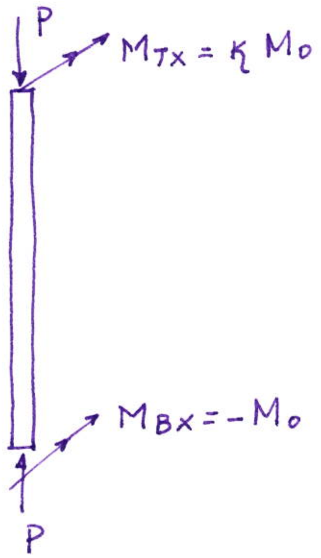
SLOPE-DEFLECTION FOR BEAM-COLUMNS



$$EI\,v^{iv} + Pv'' = 0$$

$$v^{iv} + F_v^2\, v'' = 0 \,.$$

$$\therefore v = \frac{M_0}{P} \times \left[\frac{\kappa - \cos F_v L}{\sin F_v L} \cdot \sin F_v z + \cos F_v z + \frac{z}{L}(1-\kappa) - 1 \right]$$

$$\therefore v = \frac{M_0}{P}\left[\frac{\kappa}{\sin F_v L} \cdot \sin(F_v z) - \frac{\kappa z}{L} \right] - \frac{M_0}{P}\left[\frac{\sin F_v z}{\tan F_v L} - \cos F_v z - \frac{z}{L} + 1 \right]$$

Let $\boxed{M_A = M_0}$ & $\boxed{M_B = -\kappa M_0}$ $\boxed{F_v \longrightarrow k = \sqrt{P/EI}}$

bottom top

$$F_v L = kL = \pi\sqrt{\frac{P}{P_{cr}}}$$

transformation of variables

$$\therefore v = \frac{-M_B}{EI k^2}\left[\frac{\sin(kz)}{\sin(kL)} - \frac{z}{L} \right] - \frac{M_A}{EI k^2}\left[\frac{\sin kz}{\tan kL} - \cos kz - \frac{z}{L} + 1 \right]$$

Take one derivative:

$$v' = -\frac{M_B}{EI k^2}\left[\frac{\cos(kz)}{\sin(kL)} - \frac{1}{kL} \right] - \frac{M_A}{EI k}\left[\frac{\cos kz}{\tan kL} + \sin kz - \frac{1}{kL} \right]$$

We realize that:

$\theta_A = v'$ @ $z = 0$ and $\theta_B = v'$ @ $z = L$

$$\therefore \theta_A = -\frac{M_B}{EIk}\cdot\left[\frac{1}{\sin kL} - \frac{1}{kL}\right] - \frac{M_A}{EIk}\cdot\left[\frac{1}{\tan kL} - \frac{1}{kL}\right]$$

$$\theta_A = \frac{M_B \cdot L}{EI}\left[\frac{\sin kL - kL}{(kL)^2 \sin kL}\right] + \frac{M_A L}{EI}\left[\frac{\sin kL - kL\cos kL}{(kL)^2 \sin kL}\right]$$

Similarly,

$$\theta_B = \frac{M_B \cdot L}{EI}\left[\frac{\sin kL - kL\cos kL}{(kL)^2 \sin kL}\right] + \frac{M_A L}{EI}\left[\frac{\sin kL - kL}{(kL)^2 \sin kL}\right]$$

$$\therefore \begin{Bmatrix}\theta_A\\ \theta_B\end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix}\begin{Bmatrix}M_A\\ M_B\end{Bmatrix}$$

$$f_{11} = f_{22} = \frac{L}{EI}\left[\frac{\sin kL - kL\cos kL}{(kL)^2 \sin kL}\right]$$

$$f_{12} = f_{21} = \frac{L}{EI}\left[\frac{\sin kL - kL}{(kL)^2 \sin kL}\right]$$

$$\therefore \begin{Bmatrix}M_A\\ M_B\end{Bmatrix} = \underbrace{\begin{bmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix}^{-1}}_{\text{stiffness matrix}}\begin{Bmatrix}\theta_A\\ \theta_B\end{Bmatrix}$$

