Problem No. 4.

The differential equation for the problem is:

$$\phi^{IV} - \lambda_1 \phi'' - \lambda_2 \phi = 0$$

where $\lambda_1 = \dfrac{G K_T}{E I_W}$ $\qquad \lambda_2 = \dfrac{M_0^2}{E^2 I_y I_W}$

λ_1 = constant & $\lambda_2 \rightarrow$ eigenvalue or unknown we are looking

For the W27x94 section:

$K_T = 4.03$ in^4 $\qquad r_y = 2.12$ in $\qquad G = 11200$ ksi

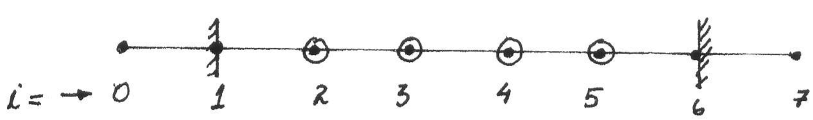
$I_W = 21300$ in^6 $\qquad I_y = 124$ in^4 $\qquad E = 29000$ ksi

Length of beam = 200 r_y = 424 in.

Divide into 5 segments for this solution:

$\therefore h = \dfrac{424}{5} = 84.8$ in.

The original differential equation is valid at each station



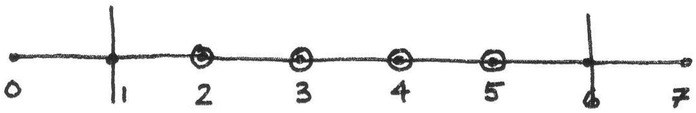
$$\therefore \phi_i^{IV} - \lambda_1 \phi_i'' - \lambda_2 \phi_i = 0 \longrightarrow \text{at each station } i$$

where, $\phi_i^{IV} = \dfrac{1}{h^4}\left\{\phi_{i-2} - 4\phi_{i-1} + 6\phi_i - 4\phi_{i+1} + \phi_{i+2}\right\}$

$$\phi_i'' = \frac{1}{h^2}\left\{\phi_{i-1} - 2\phi_i + \phi_{i+1}\right\}$$

Differential equation becomes:

$$\therefore\ \frac{1}{h^4}\left\{\phi_{i-2}\right\} + \phi_{i-1}\left\{-\frac{4}{h^4} - \frac{\lambda_1}{h^2}\right\} + \phi_i\left\{\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right\} + \phi_{i+1}\left\{\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right\} + \phi_{i+2}\cdot\frac{1}{h^4} = 0$$



@ $i=2$

$$\frac{1}{h^4}\phi_0 + \cancelto{0}{\phi_1\left[-\frac{4}{h^4} - \frac{\lambda_1}{h^2}\right]} + \phi_2\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_3\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_4\cdot\frac{1}{h^4} = 0$$

@ $i=3$

$$\cancelto{0}{\frac{1}{h^4}\phi_1} + \phi_2\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_3\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_4\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_5\cdot\frac{1}{h^4} = 0$$

@ $i=4$

$$\frac{1}{h^4}\phi_2 + \phi_3\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_4\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_5\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \cancelto{0}{\phi_6\cdot\frac{1}{h^4}} = 0$$

@ $i=5$

$$\frac{1}{h^4}\phi_3 + \phi_4\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_5\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \cancelto{0}{\phi_6\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right]} + \phi_7\cdot\frac{1}{h^4} = 0$$

$\phi_1 = \phi_6 = 0$

$\phi_0 = \phi_2$ & $\phi_5 = \phi_7$

$$\phi_2\left[\frac{7}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_3\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_4\cdot\frac{1}{h^4} = 0$$

$$\phi_2\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_3\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_4\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_5\cdot\frac{1}{h^4} = 0$$

$$\phi_2\left[\frac{1}{h^4}\right] + \phi_3\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_4\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_5\left[-\frac{4}{h^4} - \frac{\lambda_1}{h^2}\right] = 0$$

$$\phi_3\left[\frac{1}{h^4}\right] + \phi_4\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_5\left[\frac{7}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] = 0$$

