns. For further discussion see Refs. 6.8 and 6.13.
represents the highest theoretically possible load which can be supported by the frame.

Another reference load can be obtained by defining a limiting load \( P_{zz} \) (Figs. 6.10 and 6.16) at which elastic behavior is terminated and it is the second-order equivalent of \( P_E \). This load can be computed for \( \sigma_{\text{max}} = \sigma_\varepsilon \) (a local or lateral-torsional buckling stress), for \( \sigma_{\text{max}} = \sigma_\gamma \) (or \( \sigma_{\text{max}} = \sigma_\varepsilon - \sigma_\gamma \), if residual stresses are to be included) or for \( M_{\text{max}} = M_{P_E} \), as was done in connection with the example of Fig. 6.8. The load \( P_{zz} \) represents a lower bound to the maximum strength of the frame. A further reference load is \( P_{st} \) which is the load at the intersection of the second-order elastic and rigid plastic curves (Fig. 6.16). This load is always an upper bound to \( P_N \).

The performance of a second-order elastic analysis is somewhat more complicated than the first-order analysis. However, the same methods and computational tools can be employed in the solution, although a computer becomes indispensable even for smaller frames. In the analysis the small deflection theory is used, and axial and shearing deformations are neglected. Small deflection theory (that is, \( \Phi = -y'' \)) gives valid results for loads up to the rigid-plastic failure load. In the vicinity of \( P_E \) the deformations become large (of the order of magnitude of the major frame dimensions), and thus this theory is no longer valid. However, we usually do not need the second-order elastic load-deflection curve in this region anyhow. The value of \( P_E \), as obtained from the usual small deflection theory, is equal to its value from a buckling analysis (Sec. 4.3, and the example of Fig. 6.8).

Shearing deformations become important for only relatively short and deep members, and therefore for the usual frames they can be neglected. Axial deformations (that is, the axial shortening or elongation of the members) become important for slender frames for which the height is more than eight times the width of the structure.\(^{9,10}\)

RIGID-PLASTIC ANALYSIS\(^{11,6,7,11,8,3,8,9}\)

The load computed on the assumption that the frame can carry no further increase of load after a development of a mechanism is the plastic collapse load or the plastic failure load. In this analysis it is assumed that the \( M-\Phi \) relationship is that shown in Fig. 6.11. As a consequence of this assumption no deflections occur until \( M_{P_E} \) has been reached at sufficient locations in the frame and the frame can move as a mechanism. The load corresponding to the first movement of this mechanism is \( P_{st} \) (Fig. 6.16). Further deformation can be computed by neglecting the deflections in the equilibrium equations (first-order rigid-plastic analysis) or by including them (second-order rigid-plastic analysis). In this latter case the resulting curve is a drooping curve (Fig. 6.16), which eventually coincides with the second-order elastic-plastic curve.

One reference load obtained from the rigid-plastic analysis is the plastic collapse load \( P_{pt} \). This will give always an upper bound value for the true maximum load.\(^4\) Other reference loads, obtained as the points of intersection between the second-order rigid-plastic and the first-order elastic and the second-order elastic curves are \( P_{st} \) and \( P_{tt} \), respectively.\(^{6,1,6,10}\) Both of these loads will be upper bounds, that is, \( P_{st} > P_{tt} > P_N \).

Methods for determining the first-order rigid-plastic failure load \( P_{pl} \) are covered in texts on plastic analysis and design (Refs. 1.6 through 1.8, 3.2, and 3.9). These methods can be extended to cover also the case of second-order rigid-plastic analysis. First-order rigid-plastic analysis is of great practical significance because the maximum load of frames with low axial loads (single-story gabled frames, two- or three-story rectangular rigid frames) is nearly equal to the rigid-plastic failure load (that is, \( P_M \approx P_{pt} \)).

**MERCHANT’S APPROXIMATION**

In Fig. 6.16 we have shown six reference loads which help to bracket the true maximum load \( P_N \). In general, four of these, \( P_{st} , P_{tt} , P_{at} \), and \( P_{tt} \), are upper bounds and \( P_{st} \) is a lower bound of \( P_N \). These reference loads are obtained with varying degrees of effort. The value of \( P_{st} \) is determined by the least amount of effort for a complex frame, and \( P_{st} \) and \( P_{tt} \) give usually the best bounding of \( P_N \). For the frame of Fig. 6.8, for example, we found that \( P_{st} = 0.332P_r \) and \( P_{tt} = 0.356P_r \), and so \( 0.332P_r < P_N < 0.356P_r \). In this case the spread is quite modest (about 7 per cent), so that no further analysis would be necessary.

Since \( P_{st} \) and \( P_{tt} \) are readily determined for frames of reasonable size, and since \( P_M \) is lower than these loads, Merchant and others have suggested using the following empirical equation for approximating \( P_M \) by \( P_M \) \(^{9,6,1,1,4,6,10,6,10}\)

\[
\frac{P_{st}}{P_{pl}} + \frac{P_{tt}}{P_N} = 1.0
\]

(6.55)

If \( P_N \ll P_{st} \), as is the case for lightly loaded small structures, \( P_M = P_{st} \), as was pointed out previously. We can rearrange Eq. (6.55) also into the following form:

\[
P_M = \frac{P_{pl}P_N}{P_{tt} + P_{pl}}
\]

(6.56)

For the frame of Fig. 6.8 (Sec. 6.1), \( P_{st} = 0.426P_r \), \( P_{tt} = 1.236P_r \) and thus \( P_M = 0.316P_r \) from Eq. (6.56). The best estimate of \( P_M \) from Sec. 6.1 was 0.332P_r, and thus \( P_{tt} \) from Eq. (6.56) is a good lower-bound estimate. Similar

\(^4\) This will be strictly so only if strain hardening is neglected. In frames where the axial loads are small it is possible that \( P_M > P_{pt} \). For a further study of the effect of strain hardening in frames see Refs. 6.11 through 6.13 and Refs. 3.27 and 3.48, for example.
conclusions were reached by others who compared theoretically computed or experimentally determined values of $P_H$ with the predictions of Eq. (6.56), although $P_{Hd}$ was found to be somewhat overconservative in most cases.\footnote{1,4,11,6,34} It was noted that if the ratio $P_\sigma/P_H$ becomes larger than about 0.3, that is, $P_H$ is less than about 3 times $P_\sigma$, the scatter of points away from Eq. (6.56) becomes considerable.\footnote{6,1} The derivation is conservative, and so we might use Eq. (6.56) to give a rapid, but safe, estimate of $P_H$. Since better estimates can be achieved with the computer, the attractiveness of Merchant’s method is greatly diminished.

**ELASTIC-PLASTIC ANALYSIS**

A better estimate of $P_H$ can be obtained, at the expense of considerably more effort, by making a second-order elastic-plastic analysis (Fig. 6.17). In such an analysis we assume that the $M$-$\Phi$ relations are approximated as shown in Fig. 6.13. We start first with an elastic second-order analysis, and from this we determine the load at which the first hinge forms ($P_{Hs1}$ in Fig. 6.17). The next analysis is made for incremental loads (above $P_{Hs1}$) and assuming that the first plastic hinge is a real hinge, incapable of resisting moment. The next plastic hinge is then found and the process is repeated until the full mechanism has developed when the last hinge has formed. At this point, which corresponds to the load $P_j$ in Fig. 6.17, the second-order elastic-plastic curve joins the second-order rigid-plastic curve. The maximum load from the second-order elastic-plastic analysis, $P_{Hs}$ in Fig. 6.17, is now the best estimate of $P_H$, albeit still an upper bound. The method of solution was illustrated previously for the frame of Fig. 6.8. It follows closely that of the first-order elastic-plastic analysis (except for the consideration of the deformations in the equilibrium equation) which is discussed in detail in texts on plastic analysis and design.

The first-order elastic-plastic curve is also shown in Fig. 6.17. It levels off at $P = P_\sigma$. For frames with small axial loads this curve will closely approximate actual frame behavior.\footnote{1,1} Its intersection with the second-order rigid-plastic curve gives one more upper-bound estimate $P_{Hs}$ for the actual maximum load.

