Chapter 1. Introduction to Structural Stability

OUTLINE

- Definition of stability
- Types of instability
- Methods of stability analyses
- Examples – small deflection analyses
- Examples – large deflection analyses
- Examples – imperfect systems
- Design of steel structures
STABILITY DEFINITION

- Change in geometry of a structure or structural component under compression – resulting in loss of ability to resist loading is defined as instability in the book.

- Instability can lead to catastrophic failure → must be accounted in design. Instability is a strength-related limit state.

- Why did we define instability instead of stability? Seem strange!

- Stability is not easy to define.
  - Every structure is in equilibrium – static or dynamic. If it is not in equilibrium, the body will be in motion or a mechanism.
  - A mechanism cannot resist loads and is of no use to the civil engineer.
  - Stability qualifies the state of equilibrium of a structure. Whether it is in stable or unstable equilibrium.

STABILITY DEFINITION

- Structure is in stable equilibrium when small perturbations do not cause large movements like a mechanism. Structure vibrates about it equilibrium position.

- Structure is in unstable equilibrium when small perturbations produce large movements – and the structure never returns to its original equilibrium position.

- Structure is in neutral equilibrium when we can’t decide whether it is in stable or unstable equilibrium. Small perturbation cause large movements – but the structure can be brought back to its original equilibrium position with no work.

- Thus, stability talks about the equilibrium state of the structure.

- The definition of stability had nothing to do with a change in the geometry of the structure under compression – seems strange!
STABILITY DEFINITION

- **BUCKLING Vs. STABILITY**

  - Change in geometry of structure under compression – that results in its ability to resist loads – called *instability*.
  - Not true – this is called *buckling*.
  - **Buckling** is a phenomenon that can occur for structures under compressive loads.
    - The structure deforms and is in stable equilibrium in state-1.
    - As the load increases, the structure suddenly changes to deformation state-2 at some critical load $P_{cr}$.
    - The structure buckles from state-1 to state-2, where state-2 is orthogonal (has nothing to do, or independent) with state-1.
  - What has buckling to do with stability?
    - The question is - Is the equilibrium in state-2 stable or unstable?
    - Usually, state-2 after buckling is either neutral or unstable equilibrium.
Thus, there are two topics we will be interested in this course:

- **Buckling** – Sudden change in deformation from state-1 to state-2
- **Stability of equilibrium** – As the loads acting on the structure are increased, when does the equilibrium state become unstable?

The equilibrium state becomes unstable due to:

- Large deformations of the structure
- Inelasticity of the structural materials

We will look at both of these topics for:

- Columns
- Beams
- Beam-Columns
- Structural Frames
TYPES OF INSTABILITY

Structure subjected to compressive forces can undergo:

1. Buckling – bifurcation of equilibrium from deformation state-1 to state-2.
   - Bifurcation buckling occurs for columns, beams, and symmetric frames under gravity loads only

2. Failure due to instability of equilibrium state-1 due to large deformations or material inelasticity
   - Elastic instability occurs for beam-columns, and frames subjected to gravity and lateral loads.
   - Inelastic instability can occur for all members and the frame.
   - We will study all of this in this course because we don’t want our designed structure to buckle or fail by instability – both of which are strength limit states.

TYPES OF INSTABILITY

BIFURCATION BUCKLING

- Member or structure subjected to loads. As the load is increased, it reaches a critical value where:
  - The deformation changes suddenly from state-1 to state-2.
  - And, the equilibrium load-deformation path bifurcates.

- Critical buckling load when the load-deformation path bifurcates
  - Primary load-deformation path before buckling
  - Secondary load-deformation path post buckling
  - Is the post-buckling path stable or unstable?
SYMMETRIC BIFURCATION

- Post-buckling load-deform. paths are *symmetric* about load axis.
  - If the load capacity increases after buckling then *stable* symmetric bifurcation.
  - If the load capacity decreases after buckling then *unstable* symmetric bifurcation.

ASYMMETRIC BIFURCATION

- Post-buckling behavior that is asymmetric about load axis.
INSTABILITY FAILURE

- There is no bifurcation of the load-deformation path. The deformation stays in state-1 throughout.
- The structure stiffness decreases as the loads are increased. The change in stiffness is due to large deformations and/or material inelasticity.
  - The structure stiffness decreases to zero and becomes negative.
  - The load capacity is reached when the stiffness becomes zero.
  - Neutral equilibrium when stiffness becomes zero and unstable equilibrium when stiffness is negative.
- Structural stability failure – when stiffness becomes negative.

INSTABILITY FAILURE

- FAILURE OF BEAM-COLUMNS

No bifurcation. Instability due to material and geometric nonlinearity.
INSTABILITY FAILURE

- Snap-through buckling

- Shell Buckling failure – very sensitive to imperfections
Chapter 1. Introduction to Structural Stability

OUTLINE

- Definition of stability
- Types of instability
- Methods of stability analyses
- Examples – small deflection analyses
- Examples – large deflection analyses
- Examples – imperfect systems
- Design of steel structures

METHODS OF STABILITY ANALYSES

- **Bifurcation approach** – consists of writing the equation of equilibrium and solving it to determine the onset of buckling.

- **Energy approach** – consists of writing the equation expressing the complete potential energy of the system. Analyzing this total potential energy to establish equilibrium and examine stability of the equilibrium state.

- **Dynamic approach** – consists of writing the equation of dynamic equilibrium of the system. Solving the equation to determine the natural frequency ($\omega$) of the system. Instability corresponds to the reduction of $\omega$ to zero.
STABILITY ANALYSES

- Each method has its advantages and disadvantages. In fact, you can use different methods to answer different questions
  - The bifurcation approach is appropriate for determining the critical buckling load for a (perfect) system subjected to loads.
    - The deformations are usually assumed to be small.
    - The system must not have any imperfections.
    - It cannot provide any information regarding the post-buckling load-deformation path.
  - The energy approach is the best when establishing the equilibrium equation and examining its stability
    - The deformations can be small or large.
    - The system can have imperfections.
    - It provides information regarding the post-buckling path if large deformations are assumed
    - The major limitation is that it requires the assumption of the deformation state, and it should include all possible degrees of freedom.

STABILITY ANALYSIS

- The dynamic method is very powerful, but we will not use it in this class at all.
  - Remember, it though when you take the course in dynamics or earthquake engineering
  - In this class, you will learn that the loads acting on a structure change its stiffness. This is significant – you have not seen it before.

- What happens when an axial load is acting on the beam.
  - The stiffness will no longer remain 4EI/L and 2EI/L.
  - Instead, it will decrease. The reduced stiffness will reduce the natural frequency and period elongation.
  - You will see these in your dynamics and earthquake engineering class.
STABILITY ANALYSIS

- For any kind of buckling or stability analysis – need to draw the free body diagram of the deformed structure.
- Write the equation of static equilibrium in the deformed state.
- Write the energy equation in the deformed state too.
- This is central to the topic of stability analysis.
- No stability analysis can be performed if the free body diagram is in the undeformed state.

BIFURCATION ANALYSIS

- Always a small deflection analysis
- To determine $P_{cr}$ buckling load
- Need to assume buckled shape (state 2) to calculate

Example 1 – Rigid bar supported by rotational spring

Step 1 - Assume a deformed shape that activates all possible d.o.f.
BIFURCATION ANALYSIS

- Write the equation of static equilibrium in the deformed state
  \[ \sum M = 0 \quad \therefore -k\theta + PL\sin\theta = 0 \]
  \[ \therefore P = \frac{k\theta}{L\sin\theta} \]
  For small deformations, \( \sin\theta = \theta \)
  \[ \therefore P = \frac{k\theta}{L\theta} = \frac{k}{L} \]
- Thus, the structure will be in static equilibrium in the deformed state when \( P = P_{cr} = k/L \)
- When \( P<P_{cr} \), the structure will not be in the deformed state. The structure will buckle into the deformed state when \( P = P_{cr} \)

Example 2 - Rigid bar supported by translational spring at end

Assume deformed state that activates all possible d.o.f.
Draw FBD in the deformed state
BIFURCATION ANALYSIS

Write equations of static equilibrium in deformed state

\[ + \sum M_o = 0 \Rightarrow -(kL \sin \theta) \times L + PL \sin \theta = 0 \]
\[ \therefore P = \frac{kL^2 \sin \theta}{L \sin \theta} \]

For small deformations \( \sin \theta = \theta \)
\[ \therefore P_{cr} = \frac{kL^2 \theta}{L \theta} = kL \]

Thus, the structure will be in static equilibrium in the deformed state when \( P = P_{cr} = kL \). When \( P < P_{cr} \), the structure will not be in the deformed state. The structure will buckle into the deformed state when \( P = P_{cr} \).

Example 3 – Three rigid bar system with two rotational springs

Assume deformed state that activates all possible d.o.f.
Draw FBD in the deformed state

Assume small deformations. Therefore, \( \sin \theta = \theta \)
BIFURCATION ANALYSIS

Write equations of static equilibrium in deformed state

\[ \begin{align*}
&+ \sum M_B = 0 \quad \therefore k(2\theta_1 - \theta_2) - PL \sin \theta_1 = 0 \\
&+ \sum M_C = 0 \quad \therefore -k(2\theta_2 - \theta_1) + PL \sin \theta_2 = 0
\end{align*} \]

Equations of Static Equilibrium

\[ k(2\theta_1 - \theta_2) - PL \theta_1 = 0 \quad \therefore \begin{bmatrix} 2k - PL & -k \\ -k & 2k - PL \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Therefore either \( \theta_1 \) and \( \theta_2 \) are equal to zero or the determinant of the coefficient matrix is equal to zero.

When \( \theta_1 \) and \( \theta_2 \) are not equal to zero – that is when buckling occurs – the coefficient matrix determinant has to be equal to zero for equil.

Take a look at the matrix equation. It is of the form \([A] \{x\} = \{0\}\). It can also be rewritten as \(((K)-\lambda[I])\{x\} = \{0\}\)

\[ \begin{bmatrix} 2k/L & -k/L \\ -k/L & 2k/L \end{bmatrix} - P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
BIFURCATION ANALYSIS

- This is the classical eigenvalue problem. \( ([K]-\lambda[I])\{x\} = 0 \).
- We are searching for the eigenvalues \( \lambda \) of the stiffness matrix [K]. These eigenvalues cause the stiffness matrix to become singular.
  - Singular stiffness matrix means that it has a zero value, which means that the determinant of the matrix is equal to zero.

\[
\begin{vmatrix}
2k - PL & -k \\
-k & 2k - PL
\end{vmatrix} = 0
\]

\[
\therefore (2k - PL)^2 - k^2 = 0
\]

\[
\therefore (2k - PL + k) \cdot (2k - PL - k) = 0
\]

\[
\therefore (3k - PL) \cdot (k - PL) = 0
\]

\[
\therefore P_{cr} = \frac{3k}{L} \text{ or } \frac{k}{L}
\]

- Smallest value of \( P_{cr} \) will govern. Therefore, \( P_{cr} = k/L \)

BIFURCATION ANALYSIS

- Each eigenvalue or critical buckling load \( P_{cr} \) corresponds to a buckling shape that can be determined as follows.
- \( P_{cr} = k/L \). Therefore substitute in the equations to determine \( \theta_1 \) and \( \theta_2 \)

\[
\begin{align*}
\bar{k} (2\theta_1 - \theta_2) - PL \theta_1 &= 0 \\
\text{Let } P = P_{cr} = \frac{k}{L} \\
\therefore k(2\theta_1 - \theta_2) - k\theta_1 &= 0 \\
\therefore k\theta_1 - k\theta_2 &= 0 \\
\therefore \theta_1 &= \theta_2
\end{align*}
\]

- All we could find is the relationship between \( \theta_1 \) and \( \theta_2 \). Not their specific values. Remember that this is a small deflection analysis. So, the values are negligible. What we have found is the buckling shape – not its magnitude.
- The buckling mode is such that \( \theta_1 = \theta_2 \rightarrow \text{Symmetric buckling mode} \)
BIFURCATION ANALYSIS

- Second eigenvalue was $P_{cr} = 3k/L$. Therefore substitute in the equations to determine $\theta_1$ and $\theta_2$

\[
\begin{align*}
    k(2\theta_1 - \theta_2) - PL \theta_1 &= 0 \\
    \text{Let } P &= P_{cr} = \frac{3k}{L} \\
    \therefore k(2\theta_1 - \theta_2) - 3k\theta_1 &= 0 \\
    \therefore -k\theta_1 - k\theta_2 &= 0 \\
    \therefore \theta_1 = -\theta_2 \\

    -k(2\theta_2 - \theta_1) + PL \theta_2 &= 0 \\
    \text{Let } P &= P_{cr} = \frac{3k}{L} \\
    \therefore -k(2\theta_2 - \theta_1) + 3k\theta_2 &= 0 \\
    \therefore k\theta_1 + k\theta_2 &= 0 \\
    \therefore \theta_1 = -\theta_2
\end{align*}
\]

- All we could find is the relationship between $\theta_1$ and $\theta_2$. Not their specific values. Remember that this is a small deflection analysis. So, the values are negligible. What we have found is the buckling shape – not its magnitude.

- The buckling mode is such that $\theta_1 = -\theta_2 \rightarrow$ Antisymmetric buckling mode

---

Chapter 1. Introduction to Structural Stability

OUTLINE

- Definition of stability
- Types of instability
- Methods of stability analyses
- Bifurcation analysis examples – small deflection analyses
- Energy method
  - Examples – small deflection analyses
  - Examples – large deflection analyses
  - Examples – imperfect systems
- Design of steel structures
We will currently look at the use of the energy method for an elastic system subjected to conservative forces.

Total potential energy of the system – $\Pi$ – depends on the work done by the external forces ($W_e$) and the strain energy stored in the system ($U$).