$$\begin{Bmatrix} M_A \\ M_B \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix}$$

where
$$C_{11} = C_{22} = \frac{EI}{L}\left[\frac{kL\cos kL - (kL)^2\cos kL}{2 - 2\cos(kL) - kL\sin kL}\right]$$

$$C_{21} = C_{12} = \frac{EI}{L}\left[\frac{(kL)^2 - kL\sin kL}{2 - 2\cos kL - kL\sin kL}\right]$$

$$\therefore\ M_A = \frac{EI}{L}\left(S_{ii}\,\theta_A + S_{ij}\,\theta_B\right)$$

$$\&\ M_B = \frac{EI}{L}\left(S_{ij}\,\theta_A + S_{jj}\,\theta_B\right)$$

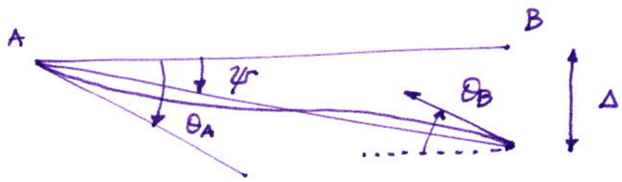
where,
$$S_{ii} = S_{jj} = \frac{C_{11}L}{EI} = \frac{C_{22}L}{EI}$$

$$S_{ij} = S_{ji} = \frac{C_{12}L}{EI} = \frac{C_{21}L}{EI}$$

Stability coefficients S_{ii}, S_{ij} are $f(kL) \rightarrow f\left(\pi\sqrt{\frac{P}{P_x}}\right)$

See graph of <u>S vs. kL</u>

What happens when there is <u>SWAY</u>

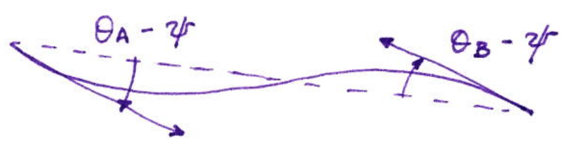


Chord rotation $= \psi = \dfrac{\Delta}{L}$

@ A, rotation w.r.t. chord $= \theta_A - \psi$

@ B, rotation w.r.t. chord $= \theta_B - \psi$

$\therefore$ We can treat the member with SWAY as a member w/o SWAY by replacing end rotations with $\theta_A - \psi$ and $\theta_B - \psi$



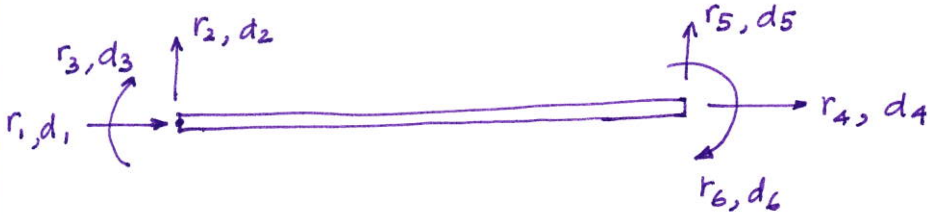
$$\therefore \quad M_A = \frac{EI}{L}\left[s_{ii}(\theta_A - \psi) + s_{ij}(\theta_B - \psi) \right]$$

$$M_B = \frac{EI}{L}\left[s_{ij}(\theta_A - \psi) + s_{jj}(\theta_B - \psi) \right]$$

$$\therefore M_A = \frac{EI}{L}\left[s_{ii}\,\theta_A + s_{ij}\,\theta_B - (s_{ii} + s_{ij})\,\psi \right]$$

$$M_B = \frac{EI}{L}\left[s_{ij}\,\theta_A + s_{jj}\,\theta_B - (s_{ii} + s_{ij})\,\psi \right]$$