A further insight into the second-order elastic-plastic analysis can be gained from Fig. 6.18. The solid curve gives the second-order elastic-plastic load-deflection curve. The branch 0-1 corresponds to the fully elastic frame. This curve becomes eventually asymptotic to the load $P_\sigma$ if infinite elastic behavior is assumed. However, a hinge forms at point 1, and the frame behaves thereafter (until the next hinge develops) as if one real hinge existed in it (branch 1-2). If unlimited elasticity is assumed after the first hinge has formed, then branch 1-2 continues and becomes asymptotic to a load $P_{Hs}$. This load is less than $P_\sigma$, and it is equal to the buckling load of a deteriorated frame which now has one real hinge in it. After the formation of the second hinge, the new curve is branch 2-3, and this curve, if continued, becomes asymptotic to a new buckling load $P_{Hs}$ (Fig. 6.18). This deterioration continues until finally the last hinge forms at load $P_j$ where the frame is a mechanism which has no stability.\footnote{6,1}

**THE LAST HINGE METHOD OF ANALYSIS**

The closest estimate of $P_H$ results from the second-order elastic-plastic analysis (the load $P_{Hs}$ in Fig. 6.17). It is possible to obtain one point on this curve without performing a complete elastic-plastic analysis. This load is $P_j$, the point at which a mechanism has just been formed and where the second-order rigid-plastic and the second-order elastic-plastic solution merge (Fig. 6.17). The method of obtaining $P_j$ is called the last hinge analysis, and it forms the basis of various discussions on the strength of rigid frames.
Fig. 6.19. Last hinge analysis, \( P_s < P_{p2} \)

(Refs. 6.1 and 6.17 through 6.21). It is an iterative method and is performed as follows:

We first determine the rigid-plastic load-deformation curve by a second-order analysis [for example, Eq. (6.40) for the example of Fig. 6.8]. The rigid-plastic load will be a function of the deformation; that is, \( P_p = f(\psi) \). The point \( P_s \) will be on this curve.

The first step in the iteration will be to compute \( P_{p1} \) (the first-order rigid-plastic load) and then to calculate the corresponding deflection \( \psi_{p1} \) at the instant the mechanism has formed. The value of \( \psi_{p1} \) is the deflection corresponding to \( P_{p1} \) on the first-order elastic-plastic curve (Fig. 6.19). The value of \( \psi_{p1} \) is computed from the known moment diagram for \( P_{p1} \) by assuming that between hinges the frame members are fully elastic and that continuity exists at the last hinge to form.\(^{0.6.1.7.1.5}\)

The next step in the iteration is to compute a new value of \( P_p = f(\psi_{p1}) \) and a new deflection for this load \( \psi_{p2} \). The operation is repeated until two subsequent loads are nearly equal. At this time \( P_p = P_s \) and \( \psi = \psi_{p2} \) (Fig. 6.19). The iteration converges very rapidly, requiring seldom more than three or four cycles of computation. The procedure ensures that the point \( P_s, \psi \) lies on the second-order rigid-plastic curve and that it corresponds always to the instant when the mechanism has just formed.

The load \( P_s \) is either less than \( P_{p2} \), the maximum point on the second-order elastic-plastic curve (Fig. 6.19), or it is equal to \( P_{p2} \) (Fig. 6.20). Whether \( P_{p1} > P_s \) or \( P_{p2} = P_s \) can be ascertained by making a buckling analysis on the deteriorated frame in which all but the last hinge have formed (Fig. 6.18). If the deteriorated critical load \( P_d \) is larger than \( P_s \), the last branch of the elastic-plastic curve has a positive slope, and thus \( P_s = P_{p2} \) (Fig. 6.20). If \( P_d = P_s \), then \( P_s = P_{p2} \) (Fig. 6.19).

**EXAMPLES ON THE LAST HINGE METHOD**

We shall now illustrate the last hinge method on two examples. One will be a frame with low axial loads, for which we shall show that \( P_s = P_{p1} \), and the other will be a heavily loaded frame for which \( P_s \) is considerably lower than the first-order rigid-plastic collapse load \( P_{p1} \).

The structure which is considered in these two examples is given in Fig. 6.21. The plastic moment \( M_p = Z\sigma_f \) of each member is equal; however, the beam members are twice as deep and have twice the moment of inertia of the column members. The slenderness ratio of the columns is 60, and the material of the frame is steel with \( \sigma_f = 50 \text{ ksi} \).

The loading for the first problem to be solved is shown in Fig. 6.22(a). Two equal vertical loads \( P \) act at the center of the two beams, and a horizontal load \( 0.1P \) is specified at the roof level. The loads are proportional, and they are assumed to vary from zero to their final value.

**Equilibrium equations.** The deformed shape of the structure and the positive values of the moments and reactions are shown in Fig. 6.22(b). Moments are positive when acting clockwise at the ends of the members. It will be assumed that the axial shortening of the members is small so that it can be ignored. Thus the top of each of the three vertical members deflects
an amount $v$ in the horizontal direction. It will also be assumed that the axial force in the horizontal members (beams) is of a smaller order of magnitude than that in the vertical members (columns), and therefore, secondary moments in the beams can be ignored.

The equilibrium equations are, from Eq. 6.22(b), equal to

$$H_i + H_4 + H_6 = 0.1P \quad \text{and} \quad V_1 + V_4 + V_6 = 2P \quad \text{(6.57)}$$

The shear equilibrium for the columns is

$$M_{3i} + H_1 L + V_1 v = 0; \quad M_{34} + H_2 L + V_4 v = 0; \quad M_{36} + H_6 L + V_6 v = 0$$

Addition of these three equations and substitution of Eq. (6.57) leads to

$$P(0.1 + 2\rho) + \frac{1}{L} (M_{31} + M_{34} + M_{36}) = 0 \quad \text{(6.58)}$$

where $\rho = v/L$.

The joint equilibrium equations are

$$M_{31} + M_{32} = 0; \quad M_{32} + M_{34} = 0; \quad M_{34} + M_{36} + M_{36} = 0;$$

$$M_{36} + M_{36} = 0; \quad M_{36} + M_{39} = 0 \quad \text{(6.59)}$$

From the relationships given in Eq. 6.22(c) we find that

$$V_{31} = -\frac{1}{L} (M_{31} + M_{32}); \quad V_{34} = -\frac{1}{L} (M_{34} + M_{36}) \quad \text{and} \quad V_{36} - P - V_{34} = 0$$

from which

$$P + \frac{1}{L} (M_{32} + M_{31} - M_{34} - M_{36}) = 0 \quad \text{(6.60)}$$

Similarly, for span 4-6-7,

$$P + \frac{1}{L} (M_{45} + M_{46} - M_{48} - M_{49}) = 0 \quad \text{(6.61)}$$

These latter two equilibrium equations are convenient for the subsequent rigid-plastic analysis. They were obtained by omitting the terms involving the product of the axial force and the deflection in the beams.

**Rigid-plastic analysis.** The frame of Fig. 6.22(a) can fail by the three plastic mechanisms shown in Fig. 6.23: the (1) beam mechanism [Fig. 6.23(a)], (2) sway mechanism [Fig. 6.23(b)], and (3) combined mechanism [Fig. 6.23(c)]. We shall assume rigid-plastic behavior (Fig. 6.11) and that the axial loads are so small that no reduction in $M_p$ results. This assumption will be checked subsequently.

By substituting the moments from Fig. 6.23(a) into the equilibrium equations for the beams [Eqs. (6.60) and (6.61)], we obtain the following collapse load for the beam mechanism:

$$P_p = \frac{4M_p}{L} \quad \text{(6.62)}$$
From Eq. (6.60), \( P + (1/L)(M_{22} - M_{3} - M_{7} - P) = 0 \), and thus
\[
M_{12} = 3M_{F} - PL \quad (6.64)
\]

Similarly, from Eq. (6.61) we get
\[
M_{19} = 3M_{F} - PL \quad (6.65)
\]

From Eqs. (6.59), \( M_{43} + M_{46} + M_{6} = M_{p} + M_{46} + 3M_{F} - PL = 0 \), or
\[
M_{46} = PL - 4M_{F} \quad (6.66)
\]

Finally, from Eqs. (6.58) and (6.66) we find that
\[
P_F = \frac{8M_F}{L(2.1 + 2\rho)} \quad (6.67)
\]

Now we have three collapse loads [Eqs. (6.62), (6.63), and (6.67)]. The one giving the lowest value of \( P_F \) is the correct one. By introducing the nondimensional ratio \( M_F/P_FL = (2\pi)(r/d)(1/(L/r)) = (2)(1.1)(0.43)(a) = 0.01577 \), we can compare the three collapse loads:

Beam mechanism:
\[
P_F = 0.0632
\]