$\Pi = U - W_e$.

For the system to be in equilibrium, its total potential energy $\Pi$ must be stationary. That is, the first derivative of $\Pi$ must be equal to zero.

Investigate higher order derivatives of the total potential energy to examine the stability of the equilibrium state, i.e., whether the equilibrium is stable or unstable.

The energy method is the best for establishing the equilibrium equation and examining its stability:

- The deformations can be small or large.
- The system can have imperfections.
- It provides information regarding the post-buckling path if large deformations are assumed.
- The major limitation is that it requires the assumption of the deformation state, and it should include all possible degrees of freedom.
ENERGY METHOD

Example 1 – Rigid bar supported by rotational spring

Assume small deflection theory

Step 1 - Assume a deformed shape that activates all possible d.o.f.

ENERGY METHOD – SMALL DEFLECTIONS

Write the equation representing the total potential energy of system

\[ \Pi = U - W_e \]

\[ U = \frac{1}{2} k \theta^2 \]

\[ W_e = P L (1 - \cos \theta) \]

\[ \Pi = \frac{1}{2} k \theta^2 - P L (1 - \cos \theta) \]

\[ \frac{d\Pi}{d\theta} = k \theta - P L \sin \theta \]

For equilibrium: \( \frac{d\Pi}{d\theta} = 0 \)

Therefore, \( k \theta - P L \sin \theta = 0 \)

For small deflections: \( k \theta - P L \theta = 0 \)

Therefore, \( P_\alpha = \frac{k}{L} \)
ENERGY METHOD – SMALL DEFLECTIONS

- The energy method predicts that buckling will occur at the same load $P_{cr}$ as the bifurcation analysis method.
- At $P_{cr}$, the system will be in equilibrium in the deformed.
- Examine the stability by considering further derivatives of the total potential energy.
  - This is a small deflection analysis. Hence $\theta$ will be $\rightarrow$ zero.
  - In this type of analysis, the further derivatives of $\Pi$ examine the stability of the initial state-1 (when $\theta = 0$)

\[
\Pi = \frac{1}{2} k \theta^2 - P L (1 - \cos \theta)
\]
\[
\frac{d\Pi}{d\theta} = k \theta - P L \sin \theta = k \theta - P L \theta
\]
\[
\frac{d^2\Pi}{d\theta^2} = k - P L
\]

When $P < P_{cr}$, $\frac{d^2\Pi}{d\theta^2} > 0$ : Stable equilibrium
When $P > P_{cr}$, $\frac{d^2\Pi}{d\theta^2} < 0$ : Unstable equilibrium
When $P = P_{cr}$, $\frac{d^2\Pi}{d\theta^2} = 0$ : Not sure

ENERGY METHOD – SMALL DEFLECTIONS

- In state-1, stable when $P<P_{cr}$, unstable when $P>P_{cr}$
- No idea about state during buckling.
- No idea about post-buckling equilibrium path or its stability.

\[
\begin{align*}
\Pi &= \frac{1}{2} k \theta^2 - P L (1 - \cos \theta) \\
\frac{d\Pi}{d\theta} &= k \theta - P L \sin \theta = k \theta - P L \theta \\
\frac{d^2\Pi}{d\theta^2} &= k - P L
\end{align*}
\]

When $P < P_{cr}$, $\frac{d^2\Pi}{d\theta^2} > 0$ : Stable equilibrium
When $P > P_{cr}$, $\frac{d^2\Pi}{d\theta^2} < 0$ : Unstable equilibrium
When $P = P_{cr}$, $\frac{d^2\Pi}{d\theta^2} = 0$ : Not sure
**ENERGY METHOD – LARGE DEFLECTIONS**

- Example 1 – Large deflection analysis (rigid bar with rotational spring)

\[ \Pi = U - W_e \]
\[ U = \frac{1}{2} k \theta^2 \]
\[ W_e = P L (1 - \cos \theta) \]
\[ \Pi = \frac{1}{2} k \theta^2 - P L (1 - \cos \theta) \]
\[ \frac{d \Pi}{d \theta} = k \theta - P L \sin \theta \]

For equilibrium: \[ \frac{d \Pi}{d \theta} = 0 \]

Therefore, \[ k \theta - P L \sin \theta = 0 \]

Therefore, \[ P = \frac{k \theta}{L \sin \theta} \] for equilibrium

The post-buckling \( P - \theta \) relationship is given above

**ENERGY METHOD – LARGE DEFLECTIONS**

- Large deflection analysis
  - See the post-buckling load-displacement path shown below
  - The load carrying capacity increases after buckling at \( P_{cr} \)
  - \( P_{cr} \) is where \( \theta \rightarrow 0 \)
ENERGY METHOD – LARGE DEFLECTIONS

- Large deflection analysis – Examine the stability of equilibrium using higher order derivatives of $\Pi$

\[
\Pi = \frac{1}{2} k \theta^2 - P L \cos \theta \\
d\Pi \over d\theta = k \theta - P L \sin \theta \\
d^2 \Pi \over d\theta^2 = -k - P L \cos \theta
\]

But, $P = {k \theta \over L \sin \theta}$

\[
\therefore d^2 \Pi \over d\theta^2 = k - {k \theta \over L \sin \theta} L \cos \theta \\
\therefore d^2 \Pi \over d\theta^2 = k(1 - {\theta \over \tan \theta}) \\
\therefore d^2 \Pi \over d\theta^2 > 0 \text{ Always (i.e., all values of } \theta) \\
\therefore \text{Always STABLE} \\
\text{But, } d^2 \Pi \over d\theta^2 = 0 \text{ for } \theta = 0
\]

- At $\theta = 0$, the second derivative of $\Pi = 0$. Therefore, inconclusive.

- Consider the Taylor series expansion of $\Pi$ at $\theta = 0$

\[
\Pi = \Pi|_{\theta = 0} + \left. {d \Pi \over d\theta} \right|_{\theta = 0} \theta + \left. {d^2 \Pi \over 2! d\theta^2} \right|_{\theta = 0} \theta^2 + \left. {1 \over 3!} {d^3 \Pi \over d\theta^3} \right|_{\theta = 0} \theta^3 + \left. {1 \over 4!} {d^4 \Pi \over 4! d\theta^4} \right|_{\theta = 0} \theta^4 + \ldots
\]

- Determine the first non-zero term of $\Pi$,

\[
\Pi|_{\theta = 0} = 0 \\
{d \Pi \over d\theta}|_{\theta = 0} = 0 \\
{d^2 \Pi \over d\theta^2}|_{\theta = 0} = 0 \\
{d^3 \Pi \over d\theta^3}|_{\theta = 0} = P L \sin \theta = 0 \\
{d^4 \Pi \over d\theta^4}|_{\theta = 0} = P L \cos \theta = 0\]

\[
{1 \over 24} k \theta^4 > 0
\]

- Since the first non-zero term is $> 0$, the state is stable at $P = P_{cr}$ and $\theta = 0$
ENERGY METHOD – LARGE DEFLECTIONS

Rigid bar with rotational spring

End rotation $\theta$

Load $P/P_{cr}$

STABLE

STABLE

STABLE

ENERGY METHOD – IMPERFECT SYSTEMS

- Consider example 1 – but as a system with imperfections
  - The initial imperfection given by the angle $\theta_0$ as shown below

  \[ L \cos(\theta_0) \]

  \[ L \cos(\theta_0) - \cos(\theta_0) \]

- The free body diagram of the deformed system is shown below
ENERGY METHOD – IMPERFECT SYSTEMS

\[ \Pi = U - W_e \]

\[ U = \frac{1}{2} k (\theta - \theta_0)^2 \]

\[ W_e = P L (\cos \theta_0 - \cos \theta) \]

\[ \Pi = \frac{1}{2} k (\theta - \theta_0)^2 - P L (\cos \theta_0 - \cos \theta) \]

\[ \frac{d \Pi}{d \theta} = k (\theta - \theta_0) - P L \sin \theta \]

For equilibrium, \( \frac{d \Pi}{d \theta} = 0 \)

Therefore, \( k (\theta - \theta_0) - P L \sin \theta = 0 \)

Therefore, \( P = \frac{k (\theta - \theta_0)}{L \sin \theta} \) for equilibrium

The equilibrium \( P - \theta \) relationship is given above

\[ P = \frac{k (\theta - \theta_0)}{L \sin \theta} \]

\[ \therefore \frac{P}{P_{cr}} = \frac{\theta - \theta_0}{\sin \theta} \]

\( P - \theta \) relationships for different values of \( \theta_0 \) shown below:
ENERGY METHODS – IMPERFECT SYSTEMS

- As shown in the figure, deflection starts as soon as loads are applied. There is no bifurcation of load-deformation path for imperfect systems. The load-deformation path remains in the same state throughout.
- The smaller the imperfection magnitude, the closer the load-deformation paths to the perfect system load – deformation path.
- The magnitude of load is influenced significantly by the imperfection magnitude.
- All real systems have imperfections. They may be very small but will be there.
- The magnitude of imperfection is not easy to know or guess. Hence if a perfect system analysis is done, the results will be close for an imperfect system with small imperfections.

Examine the stability of the imperfect system using higher order derivatives of $\Pi$

$\Pi = \frac{1}{2} k (\theta - \theta_o)^2 - P L (\cos \theta_o - \cos \theta)$

$\frac{d \Pi}{d \theta} = k (\theta - \theta_o) - P L \sin \theta$

$\frac{d^2 \Pi}{d \theta^2} = k - P L \cos \theta$

$\therefore$ Equilibrium path will be stable

If $\frac{d^2 \Pi}{d \theta^2} > 0$

i.e., if $k - P L \cos \theta > 0$

i.e., if $P < \frac{k}{L \cos \theta}$

i.e., if $\frac{k(\theta - \theta_o)}{L \sin \theta} < \frac{k}{L \cos \theta}$

i.e., $\theta - \theta_o < \tan \theta$

Which is always true, hence always in STABLE EQUILIBRIUM
ENERGY METHOD – SMALL DEFLECTIONS

Example 2 - Rigid bar supported by translational spring at end

\[ P \]
\[ k \]
\[ L \]

Assume deformed state that activates all possible d.o.f.
Draw FBD in the deformed state

\[ \text{L. sin}\theta \]
\[ k \text{L. sin}\theta \]
\[ \text{L. cos}\theta \]
\[ \text{L (1-cos}\theta) \]

ENERGY METHOD – SMALL DEFLECTIONS

Write the equation representing the total potential energy of system

\[ \Pi = U - W_e \]
\[ U = \frac{1}{2} k (L \sin \theta)^2 = \frac{1}{2} k L^2 \theta^2 \]
\[ W_e = P L (1 - \cos \theta) \]
\[ \Pi = \frac{1}{2} k L^2 \theta^2 - P L (1 - \cos \theta) \]
\[ \frac{d\Pi}{d\theta} = k L^2 \theta - P L \sin \theta \]

For equilibrium: \[ \frac{d\Pi}{d\theta} = 0 \]

Therefore, \[ k L^2 \theta - P L \sin \theta = 0 \]
For small deflection: \[ k L^2 \theta - P L \theta = 0 \]
Therefore, \( P_{ce} = k \)
ENERGY METHOD – SMALL DEFLECTIONS

- The energy method predicts that buckling will occur at the same load $P_{cr}$ as the bifurcation analysis method.
- At $P_{cr}$, the system will be in equilibrium in the deformed. Examine the stability by considering further derivatives of the total potential energy.
  - This is a small deflection analysis. Hence $q$ will be $\rightarrow$ zero.
  - In this type of analysis, the further derivatives of $\Pi$ examine the stability of the initial state-1 (when $\theta = 0$)

$$\Pi = \frac{1}{2} k L^2 \theta^2 - P L (1 - \cos \theta)$$

$$\frac{d \Pi}{d \theta} = k L^2 \theta - P L \sin \theta$$

$$\frac{d^2 \Pi}{d \theta^2} = k L^2 - P L \cos \theta$$

For small deflection and $\theta = 0$
$$\frac{d^2 \Pi}{d \theta^2} = k L^2 - P L$$

When, $P < k L$ \quad $\frac{d^2 \Pi}{d \theta^2} > 0$ \quad :: STABLE

When, $P > k L$ \quad $\frac{d^2 \Pi}{d \theta^2} < 0$ \quad :: UNSTABLE

When $P = kL$ \quad $\frac{d^2 \Pi}{d \theta^2} = 0$ \quad :: INDETERMINATE

ENERGY METHOD – LARGE DEFLECTIONS

- Write the equation representing the total potential energy of system

$$\Pi = U - W_e$$

$$U = \frac{1}{2} k (L \sin \theta)^2$$

$$W_e = P L (1 - \cos \theta)$$

$$\Pi = \frac{1}{2} k L^2 \sin^2 \theta - P L (1 - \cos \theta)$$

$$\frac{d \Pi}{d \theta} = k L^2 \sin \theta \cos \theta - P L \sin \theta$$

For equilibrium; $\frac{d \Pi}{d \theta} = 0$

Therefore, \quad $k L^2 \sin \theta \cos \theta - P L \sin \theta = 0$

Therefore, \quad $P = k L \cos \theta$ for equilibrium

The post-buckling $P - \theta$ relationship is given above
ENERGY METHOD – LARGE DEFLECTIONS

- Large deflection analysis
  - See the post-buckling load-displacement path shown below
  - The load carrying capacity decreases after buckling at $P_{cr}$
  - $P_{cr}$ is where $\theta \to 0$

![Diagram showing load P/Pc and end rotation $\theta$](image)

$$P = k L \cos \theta \quad \text{for equilibrium}$$
$$\frac{P}{P_{cr}} = \cos \theta$$