↳ Slope - deflection equations for members with <u>SWAY</u>



$r_i \longrightarrow$ Forces

$d_i \longrightarrow$ Deformations

$r_1 = P \qquad\qquad r_4 = -P$

$$r_2 = -\frac{M_{AB} + M_{BA} + P\Delta}{L} \qquad r_5 = \frac{M_{AB} + M_{BA} + P\Delta}{L}$$

$r_6 = M_{BA}$

$r_3 = M_{AB}$

global forces:

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\Delta/L & -1/L & -1/L \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ \Delta/L & 1/L & 1/L \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} P \\ M_A \\ M_B \end{Bmatrix}$$

or

$$\{r\} = [T]_F \{F\}$$

deformations

$$\theta_A = d_3 + \frac{d_5 - d_2}{L} \qquad e = -(d_4 - d_1)$$

$$\theta_B = d_6 + \frac{d_5 - d_2}{L} \qquad \frac{\Delta}{L} = \frac{d_5 - d_2}{L}$$

$$\begin{Bmatrix} e \\ \theta_A \\ \theta_B \\ \Delta/L \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1/L & 1 & 0 & 1/L & 0 \\ 0 & -1/L & 0 & 0 & 1/L & 1 \\ 0 & -1/L & 0 & 0 & 1/L & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}$$

element deform.

global deform.

$$[D] = [T]_d \{d\}$$

$$\{F\} = [k]_e \{D\} \implies \{F\} = [k]_e [T]_d \{d\}$$

$$\therefore \underbrace{\{r\}}_{\text{global}} = [T]_F \{F\} = \underbrace{[T]_F [k]_e [T]_d}_{\text{Element Stiffness Matrix}} \underbrace{\{d\}}_{\text{global}}.$$

$$\therefore \{r\} = [k]\{d\} \implies \underset{6\times6}{[k]} = \underset{6\times3}{[T]_F}\ \underset{3\times4}{[k]_e}\ \underset{4\times6}{[T]_d}$$

$$[k] = \frac{EI}{L}\begin{bmatrix} A/I & 0 & 0 & -A/I & 0 & 0 \\ 0 & \frac{2(S_{ii}+S_{ij})-(kL)^2}{L^2} & -\frac{(S_{ii}+S_{ij})}{L} & 0 & -2(S_{ii}+S_{ij})+(kL)^2 & -\frac{S_{ii}+S_{ij}}{L} \\ 0 & -\frac{S_{ii}+S_{ij}}{L} & S_{ii} & 0 & \frac{S_{ii}+S_{ij}}{L} & S_{ij} \\ -A/I & 0 & 0 & A/I & 0 & 0 \\ 0 & -\frac{2(S_{ii}+S_{ij})+(kL)^2}{L^2} & \frac{S_{ii}+S_{ij}}{L} & 0 & \frac{2(S_{ii}+S_{ij})-(kL)^2}{L^2} & \frac{S_{ii}+S_{ij}}{L} \\ 0 & -\left(\frac{S_{ii}+S_{ij}}{L}\right) & S_{ij} & 0 & \frac{S_{ii}+S_{ij}}{L} & S_{ii} \end{bmatrix}$$

Substituting the expressions for the stability functions S_{ii}, S_{ij}

$$[k] = \frac{EI}{L}\begin{bmatrix} A/I & 0 & 0 & -A/I & 0 & 0 \\ 0 & \frac{12}{L^2}\phi_1 & -\frac{6}{L}\phi_2 & 0 & -\frac{12}{L^2}\phi_1 & -\frac{6}{L}\phi_2 \\ 0 & -\frac{6}{L}\phi_2 & 4\phi_3 & 0 & \frac{6}{L}\phi_2 & 2\phi_4 \\ -A/I & 0 & 0 & A/I & 0 & 0 \\ 0 & -\frac{12}{L^2}\phi_1 & \frac{6}{L}\phi_2 & 0 & \frac{12}{L^2}\phi_1 & \frac{6}{L}\phi_2 \\ 0 & -\frac{6}{L}\phi_2 & 2\phi_4 & 0 & \frac{6}{L}\phi_2 & 4\phi_3 \end{bmatrix}$$