Sway mechanism:
\[
P_F = \frac{0.473}{1 + 20\rho}
\]

Combined mechanism:
\[
P_F = \frac{0.0601}{1 + 0.952\rho}
\]

For meaningful values of \( \rho \) the last of these will always furnish the smallest value of \( P_F \), and thus the structure of Fig. (6.22a) will fail as a combined mechanism [Fig. 6.23(c)], and
\[
P_F = \frac{0.0601}{1 + 0.952\rho} \quad (6.68)
\]

The maximum possible value of \( P_F \) is \( P_{F1} = 0.0601P_F \) when \( \rho = 0 \). This corresponds to a relatively small axial force in the columns, and \( M_F \) is not appreciably reduced by it (see Fig. 5.21). With the collapse mechanism of Fig. 6.23(c) and the failure load of Eq. (6.68), we also know the moments in the frame, that is,
\[
M_{43} = M_{46} = M_{4} = M_{6} = M_{F} \quad (6.69)
\]
\[
M_{38} = M_{34} = M_{3} = -M_{F} \quad (6.70)
\]
\[
M_{33} = -M_{34} = M_{36} = 3M_{F} - PL \quad (6.71)
\]
\[
M_{43} = PL - 4M_{F} \quad (6.72)
\]

**Last hinge analysis.** In this analysis use is made of the fact that continuity exists up to the collapse load at the joint where the last hinge forms. \(^{18}\) We do not know, however, which of the four hinges forms last, and so we shall
try each one in turn. The one which furnishes the largest deflection is the correct one.\footnote{Another method is outlined in Refs. 1.6 and 1.7 whereby any one hinge is chosen to be the last, the corresponding hinge rotations are computed, and adjustments are made for negative rotations by rigid-body rotations.}

The rotations at the ends of each member are shown in Fig. 6.24. Since allowances must be made later for hinge rotations, we have denoted separate rotations at each joint, depending on the members which frame into it (that is, at joint 2 we have rotations \( \theta_{21} \) and \( \theta_{25} \)). The slope deflection equations for any member \( ij \) are \((C = 4, S = 2, \text{ that is, no axial force effect})\)

\[
M_{ij} = \frac{EJ}{L} (4\theta_{ij} + 2\theta_{ji} - 6\rho_{ij}) \quad (6.73)
\]

\[
M_{ji} = \frac{EJ}{L} (2\theta_{ji} + 4\theta_{ij} - 6\rho_{ij}) \quad (6.74)
\]

These two equations can be solved for the rotations, that is,

\[
\theta_{ij} = \rho_{ij} + \frac{L}{6EI} (2M_{ij} - M_{ji}) \quad (6.75)
\]

\[
\theta_{ji} = \rho_{ji} + \frac{L}{6EI} (2M_{ji} - M_{ij})
\]

SEC. 6.2

The end rotations of all members in the frame are listed in Fig. 6.24 for the moments given by Eqs. (6.69) through (6.72).

No hinges occur at point 2 and between members 4-5 and 4-6 in the frame, and thus \( \theta_{21} = \theta_{24} \) and \( \theta_{45} = \theta_{46} \); therefore (Fig. 6.24)

\[
\frac{v}{L} + \frac{M_{p}L}{3EI} (-3 + \frac{P}{P_{F}}) \frac{P_{F}L}{M_{p}} = \frac{v}{L} + \frac{M_{p}L}{12EI} (7 - 2\frac{P}{P_{F}}) \frac{P_{F}L}{M_{p}}
\]

or

\[
\frac{v}{L} = \frac{M_{p}L}{12EI} \left[ -\frac{19}{12} + \frac{1}{2} \left( \frac{P}{P_{F}} \right) \left( \frac{P_{F}L}{M_{p}} \right) \right]
\]

Similarly, from \( \theta_{45} = \theta_{46} \) we obtain

\[
\frac{v_{3}}{L} = \frac{v}{L} + \frac{M_{p}L}{12EI} \left[ -\frac{23}{12} + \frac{1}{2} \left( \frac{P}{P_{F}} \right) \left( \frac{P_{F}L}{M_{p}} \right) \right]
\]

Next we assume that the last hinge forms at point 3, that is, \( \theta_{31} = \theta_{34} \).

By using the appropriate relationships in Fig. 6.24 and Eq. (6.77), we find that

\[
\frac{v}{L} = \frac{M_{p}L}{EI} \left[ \frac{11}{6} - \frac{13}{24} \left( \frac{P}{P_{F}} \right) \left( \frac{P_{F}L}{M_{p}} \right) \right]
\]

Similarly, when \( \theta_{45} = \theta_{46} \) (last hinge at 4), we obtain

\[
\frac{v}{L} = \frac{M_{p}L}{EI} \left[ \frac{3}{2} - \frac{5}{12} \left( \frac{P}{P_{F}} \right) \left( \frac{P_{F}L}{M_{p}} \right) \right]
\]

when \( \theta_{64} = \theta_{67} \) (last hinge at 6)

\[
\frac{v}{L} = \frac{M_{p}L}{EI} \left[ \frac{13}{6} - \frac{13}{24} \left( \frac{P}{P_{F}} \right) \left( \frac{P_{F}L}{M_{p}} \right) \right]
\]

and when \( \theta_{78} = \theta_{79} \) (last hinge at 7)

\[
\frac{v}{L} = \frac{M_{p}L}{EI} \left[ \frac{7}{6} - \frac{1}{4} \left( \frac{P}{P_{F}} \right) \left( \frac{P_{F}L}{M_{p}} \right) \right]
\]

The largest of the four deflections given by Eqs. (6.79) through (6.82) is the correct one.

We start the iterative determination of the load \( P_{F} \) by assuming that \( \rho = v/L = 0 \). Then from Eq. (6.68) \( P = 0.0601P_{F} \). Substituting this value of \( P \) and the constant nondimensional ratios \( M_{p}L/EI = (\sigma r/E)(2f)(L/r)(r/d) = 0.0946 \), and \( P_{F}L/M_{p} = 1/0.01577 \) successively into Eqs. (6.79) through (6.82), we find that \( v/L \) from Eq. (6.82) is the largest, that is, the last hinge forms at point 7 on the frame. The value of \( v/L = 0.0202 \) radians is thus the frame deflection which corresponds to \( P = 0.0601P_{F} \). This rotation is now substituted into Eq. (6.68), that is,

\[
\frac{P_{F}}{P_{T}} = \frac{0.0601}{1 + 0.952\rho} = \frac{0.0601}{1 + 0.952(0.0202)} = 0.0590
\]
This now represents a new value of $P$ with which we start a new cycle of computations. From Eq. (6.82) we obtain for $P = 0.0590P_r$ a new rotation, $\psi/L = 0.0220$ radians. The new collapse load $[P_r]$ from Eq. (6.68) for $\rho = 0.0220$ is equal to $0.0588P_r$. This value of $P$ is sufficiently close to $0.0590P_r$, so no further calculation is required.

From the last hinge analysis, then, we note that

$$P_f = 0.059P_r$$

Since $P_f < P_m < P_r$ (Fig. 6.17), the value of $P_m$ for the structure of Fig. 6.22(a) lies between $0.059P_r$ and $0.060P_r$. There is a very small difference between these two bounds, and therefore the maximum load is approximated with sufficient accuracy by the first-order rigid-plastic collapse load. This is usually the case for frames with low axial loads.

For frames under relatively heavy axial loads the conclusion above no longer applies. This is illustrated on Fig. 6.26 for the frame of Fig. 6.21 and the loading of Fig. 6.25(a). For this problem the rigid-plastic collapse load $P_m$ is equal to $0.325P_r$, and the load $P_f$ is less than half that; that is, $P_f = 0.153P_r$. The failure mechanism is a sway mechanism [Fig. 6.23(b)], and the last hinge forms at the point 2 in the frame. The deteriorated structure

\[\text{Fig. 6.26. Last hinge load of the frame of Fig. 6.21}\]

[shown in Fig. 6.25(b)] has a critical load equal to $0.082P_r$, which is smaller than $P_f$. Thus the best estimate of $P_m$, that is, $P_m$ (Fig. 6.19) is larger than $P_r$, and so $P_f = 0.153P_r$ is a lower bound of the maximum load.\footnote{7}

**SUMMARY OF FRAME ANALYSIS**

In the preceding parts of this section a profusion of frame reference loads were described. All these have at one time or another been used in frame analyses and an understanding of them is essential for following the literature.