Rigid bar with translational spring

End rotation $\theta$

Load $P/P_{cr}$

$\cos \theta = \frac{1}{2} k L^2 \sin^2 \theta - P L (1 - \cos \theta)$

$$\frac{d \Pi}{d \theta} = k L^2 \sin \theta \cos \theta - P L \sin \theta$$

$$\frac{d^2 \Pi}{d \theta^2} = k L^2 \cos 2 \theta - P L \cos \theta$$

For equilibrium $P = k L \cos \theta$

$$\therefore \frac{d^2 \Pi}{d \theta^2} = k L^2 \cos 2 \theta - k L^2 \cos^2 \theta$$

$$\therefore \frac{d^3 \Pi}{d \theta^3} = k L^2 (\cos^2 \theta - \sin^2 \theta) - k L^2 \cos^2 \theta$$

$$\therefore \frac{d^4 \Pi}{d \theta^4} = -k L^2 \sin^2 \theta$$

$$\therefore \frac{d^5 \Pi}{d \theta^5} < 0 \quad \text{ALWAYS. HENCE UNSTABLE}$$
ENERGY METHOD – LARGE DEFLECTIONS

- At $\theta=0$, the second derivative of $\Pi=0$. Therefore, inconclusive.
- Consider the Taylor series expansion of $\Pi$ at $\theta=0$

$$\Pi = \Pi_0 + \frac{d \Pi}{d \theta} \bigg|_{\theta=0} \theta + \frac{1}{2!} \left( \frac{d^2 \Pi}{d \theta^2} \right) \bigg|_{\theta=0} \theta^2 + \frac{1}{3!} \left( \frac{d^3 \Pi}{d \theta^3} \right) \bigg|_{\theta=0} \theta^3 + \frac{1}{4!} \left( \frac{d^4 \Pi}{d \theta^4} \right) \bigg|_{\theta=0} \theta^4 + \cdots + \frac{1}{n!} \left( \frac{d^n \Pi}{d \theta^n} \right) \bigg|_{\theta=0} \theta^n$$

- Determine the first non-zero term of $\Pi$,

$$\Pi = \frac{1}{2} k L^2 \sin^2 \theta - P L (1 - \cos \theta) = 0$$

$$\frac{d \Pi}{d \theta} = \frac{1}{2} k L^2 \sin 2\theta - P L \sin \theta = 0$$

$$\frac{d^2 \Pi}{d \theta^2} = k L^2 \cos 2\theta - P L \cos \theta = 0$$

$$\frac{d^3 \Pi}{d \theta^3} = -2k L^2 \sin 2\theta + P L \sin \theta = 0$$

$$\frac{d^4 \Pi}{d \theta^4} = -4k L^2 \cos 2\theta + P L \cos \theta$$

$$\therefore \frac{d^4 \Pi}{d \theta^4} = -4k L^2 + k L^2 = -3k L^2$$

$$\therefore \frac{d^4 \Pi}{d \theta^4} < 0$$

$$\therefore \text{UNSTABLE at } \theta = 0 \text{ when buckling occurs}$$

- Since the first non-zero term is $< 0$, the state is unstable at $P=P_{cr}$ and $\theta=$

**Graph:**

- Rigid bar with translational spring
- Load $P/P_{cr}$ vs. End rotation $\theta$
- UNSTABLE
- $P_{cr}$ occurs when buckling happens

8/25/2014
ENERGY METHOD - IMPERFECTIONS

Consider example 2 – but as a system with imperfections
- The initial imperfection given by the angle $\theta_0$ as shown below
- The free body diagram of the deformed system is shown below

\[
\Pi = U - W_e
\]
\[
U = \frac{1}{2} k L^2 (\sin \theta - \sin \theta_0)^2
\]
\[
W_e = P L (\cos \theta_0 - \cos \theta)
\]
\[
\Pi = \frac{1}{2} k L^2 (\sin \theta - \sin \theta_0)^2 - P L (\cos \theta_0 - \cos \theta)
\]
\[
d\frac{d\Pi}{d\theta} = k L^2 (\sin \theta - \sin \theta_0) \cos \theta - P L \sin \theta
\]
For equilibrium; \( \frac{d\Pi}{d\theta} = 0 \)
Therefore, \( k L^2 (\sin \theta - \sin \theta_0) \cos \theta - P L \sin \theta = 0 \)
Therefore, \( P = k L \cos \theta \left( 1 - \frac{\sin \theta_0}{\sin \theta} \right) \) for equilibrium
The equilibrium $P - \theta$ relationship is given above.
ENERGY METHOD - IMPERFECTIONS

$P = k L \cos \theta \left(1 - \frac{\sin \theta}{\sin \varphi}\right)$

$\therefore \frac{P}{P_{cr}} = \cos \theta \left(1 - \frac{\sin \theta}{\sin \varphi}\right)$

$P_{max} \Rightarrow \frac{dP}{d\varphi} = 0$

$\therefore k L(-\sin \varphi + \frac{\sin \theta}{\sin^2 \varphi}) = 0$

$\therefore \sin \theta_0 = \sin^3 \varphi$

$\therefore P_{max} = k L \cos^3 \varphi$

As shown in the figure, deflection starts as soon as loads are applied. There is no bifurcation of load-deformation path for imperfect systems. The load-deformation path remains in the same state through-out.

The smaller the imperfection magnitude, the close the load-deformation paths to the perfect system load–deformation path.

The magnitude of load, is influenced significantly by the imperfection magnitude.

All real systems have imperfections. They may be very small but will be there.

The magnitude of imperfection is not easy to know or guess. Hence if a perfect system analysis is done, the results will be close for an imperfect system with small imperfections.

However, for an unstable system – the effects of imperfections may be too large.
Energy Methods – Imperfect Systems

Examine the stability of the imperfect system using higher order derivatives of $\Pi$

$$\Pi = \frac{1}{2} k L^2 (\sin \theta - \sin \theta_b)^2 - P L (\cos \theta_b - \cos \theta)$$

$$\frac{d \Pi}{d \theta} = k L^2 (\sin \theta - \sin \theta_b) \cos \theta - P L \sin \theta$$

$$\frac{d^2 \Pi}{d \theta^2} = k L^2 (\cos 2 \theta + \sin \theta_b \sin \theta) - P L \cos \theta$$

For equilibrium $P = k L \left(1 - \frac{\sin \theta_b}{\sin \theta}\right)$

$$\therefore \frac{d^2 \Pi}{d \theta^2} = k L^2 (\cos 2 \theta + \sin \theta_b \sin \theta) - k L^2 \left(1 - \frac{\sin \theta_b}{\sin \theta}\right) \cos^2 \theta$$

$$\therefore \frac{d^2 \Pi}{d \theta^2} = k L^2 \left[ \cos^2 \theta - \sin^2 \theta + \sin \theta_b \sin \theta - \cos^2 \theta + \frac{\sin \theta_b \cos^2 \theta}{\sin \theta} \right]$$

$$\therefore \frac{d^2 \Pi}{d \theta^2} = k L^2 \left[-\sin^2 \theta + \sin \theta_b \sin \theta + \frac{\sin \theta_b \cos^2 \theta}{\sin \theta} \right]$$

$$\therefore \frac{d^2 \Pi}{d \theta^2} = k L^2 \left[-\sin^2 \theta + \sin \theta_b (\sin^2 \theta + \cos^2 \theta) \right]$$

$$\therefore \frac{d^2 \Pi}{d \theta^2} = k L^2 \left[-\sin^2 \theta + \sin \theta_b \frac{\sin \theta}{\sin \theta} \right]$$

When $P < P_{\text{max}}$:

- $\frac{d^2 \Pi}{d \theta^2} > 0$: Stable
- $\frac{d^2 \Pi}{d \theta^2} < 0$: Unstable

When $P > P_{\text{max}}$:

- $\frac{d^2 \Pi}{d \theta^2} > 0$: Stable
- $\frac{d^2 \Pi}{d \theta^2} < 0$: Unstable

$P = k L \cos \theta \left(1 - \frac{\sin \theta_b}{\sin \theta}\right)$ and $P_{\text{max}} = k L \cos^3 \theta$

When $P < P_{\text{max}}$:

- $k L \cos \theta \left(1 - \frac{\sin \theta_b}{\sin \theta}\right) < k L \cos^3 \theta$

  - $1 - \frac{\sin \theta_b}{\sin \theta} < \cos^2 \theta$

  - $1 - \frac{\sin \theta_b}{\sin \theta} < 1 - \sin^2 \theta$

  - $\sin \theta_b < \sin^3 \theta$

  - $\frac{d^2 \Pi}{d \theta^2} = k L^2 \left[\frac{\sin \theta_b - \sin^3 \theta}{\sin \theta}\right] > 0$

When $P > P_{\text{max}}$:

- $k L \cos \theta \left(1 - \frac{\sin \theta_b}{\sin \theta}\right) > k L \cos^3 \theta$

  - $1 - \frac{\sin \theta_b}{\sin \theta} > \cos^2 \theta$

  - $1 - \frac{\sin \theta_b}{\sin \theta} > 1 - \sin^2 \theta$

  - $\sin \theta_b < \sin^3 \theta$

  - $\frac{d^2 \Pi}{d \theta^2} = k L^2 \left[\frac{\sin \theta_b - \sin^3 \theta}{\sin \theta}\right] < 0$
Chapter 2. – Second-Order Differential Equations

This chapter focuses on deriving second-order differential equations governing the behavior of elastic members

- 2.1 – First order differential equations
- 2.2 – Second-order differential equations

2.1 First-Order Differential Equations

- Governing the behavior of structural members
  - Elastic, Homogenous, and Isotropic
  - Strains and deformations are really small – small deflection theory
  - Equations of equilibrium in undeformed state

- Consider the behavior of a beam subjected to bending and axial forces
2.1 First-Order Differential Equations

- Assume tensile forces are positive and moments are positive according to the right-hand rule.
- Longitudinal stress due to bending
  \[ \sigma = \frac{P}{A} + \frac{M_y}{I_y} y - \frac{M_x}{I_x} x \]
- This is true when the x-y axis system is a centroidal and principal axis system.
  \[ \int_A y \, dA = \int_A x \, dA = \int_A x \, y \, dA = 0 \quad \therefore \text{Centroidal axis} \]
  \[ \int_A dA = A; \quad \int_A x^2 \, dA = I_x; \quad \int_A y^2 \, dA = I_y \]
  \[ I_x \text{ and } I_y \text{ are principal moments of inertia} \]

2.1 First-Order Differential Equations

- The corresponding strain is \[ \varepsilon = \frac{P}{AE} + \frac{M_x}{EI_x} y - \frac{M_y}{EI_y} x \]
  \[ \text{If } P = M_y = 0, \text{ then } \varepsilon = \frac{M_x}{EI_x} y \]
- Plane-sections remain plane and perpendicular to centroidal axis before and after bending.
- The measure of bending is curvature \( \phi \) which denotes the change in the slope of the centroidal axis between two point \( dz \) apart
  \[ \tan \phi_y = \frac{\varepsilon}{y} \]
  \[ \therefore \phi_y = \frac{\varepsilon}{y} \]
  \[ \therefore \phi_y = \frac{M_y}{EI_y} \]
  \[ \therefore M_x = E I_x \phi_y \quad \text{and similarly} \quad M_y = E I_y \phi_y \]
2.1 First-Order Differential Equations

- Shear Stresses due to bending

\[ \tau_t = \frac{V_y}{I_y} \int_0^s y t \, ds \]
\[ \tau_t = \frac{V_x}{I_x} \int_0^s x t \, ds \]

- Differential equations of bending
  - Assume principle of superposition
    - Treat forces and deformations in y-z and x-z plane separately
    - Both the end shears and \( q_y \) act in a plane parallel to the y-z plane through the shear center S

\[ \frac{dV_y}{dz} = -q_y \]
\[ \frac{dM_y}{dz} = V_y \]
\[ \therefore \frac{d^2M_y}{dz^2} = -q_y \]
\[ \frac{d^2(E I_y \phi_y)}{dz^2} = -q_y \]
\[ \therefore E I_y \phi_y = -q_y \]
2.1 First-Order Differential Equations

- Differential equations of bending

\[ E I \phi_y'' = -q_y \]
\[ \phi_y = \frac{v''}{\left[1 + (v')^2\right]^{1/2}} \]

For small deflection
\[ \phi_y = -v'' \]
\[ \therefore E I, v'' = q_s \]

Similarly \[ E I, u'' = q_s \]

- Fourth-order differential equations using first-order force-deformation theory

Torsion behavior – Pure and Warping Torsion

- Torsion behavior – uncoupled from bending behavior

- Thin walled open cross-section subjected to torsional moment
  - This moment will cause twisting and warping of the cross-section.
  - The cross-section will undergo pure and warping torsion behavior.
  - Pure torsion will produce only shear stresses in the section
  - Warping torsion will produce both longitudinal and shear stresses
  - The internal moment produced by the pure torsion response will be equal to \( M_{sv} \) and the internal moment produced by the warping torsion response will be equal to \( M_w \).
  - The external moment will be equilibrated by the produced internal moments

\[ M_Z = M_{SV} + M_W \]
Pure and Warping Torsion

\[ M_Z = M_{SV} + M_W \]

Where,

- \( M_{SV} = G K_T \phi' \) and \( M_W = -E I_w \phi'''' \)
- \( M_{SV} \) = Pure or Saint Venant’s torsion moment
- \( K_T = J = \) Torsional constant =
- \( \phi \) is the angle of twist of the cross-section. It is a function of \( z \).
- \( I_w \) is the warping moment of inertia of the cross-section. This is a new cross-sectional property you may not have seen before.

\[ M_Z = G K_T \phi' - E I_w \phi'''' \] \hspace{1cm} (3), differential equation of torsion