Symmetry

The expressions for ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are given as:

$$\phi_1 = \frac{1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}\left[\pm (kL)^2\right]^n}{12\,\phi}$$

$$\phi_2 = \frac{\frac{1}{2} + \sum_{1}^{\infty} \frac{1}{(2n+2)!}\left[\pm (kL)^2\right]^n}{6\,\phi}$$

$$\phi_3 = \frac{\frac{1}{3} + \sum_{1}^{\infty} \frac{2(n+1)}{(2n+3)!}\left[\pm (kL)^2\right]^n}{4\,\phi}$$

$$\phi_4 = \frac{\frac{1}{6} + \sum_{1}^{\infty} \frac{1}{(2n+3)!}\left[\pm (kL)^2\right]^n}{2\,\phi}$$

where,

$$\phi = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{2(n+1)}{(2n+4)!}\left[\pm (kL)^2\right]^n$$

- Chen has shown that these power series expressions are convenient & efficient to use in numerical analysis.
- No difficulties will arise if P is compressive, tensile, or zero
- The series will converge to a high degree of accuracy if $n = 10$ is used.
- If the axial force is small, use a Taylor series expansion for the ϕ_i, and retain only 2 terms of the series.

$$[K] = \begin{bmatrix} A/I & 0 & 0 & -A/I & 0 & 0 \\ 0 & 12/L^2 & -6/L & 0 & -12/L^2 & -6/L \\ 0 & -6/L & 4 & 0 & 6/L & 2 \\ -A/I & 0 & 0 & A/I & 0 & 0 \\ 0 & -12/L^2 & 6/L & 0 & 12/L^2 & 6/L \\ 0 & -6/L & 2 & 0 & 6/L & 4 \end{bmatrix} \cdot \frac{EI}{L}$$

$$\pm \frac{P}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6/5 & -L/10 & 0 & -6/5 & -L/10 \\ 0 & -L/10 & 2L^2/15 & 0 & L/10 & -L^2/30 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6/5 & L/10 & 0 & 6/5 & L/10 \\ 0 & -L/10 & -L^2/30 & 0 & L/10 & 2L^2/15 \end{bmatrix}$$