\footnote{7 The details of the computation of $P_f$ are left as an exercise. In the calculations the axial force in the beams was neglected and the axial force in the three beam-columns was assumed to be $V_z = 1.250P$, $V_y = 1.000P$, and $V_x = 1.075P$. The values of $C$ and $S$ were taken to be that for the central beam-column, that is, corresponding to $1.000P$. The reduction of the plastic moment for the beam-columns was included in the calculation. The values of $M_{pl}$ for $V_z/P_r$, $V_y/P_r$, and $V_x/P_r$ could have been taken from Fig. 5.21. Instead an approximate equation was used from the AISC specifications (p. 5-64, AISC Steel Design Manual).\footnote{80} This approximate equation is $80$

$$M_{pl} = M_P \left[ 1 - G \left( \frac{V_z}{P_r} \right) \right]$$

where $B$ and $G$ are dependent on the adjusted slenderness ratio $L/\sqrt{\lambda F/35}$.}
and obtaining a general concept of planar frame behavior. Many of the suggestions advanced are an outgrowth of a desire to simplify the calculations to make them amenable to hand computation. These efforts seem now (1967) pointless and hopeless. There is no doubt that the only profitable way of analysis for complex, or even moderately complex frames is by computer. The examples chosen above were very simple and were selected to illustrate behavior. No attempt was made at achieving computational efficiency.

One of the questions in the analysis of any frame is the degree of sophistication required in the analysis. This degree of sophistication depends on the type of the frame, the answers sought, and the available computer capacity.

There are essentially five classes of planar frames, which are grouped according to the degree of required sophistication of the analysis. The first class of problem encompasses continuous beams, one- or two-story and one- or two-bay frames, and shed-type frames of one or more bays (flat-roofed, gabled, or sawtooth frames) for which the combination of plastic hinges gives relatively few and easily recognizable plastic mechanisms, for which the axial load in any member does not exceed 0.15 to 0.2 \( P_r \), and which have beam-columns with relatively low slenderness ratios (say, 30 or less). For these structures the maximum load is very nearly equal to the maximum load according to the first-order rigid-plastic analysis \( P_{pl} \) in Fig. 6.16). This load is best computed by the first-order rigid-plastic method according to the common methods of plastic analysis (Refs. 1.6 through 1.8, 3.2, and 3.9).

The validity of the statement that \( P_{pl} \) can be reached and even exceeded is impressively documented by many full-scale tests.1.41,1.42

The second class of frames, in increasing order of complexity, are the frames with low axial forces and low slenderness ratios, as above—but for which the governing plastic mechanism is not readily determined. The maximum load, however, is still approximated to a very good degree by the first-order rigid-plastic analysis load \( P_{pl} \). The most efficient way to obtain \( P_{pl} \) is to perform a first-order elastic-plastic analysis by computer. A method for this is described, and a tested computer program is published by Wang.1.10 This analysis will give the whole load-deformation curve, including the value of \( P_{pl} \) and the location of the final plastic hinges. Thus the search for the governing mechanism and the frustrations of a moment check are avoided.

The third class of frames encompasses in general the multistory frames where axial loads and secondary deflections play an important role. The most efficient analysis of such a frame is by the second-order elastic-plastic analysis, giving the whole load-deformation path up to and including the maximum load \( P_{pl} \) (Fig. 6.17). This method is preferable to the last hinge method since no prior knowledge of the controlling mechanism is required. Byproducts of such an analysis are also the order of hinge-formation and the hinge rotations.6.8 The method of analysis is necessarily iterative since the axial forces are unknown at the start. Axial stiffness reductions are used, and the equilibrium is formulated on the deformed structure. However, axial shortening and shear deformations are ignored. Methods for an efficient formulation of the equations, hints for pitfalls in programming and computation, computer programs, and fairly complex examples can be found, for example, in Refs. 1.5, 4.14, 6.8, 6.25, 6.26, and 6.27. It is expected that this type of analysis will be further refined and simplified for routine use. The studies of the frame examples have indicated that the predominant second-order effect is due to the moments from the lateral deflections at each story level. In most cases the first-order load \( P_{pl} \) would have been unacceptably unconservative.

A further refinement of analysis leads to the fourth class of frames in which axial shortening,6.8 shear deformations, residual stress,6.90 and strain hardening6.113 are taken into account. Whereas the previously discussed types of frames, analyzed by the second-order elastic-plastic method, are still possibly amenable to routine analysis as part of a design procedure, as proposed in Ref. 6.25, the types of analysis considered here are mainly research tools. In one such research study it was demonstrated that the effect of axial shortening did not significantly reduce the second-order elastic-plastic collapse load for slender single bay frames up to 15 stories high.6.8 There are still a variety of problems left to consider in the area of such refined analysis, particularly with regard to the role of shear deformations, strain-hardening, and elastic unloading of previously yielded portions of the frame.

The fifth class of problems relates to frames subjected to high repetitive loads or to dynamic loads due to bomb blast or earthquake. A very large body of literature exists on the elastic first-order analysis of such frames by computer (see, for example, Refs. 1.13 through 1.16), and work has been done or is in progress on first-order elastic-plastic analyses (Refs. 6.28 and 6.29, for example). However, second-order effects have not been incorporated yet into such analyses, although tests show the desirability of considering at least the story deflection effect.6.20

We can see that not all the answers are yet in, even for planar frames. This has been a very active area of research in the 1960's, and we can expect that the near future will bring to focus an even clearer and more complete picture of how frames behave under load.

Our discussion on the reference loads was concerned entirely with planar frames. The procedures have been extended, as far as first-order rigid-plastic theory1.8,6.10,6.20 and first-order elastic theory is concerned, to three-dimensional frames.

The examples in this article were steel structures. However, the conclusions reached are not restricted to material, as long as the assumptions regarding the \( M-\Phi \) curves (Figs. 6.11 and 6.13) are valid. Insofar as any other material, say, reinforced or prestressed concrete, fulfills these conditions, the methods outlined here can be used.6.20
Finally, the various methods of analysis should not be used indiscriminately, but the analyst should ensure that the assumptions underlying them can be reasonably expected to hold. This applies especially to those reference loads which depend on the plastic portion of the \( M-\Phi \) curves. This includes all reference loads except \( P_{E1}, P_{E2}, \) and \( P_{E3} \). If the member cannot perform as expected in the plastic range, then the limiting load must be the elastic limit load \( (P_{EL} \) or \( P_{EL}) \), which is based on a critical stress controlled by lateral-torsional or local buckling. Thus in an analysis, not only the overall frame behavior, but also the behavior of the members must be analyzed. Such procedures were given in Chapters 3, 4, and 5 previously.

**EXPERIMENTAL VERIFICATION**

Experiments play a vital role, parallel to theoretical research, in the study of structural behavior in the inelastic range. Experiments are used to verify theoretical predictions, or they serve as a starting point in formulating analytical models. We have discussed many experiments on individual members and groups of members in Chapters 3, 4, and 5. We have also noted the experimental vindication of the first-order rigid-plastic maximum load as the correct collapse load for small frames and continuous beams. We.

Experimental verification of the theoretically predicted behavior of multi-story frames is, on the other hand, extremely complex, costly, and mostly impractical. We have to take on faith the assumption that the more easily predictable behavior of the individual parts also applies to the whole frame.

Despite the immense difficulties involved, there have been some experiments which showed that the second-order elastic-plastic analysis correctly predicts the behavior of frames under relatively large vertical loads. These tests were performed at the Fritz Engineering Laboratory of Lehigh University during 1964 and 1965. Laterally braced frames of steel were tested to collapse under heavy axial loads and horizontal loads simulating a wind load. The bay width was 15 ft and the story height 10 ft. Three-story one-bay and three-story two-bay frames were tested. The experimental setup for one of these frames is shown in Fig. 6.27. The test frame is the single frame within the outer supporting frame. Also visible are the loading beams, the jacks, and the vee-shaped frames used for assuring vertical loading. This particular test frame is braced by diagonal bracing.