Pure Torsion Differential Equation

- Lets look closely at pure or Saint Venant’s torsion. This occurs when the warping of the cross-section is unrestrained or absent

\[ \gamma \, dz = r \, d\phi \]
\[ \therefore \gamma = \frac{r \, d\phi}{dz} = r \, \phi' \]
\[ \therefore \tau = G \, r \, \phi' \]

\[ : M_{SV} = \int_A \tau \, r \, dA = G \, \phi' \int_A r^2 \, dA \]
\[ : M_{SV} = G K_T \phi' \]

where, \( K_T = J = \int_A r^2 \, dA \)

- For a circular cross-section – warping is absent. For thin-walled open cross-sections, warping will occur.
- The out of plane warping deformation \( w \) can be calculated using an equation I will not show.
Pure Torsion Stresses

The torsional shear stresses vary linearly about the center of the thin plate

\[ \tau_{sv} = G \cdot r \cdot \phi' \]

\[ (\tau_{sv})_{\text{max}} = G \cdot t \cdot \phi' \]

Warping deformations

- The warping produced by pure torsion can be restrained by the: (a) end conditions, or (b) variation in the applied torsional moment (non-uniform moment)
- The restraint to out-of-plane warping deformations will produce longitudinal stresses (\(\sigma_w\)), and their variation along the length will produce warping shear stresses (\(\tau_w\)).
Warping Torsion Differential Equation

- Lets take a look at an approximate derivation of the warping torsion differential equation.
  - This is valid only for I and C shaped sections.
    \[ u_f = \phi \left( \frac{h}{2} \right) \]
    where \( u_f \) = flange lateral displacement
    \( M_f \) = moment in the flange
    \( V_f \) = Shear force in the flange
    \[ E I_f u''_f = -M_f \] .......borrowing d.e. of bending
    \[ E I_f u''_w = -V_f \]
    \( M_w = V_f h \)
    \[ \therefore M_w = -E I_f u''_w h \]
    \[ \therefore M_w = -E I_f \frac{h^2}{2} \phi'' \]
    \[ \therefore M_w = -E I_w \phi'' \]
    where \( I_w \) is warping moment of inertia \( \rightarrow \) new section property

Torsion Differential Equation Solution

- Torsion differential equation \( M_Z = M_{SV} + M_W = G K_T \phi' - E I_W \phi'' \)
- This differential equation is for the case of concentrated torque
  \[ G K_T \phi' - E I_W \phi'' = M_Z \]
  \[ \therefore \phi'' = \frac{G K_T}{E I_W} \phi' = -\frac{M_Z}{E I_W} \]
  \[ \therefore \phi'' + \lambda^2 \phi' = -\frac{M_Z}{E I_W} \]
  \[ \therefore \phi = C_1 + C_2 \cosh \lambda z + C_3 \sinh \lambda z + \frac{M_z z}{\lambda^2 E I_W} \]
- Torsion differential equation for the case of distributed torque
  \[ m_z = -\frac{dM_z}{dz} \]
  \[ G K_T \phi'' + E I_W \phi'' = -m_z \]
  \[ \therefore \phi'' + \frac{G K_T}{E I_W} \phi'' = -\frac{m_z}{E I_W} \]
  \[ \therefore \phi'' + \lambda^2 \phi'' = -\frac{m_z}{E I_W} \]
  \[ \therefore \phi = C_4 + C_5 z + C_6 \cosh \lambda z + C_7 \sinh \lambda z + \frac{m_z z^2}{2 G K_T} \]
- The coefficients \( C_1 \ldots C_6 \) can be obtained using end conditions
Torsion Differential Equation Solution

- Torsionally fixed end conditions are given by \( \phi = \phi' = 0 \)
- These imply that twisting and warping at the fixed end are fully restrained. Therefore, equal to zero.
- Torsionally pinned or simply-supported end conditions given by:
  \( \phi = \phi'' = 0 \)
- These imply that at the pinned end twisting is fully restrained (\( \phi=0 \)) and warping is unrestrained or free. Therefore, \( \sigma_w=0 \rightarrow \phi''=0 \)
- Torsionally free end conditions given by \( \phi' = \phi'' = \phi''' = 0 \)
- These imply that at the free end, the section is free to warp and there are no warping normal or shear stresses.
- Results for various torsional loading conditions given in the AISC Design Guide 9 – can be obtained from my private site

Warping Torsion Stresses

- Restraint to warping produces longitudinal and shear stresses
  \[
  \sigma_w = E \left( W_n \phi' \right) \\
  \tau_w = -E \left( S_w \phi'' \right)
  \]
  where,
  \( W_n = \text{NormalizedUnit Warping~Section Property} \)
  \( S_w = \text{Warping StaticalMoment~Section Property} \)
- The variation of these stresses over the section is defined by the section property \( W_n \) and \( S_w \)
- The variation of these stresses along the length of the beam is defined by the derivatives of \( \phi \).
- 
  **Note that a major difference between bending and torsional behavior is**
  - The stress variation along length for torsion is defined by derivatives of \( \phi \), which cannot be obtained using force equilibrium.
  - The stress variation along length for bending is defined by derivatives of \( \nu \), which can be obtained using force equilibrium (M, V diagrams).
Torsional Stresses

![Diagram of Torsional Stresses](image)

**Figure 4.2.**

**Figure 4.3.**
Torsional Section Properties for I and C Shapes

\[ W_1, M_1, S_1, \text{ and } H-P \text{-Shapes} \]

\[ \phi \text{ and derivatives for concentrated torque at midspan} \]

\[ \text{Case 3 } \phi \left( \frac{W}{y} \right) \]

\[ \text{Case 3 } \psi \left( \frac{W}{y} \right) \]

\[ \text{Case 3 } \omega \left( \frac{W}{y} \right) \]
Summary of first order differential equations

\[- E I_x v'' = M_x \quad \ldots \ldots (1)\]
\[E I_y u'' = M_y \quad \ldots \ldots (2)\]
\[G K_I \phi' - E I_w \phi'' = M_z \quad \ldots \ldots (3)\]

NOTES:
(1) Three uncoupled differential equations
(2) Elastic material – first order force-deformation theory
(3) Small deflections only
(4) Assumes – no influence of one force on other deformations
(5) Equations of equilibrium in the undeformed state.

Chapter 2. – Second-Order Differential Equations

- This chapter focuses on deriving second-order differential equations governing the behavior of elastic members
- 2.1 – First order differential equations
- 2.2 – Second-order differential equations
2.2 Second-Order Differential Equations

- Governing the behavior of structural members
  - Elastic, Homogenous, and Isotropic
  - Strains and deformations are really small – small deflection theory
  - Equations of equilibrium in deformed state
  - The deformations and internal forces are no longer independent. They must be combined to consider effects.

- Consider the behavior of a member subjected to combined axial forces and bending moments at the ends. No torsional forces are applied explicitly – because that is very rare for CE structures.

Member model and loading conditions

- Member is initially straight and prismatic. It has a thin-walled open cross-section
- Member ends are pinned and prevented from translation.
- The forces are applied only at the member ends
- These consist only of axial and bending moment forces $P, M_{TX}, M_{TY}, M_{BX}, M_{BY}$
- Assume elastic behavior with small deflections
- Right-hand rule for positive moments and reactions and $P$ assumed positive.
Member displacements (cross-sectional)

- Consider the middle line of thin-walled cross-section
- x and y are principal coordinates through centroid C
- Q is any point on the middle line. It has coordinates (x, y).
- Shear center S coordinates are \((x_0, y_0)\)
- Shear center S displacements are \(u, v,\) and \(\phi\)

Displacements of Q are:
\[
\begin{align*}
    u_Q &= u + a \phi \sin \alpha \\
    v_Q &= v - a \phi \cos \alpha \\
\end{align*}
\]
where a is the distance from S.
- But, \(\sin \alpha = (y_0 - y) / a\)
- \(\cos \alpha = (x_0 - x) / a\)
- Therefore, displacements of Q:
\[
\begin{align*}
    u_Q &= u + \phi (y_0 - y) \\
    v_Q &= v - \phi (x_0 - x) \\
\end{align*}
\]
- Displacements of centroid C:
\[
\begin{align*}
    u_C &= u + \phi (y_0) \\
    v_C &= v - \phi (x_0) \\
\end{align*}
\]
Internal forces – second-order effects

- Consider the free body diagrams of the member in the deformed state.
- Look at the deformed state in the x-z and y-z planes in this Figure.
- The internal resisting moment at a distance \( z \) from the lower end are:
  
  \[
  M_x = - M_{BX} + R_y z + P v_c \\
  M_y = - M_{BY} + R_x z - P u_c 
  \]

- The end reactions \( R_x \) and \( R_y \) are:
  
  \[
  R_x = (M_{TX} + M_{BY}) / L \\
  R_y = (M_{TX} + M_{BX}) / L 
  \]

- Therefore,

  \[
  M_x = - M_{BX} + \frac{z}{L} (M_{TX} + M_{BX}) + P (y - \phi x_0) \\
  M_y = - M_{BY} + \frac{z}{L} (M_{TY} + M_{BY}) - P (u + \phi y_0) 
  \]
In the deformed state, the cross-section is such that the principal coordinate systems are changed from $x$-$y$-$z$ to the $\xi$-$\eta$-$\zeta$ system.
Internal forces in the deformed state

- The internal forces $M_x$ and $M_y$ must be transformed to these new $\xi$–$\eta$–$\zeta$ axes.
- Since the angle $\phi$ is small
- $M_\xi = M_x + \phi M_y$
- $M_\eta = M_y - \phi M_x$

\[
M_x = -M_{Bx} + \frac{z}{L} (M_{Tx} + M_{Bx}) + P(v - \phi x_0)
\]

\[
M_y = -M_{By} + \frac{z}{L} (M_{Ty} + M_{By}) - P(u + \phi y_0)
\]

\[
\therefore M_\xi = -M_{Bx} + \frac{z}{L} (M_{Tx} + M_{Bx}) + P v - \phi \left( P x_0 + M_{By} - \frac{z}{L} (M_{Ty} + M_{By}) \right)
\]

\[
\therefore M_\eta = -M_{By} + \frac{z}{L} (M_{Ty} + M_{By}) + P u + \phi \left( -P y_0 + M_{Bx} - \frac{z}{L} (M_{Tx} + M_{Bx}) \right)
\]
Twisting component of internal forces

- Twisting moments $M_\phi$ are produced by the internal and external forces.

- There are four components contributing to the total $M_\phi$:
  1. Contribution from $M_x$ and $M_y - M_{\phi 1}$
  2. Contribution from axial force $P - M_{\phi 2}$
  3. Contribution from normal stress $\sigma - M_{\phi 3}$
  4. Contribution from end reactions $R_x$ and $R_y - M_{\phi 4}$

- The total twisting moment $M_\phi = M_{\phi 1} + M_{\phi 2} + M_{\phi 3} + M_{\phi 4}$

Twisting component – 1 of 4

- Twisting moment due to $M_x$ & $M_y$
  
  \[ M_{\phi 1} = M_x \sin (du/dz) + M_y \sin (dv/dz) \]

- Therefore, due to small angles, $M_{\phi 1} = M_x \ du/dz + M_y \ dv/dz$

  \[ M_{\phi 1} = M_x \ u' + M_y \ v' \]
The axial load $P$ acts along the original vertical direction.

In the deformed state of the member, the longitudinal axis $\zeta$ is not vertical. Hence $P$ will have components producing shears.

These components will act at the centroid where $P$ acts and will have values as shown above — assuming small angles.

These shears will act at the centroid $C$, which is eccentric with respect to the shear center $S$. Therefore, they will produce secondary twisting.

$$M_{\zeta z} = P \left( y_0 \frac{du}{dz} - x_0 \frac{dv}{dz} \right)$$

Therefore, $M_{\zeta z} = P \left( y_0 u' - x_0 v' \right)$
Twisting component – 3 of 4

- The end reactions (shears) $R_x$ and $R_y$ act at the shear center $S$ at the ends. But, along the member ends, the shear center will move by $u$, $v$, and $\phi$.
- Hence, these reactions will also have a twisting effect produced by their eccentricity with respect to the shear center $S$.

$$M_{\zeta 4} + R_y u + R_x v = 0$$

Therefore,

$$M_{\zeta 4} = -(M_{TY} + M_{BY}) \frac{v}{L} - (M_{TX} + M_{BX}) \frac{u}{L}$$

Twisting component – 4 of 4

- Wagner’s effect or contribution – complicated.
- Two cross-sections that are $d\zeta$ apart will warp with respect to each other.
- The stress element $\sigma \, dA$ will become inclined by angle $(a \, d\phi/d\zeta)$ with respect to $d\zeta$ axis.
- Twist produced by each stress element about $S$ is equal to

$$dM_{\zeta 3} = -a(\sigma \, dA) \left( a \, \frac{d\phi}{d\zeta} \right)$$

$$\therefore M_{\zeta 3} = -a \int d\phi \sigma \, a^2 \, dA$$

Fig. 2.37. Twisting due to the differential warping of two adjacent cross sections
Twisting component – 4 of 4

\[\int_A \sigma a^2 dA = \bar{K}\]

\[\therefore M_\xi = -\bar{K} \frac{d\phi}{d\zeta}\]

\[\therefore M_\chi = -\bar{K} \frac{d\phi}{dz} \quad \text{for small angles}\]
Total Twisting Component

\[ M_z = M_{z1} + M_{z2} + M_{z3} + M_{z4} \]

\[ M_{z1} = M_x u' + M_y v' \]
\[ M_{z2} = P (y_0 u' - x_0 v') \]
\[ M_{z4} = - (M_{TY} + M_{BY}) v/L - (M_{TX} + M_{BX}) u/L \]
\[ M_{z3} = -K \phi' \]