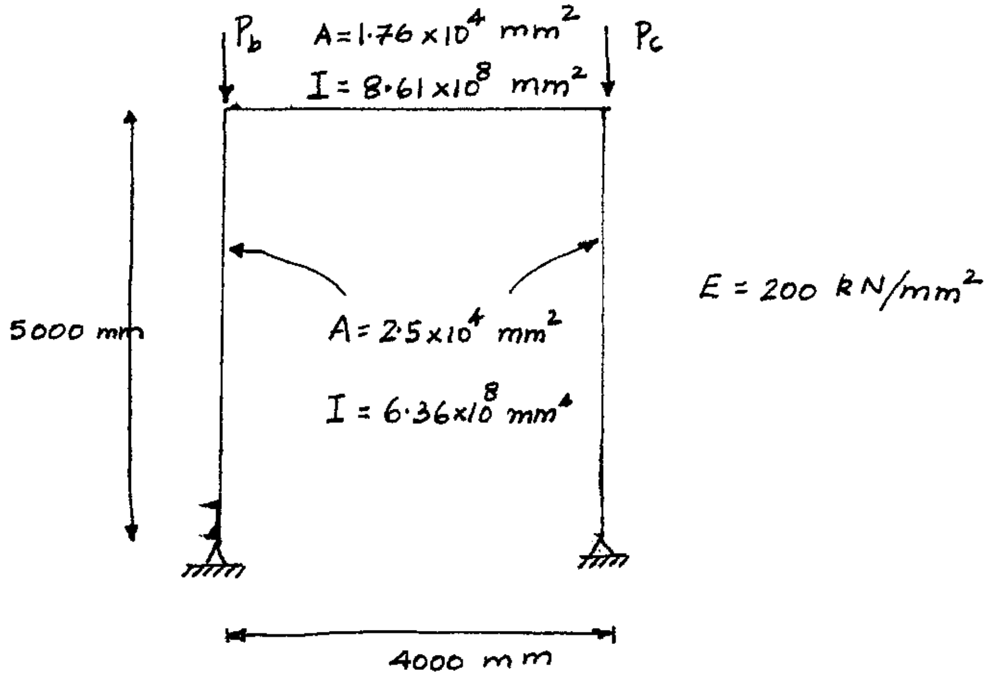
$$[K] = [K]_o + [K]_g$$

original stiffness matrix + geometric stiffness matrix

- The geometric stiffness matrix accounts for the effects of axial force on bending stiffness of the member.

- You can also use it for nonlinear (elastic) force-deformation analysis of frames

- <u>Example:</u>



If $P_b = P_c$, find the critical buckling load $[P_{cr}]$ for the structure.

- Solve the eigenvalue problem to get $P = P_{cr}$ that makes $[K]_G$ so large that it offsets $[K]_0$ and makes the stiffness matrix singular.
- We need to perform eigenvalue analysis on 2 matrices $[K]_0$ and $[K]_G$. The eigenvalue will be the coefficient of $[K]_G$.

Important

- <u>Normalize</u> ALL THE LOADS ACTING ON THE STRUCTURE

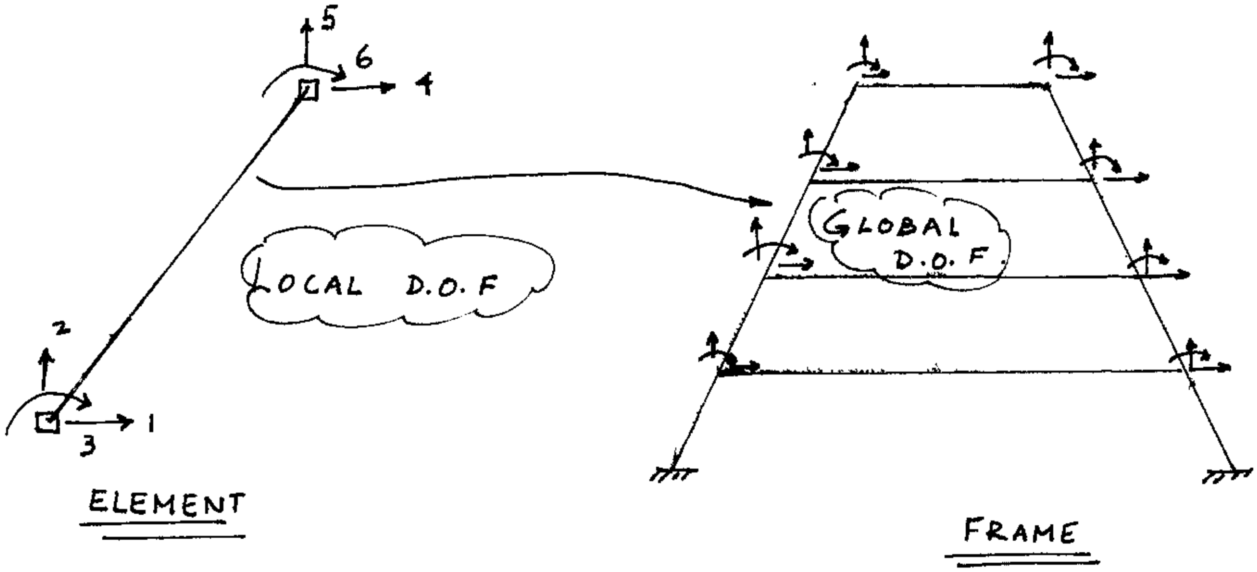
(1) Proportion all the loads acting on the structure so that the max. value is equal to <u>unity</u> in the units (lb, kip, N, kN, etc.) you are using.