This series of tests demonstrated that not only can the second-order elastic-plastic collapse load be closely predicted but that the load-deformation path can also be reproduced with remarkable accuracy. This is illustrated in Fig. 6.29 for the frame of Fig. 6.28, which was a fixed base portal frame loaded by heavy vertical loads to simulate the loads from stories above. The horizontal load \( H \) was applied after the axial loads, and the solid lines connecting the circles in Fig. 6.29 represent the relationship between the applied force \( H \) and the experimentally observed deflection \( \Delta \) of the top of the frame. The dashed lines indicate the theoretical prediction by second-order elastic-plastic theory. After the attainment of the maximum load it was necessary to consider strain hardening in the analysis. The frame could not support the load according to the first-order rigid-plastic theory, but the more refined theory gave excellent correlation with the experimental curve.
amount of literature dealing with experimental behavior and theoretical prediction is in existence. Connections are highly statically indeterminate, and the distribution of the forces within them depends upon the relative deformations of the component parts and the fasteners. In most cases it is almost impossible to perform a rigorous analysis. The usual analyses are based on many simplifying assumptions, and local yielding is permitted to facilitate the attainment of a statically admissible stress system. These analyses and design procedures rest heavily on extensive experimental background and successful and unsuccessful past experience.

The design of connections is treated extensively in most texts of structural steel design (see, for example, Refs. 1.19 and 6.37 through 6.39). These texts also give many additional references on the research on connections. We shall not discuss these design methods nor the research background here; we should note, however, that connections have many buckling and instability problems similar to the ones we encountered with beams, columns, and beam-columns (e.g., lateral-torsional buckling and plate buckling, both in the elastic and the inelastic range).

The performance of connections under load has an effect on the overall behavior of the frame. The methods of frame analysis discussed in this chapter assume that the connections are fully rigid, that is, the original angles between two or more members joined together by the connection remain virtually unchanged. This means that general yielding, characterizing a plastic hinge, will form outside of the connection and in the members. For these frames the connection is the stronger link in the member-joint assembly. Design methods based on this premise have been developed, and these have been experimentally verified (for background see Refs. 1.6, 1.41, 5.47, and 6.37).

With these rigid connections, which are designed to be stronger than the members they connect, we can apply the rigid-elastic and elastic-plastic second-order analyses to determine the collapse loads. However, if the connections are not designed to be fully rigid or if they cannot support the forces resulting from the full plastification of the connected members, these methods of analysis cannot be applied without modification. The limiting load is \( P_{25} \) or \( P_{20} \) (Fig. 6.17) if the connections are fully rigid only up to the elastic limit. If the connections are not fully rigid, as for example, the AISC Type 2 (simple framing) and Type 3 (semirigid framing) construction,\(^{(6.30)}\) then the degree of rigidity of the connections must be considered in the analysis. The force distribution in the elastic range will be affected and the stability of the frame is influenced. Methods for determining the moment distribution of frames with semirigid connections are given by Lothers,\(^{(6.39)}\) and frame buckling loads for semirigid and simple framing are discussed by Driscoll\(^{(6.40)}\) and DeFalco and Marino,\(^{(6.41)}\) respectively. The degree of fixity of the column bases also can affect the frame buckling loads.\(^{(6.42)}\)

In addition to the testing of full-scale structures, it is also possible to use small-scale models to study multistory frame behavior.\(^{(6.18,6.38)}\)

**CONNECTIONS**

The discussion of frame behavior would not be complete without a few words about connections. The members of a structure may be designed with the utmost refinement, but if little attention is paid to connections or if they are incorrectly designed, the structure as a whole is poorly designed.

The behavior of connections under loads is very complex, and a vast
This brief discussion was intended to point to the importance of connections in analyzing the behavior of frames. Connections are of equal importance to the other parts of the frame, and they deserve a special and detailed study.

6.3. FRAME DESIGN

INTRODUCTION

In the previous article we dealt with the analysis of the strength of frames for which we knew all the dimensions and properties of the frame as well as the character of the loads. We were interested in finding an estimate of the maximum intensity of the loading. In design the process is reversed. The overall dimensions as well as the maximum intensity of the loads are known, and we want to find the frame dimensions (more specifically, the sizes and the properties of the individual members) of the frame which can safely and economically support the given loads.

The usual process of frame design is one of trial and error; the frame properties are assumed, and an analysis is performed to see whether the chosen frame fulfills the conditions of the design. Adjustments are then made in the frame until the designer, in his judgment, is satisfied that the frame which he designed performs in the manner that the design conditions require.

In the following discussion we shall not be able to discuss very thoroughly the various steps of frame design. This is a separate and detailed study covered in texts on structural design.

We shall concern ourselves only with a brief review of some of the most important design procedures.

The design of frames proceeds roughly as follows:15-47

1. Determination of the overall geometry and loading.
2. Estimation of the initial sizes of the members in the frame from rough approximations (preliminary design).
3. Determination of the final member sizes to carry the expected frame loads by repeated analyses.
4. Checking and adjustment of the member sizes to ascertain that the assumptions on which the analyses in step 3 were based are indeed fulfilled (lateral-bracing spacing, width-thickness ratios, etc.).
5. Design of details (lateral bracing, connections, etc.).

In the previous discussion in this chapter we dealt with step 3 above. The topics in Chapters 3, 4, and 5 considered step 4 in the design process.

The three most prevalent design procedures are: (1) plastic design, (2) ultimate strength design, and (3) allowable stress design. We shall turn our attention briefly to the way in which these procedures are used and prescribed in specifications. This discussion will be restricted to the design of steel framed structures for buildings. In the United States the design of such buildings is controlled by the Specifications for the Design, Fabrication and Erection of Structural Steel for Buildings of the American Institute of Steel Construction (AISC specifications, Ref. 1.20). Parallels from other specifications in the United States and other countries, as well as parallels from specifications for concrete and aluminum structures, will be readily apparent to the readers familiar with such specifications. We shall consider here only the overall design of the frame.

PLASTIC DESIGN

In plastic design (or limit design, maximum load design, or collapse load design) it is assumed that the structural frame will ultimately form a mechanism according to the rigid-plastic or the elastic-plastic theory. That is, the \( M - P \) relationship of the members can be approximated by the curves given in Fig. 6.11 or 6.13. Under certain conditions both steel and concrete structures can be assumed to fulfill these conditions. For example, if the provisions with respect to bracing spacing and width-thickness ratios which were given in Sec. 3.3 for steel beams are met, steel structures can be designed by plastic design. Thus if the designer decides, at the outset, that he will assure that the assumptions of plastic theory are to be fulfilled, then the frame can be designed by the plastic method.

Plastic design then proceeds by assuming the frame dimensions in a preliminary analysis and by computing the reference load (or loads) as discussed in Sec. 6.2. The best estimate of \( P_X \) is to be obtained, consistent with the available time and computer (\( P_{FR}, P_{X}, \text{ or } P_X \)). The obtained maximum load is then divided by the working load to get the load factor, that is, \( P_x/P_{FR} = L.F. \). If the load factor is slightly more than or equal to the required load factor according to the specification, the design is satisfactory.8 If it is not, a new design is tried. The final structure, to be satisfactory, must be checked also to fulfill the requirements for local effects (bracing spacing, local buckling, etc.).

In the 1963 AISC specifications plastic design is permitted for continuous beams and for one- and two-story rigid frames only.5 For such frames the second-order deflection effects are negligible (see example of Fig. 6.22), and the design may be based on the first-order rigid-plastic load \( P_{FR} \) (Fig. 6.17). Basic theorems (upper- and lower-bound theorems),12 analysis proceeds.

8 The load factors required by the AISC specifications (Ref. 1.20 Part 2) are 1.70 for continuous beams, 1.85 for frames under vertical loads only, and 1.40 for frames under combined vertical (gravity) and horizontal (wind or earthquake) loads.

5 This limitation was set because of the lack of adequate research data on frames with heavy axial loads. Since the appearance of the specifications (1963) much of this new research has been completed.5,45
dures (stistical method, mechanism method, moment balancing procedures), optimization procedures, experimental verifications, and design aids, for such steel structures are amply available, and thus the plastic design of one- and two-story rigid frames can be very rapid.

Research since the early 1950's has concentrated on the analysis and design of frames where it is no longer possible to base the design on \( P_{a1} \) (Refs. 5.47, 6.1, 6.11, 6.14, 6.15, 6.16, 6.18, for example). For such frames the load factor is computed from either \( P_{a2} \) or \( P_{a1} \). A variant to this procedure is suggested in Ref. 5.47, whereby the frame is subdivided into subassemblages consisting of one column and all the beams framing its ends, and then each subassemblage is designed to just collapse under the loads acting on it (see Refs. 5.45 through 5.49).