Therefore,

\[ M_z = M_x u' + M_y v' + P (y_0 u' - x_0 v') - (M_{TY} + M_{BY}) v/L - (M_{TX} + M_{BX}) u/L - K \phi' \]

\[ M_\phi = -M_{BY} + \frac{z}{L} (M_{TY} + M_{BY}) + P u + \phi \left( -P y_0 + M_{BX} - \frac{z}{L} (M_{TX} + M_{BX}) \right) \]

Total Twisting Component

\[ M_z = M_{z1} + M_{z2} + M_{z3} + M_{z4} \]

\[ M_{z1} = M_x u' + M_y v' \quad M_{z2} = P (y_0 u' - x_0 v') \quad M_{z3} = -K \phi' \]
\[ M_{z4} = - (M_{TY} + M_{BY}) v/L - (M_{TX} + M_{BX}) u/L \]

Therefore,

\[ M_z = M_x u' + M_y v' + P (y_0 u' - x_0 v') - (M_{TY} + M_{BY}) v/L - (M_{TX} + M_{BX}) u/L - K \phi' \]

\[ M_\phi = -M_{BY} + \frac{z}{L} (M_{TY} + M_{BY}) + P u + \phi \left( -P y_0 + M_{BX} - \frac{z}{L} (M_{TX} + M_{BX}) \right) \]
Thus, now we have the internal moments about the \( \xi - \eta - \zeta \) axes for the deformed member cross-section.

\[
M_\xi = -M_{rx} + \frac{z}{L}(M_{tx} + M_{rx}) + P v - \phi \left( P x_0 + M_{by} - \frac{z}{L}(M_{ty} + M_{by}) \right)
\]

\[
M_\eta = -M_{by} + \frac{z}{L}(M_{ty} + M_{by}) - P u + \phi \left( -P y_0 + M_{bx} - \frac{z}{L}(M_{tx} + M_{bx}) \right)
\]

\[
M_\zeta = (-M_{rx} - \frac{z}{L}(M_{rx} + M_{tx}) + P y_0) u' + (-M_{by} - \frac{z}{L}(M_{by} + M_{ty}) - P x_0) v' - (M_{ty} + M_{by}) \frac{v}{L} - (M_{tx} + M_{bx}) \frac{u}{L} \bar{K} \phi'
\]

The internal moments \( M_\xi, M_\eta, \) and \( M_\zeta \) will still produce flexural bending about the centroidal principal axis and twisting about the shear center.

The flexural bending about the principal axes will produce linearly varying longitudinal stresses.

The torsional moment will produce longitudinal and shear stresses due to warping and pure torsion.

The differential equations relating moments to deformations are still valid. Therefore,

\[
M_\xi = - E I_x v'' \quad (I_x = l_x)
\]

\[
M_\eta = E I_\eta u'' \quad (I_\eta = l_\eta)
\]

\[
M_\zeta = G K_T \phi' - E I_w \phi''
\]
Internal Moment – Deformation Relations

Therefore,

\[ M_z = -E I_x \phi' = -M_{by} + \frac{z}{L}(M_{tx} + M_{bx}) + P\phi \left( P x_0 + M_{by} - \frac{z}{L}(M_{tx} + M_{bx}) \right) \]

\[ M_y = E I_y \phi'' = -M_{by} + \frac{z}{L}(M_{ty} + M_{by}) - P u + \phi \left( -P y_0 + M_{bx} - \frac{z}{L}(M_{tx} + M_{bx}) \right) \]

\[ M_z = G K T \phi' - E I_w \phi'' = (-M_{by} - \frac{z}{L}(M_{bx} + M_{tx}) + P y_0)u' + \]

\[ (-M_{by} - \frac{z}{L}(M_{by} + M_{ty}) - P x_0)\phi' - (M_{ty} + M_{by})u' - (M_{tx} + M_{bx})\frac{u}{L} - K \phi' \]

Second-Order Differential Equations

You end up with three coupled differential equations that relate the applied forces and moments to the deformations \( u, v, \) and \( \phi. \)

Therefore,

1. \[ E I_x \phi'' + P \phi' - \phi \left( P x_0 + M_{by} - \frac{z}{L}(M_{tx} + M_{bx}) \right) = M_{bx} - \frac{z}{L}(M_{tx} + M_{bx}) \]

2. \[ E I_y \phi'' + P u - \phi \left( -P y_0 + M_{bx} - \frac{z}{L}(M_{tx} + M_{bx}) \right) = -M_{by} + \frac{z}{L}(M_{ty} + M_{by}) \]

3. \[ E I_w \phi'' - (G K T + K) \phi' + u' \left( -M_{bx} - \frac{z}{L}(M_{bx} + M_{tx}) + P y_0 \right) \]

\[ -v' \left( M_{by} + \frac{z}{L}(M_{by} + M_{ty}) + P x_0 \right) - \frac{v}{L}(M_{ty} + M_{by}) - \frac{u}{L}(M_{tx} + M_{bx}) = 0 \]

These differential equations can be used to investigate the elastic behavior and buckling of beams, columns, beam-columns and also complete frames — that will form a major part of this course.
Chapter 3. Structural Columns

- 3.1 Elastic Buckling of Columns
- 3.2 Elastic Buckling of Column Systems – Frames
- 3.3 Inelastic Buckling of Columns
- 3.4 Column Design Provisions (U.S. and Abroad)

3.1 Elastic Buckling of Columns

Start out with the second-order differential equations derived in Chapter 2. Substitute $P=P$ and $M_{TY} = M_{BY} = M_{TX} = M_{BX} = 0$

Therefore, the second-order differential equations simplify to:

1. $EI_x v'' + P v - \phi (P x_0) = 0$
2. $EI_y u'' + P u - \phi (-P y_0) = 0$
3. $EI_w \phi'' - (G K_T + K) \phi' + u' (P y_0) - v' (P x_0) = 0$

This is all great, but before we proceed any further we need to deal with Wagner’s effect – which is a little complicated.
Wagner's effect for columns

\[ \bar{K} \phi' = \int_A \sigma a^2 \phi' dA \]

where,

\[ \sigma = -\frac{P}{A} + \frac{M_z}{I_z} - \frac{M_y}{I_y} + EW_\phi \phi' \]

\[ M_z = P (v - \phi x_0) \]

\[ M_y = -P (u + \phi y_0) \]

\[ \therefore \bar{K} \phi' = \int_A \left[ -\frac{P}{A} + \frac{P (v - \phi x_0) y}{I_z} - \frac{P (u + \phi y_0) x}{I_y} + EW_\phi \phi' \right] \phi' a^2 dA \]

Neglecting higher order terms:

\[ \bar{K} \phi' = -\frac{P}{A} \phi' \int_A a^2 dA \]

---

Wagner's effect for columns

**But,** \( a^2 = (x_0 - x)^2 + (y_0 - y)^2 \)

\[ \therefore \int_A a^2 dA = \int_A (x_0 - x)^2 + (y_0 - y)^2 dA \]

\[ \therefore \int_A a^2 dA = \int_A \left[ x_0^2 + y_0^2 + x^2 + y^2 - 2x_0 x - 2y_0 y \right] dA \]

\[ \therefore \int_A a^2 dA = \left[ x_0^2 + y_0^2 \right] dA + \int_A x^2 dA + \int_A y^2 dA - 2x_0 \int_A x dA - 2y_0 \int_A y dA \]

\[ \therefore \int_A a^2 dA = (x_0^2 + y_0^2) A + I_x + I_y \]

Finally,

\[ \therefore \bar{K} \phi' = -\frac{P}{A} \left[ (x_0^2 + y_0^2) A + I_x + I_y \right] \phi' \]

\[ \therefore \bar{K} \phi' = -P \left[ (x_0^2 + y_0^2) + \frac{I_x + I_y}{A} \right] \phi' \]

Let \( \tau_0^2 = \left[ (x_0^2 + y_0^2) + \frac{I_x + I_y}{A} \right] \]

\[ \therefore \bar{K} \phi' = -P \tau_0^2 \phi' \]
Second-order differential equations for columns

- Simplify to:

\[ \begin{align*}
1 & \quad E I_x v'' + P v - \phi(P x_0) = 0 \\
2 & \quad E I_y u'' + P u + \phi(P y_0) = 0 \\
3 & \quad E I_w \phi'' + (P \overline{r}^2 - G K_T) \phi' + u' (P y_0) - v' (P x_0) = 0
\end{align*} \]

- Where

\[ F_0^2 = x_0^2 + y_0^2 + \frac{I_x + I_y}{A} \]

Column buckling – doubly symmetric section

- For a doubly symmetric section, the shear center is located at the centroid \( x_0 = y_0 = 0 \). Therefore, the three equations become uncoupled

\[ \begin{align*}
1 & \quad E I_x v'' + P v = 0 \\
2 & \quad E I_y u'' + P u = 0 \\
3 & \quad E I_w \phi'' + (P \overline{r}^2 - G K_T) \phi' = 0
\end{align*} \]

- Take two derivatives of the first two equations and one more derivative of the third equation.

\[ \begin{align*}
1 & \quad E I_x v''' + P v''' = 0 \\
2 & \quad E I_y u''' + P u''' = 0 \\
3 & \quad E I_w \phi''' + (P \overline{r}^2 - G K_T) \phi'' = 0
\end{align*} \]

Let \( F_v^2 = \frac{P}{E I_x} \), \( F_u^2 = \frac{P}{E I_y} \), \( F_\phi^2 = \frac{P \overline{r}^2 - G K_T}{E I_w} \)}
Column buckling – doubly symmetric section

1. \( v'''' + F_v^2 u'' = 0 \)
2. \( u'''' + F_u^2 u'' = 0 \)
3. \( \phi'''' + F_\phi^2 \phi'' = 0 \)

- All three equations are similar and of the fourth order. The solution will be of the form \( C_1 \sin \lambda z + C_2 \cos \lambda z + C_3 z + C_4 \)
- Need four boundary conditions to evaluate the constant \( C_1 \ldots C_4 \)
- For the simply supported case, the boundary conditions are:
  \( u = u'' = 0; \ v = v'' = 0; \ f = f'' = 0 \)
- Let's solve one differential equation – the solution will be valid for all three.

### Solution
\[
\begin{align*}
v &= C_1 \sin \lambda z + C_2 \cos \lambda z + C_3 z + C_4 \\
\therefore v'' &= -C_1 \lambda^2 \sin \lambda z - C_2 \lambda^2 \cos \lambda z \\
\text{Boundary conditions:} \\
\begin{align*}
v(0) &= v''(0) = v(L) = v''(L) = 0 \\
C_2 + C_4 &= 0 \\
C_2 &= 0 \\
C_1 \sin F_v L + C_2 \cos F_v L + C_3 L + C_4 &= 0 \\
-C_1 F_v^2 \sin F_v L - C_2 F_v^2 \cos F_v L &= 0 \\
\begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\sin F_v L & \cos F_v L & L & 1 \\
-F_v^2 \sin F_v L & -F_v^2 \cos F_v L & 0 & 0
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix} &= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\end{align*}
\]

The coefficient matrix \( \begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\sin F_v L & \cos F_v L & L & 1 \\
-F_v^2 \sin F_v L & -F_v^2 \cos F_v L & 0 & 0
\end{bmatrix} \begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \)

- \( F_v L = n \pi \)
- \( F_v = \sqrt{\frac{P}{E I_v}} = \frac{n \pi}{L} \)
- \( P_v = \frac{n^2 \pi^2 E I_v}{L^2} \)
- Smallest value of \( n = 1 \):
  \[
  P_v = \pi^2 E I_v \]

\[
F_v = \sqrt{\frac{P}{E I_v}} = \frac{n \pi}{L}
\]

\[
P_v = \frac{n^2 \pi^2 E I_v}{L^2}
\]

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\sin F_v L & \cos F_v L & L & 1 \\
-F_v^2 \sin F_v L & -F_v^2 \cos F_v L & 0 & 0
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
Summary

\[
\begin{align*}
P_x &= \frac{\pi^2 EI_y}{L^2} \\
P_y &= \frac{\pi^2 EI_z}{L^2} \\
\phi &= \frac{\pi^2 EI_w + G K_T}{L^2} \frac{1}{r_0^2}
\end{align*}
\]

Similarly, \(\sin F_x L = 0\)

\[
\therefore F_x = \frac{n \pi}{L}
\]

Similarly, \(\sin F_y L = 0\)

\[
\therefore F_y = \frac{n \pi}{L}
\]

\[
\therefore F_\phi = \sqrt{\frac{P \pi^2 L - G K_T}{E I_y}} = \frac{n \pi}{L}
\]

Smallest value of \(n = 1\):

\[
P_x = \frac{\pi^2 EI_y}{L}
\]

\[
P_y = \frac{\pi^2 EI_z}{L}
\]

\[
\phi = \left(\frac{\pi^2 EI_w + G K_T}{L^2}\right) \frac{1}{r_0^2}
\]

\[
P = \left(\frac{\pi^2 EI_y + G K_T}{L^2}\right) \frac{1}{r_0^2}
\]

Thus, for a doubly symmetric cross-section, there are three distinct buckling loads \(P_x\), \(P_y\), and \(P_z\).

The corresponding buckling modes are:

\[
v = C_1 \sin(\pi z/L), \quad u = C_2 \sin(\pi z/L), \quad \phi = C_3 \sin(\pi z/L).
\]

These are, flexural buckling about the x and y axes and torsional buckling about the z axis.

As you can see, the three buckling modes are uncoupled. You must compute all three buckling load values.