(2) All the length based and force based units must be consistent

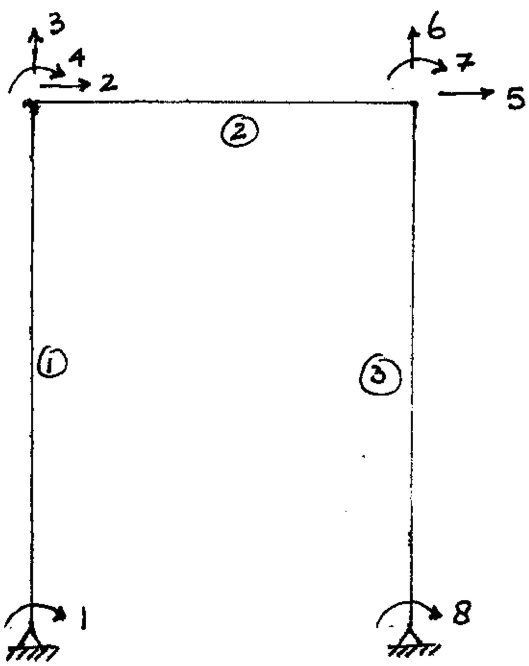
$$E\,,\ I\,,\ A\,,\ L\,,\ x_i\,,\ y_i \longrightarrow \text{CONSISTENT UNITS.}$$

(3) The critical buckling load will be the smallest value from the eigenvalue analysis.

- The corresponding eigenvector will be the <u>mode</u>



The local d.o.f. are related the global d.o.f. through id array.



ID ARRAY

LOCAL D.O.F.	ELEMENT 1	ELEMENT 2	ELEMENT 3
1	0	2	5
2	0	3	6
3	1	4	7
4	2	5	0
5	3	6	0
6	4	7	8

Frame with global d.o.f.

Develop a computer program to

(1) ASSEMBLE structure stiffness matrix $[K]_0$ using id array and element stiffness matrix

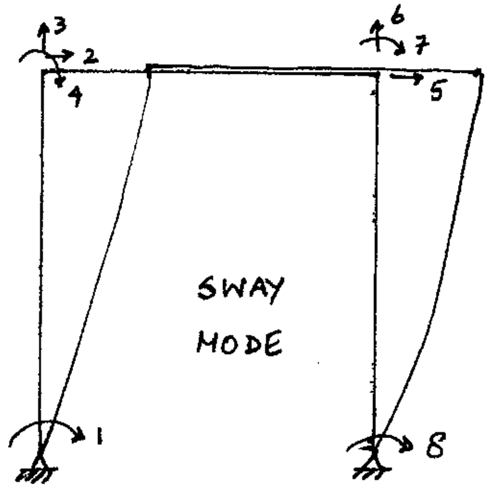
(2) ASSEMBLE geometric stiffness matrix $[K]_G$ using id array and element geometric stiffness matrix

(3) Solve eigenvalue problem

(4) Note first two eigenvalues and sketch corresponding buckling modes:

- $P_{cr})_1 = \underline{10.34 \text{ kN}}$ $\equiv$

$$\begin{Bmatrix} 0.000287 \\ 1 \\ 0.00517 \\ 0 \\ 1 \\ -0.00517 \\ 0 \\ 0.000287 \end{Bmatrix} \begin{array}{l} \to 1 \\ \to 2 \\ \to 3 \\ \to 4 \\ \to 5 \\ \to 6 \\ \to 7 \\ \to 8 \end{array}$$



- $P_{cr})_2 = \underline{97 \text{ kN}}$ $\equiv$

$$\begin{array}{r} 1 \leftarrow \\ 2 \leftarrow \\ 3 \leftarrow \\ 4 \leftarrow \\ 5 \leftarrow \\ 6 \leftarrow \\ 7 \leftarrow \\ 8 \leftarrow \end{array} \begin{Bmatrix} 0.185 \\ 1 \\ 0 \\ -0.1 \\ -1 \\ 0 \\ 0.1 \\ -0.185 \end{Bmatrix}$$