**ULTIMATE STRENGTH DESIGN**

This name for a design method is actually a misnomer, because it is not based on the ultimate or maximum load of the frame, but on the load at which the first hinge forms in the structure (\( P_{a2} \) or \( P_{a1} \) in Fig. 6.17). The design is thus based on the ultimate strength of the section and not the frame. The name derives from concrete design, where full plastification is assumed to take place at the point of maximum moment in the frame. The moment distribution in the frame is determined by a first-order elastic analysis, and thus the design is based on \( P_{a1} \) (Fig. 6.17). For concrete frames, where the members are usually quite stocky, this is a very reasonable procedure. Whereas the design here is based on a lower-bound estimate of the maximum load—and thus a heavier frame results—the stringent requirements of plastic design with respect to lateral-torsional and local buckling need not be met (no hinge rotations).

Ultimate strength design for steel buildings is not explicitly permitted by the AISC specifications, but it is implicitly allowed in the design of beams (Sec. 1.5.1.4.1 in Ref. 1.20). The allowable maximum bending stress \( \sigma_{sa} \) is computed as follows:

\[
M_{\max} = \frac{P_y}{S(F.S.)}
\]

\[
M_{\max} = \frac{\sigma_{sa} - M_p}{S(F.S.)} = \frac{Z \sigma_{fx}}{S(F.S.)} = \frac{f \sigma_{fx}}{S(F.S.)}
\]

If the shape factor is \( f = 1.12 \) (median value for wide-flange shapes) and the factor of safety is F.S. = 1.70, then \( \sigma_{sa} = 0.66 \sigma_{fx} \), as allowed by the AISC. Further stipulations with respect to lateral bracing and width-thickness ratios, as well as permission of a partial redistribution of moment, make this provision a mixture of both plastic and ultimate strength design.

**ALLOWABLE STRESS DESIGN**

The load factor for this design method is based on the second-order elastic limit load \( P_{a2} \) (Fig. 6.17) which is computed as the load corresponding to the initiation of yielding (\( M_{\max} = \frac{P_y}{S(F.S.)} \)) and not to the load corresponding to the formation of the first hinge (\( M_{\max} = \frac{P_y}{Z \sigma_{fy}} \)). The limiting condition is usually expressed as an allowable stress, \( \sigma_{fy}/F.S. \) or a critical stress (lateral-torsional or local buckling) \( \sigma_{cr}/F.S. \), whichever is smaller.

If the design condition is \( P_{a2}/F.S. = \sigma_{fy} \), and \( P_{a2} \) is computed by second-order analysis, the resulting design would always be conservative with respect to the true value of \( P_{a2} \). In actual design practice it is not usually customary to perform a second-order analysis of the assumed structure, but only a first-order analysis is made. Thus the design is really based on \( P_{a1} \) (Fig. 6.17). For frames with small axial loads \( P_{a1} \leq P_{a2} \) and so no harm results. However, if the axial load is relatively large, \( P_{a1} \) is not necessarily conservative with respect to the actual maximum load of the frame.

The following procedure is used to compensate for this shortcoming: (1) The moments, shears, and axial forces in the frame are determined by an elastic first-order analysis. (2) The beams are checked to insure that \( \sigma_{sa} \leq \sigma_{fy}/F.S. \) or \( \sigma_{sa} \leq \sigma_{cr}/F.S. \). (3) The beam-columns are checked to assure that (Sec. 1.6.1 in Ref. 1.20)

\[
\frac{\sigma_{fy}}{F.S.} + \frac{\sigma_{sa}}{S(F.S.)} \leq 1.0
\]

and

\[
\frac{\sigma_{sa}}{S(F.S.)} + \frac{C_{sa}}{S(F.S.)} \leq 1.0
\]

In any case both Eqs. (6.83) and (6.84) must be satisfied. The terms in these equations are

\[
\sigma_{sa} = \frac{P}{A}
\]

\[
\sigma_{sa} = \frac{M_{\max}}{S}
\]

\[
\sigma_{sa} = \frac{P_{a2}}{F(S)}
\]

where \( P_{a2} \) is the critical axial load for the beam-column in the absence of bending moment, that is [Eqs. (4.203) and (4.23)],

\[
P_{a2} = A \sigma_y \left[ 1 - \frac{\sigma_{fy}(K_s L/r_y)^2}{4 \pi^2 E} \right] \quad \text{for} \quad \frac{K_s L}{r_y} \leq \sqrt{\frac{2 \pi^2 E}{\sigma_{fy}}} \quad (6.88)
\]

\[
P_{a2} = A \sigma_y \left[ 1 - \frac{\sigma_{fy}(K_s L/r_y)^2}{4 \pi^2 E} \right] \quad \text{for} \quad \frac{K_s L}{r_y} \leq \sqrt{\frac{2 \pi^2 E}{\sigma_{fy}}} \quad (6.89)
\]
\[ P_{cr} = \frac{\pi^4 A E \sigma_y}{(K_e L/r_x)^3} \quad \text{for} \quad \frac{K_e L}{r_x} > \sqrt{\frac{2\pi^2 E}{\sigma_y}} \] (6.90)

\[ P_{cr} = \frac{\pi^4 A E \sigma_y}{(K_e L/r_y)^3} \quad \text{for} \quad \frac{K_e L}{r_y} > \sqrt{\frac{2\pi^2 E}{\sigma_y}} \] (6.91)

In any situation one pair of equations must be checked (buckling about the x or y axis), and the smaller value of \( P_{cr} \) is inserted in Eq. (6.87). The first two of these equations [Eqs. (6.88) and (6.89)] are the CRC basic column strength formula for inelastic buckling [Eq. (4.203)] and the second two equations are for elastic buckling. The effective length factors in the plane of bending (\( K_e \)) and out of the plane of bending (\( K_e \)) are determined by the methods discussed in Sec. 4.3 or by Fig. 4.19.\(^{1,9,10}\) The inclusion of the effective length factors assures that the resulting limiting load will not exceed \( P_e \) (Fig. 6.16).

Continuing now with the terms in Eqs. (6.83) and (6.84),

\[ \sigma_{es} = \frac{M_{cr}}{S(F.S.)} \quad \text{or} \quad \sigma_{ne} = \frac{M_{cr}}{S(F.S.)} \] (6.92)

\[ \sigma_n = \frac{P_e}{A(F.S.)} \] (6.93)

where

\[ P_e = \frac{\pi^4 A E \sigma_y}{(K_e L/r_x)^3} \] (6.94)

Finally, \( C_x^* = 0.85 \) for frames which can sway (no diagonal bracing or shear walls) and for braced frames

\[ C_x^* = 0.6 + 0.4\kappa \leq 0.4 \] (6.95)

where \( \kappa \) is the ratio of the smaller to the larger end moment (Fig. 5.2).

For the case of in-plane bending \( K_e = 0 \) and \( K_e = 1.0 \), Eqs. (6.83) and (6.84) reduce to Eqs. (5.42) and (5.41), respectively, if F.S. = 1.0. Thus these two equations represent, for the limiting conditions stated above, the approximate expressions derived in Sec. 5.2 for the elastic limit load of beam-columns loaded by terminal couples. But the equations in the AISC specifications are actually much more than beam-column interaction equations. They are designed to prevent frame instability (in and out of the plane of bending through \( K_e \) and \( K_e \)) and second-order bending [through the term \( C_x^*/(1 - \sigma_n/\sigma_y) \)]. There is no need to apply them if the design load is based on a second-order elastic analysis. However, for a first-order analysis, these equations provide an empirical basis for accounting for a variety of effects. There is no really rational way in which the resulting design load can be tied in with the second-order elastic limit load \( P_{cr} \). But it has been shown for a variety of examples that the resulting structure is conservatively designed.\(^{1,9,10,11}\)

The factor of safety used in connection with Eqs. (6.83) and (6.89) are: In Eq. (6.83), F.S. = 1.67; in Eq. (6.87), F.S. = 5/3 + (3/8\( C_x \))(KL/r) - (1/8C_x^2)(KL/r)^3, where \( C_x = \sqrt{2\pi^2 E/\sigma_y} \) for \( KL/r < \) \( C_x \) and F.S. = \( \frac{4}{2} \) = 1.92 for \( KL/r \geq \) \( C_x \); In Eq. (6.92) F.S. = 1.67; finally in Eq. (6.93), F.S. = \( \frac{4}{2} \) = 1.92. The provisions of the AISC specifications (Ref. 3.52) and the Aluminum Specifications (Ref. 3.53) are similar to those of the AISc, but with different factors of safety. The German and British specifications are also similar, but in the former a second-order analysis is permitted as an alternate procedure.