The smallest of three buckling loads will govern the buckling of the column.
Column buckling – boundary conditions

Consider the case of fix-fix boundary conditions:

\[ v'' + F_i^2 v^n = 0 \]

**Solution is**

\[ v = C_1 \sin F_i z + C_2 \cos F_i z + C_3 z + C_4 \]

\[ \therefore v' = C_1 F_i \cos F_i z - C_2 F_i \sin F_i z + C_3 \]

**Boundary conditions:**

\[ v(0) = v'(0) = v(L) = v'(L) = 0 \]

\[ \therefore C_2 + C_4 = 0 \]

\[ \therefore v(0) = 0 \]

\[ C_1 F_i + C_3 = 0 \]

\[ \therefore v'(0) = 0 \]

\[ C_1 \sin F_i L + C_2 \cos F_i L + C_3 L + C_4 \]

\[ \therefore v(L) = 0 \]

\[ C_1 F_i \cos F_i L - C_2 F_i \sin F_i L + C_3 \]

\[ \therefore v'(L) = 0 \]

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
F_i & 0 & 1 & 0 \\
\sin F_i L & \cos F_i L & L & 1 \\
F_i \cos F_i L & -F_i \sin F_i L & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

The coefficient matrix \( = 0 \)

\[ 
\therefore F_i L \sin F_i L - 2 \cos F_i L + 2 = 0 
\]

\[ 2 \sin \frac{F_i L}{2} \left[ F_i L \cos \frac{F_i L}{2} + 2 \sin \frac{F_i L}{2} \right] = 0 \]

\[ \therefore \frac{F_i L}{2} = n \pi \]

\[ \therefore F_i = \frac{2 n \pi}{L} \]

\[ P_i = \frac{4 n^2 \pi^2}{L^2} E I_s \]

**Smallest value of** \( n = 1 \):

\[ P_i = \frac{\pi^2 E I_s}{(0.5 L)^2} = \frac{\pi^2 E I_s}{(K L)^2} \]

### Column Boundary Conditions

- The critical buckling loads for columns with different boundary conditions can be expressed as:

\[
\begin{align*}
P_s &= \frac{\pi^2 E I_s}{(K_x L)^2} \\
P &= \frac{\pi^2 E I_s}{(K_y L)^2} \\
P_3 &= \left[ \frac{\pi^2 E I_s}{(K_z L)^2} + G K_f \right] \frac{1}{I_o}
\end{align*}
\]

- Where, \( K_x \), \( K_y \), and \( K_z \) are functions of the boundary conditions:
  - \( K=1 \) for simply supported boundary conditions
  - \( K=0.5 \) for fix-fix boundary conditions
  - \( K=0.7 \) for fix-simple boundary conditions
Column buckling – example.

- Consider a wide flange column W27 x 84. The boundary conditions are:
  \( \nu=\nu''=u''=\phi''=0 \) at \( z=0 \), and \( \nu=\nu''=u''=\phi''=0 \) at \( z=L \)
- For flexural buckling about the \( x \)-axis – simply supported – \( K_x=1.0 \)
- For flexural buckling about the \( y \)-axis – fixed at both ends – \( K_y = 0.5 \)
- For torsional buckling about the \( z \)-axis – pin-fix at two ends - \( K_z=0.7 \)

\[
P_x = \frac{\pi^2 EI_x}{(K_x L)^2} = \frac{\pi^2 EA r_x^2}{(K_x L)^2} = \frac{\pi^2 EA}{K_x \frac{L}{r_x}}
\]

\[
P_y = \frac{\pi^2 EI_y}{(K_y L)^2} = \frac{\pi^2 EA r_y^2}{(K_y L)^2} = \frac{\pi^2 EA}{K_y \frac{L}{r_y}}
\]

\[
P_z = \left[ \frac{\pi^2 EI_z}{(K_z L)^2} + G K_r \right] \frac{1}{\frac{1}{r_i^2} + \frac{2}{r_i}} = \left[ \frac{\pi^2 EI_z}{K_z \frac{L}{r_i}} + G K_r \frac{r_i^2}{r_i} \right] \frac{A}{r_i^2 (I_x + I_y)}
\]

\[
\frac{P}{P_t} = \frac{\pi^2 EA}{K_x \frac{L}{r_x}} \times \frac{1}{A \sigma_t} = \frac{\pi^2 E}{\sigma_t \left( K_x \frac{L}{r_x} \right)} \frac{L^2}{r_x^2} = 5823.066
\]

\[
\frac{P}{P_t} = \frac{\pi^2 EA}{K_y \frac{L}{r_y}} \times \frac{1}{A \sigma_t} = \frac{\pi^2 E (r_i / r_y)^2}{\sigma_t \left( K_y \frac{L}{r_y} \right)} \frac{L^2}{r_y^2} = 791.02
\]

\[
\frac{P}{P_t} = \left[ \frac{\pi^2 EI_z}{K_z \frac{L}{r_i}} + G K_r \frac{r_i^2}{r_i} \right] \frac{1}{r_i^2 (I_x + I_y) \times \sigma_t} = \frac{\pi^2 EI_z}{r_i^2 (I_x + I_y) \times \sigma_t}
\]

\[
\frac{P}{P_t} = \frac{578.26}{0.2333} = 2.49
\]
- When L is such that \( L/r_x < 31 \); torsional buckling will govern
- \( r_x = 10.69 \) in. Therefore, \( L/r_x = 31 \rightarrow L=338 \) in.=28 ft.
- Typical column length =10 – 15 ft. Therefore, typical \( L/r_x= 11.2 – 16.8 \)
- Therefore elastic torsional buckling will govern.
- But, the predicted load is much greater than \( P_Y \). Therefore, inelastic buckling will govern.

- Summary – Typically must calculate all three buckling load values to determine which one governs. However, for common steel buildings made using wide flange sections – the minor (y-axis) flexural buckling usually governs.
- In this problem, the torsional buckling governed because the end conditions for minor axis flexural buckling were fixed. This is very rarely achieved in common building construction.
Well, what if the column has only one axis of symmetry. Like the x-axis or the y-axis or so.

As shown in this figure, the y – axis is the axis of symmetry.

The shear center S will be located on this axis.

Therefore $x_0= 0$.

The differential equations will simplify to:

1. $EI_x v'' + P v = 0$
2. $EI_x u'' + P u + \phi (P y_0) = 0$
3. $EI_u \phi'' + (P \bar{\tau}_0^2 - G K_y) \phi' + u' (P y_0) = 0$

The first equation for flexural buckling about the x-axis (axis of non-symmetry) becomes uncoupled.

Equations (2) and (3) are still coupled in terms of $u$ and $\phi$.

These equations will be satisfied by the solutions of the form

$u = C_2 \sin \left(\frac{\pi z}{L}\right)$ and $\phi = C_3 \sin \left(\frac{\pi z}{L}\right)$
Column Buckling – Singly Symmetric Columns

\[ EI_y \ u'' + P \ u + \phi (P \ y_0) = 0 \]  \hspace{1cm} (2)

\[ EI_u \ \phi'' + (P \bar{y}_0^2 - G \ K_f) \ \phi' + u'(P \ y_0) = 0 \]  \hspace{1cm} (3)

\therefore \ E I_y \ u'' + P \ u'' + \phi'' (P \ y_0) = 0

\[ EI_u \ \phi'' + (P \bar{y}_0^2 - G \ K_f) \ \phi'' + u''(P \ y_0) = 0 \]

Let, \[ u = C_2 \sin \frac{\pi z}{L}; \ \ \ \ \phi = C_3 \sin \frac{\pi z}{L} \]

Therefore, substituting these in equations 2 and 3

\[ EI_y \left( \frac{\pi}{L} \right)^4 \ C_2 \sin \frac{\pi z}{L} - P \ C_2 \left( \frac{\pi}{L} \right)^2 \sin \frac{\pi z}{L} - P \ y_0 \left( \frac{\pi}{L} \right)^2 \ C_3 \sin \frac{\pi z}{L} = 0 \]

\[ EI_u \left( \frac{\pi}{L} \right)^4 \ C_3 \sin \frac{\pi z}{L} - (P \bar{y}_0^2 - G \ K_f) \left( \frac{\pi}{L} \right)^2 \ C_3 \sin \frac{\pi z}{L} - P \ y_0 \left( \frac{\pi}{L} \right)^2 \ C_2 \sin \frac{\pi z}{L} = 0 \]

\[ \therefore \] \[ \left[ EI_y \left( \frac{\pi}{L} \right)^2 - P \right] C_2 - P \ y_0 \ C_3 = 0 \]

and \[ \left[ EI_u \left( \frac{\pi}{L} \right)^2 - (P \bar{y}_0^2 - G \ K_f) \right] C_3 - P \ y_0 \ C_2 = 0 \]

Let, \[ P_y = \frac{\pi^2 EI_y}{L^2} \] and \[ P_\phi = \left( \frac{\pi^2 EI_u}{L^2} + G K_f \right) \frac{1}{\bar{y}_0^2} \]

\[ \left[ P_y - P \right] C_2 - P \ y_0 \ C_3 = 0 \]

\[ \left[ P_\phi - P \bar{y}_0^2 \right] C_3 - P \ y_0 \ C_2 = 0 \]

\[ \left[ P_y - P \right] - P \ y_0 \ \left\{ \begin{array}{c} C_2 \\ C_3 \end{array} \right\} = \left\{ 0 \right\} \]

\[ \left[ P_\phi - P \bar{y}_0^2 \right] - P \ y_0 \ \left\{ \begin{array}{c} C_2 \\ C_3 \end{array} \right\} = 0 \]
**Column Buckling – Singly Symmetric Columns**

\[
\therefore (P_y - P)(P_{y} - P) \frac{r_0^2}{E_0} - P^2 f^2_0 = 0
\]

\[
\therefore \left[ P_y P_y - P(P_y + P_y) + P^2 \right] \frac{r_0^2}{E_0} - P^2 f^2_0 = 0
\]

\[
\therefore P^2 (1 - \frac{f^2_0}{r_0^2}) - P(P_y + P_y) + P P_y = 0
\]

\[
(P_y + P_y) \pm \sqrt{(P_y + P_y)^2 - 4 P_y P_y (1 - \frac{f^2_0}{r_0^2})}
\]

\[
\therefore P = \frac{(P_y + P_y) \pm \sqrt{(P_y + P_y)^2 - 4 P_y P_y (1 - \frac{f^2_0}{r_0^2})}}{2 (1 - \frac{f^2_0}{r_0^2})}
\]

\[
\therefore P = \frac{(P_y + P_y) \pm \sqrt{4 P_y P_y (1 - \frac{f^2_0}{r_0^2}) - (P_y + P_y)^2}}{2 (1 - \frac{f^2_0}{r_0^2})}
\]

Thus, there are two roots for \(P\)

Smaller value will govern

\[
\therefore P = \frac{(P_y + P_y) \pm \sqrt{4 P_y P_y (1 - \frac{f^2_0}{r_0^2}) - (P_y + P_y)^2}}{2 (1 - \frac{f^2_0}{r_0^2})}
\]

---

**Column Buckling – Singly Symmetric Columns**

- The critical buckling load will be the lowest of \(P_x\) and the two roots shown on the previous slide.
- If the flexural torsional buckling load governs, then the buckling mode will be \(C_2 \sin \left(\frac{\pi z}{L}\right) \times C_3 \sin \left(\frac{\pi z}{L}\right)\)
- This buckling mode will include both flexural and torsional deformations – hence flexural-torsional buckling mode.
Column Buckling – Asymmetric Section

- No axes of symmetry: Therefore, shear center S \((x_o, y_o)\) is such that neither \(x_o\) not \(y_o\) are zero.

\[E I_y v'' + P v - \phi (P x_0) = 0 \quad \text{……………………………}(1)\]
\[E I_y u'' + P u + \phi (P y_0) = 0 \quad \text{……………………………}(2)\]
\[E I_w \phi'' + (P \tau_{0z}^2 - G K_y) \phi' + u' (P y_0) - v' (P x_0) = 0 \quad \ldots (3)\]

- For simply supported boundary conditions: \((u, u'', v, v'', f, f'=0)\), the solutions to the differential equations can be assumed to be:
  - \(u = C_1 \sin \left(\frac{\pi z}{L}\right)\)
  - \(v = C_2 \sin \left(\frac{\pi z}{L}\right)\)
  - \(\phi = C_3 \sin \left(\frac{\pi z}{L}\right)\)

These solutions will satisfy the boundary conditions noted above.