$$\begin{bmatrix} \frac{7}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2 & -\frac{4}{h^4} - \frac{\lambda_1}{h^2} & \frac{1}{h^4} & 0 \\ -\frac{4}{h^4} - \frac{\lambda_1}{h^2} & \frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2 & -\frac{4}{h^4} - \frac{\lambda_1}{h^2} & \frac{1}{h^4} \\ \frac{1}{h^4} & -\frac{4}{h^4} - \frac{\lambda_1}{h^2} & \frac{6}{h^4} + 2\frac{\lambda_1}{h^2} - \lambda_2 & -\frac{4}{h^4} - \frac{\lambda_1}{h^2} \\ 0 & \frac{1}{h^4} & -\frac{4}{h^4} - \frac{\lambda_1}{h^2} & \frac{7}{h^4} + 2\frac{\lambda_1}{h^2} - \lambda_2 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = 0$$

Multiplying all equations by h^4

$$\begin{bmatrix} 7 + 2\lambda_1 h^2 - \lambda_2 h^4 & -4 - \lambda_1 h^2 & 1 & 0 \\ -4 - \lambda_1 h^2 & 6 + 2\lambda_1 h^2 - \lambda_2 h^4 & -4 - \lambda_1 h^2 & 1 \\ 1 & -4 - \lambda_1 h^2 & 6 + 2\lambda_1 h^2 - \lambda_2 h^4 & -4 - \lambda_1 h^2 \\ 0 & 1 & -4 - \lambda_1 h^2 & 7 + 2\lambda_1 h^2 - \lambda_2 h^4 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix}$$

Solve using inverse iteration:

$$\lambda_1 = \frac{G K_T}{E I_w} \checkmark \qquad \lambda_2 = \frac{M_0^2}{E^2 I_y I_w}$$

$$[A] - \lambda_2 [B] = \{0\}.$$

$$\therefore [A] = \begin{bmatrix} 7 + 2\lambda_1 h^2 & -4 - \lambda_1 h^2 & 1 & 0 \\ -4 - \lambda_1 h^2 & 6 + 2\lambda_1 h^2 & -4 - \lambda_1 h^2 & 1 \\ 1 & -4 - \lambda_1 h^2 & 6 + 2\lambda_1 h^2 & -4 - \lambda_1 h^2 \\ 0 & 1 & -4 - \lambda_1 h^2 & 7 + 2\lambda_1 h^2 \end{bmatrix}$$

$$[B] = h^4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [A] - \lambda_2 [B] = \{0\}$$

$$\text{Length} = 200\, r_y = 200 \times 2.12 \text{ in.} = 424 \text{ in.}$$

$$\lambda_1 = \frac{GK_T}{EI_w} = \frac{11200 \times 0.492}{29000 \times 21200} = 7.342 \times 10^{-5}$$

$$\lambda_2 = \frac{M_o^2}{E^2 I_y I_w}$$

$$h = \frac{L}{5} = \frac{424}{5} = 84.8 \text{ in}$$

$$\therefore \begin{bmatrix} 8.055932 & -4.52766 & 1 & 0 \\ -4.52766 & 7.055932 & -4.52766 & 1 \\ 1 & -4.52766 & 7.055932 & -4.52766 \\ 0 & 1 & -4.52766 & 8.055932 \end{bmatrix} - h^4 \lambda_2 [I] = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Using Matlab eigenvalue command

main eigenvalue

$$h^4 \lambda_2 = \begin{Bmatrix} 0.8107 \\ 4.0175 \\ 9.7735 \\ 15.6220 \end{Bmatrix}$$

0.8107 → eigenvector $= \begin{matrix} -0.3095 \\ -0.6358 \\ -0.6358 \\ -0.3095 \end{matrix}$

$4.0175, 9.7735, 15.6220$: other eigenvalues

$$h^4 \cdot \frac{M_o^2}{E^2 I_y I_w} = 0.8107$$

$$\therefore M_o^2 = 0.8107 \times E^2 \times 124 \times \frac{21200}{84.8^4}$$

$$\therefore M_o = 5887 \text{ k-in}$$

$$M_o = 490.607 \text{ k-ft}$$

This is just with using 5 segments along the length.