The allowable stress design provision of the AISc is not, as we have seen, a pure elastic limit analysis, but a mixture of a variety of things, which cannot be explicitly tied down to the actual maximum strength of the structure except by direct comparison of two computed loads (that is, \( P_{cr} \) compared with as good an estimate of \( P_e \) as we can get). This does not detract, however, from the usefulness of the method. Engineers are used to it and have applied it for decades to successfully design not only small but truly monumental structures (for example, the Empire State Building).

DESIGN FOR DEFLECTIONS

The three design procedures discussed above (plastic design, ultimate strength design, and allowable stress design) are based on an estimate of the strength of the structure. In all structures it is also necessary to limit deflections under working loads to certain specified maximum values. The design of a structure to limit deflections is a separate design problem which may often result in the controlling design. The deflection analysis is usually made on the assumption of an elastic structure.

THE AISC ALLOWABLE LOAD FOR THE FRAME OF FIG. 6.8.

We shall now determine the allowable load according to the AISC specifications for the frame of Fig. 6.8. We recall from Sec. 6.2 and Fig. 6.15 that for this frame \( P_{cr} = P_{cr} = 0.332P_y \), and that this load is a reasonable estimate of the actual maximum load. Only in-plane effects are considered in the analysis of this frame, and therefore the allowable stress in bending is equal to \( \sigma_{es} = \sigma_y/F.S. \), or \( \sigma_{es} = 50/1.67 = 30 \) ksi.

The effective length factor \( K_e \) can be obtained from Sec. 4.3 (Fig. 4.19). In this figure \( G = (2I_l/L_x)(2I_l/L_y) \). For the top of the beam-columns, \( G = (I/l)(2l/2l) = 1.0 \). For the bottom of the beam-columns \( G = (I/l)/(\infty/2l) = 0 \), if we set the stiffness of the fixed base foundation equal to infinity.\(^{11}\) From Fig. 4.19 we find that \( K_e = 1.15 \) for \( G = 1.0 \) and \( G = 0 \). The effective slender ratio is then \( K_e L/r = (1.15)(60) = 69.0 \). Instead of

\(^{11}\) In the AISC specifications (p. 5-118 in the AISC manual) it is recommended that for fixed bases \( G = 1.0 \) be used to account for the fact that in practice it is almost impossible to construct truly fixed bases. In this case \( K_e = 1.31 \), giving a more conservative final answer than that which is obtained above.
using Eqs. (6.87) through (6.91) and Eqs. (6.89) and (6.90) to compute \( \sigma_{ea} \) and \( \sigma_{es} \), we shall use the values for these stresses from the AISC manual. From page 5-92 of this manual we find that

\[
\sigma_{ea} = 21.12 \text{ ksi} \quad \text{or} \quad \frac{\sigma_{ea}}{\sigma_y} = \frac{21.12}{50} = 0.422
\]

and from page 5-93,

\[
\sigma_{es} = 31.32 \text{ ksi} \quad \text{or} \quad \frac{\sigma_{es}}{\sigma_y} = \frac{31.32}{50} = 0.626
\]

Since the frame of Fig. 6.8 can sway, use \( C_s^* = 0.85 \).

The two interaction equations [Eqs. (6.83) and (6.84)] can now be written as

\[
\frac{\sigma_z}{\sigma_y} + \frac{\sigma_y}{\sigma_y} = 1.0
\]

or

\[
\frac{\sigma_z}{\sigma_y} + \frac{\sigma_y}{\sigma_y} = 0.6
\]

(6.96)

and

\[
\frac{0.85(\sigma_z/\sigma_y)}{0.422 + (30/50)(1 - \sigma_z/0.626\sigma_y)} = 1.0
\]

or

\[
3.97 \left( \frac{\sigma_z}{\sigma_y} \right) + 1.42 \left( \frac{\sigma_y}{\sigma_y} \right) = 1.0
\]

(6.97)

The values of \( \sigma_z \) and \( \sigma_y \) in Eqs. (6.96) and (6.97) are for the maximum moment and axial force obtained by a first-order elastic analysis. From Eqs. (6.22) and (6.23) we already know that the moments at the top and at the bottom of the two beam-columns are equal to \( M_{Ab} = M_{Bp} = -0.0214PL \) and \( M_{Ab} = M_{Bp} = -0.0286PL \) according to such an analysis. The maximum moment thus occurs at the bottom of these members, and, therefore,

\[
\frac{\sigma_y}{\sigma_y} = \frac{0.0286PL}{S\sigma_y} = 0.0286 \frac{P}{P_y} \frac{dL}{2L} = 1.995 \frac{P}{P_y}
\]

where \( P \) is the load on the frame as shown in Fig. 6.8. The maximum axial load occurs in the leeward beam-column (member CD), and it is equal to [Eq. (6.4)]

\[
R_B = P \left( 1 + \frac{0.1}{2} \right) + \frac{M_{Ab} + M_{Bp}}{2L} = 1.021P
\]

Thus

\[
\frac{\sigma_z}{\sigma_y} = \frac{R_B}{2L} = \frac{R_B}{2L} = 1.021 \frac{P}{P_y}
\]

If we now set \( \frac{\sigma_z}{\sigma_y} = 1.021 \left( \frac{P}{P_y} \right) \) and \( \frac{\sigma_y}{\sigma_y} = 1.995 \left( \frac{P}{P_y} \right) \) into Eqs. (6.96) and (6.97), we can compute the allowable AISC load, that is, from Eq. (6.96),

\[
1.021 \frac{P}{P_y} + 1.995 \frac{P}{P_y} = 0.6, \quad \text{or} \quad \frac{P}{P_y} = 0.199
\]

and from Eq. (6.97)

\[
3.97 \left( 1.021 \frac{P}{P_y} \right) + 1.42 \left( 1.995 \frac{P}{P_y} \right) = 1.0, \quad \text{or} \quad \frac{P}{P_y} = 0.145
\]

The smaller value from these two equations is \( P \) corresponding to Eq. (6.97), or

\[
P_{AISC} = 0.145P_y
\]

The factor of safety with respect to the maximum load is now F.S. = 0.332/0.145 = 2.29. If we assume that the horizontal load is due to wind, then the AISC specifications permit an increase of \( \frac{1}{4} \) in the allowable stresses \( \sigma_{ea} \), \( \sigma_{es} \), and \( \sigma_y \) and then \( P_{AISC} = (0.145P_y) (\frac{3}{4}) = 0.194P_y \). The factor of safety is then F.S. = 0.332/0.194 = 1.71. This is a reasonable factor of safety, although if it is compared with the factor 1.40 permitted in the AISC specifications for plastic design, it seems somewhat high.

6.4. SUMMARY

In this chapter we first discussed methods whereby the actual strength of frames can be determined by a semigraphical method. The method is involved for even relatively simple frames, and thus we turned our attention to the determination of various reference loads by which the actual maximum load can be estimated. An upper bound of the strength is obtained if the concepts of rigid-plastic or elastic-plastic theory are used, and a lower-bound estimate results if we use the elastic limit load and second-order elastic theory as the basis for determining the reference load. Finally three design methods were examined in the light of the analysis procedures discussed earlier. The AISC specifications for steel buildings were used as the basis of this discussion.

REFERENCES


**PROBLEMS**

6.1. Work through the details of the example presented in the text in connection with Fig. 6.8.

6.2. Work through the details of the example presented in the text in connection with Fig. 6.21.

6.3. Review Refs. 6.1, 6.2, and 6.11.

6.4. Determine the frame reference loads described in Sec. 6.2 and derive and plot the various appropriate load-deformation curves for the following frames. In each case compute also the allowable load according to the AISC specifications, noting that the horizontal loads are due to wind and therefore the AISC permits a 1/3 increase in the allowable stresses.

   For each of these three frames the vertical load remains constant while the horizontal load increases from zero. The data are sufficient to work the problems in a completely nondimensional manner. The load parameter is thus \( H/P_r \), where \( P_r = A \sigma_r \), and the deformation parameter is \( \Delta/L \), where \( \Delta \) is the horizontal deflection of the top of the frame and \( L \) is the story height. In each problem assume that axial shortening is negligible, the axial load in the horizontal members is small so that \( C = 4 \) and \( S = 2 \), hinges form at a point and the reduced plastic moment is given by Eq. (6.34). The shape factor \( f = Z/S = 1.100 \), \( r_o/d = 0.43 \), and \( E = 30,000 \) ksi in each problem. Consider only in-plane behavior.