Column Buckling – Asymmetric Section

- Substitute the solutions into the d.e. and assume that it satisfied too:

\[
E I_y \left[-C_1 \left(\frac{\pi}{L}\right)^2 \sin \left(\frac{\pi z}{L}\right) + P \left(C_2 \sin \left(\frac{\pi z}{L}\right) - P x_0 \right) C_1 \sin \left(\frac{\pi z}{L}\right) = 0 \right.
\]
\[
E I_y \left[-C_2 \left(\frac{\pi}{L}\right)^2 \cos \left(\frac{\pi z}{L}\right) + P \left(C_1 \cos \left(\frac{\pi z}{L}\right) + P y_0 \right) C_2 \cos \left(\frac{\pi z}{L}\right) = 0 \right.
\]
\[
E I_w \left[-C_3 \left(\frac{\pi}{L}\right)^2 \cos \left(\frac{\pi z}{L}\right) + \left(\frac{P \tau_{0z}^2 - G K_y}{L}\right) C_3 \left(\frac{\pi}{L}\right)^2 \cos \left(\frac{\pi z}{L}\right) + P y_0 \right) - P x_0 \right] C_3 \left(\frac{\pi}{L}\right)^2 \cos \left(\frac{\pi z}{L}\right) = 0
\]

\[
\begin{pmatrix}
\frac{\pi^2}{L^2} E I_x + P & 0 & -P x_0 \\
0 & \frac{\pi^2}{L^2} E I_y + P & P y_0 \\
-P x_0 & P y_0 & \frac{\pi^2}{L^2} E I_w + (P \tau_{0z}^2 - G K_y)
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2 \\
C_3
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]
Column Buckling – Asymmetric Section

\[
\begin{pmatrix}
-P_x + P & 0 & -P_x \\
0 & -P_y + P & P_y \\
-P_x & P_y & -P_x + P
\end{pmatrix}
\begin{pmatrix}
C_1 \sin \left( \frac{\pi z}{L} \right) \\
C_2 \sin \left( \frac{\pi z}{L} \right) \\
\pi \frac{C_3 \cos \left( \frac{\pi z}{L} \right)}{L}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

where,

\[P_x = \left( \frac{\pi}{L} \right)^2 E I_x, \quad P_y = \left( \frac{\pi}{L} \right)^2 E I_y, \quad P_\phi = \left( \frac{\pi^2 E I_w + G K_f}{L} \right) \frac{1}{r_0^2}\]

- Either \(C_1, C_2, C_3 = 0\) (no buckling), or the determinant of the coefficient matrix =0 at buckling.
- Therefore, determinant of the coefficient matrix is:

\[
(P - P_x)(P - P_y)(P - P_\phi) - P^2 \left( P - P_x \right) \left( \frac{x_r^2}{r_0^2} \right) - P^2 \left( P - P_y \right) \left( \frac{x_r^2}{r_0^2} \right) = 0
\]

This is the equation for predicting buckling of a column with an asymmetric section.
- The equation is cubic in \(P\). Hence, it can be solved to obtain three roots \(P_{cr1}, P_{cr2}, P_{cr3}\).
- The smallest of the three roots will govern the buckling of the column.
- The critical buckling load will always be smaller than \(P_x, P_y,\) and \(P_\phi\).
- The buckling mode will always include all three deformations \(u, v,\) and \(\phi\). Hence, it will be a flexural-torsional buckling mode.
- For boundary conditions other than simply-supported, the corresponding \(P_x, P_y,\) and \(P_\phi\) can be modified to include end condition effects \(K_x, K_y,\) and \(K_\phi\).
Column Buckling - Inelastic

A long topic

Effects of geometric imperfection

Leads to bifurcation buckling of perfect doubly-symmetric columns

\[ EI_x v'' + P v = 0 \]
\[ EI_y u'' + Pu = 0 \]

\[ M_x - P(v + v_o) = 0 \]
\[ \therefore EI_y v'' + P(v + v_o) = 0 \]
\[ \therefore v'' + F^2 v = -F^2 v_o \]
\[ \therefore v'' + F^2 v = -F^2 (\delta_o \sin \frac{\pi z}{L}) \]

Solution: \[ v_c + v_p \]
\[ v_c = A \sin(F_c z) + B \cos(F_c z) \]
\[ v_p = C \sin \frac{\pi z}{L} + D \cos \frac{\pi z}{L} \]
Effects of Geometric Imperfection

Solve for C and D first

\[ v_p^* + F_v^2 v_p = -F_v^2 \delta_o \sin \frac{\pi z}{L} \]

\[ -\left( \frac{\pi}{L} \right)^2 \left[ C \sin \frac{\pi z}{L} + D \cos \frac{\pi z}{L} \right] + F_v^2 \left[ C \sin \frac{\pi z}{L} + D \cos \frac{\pi z}{L} \right] + F_v^2 \delta_o \sin \frac{\pi z}{L} = 0 \]

\[ \sin \frac{\pi z}{L} \left[ -C \left( \frac{\pi}{L} \right)^2 + F_v^2 C + F_v^2 \delta_o \right] + \cos \frac{\pi z}{L} \left[ \left( \frac{\pi}{L} \right)^2 D + F_v^2 D \right] = 0 \]

\[ -C \left( \frac{\pi}{L} \right)^2 + F_v^2 C + F_v^2 \delta_o = 0 \quad \text{and} \quad \left[ \left( \frac{\pi}{L} \right)^2 D + F_v^2 D \right] = 0 \]

\[ C = \frac{F_v^2 \delta_o}{\left( \frac{\pi}{L} \right)^2 - F_v^2} \quad \text{and} \quad D = 0 \]

\[ \text{Solution becomes} \]

\[ v = A \sin(F_v z) + B \cos(F_v z) + \frac{F_v^2 \delta_o}{\left( \frac{\pi}{L} \right)^2 - F_v^2} \sin \frac{\pi z}{L} \]

Geometric Imperfection

Solve for A and B

Boundary conditions:

\[ v(0) = v(L) = 0 \]

\[ v(0) = B = 0 \]

\[ v(L) = A \sin F_v L = 0 \]

\[ \therefore \quad A = 0 \]

\[ \therefore \quad \text{Solution becomes} \]

\[ v = \frac{F_v^2 \delta_o}{\left( \frac{\pi}{L} \right)^2 - F_v^2} \sin \frac{\pi z}{L} \]

\[ v = \frac{F_v^2 \delta_o}{\left( \frac{\pi}{L} \right)^2} \sin \frac{\pi z}{L} = \frac{\delta_o \sin \frac{\pi z}{L}}{1 - \frac{P}{P_e}} \]

\[ \text{Total Deflection} \]

\[ v + v_o = \frac{P}{P_e} \delta_o \sin \frac{\pi z}{L} + \delta_o \sin \frac{\pi z}{L} \]

\[ = \frac{P}{P_e} \delta_o \sin \frac{\pi z}{L} + 1 - \frac{P}{P_e} \frac{\delta_o \sin \frac{\pi z}{L} - 1}{1 - \frac{P}{P_e}} \]

\[ = A_r \delta_o \sin \frac{\pi z}{L} \]

\[ A_r = \text{amplification factor} \]
Geometric Imperfection

\[ A_F = \frac{1}{P} = \text{amplification factor} \]

\[ 1 - \frac{P}{P_E} \]

\[ M_x = P(v + v_o) \]

\[ \therefore M_x = A_F(P\delta_o \sin \frac{\pi z}{L}) \]

\[ \text{i.e., } M_x = A_F \times \text{(moment due to initial crooked)} \]

Increases exponentially
Limit \( A_F \) for design
Limit \( P/P_E \) for design

Value used in the code is 0.877
This will give \( A_F = 8.13 \)
Have to live with it.

Residual Stress Effects

Figure 6.5.1 Typical residual stress pattern on rolled shapes.

Figure 6.5.3 Typical residual stress distribution in welded shapes.
Residual Stress Effects

Figure 6.5.2 Influence of residual stress on average stress-strain curve.

History of column inelastic buckling

- Euler developed column elastic buckling equations (buried in the million other things he did).
  - Take a look at: http://en.wikipedia.org/wiki/Euler
  - An amazing mathematician
- In the 1750s, I could not find the exact year.
- The elastica problem of column buckling indicates elastic buckling occurs with no increase in load.
  - \( \frac{dP}{dv} = 0 \)
For a bar fixed at the base and free at the axially loaded upper end, the load $P$ must be slightly greater than the Euler buckling load in order to cause the large deflection depicted in Figure 2-25. Note that the moving origin of coordinates is located at the loaded free end of the bar.

Figure 2-25  Axially Loaded Bar with One End Fixed and One End Free
The deflections of the bar are obtained from the differential triangle in Figure 2-25, i.e.,

\[
dy = \sin \theta \, ds = - \frac{\sin \theta \, d\theta}{\sqrt{2 \, k \sqrt{\cos \theta - \cos \alpha}}} \tag{2.200}
\]

so the transverse deflection of the loaded free end of the bar is

\[
y_a = \frac{1}{2k} \int_0^\alpha \frac{\sin \theta \, d\theta}{\sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}} \tag{2.201}
\]

which, upon substitution of relations derived earlier, can be written as

\[
y_a = \frac{2p}{k} \int_0^{\pi/2} \sin \phi \, d\phi = \frac{2p}{k} \tag{2.202}
\]

For a given k, p is determined from Equation (2.197) (or vice versa). Then, \( y_a \) is determined from Equation (2.202). Similarly, \( x_a \) is

\[
x_a = \frac{2}{k} \int_0^{\pi/2} \sqrt{1 - p^2 \sin^2 \phi} \, d\phi - L = \frac{2}{k} E(p) - L \tag{2.203}
\]

in which \( E(p) \) is the complete elliptic integral of the second kind and is also a tabulated function. Thus, the load and the coordinates of its de-
Figure 2-26 Deformed Bar
(After Timoshenko and Gere [2-6])
reproduced by permission

Figure 2-27 Deformed Bar under Various Axial Loads
(After Nemeth [2-7])
2-28 Load versus Tip Deflection Behavior for Many Loads
(After Nemeth [2-7])
Load versus Tip Deflection Behavior Near the First Buckling Load
(After Nemeth [2-7])
History of Column Inelastic Buckling

- Engesser extended the elastic column buckling theory in 1889.
- He assumed that inelastic buckling occurs with no increase in load, and the relation between stress and strain is defined by tangent modulus $E_t$.

Engesser’s tangent modulus theory is easy to apply. It compares reasonably with experimental results.

\[ P_T = \pi E_T I / (KL)^2 \]

History of Column Inelastic Buckling

- In 1895, Jasinsky pointed out the problem with Engesser’s theory.
  - If $dP/dv = 0$, then the 2\textsuperscript{nd} order moment ($Pv$) will produce incremental strains that will vary linearly and have a zero value at the centroid (neutral axis).
  - The linear strain variation will have compressive and tensile values. The tangent modulus for the incremental compressive strain is equal to $E_t$ and that for the tensile strain is $E$. 

\[ \sigma - \epsilon \text{ curve} \]
History of Column Inelastic Buckling

- In 1898, Engesser corrected his original theory by accounting for the different tangent modulus of the tensile increment.
  - This is known as the *reduced modulus* or *double modulus*
  - The assumptions are the same as before. That is, there is no increase in load as buckling occurs.
- The corrected theory is shown in the following slide

The buckling load $P_R$ produces critical stress $\sigma_R = P_r/A$

During buckling, a small curvature $d\phi$ is introduced

The strain distribution is shown.

The loaded side has $d\varepsilon_L$ and $d\sigma_L$

The unloaded side has $d\varepsilon_U$ and $d\sigma_U$

\[
d\varepsilon_L = (\bar{y}_1 + y) \, d\phi \\
d\varepsilon_U = (y - \bar{y} + y_1) \, d\phi \\
\therefore d\sigma_L = E_t (\bar{y}_1 + y) \, d\phi \\
\therefore d\sigma_U = E (y - \bar{y} + y_1) \, d\phi
\]
History of Column Inelastic Buckling

\[ d\phi = -v'' \]

\[ d\sigma_L = -E_t (\bar{y} - y_1 + y) \ v^* \]

\[ d\sigma_U = -E (y - \bar{y} + y_1) \ v^* \]

But, the assumption is \( dP = 0 \)

\[ \therefore \int_{-y_1}^{y} d\sigma_U \ dA - \int_{-(d-y)}^{d} d\sigma_L \ dA = 0 \]

\[ \therefore \int_{-y_1}^{y} E (y - \bar{y} + y_1) \ dA - \int_{-(d-y)}^{d} E_t (\bar{y} - y_1 + y) \ dA = 0 \]

\[ \therefore ES_1 - E_t S_2 = 0 \]

where, \( S_1 = \int_{-y_1}^{y} (y - \bar{y} + y_1) \ dA \)

and \( S_2 = \int_{-(d-y)}^{d} (\bar{y} - y_1 + y) \ dA \)

---

History of Column Inelastic Buckling

- \( S_1 \) and \( S_2 \) are the statical moments of the areas to the left and right of the neutral axis.
  - Note that the neutral axis does not coincide with the centroid any more.
  - The location of the neutral axis is calculated using the equation derived \( ES_1 - E_t S_2 = 0 \)

\[ M = P\nu \]

\[ \therefore M = \int_{-y_1}^{y} d\sigma_U (y - \bar{y} + y_1) \ dA - \int_{-(d-y)}^{d} d\sigma_L (\bar{y} - y_1 + y) \ dA \]

\[ \therefore M = P\nu = -v''(EI_1 + E_t I_2) \]

where, \( I_1 = \int_{-y_1}^{y} (y - \bar{y} + y_1)^2 \ dA \)

and \( I_2 = \int_{-(d-y)}^{d} (\bar{y} - y_1 + y)^2 \ dA \)
History of Column Inelastic Buckling

\[ M = Pv = -v''(EI_1 + E_rI_2) \]
\[ \therefore Pv + (EI_1 + E_rI_2)v'' = 0 \]
\[ \therefore v'' + \frac{P}{EI_1 + E_rI_2}v = 0 \]
\[ \therefore v'' + F_v^2v = 0 \]
where \[ F_v^2 = \frac{P}{EI_1 + E_rI_2} = \frac{P}{E\bar{I}_x} \]
and \[ \bar{E} = E_1 \frac{I_1}{I_x} + E_r \frac{I_2}{I_x} \]
\[ P_R = \frac{\pi^2 E\bar{I}_x}{(KL)^2} \]
\[ \bar{E} \] is the reduced or double modulus
\[ P_R \] is the reduced modulus buckling load

History of Column Inelastic Buckling

- For 50 years, engineers were faced with the dilemma that the reduced modulus theory is correct, but the experimental data was closer to the tangent modulus theory. How to resolve?
- Shanley eventually resolved this dilemma in 1947. He conducted very careful experiments on small aluminum columns.
  - He found that lateral deflection started very near the theoretical tangent modulus load and the load capacity increased with increasing lateral deflections.
  - The column axial load capacity never reached the calculated reduced or double modulus load.
- Shanley developed a column model to explain the observed phenomenon
History of Column Inelastic Buckling

\[ \psi_s = \frac{\theta_s L}{2} \quad \text{and} \quad \theta_s = \frac{1}{2d}(\varepsilon_s + \varepsilon_t) \quad (4.129) \]

By combining these two equations we can eliminate \( \theta_s \), and thus

\[ \psi_s = \frac{L}{4d}(\varepsilon_s + \varepsilon_t) \quad (4.130) \]