   Discuss the results and the effects of the assumptions for each problem.
During the erection of a building when the partially erected frame looks like that above, it becomes desirable to attach a load $P$ to the lower flange of each girder. This load is applied at midlength of the girder, and it passes through the plane of the web (causing no torsion). Using a safety factor of 1.2 against collapse, determine the maximum force $P$ which the frame can carry. All joints are rigid, offering full restraint against bending and warping deformations. The webs lie in the $ABCD$ plane for the beams and columns and in the $AA'C'C'$ plane for the girders. Members are of A36 steel, and the sizes are as follows:

- Beams: 12 WF 40
- Columns: 12 WF 65
- Girders: 27 WF 94

Insure also that $P$ will cause no yielding (permanent deformation). This problem will present the student with a considerable latitude in exercising his judgment.

### Answers to Problems

**Problem 2.1**

$$
\sigma = \frac{P}{A} + \left( \frac{Mz_{xz} - Mz_{yz}}{I_{xz} - I_{yz}} \right)x + \left( \frac{Mz_{yz} - Mz_{xy}}{I_{yz} - I_{xy}} \right)y
$$

$$
\tau = \frac{1}{I_{xy} - I_{xz}} \left[ V_x \left( I_x \int_0^x y t \, ds - I_{xy} \int_0^x x t \, ds \right) + V_y \left( I_y \int_0^y x t \, ds - I_{xy} \int_0^y y t \, ds \right) \right]
$$

**Problem 2.2**

$$
x_0 = \frac{I_{xz} - I_{xy}}{I_{xz} - I_{xy}}; \quad y_0 = \frac{I_{xy} - I_{xz}}{I_{xy} - I_{xz}}
$$

**Problem 2.5(a)** Shear center is located 1.84 in. to the left and 3.14 in. above the intersection of the 10-in. and the 20-in. plate.

**Problem 2.5(b)** Shear center is located $\sqrt{3} \, a/6$ below horizontal plate.

**Problem 2.5(c)** Maximum value of $\omega = \pi^2; \quad I_0 = 7d^2t/12$. 

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Problem 2.5(d) Shear center is located 23d/78 to the left of the web.

Problem 2.5(e)

\[
I_w = \pi r^4 \left( \frac{2\pi^2}{3} - 4 \right)
\]

Maximum value of \( \omega_n = \pi r^4 \) and of \( S_w = 1.70r^4t \).

Problem 2.5(f) Shear center is located d/11 above cross section.

Maximum value of \( \omega_n = \pm 0.4545d^2 \) and it occurs at the lower edge of the webs.

Maximum value of \( \int y^2 ds = 8d^2t/25 \) and it occurs at the intersection of the \( y \) axis with the webs.

Maximum value of \( S_w = 25d^3t/121 \) and it occurs 0.909d from the lower edge of the webs.

\[
I_w = 0.1439d^4t
\]

Problem 2.5(g)

\[
I_w = 827,000 \text{ in.}^4
\]

Problem 2.6(a)

\[ m_s = 0.679 \text{ kip-in./in.} \]

Problem 2.8

\[ q_a = 0.621 \text{ kip/in.} \]

Problem 2.9 \( \sigma_{\text{max}} = 0.0205Q \) at the center of the beam and at the lower edge of the left web.

Problem 2.10

\[ q = 0.0722 \text{ kip/in.} \]

Problem 2.11 \( \sigma_{\text{max}} = 500w \) at edge of web at interior support.

\[ \tau_{\text{max}} = 180w \] at center of gravity of section on inside face of interior support.

Problem 3.4(a)

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7320 kip-in.</td>
<td>7700 kip-in.</td>
<td>8360 kip-in.</td>
</tr>
</tbody>
</table>

(c) 7668 kip-in.

e) 7180 kip-in. with 3 spaces

7345 kip-in. with 4 spaces

7606 kip-in. with 10 spaces

7650 kip-in. by extrapolating solution for 4 and 10 spaces

Problem 3.5

(a) \( \Pi = \frac{1}{2} \int_0^L [E(Iu''')^2 + GK_xu^3 + EI(u')^3 + q(Lx - z^2)u''u'] dz \)

(b) \[ \frac{q_a L^5}{8} = 1.15 \sqrt{EI_GK_L \left( 1 + \pi^2 \frac{EI}{GK_xL^3} \right)} \]

Problem 3.6 Same as Eq. 3.42.

Problem 3.14

\[ P_{cr} = 23.0 \text{ kip} \]

Problem 4.1(a)

For \( L/r_e = 100 \) and 14WF142 section; \( P_{cr} = 0.353A\sigma_f \)

For \( L/r_e = 40 \) and 27WF94 section; \( P_{cr} = 0.199A\sigma_f \)

Problem 4.1(b)

For \( L = 50 \text{ in.} \); \( P_{cr} = 820 \text{ kip} \)

Problem 4.1(c)

For \( L = 144 \text{ in.} \); \( P_{cr} = 0.614A\sigma_f \)

Problem 4.1(d)

For \( L/r_{\text{max}} = 40 \); \( P_{cr} = 0.172A\sigma_f \)

Problem 4.3(a) Lowest nonsway buckling mode is obtained from the following equation:

\[ \lambda = -9.47\alpha \]

Lower sway buckling mode is obtained from the following equation:

\[ \lambda = -\frac{\alpha}{12(\beta - \alpha)}(3\beta - 2\alpha + \sqrt{3\beta^2 - 6\alpha\beta + 4\alpha^2}) \]

where

\[ \alpha = \frac{C^2 - S^2}{C} \]

\[ \beta = \frac{PL^3}{EI_c} \]

Problem 4.4

\[ A = 0.423 \text{ in.}^2 \]

Problem 4.5

(a) \( K = 1.59 \)

(b) \( K = 1.65 \)

(c) \( K = 2.70 \)

Problem 4.6(a)

For \( 0 \leq \frac{P}{P_T} \leq 0.5 \)

\[ \lambda_{zE} = \lambda_{xE} = \frac{1}{\sqrt{P/P_T}} \]

For \( 0.5 \leq \frac{P}{P_T} \leq 0.909 \)

\[ \lambda_{xT} = \lambda_{zT} = \frac{1}{\sqrt{P/P_T}} \]

\[ \lambda_{xT} = 0.970 \]

\[ \lambda_{zT} = 0.975 \]
Problem 4.6(b)

\[ \lambda = \frac{1}{\sqrt{P/P_r}} \quad \text{for} \quad 0 \leq P/P_r \leq \frac{1}{4} \]
\[ \lambda = 0.750 \quad \text{for} \quad \frac{1}{4} \leq P/P_r \leq \frac{5}{8} \]
\[ \lambda = 0.707 \quad \text{for} \quad \frac{5}{8} \leq P/P_r \leq \frac{7}{8} \]
\[ \lambda = 0.25 \quad \text{for} \quad \frac{7}{8} \leq P/P_r \leq 1.0 \]

Problem 5.2(a)

\[ v = \frac{q_0 L}{P} \left( \frac{z}{L} \right) \left( -\frac{1}{P^2 L} - \frac{1}{6} \right) + \frac{\sin F_z z}{P^2 L^3 \sin F_y L} + \frac{1}{6} \left( \frac{z}{L} \right)^3 \]

\[ M = \frac{q_0 L}{P} \left( \frac{\sin F_z z}{P^2 L} - \frac{z}{L} \right) \]

where \( z \) originates at left end of beam.

Problem 5.4 Yielding occurs first when \( M_o = 891 \) kip-in.

Problem 5.5(a) \( M_o = 9,500 \) kip-in.

(b) No

Problem 5.6

\[ P_{\text{max}} = 3.2 \text{ units} \]

Problem 5.7(a)

\( (M_o)_{\text{max}} = 1,250 \) kip-in.

Problem 6.4(a) \( H_{\text{max}} = 0.0239 P_r \) according to best maximum strength estimate.

\[ H_{\text{max}} = 0.0122 P_r \] according to AISC and assuming that under working conditions \( P = 0.26 P_r / 1.3 = 0.20 P_r \).

Problem 6.4(d) \( H_{\text{max}} = 0.0173 P_r \) according to best maximum strength estimate.

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