The external moment at the midheight of the column is

\[ M_s = P\psi_s = \frac{PL}{4d}(\varepsilon_s + \varepsilon_t) \quad (4.131) \]

The forces in the two flanges due to buckling are

\[ P_1 = \frac{E_1 \varepsilon_t A}{2d} \quad \text{and} \quad P_2 = \frac{E_t \varepsilon_t A}{2d} \quad (4.132) \]

The internal moment is then

\[ M_i = \frac{dP_1}{2} + \frac{dP_2}{2} = \frac{A}{4}(E_1 \varepsilon_t + E_t \varepsilon_t) \quad (4.133) \]

With \( M_s = M_i \) we get an expression for the axial load \( P \), or

\[ P = \frac{Ad}{L}(\frac{E_1 \varepsilon_t + E_t \varepsilon_t}{\varepsilon_s + \varepsilon_t}) \quad (4.134) \]
History of Column Inelastic Buckling

In case the cell is elastic $E_i = E_0$, then

$$P_x = \frac{AEd}{L}$$

(4.135)

For the tangent modulus concept $E_i = E_0 = E_{ii}$, and so

$$P_x = \frac{AEd}{L}$$

(4.136)

When we consider the elastic unloading of the “tension” flange, then $E_i = E_0$ and $E_0 = E_0$, and thus

$$P = \frac{Aed}{L} \left( \frac{E_{ii} e_1 + E_{ies}}{e_1 + e_2} \right)$$

(4.137)

Upon substitution of $e_i$ from Eq. (4.130) and $P_x$ from Eq. (4.136) and using the abbreviation

$$\tau = \frac{E_i}{E_0}$$

(4.138)

we find that

$$P = P_x \left[ 1 + \frac{L e_2}{4de_0} \left( \frac{1}{\tau} - 1 \right) \right]$$

(4.139)

History of Column Inelastic Buckling

$$P = P_x \left[ 1 + \frac{1}{(d/2e_0) + (1 + \tau)(1 - \tau)} \right]$$

(4.143)

$$P_x = P_x \left( 1 + \frac{1 - \tau}{1 + \tau} \right)$$

(4.146)
Shaw (1949) conducted very careful tests on small aluminum columns. He found that:
- initial deflections $P_a$ occurred very near to the tangent modulus load $P_t$
- but, additional load was carried until ultimate load
- The reduced modulus $P_a$ could never be reached

Shaw's explanation:

Using equations (37) and (38) to eliminate $E_2$

For example, $F_1 = 0$ then

The plot of $P_t$ vs. $\frac{d}{t}$ shown below

- halved deflections occur when $P_t$ is reached
- buckling occurs with increasing loads
- curve approaches $P_t$ at $\frac{d}{t} = \infty$
- $P_t$ decreases with strain at $P_t$ will never be reached and the dotted curve will be followed

Then $P_t < P_a < P_r$
Column Inelastic Buckling

- Three different theories
  - Tangent modulus
  - Reduced modulus
  - Shanley model

- Tangent modulus theory assumes
  - Perfectly straight column
  - Ends are pinned
  - Small deformations
  - No strain reversal during buckling

\[ \frac{dP}{dv} = 0 \]

Slope is zero at buckling
\( \Delta P = 0 \) with increasing \( \Delta v \)

Elastic buckling analysis

\[
P_T
\]

Tangent modulus theory

- Assumes that the column buckles at the tangent modulus load such that there is an increase in \( \Delta P \) (axial force) and \( \Delta M \) (moment).
- The axial strain increases everywhere and there is no strain reversal.

\[
\Delta \varepsilon_T = \frac{P_T}{A}
\]

\[
\Delta \sigma_T = E_T \Delta \varepsilon_T
\]

\[
\phi = \frac{\Delta \varepsilon_T}{h}
\]

\[
\Delta \varepsilon_T = \phi \left( \frac{h}{2} + y \right) \quad \text{where} \ y = \text{distance from centroid}
\]

\[
\Delta \sigma_T = \phi \left( \frac{h}{2} + y \right) \cdot E_T
\]
Tangent modulus theory

- Deriving the equation of equilibrium

\[ M_x = \int \sigma \cdot y dA \]

\[ \sigma = \sigma_T + \Delta \sigma_T \]

\[ \sigma = \sigma_T + \phi(y + h/2) \cdot E_T \]

\[ \therefore M_x = \int \left( \sigma_T + \phi(y + h/2)E_T \right) \cdot y dA \]

\[ \therefore M_x = \sigma_T \int y dA + E_T \int \phi y^2 dA + (\phi h/2)E_T \int y dA \]

\[ \therefore M_x = 0 + E_T \phi I_x + 0 \]

\[ \therefore M_x = -E_T I_x \varepsilon'' \]

- The equation \( M_x - P_T \varepsilon = 0 \) becomes \(-E_T I_x \varepsilon'' - P_T \varepsilon = 0\)

- Solution is \( P_T = \pi^2 E_T I_x / L^2 \)

Example - Aluminum columns

- Consider an aluminum column with Ramberg-Osgood stress-strain curve

\[
\varepsilon = \frac{\sigma}{E} + 0.002 \left( \frac{\sigma}{\sigma_{0.2}} \right)^n
\]

\[
\therefore \frac{\partial \varepsilon}{\partial \sigma} = E + \frac{0.002 n \sigma^{n-1}}{\sigma_{0.2}^n}
\]

\[
\therefore \frac{\partial \varepsilon}{\partial \sigma} = 1 + \frac{0.002 n E \sigma^{n-1}}{\sigma_{0.2}^n}
\]

\[
\therefore \frac{\partial \varepsilon}{\partial \sigma} = \frac{E}{\sigma_{0.2}^n} \left( \frac{\sigma}{\sigma_{0.2}} \right)^{n-1}
\]

\[
\therefore \frac{\partial \varepsilon}{\partial \sigma} = 1 + \frac{0.002 n E \left( \frac{\sigma}{\sigma_{0.2}} \right)^{n-1}}{\sigma_{0.2}^n}
\]

\[
\therefore \frac{\partial \varepsilon}{\partial \sigma} = \frac{E}{\sigma_{0.2}^n} \left( \frac{\sigma}{\sigma_{0.2}} \right)^{n-1} = E_T
\]
Tangent Modulus Buckling

Ramberg-Osgood Stress-Strain

Strain (in./in.)

Stress (ksi)

0.000 0.010 0.020 0.030 0.040 0.050

ET differences

ET equation

Stress-tangent modulus relationship

0

2000

4000

6000

8000

10000

12000

0 10 20 30 40 50

Stress (ksi)

Tangent Modulus (ksi)

Tangent Modulus Buckling

Column Inelastic Buckling Curve

\[ P_T = \frac{\pi^2 E_T I_x}{L^2} \]

\[ A = \frac{\pi^2 E_T I_x}{AE^2} = \frac{\pi^2 E_T}{(KL/r)^2} \]

\[ (KL/r)_{cr} = \frac{\pi^2 E_T}{\sigma_T} \]

<table>
<thead>
<tr>
<th>n</th>
<th>(KL/r)_{cr}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>223.2521046</td>
</tr>
<tr>
<td>2</td>
<td>157.830773</td>
</tr>
<tr>
<td>4</td>
<td>128.896622</td>
</tr>
<tr>
<td>6</td>
<td>111.6260523</td>
</tr>
<tr>
<td>8</td>
<td>99.84137641</td>
</tr>
<tr>
<td>10</td>
<td>91.1422898</td>
</tr>
<tr>
<td>12</td>
<td>84.3813604</td>
</tr>
<tr>
<td>14</td>
<td>78.93150275</td>
</tr>
<tr>
<td>16</td>
<td>74.41710153</td>
</tr>
<tr>
<td>18</td>
<td>70.59690679</td>
</tr>
<tr>
<td>20</td>
<td>67.3048795</td>
</tr>
<tr>
<td>22</td>
<td>64.8136011</td>
</tr>
<tr>
<td>24</td>
<td>61.77857439</td>
</tr>
<tr>
<td>26</td>
<td>59.1743952</td>
</tr>
<tr>
<td>28</td>
<td>56.0208280</td>
</tr>
<tr>
<td>30</td>
<td>51.5097656</td>
</tr>
<tr>
<td>32</td>
<td>48.14566415</td>
</tr>
<tr>
<td>34</td>
<td>44.1419681</td>
</tr>
<tr>
<td>36</td>
<td>40.6964013</td>
</tr>
<tr>
<td>38</td>
<td>37.8912017</td>
</tr>
<tr>
<td>40</td>
<td>35.6927672</td>
</tr>
<tr>
<td>42</td>
<td>33.90256144</td>
</tr>
<tr>
<td>44</td>
<td>32.59633408</td>
</tr>
<tr>
<td>46</td>
<td>31.05440361</td>
</tr>
<tr>
<td>48</td>
<td>2.129145204</td>
</tr>
</tbody>
</table>

\[ P_T = \frac{\pi^2 E_T I_x}{L^2} \]

\[ A = \frac{\pi^2 E_T I_x}{AE^2} = \frac{\pi^2 E_T}{(KL/r)^2} \]

\[ (KL/r)_{cr} = \frac{\pi^2 E_T}{\sigma_T} \]
Residual Stress Effects

- Consider a rectangular section with a simple residual stress distribution
  - Assume that the steel material has elastic-plastic stress-strain $\sigma - \varepsilon$ curve.
  - Assume simply supported end conditions
  - Assume triangular distribution for residual stresses

One major constrain on residual stresses is that they must be such that

$$\int \sigma_r dA = 0$$

\[ \begin{align*}
\therefore \int_{-b/2}^{-b/2} (-0.5\sigma_y + \frac{2\sigma_y}{b} x) dx + \int_{0}^{b/2} (+0.5\sigma_y - \frac{2\sigma_y}{b} x) dx & = -0.5\sigma_y \frac{db}{2} + 0.5\sigma_y \frac{db}{2} + \frac{2d}\frac{\sigma_y}{b} \left( \frac{b^3}{8} \right) - \frac{2d}\frac{\sigma_y}{b} \left( \frac{b^3}{8} \right) \\
& = 0
\end{align*} \]

- Residual stresses are produced by uneven cooling but no load is present
Residual Stress Effects

- Response will be such that elastic behavior when
  \[ \sigma < 0.5\sigma_y \]
  \[ P_x = \frac{\pi^2 EI_x}{L^2} \quad \text{and} \quad P_y = \frac{\pi^2 EI_y}{L^2} \]

Yielding occurs when
\[ \sigma = 0.5\sigma_y \quad \text{i.e.,} \quad P = 0.5P_y \]
Inelastic buckling will occur after \( \sigma > 0.5\sigma_y \)

\[
\left( \sigma_y - \frac{2\sigma_y}{b} ab \right) = \sigma_y (1-2a)
\]

Total axial force corresponding to the yielded section
\[
\sigma_y (b-2ab) d + \left( \sigma_y + \frac{\sigma_y (1-2a)}{2} \right) abd \times 2
\]
\[
= \sigma_y (1-2a) bd + \sigma_y (2-2a) abd
\]
\[
= \sigma_y bd - 2abd\sigma_y + 2\sigma_y abd - 2\alpha^2 b d \sigma_y
\]
\[
= \sigma_y bd (1-2\alpha^2) = P_y (1-2\alpha^2)
\]

\[ \therefore \text{If inelastic buckling were to occur at this load} \]
\[ P_x = P_y (1-2\alpha^2) \]
\[ \therefore \alpha = \frac{1}{2} \left( 1 - \frac{P_x}{P_y} \right) \]
If inelastic buckling occurs about $x$-axis

$$P_{cr} = P_{tx} = \frac{\pi^2 E}{L^2} (2ab) \frac{d^3}{12}$$

$$\therefore P_{tx} = \frac{\pi^2 EI}{L^2} 2\alpha$$

$$\therefore P_y = P_x \times 2 \times \left[ \frac{1}{2} \left( 1 - \frac{P_y}{P_x} \right) \right]$$

$$\therefore P_{tx} = P_y \times 2 \times \left[ \frac{1}{2} \left( 1 - \frac{P_{tx}}{P_y} \right) \right]$$

Let

$$\frac{P_x}{P_y} = \frac{1}{\lambda_x} = \pi \left( \frac{E}{\sigma_x K_x} \right)^{2}$$

$$\therefore \lambda_x = \sqrt{2 \left( 1 - \frac{P_{tx}}{P_y} \right)}$$

$$\therefore \lambda_x = \frac{P_{tx}}{P_y}$$

If inelastic buckling occurs about $y$-axis

$$P_{cr} = P_{ty} = \frac{\pi^2 E}{L^2} (2ab) \frac{d^3}{12}$$

$$\therefore P_{ty} = \frac{\pi^2 EI}{L^2} (2\alpha)$$

$$\therefore P_y = P_x \times \left[ \frac{1}{2} \left( 1 - \frac{P_y}{P_x} \right) \right]$$

$$\therefore P_{ty} = P_y \times \left[ \frac{1}{2} \left( 1 - \frac{P_{ty}}{P_y} \right) \right]$$

Let

$$\frac{P_y}{P_t} = \frac{1}{\lambda_y} = \pi \left( \frac{E}{\sigma_y K_y} \right)^{2}$$

$$\therefore \lambda_y = \sqrt{2 \left( 1 - \frac{P_{ty}}{P_y} \right)}$$

$$\therefore \lambda_y = \frac{P_{ty}}{P_y}$$
Residual Stress Effects

<table>
<thead>
<tr>
<th>P/Py</th>
<th>λx</th>
<th>λy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.200</td>
<td>2.236</td>
<td>2.236</td>
</tr>
<tr>
<td>0.250</td>
<td>2.000</td>
<td>2.000</td>
</tr>
<tr>
<td>0.300</td>
<td>1.826</td>
<td>1.826</td>
</tr>
<tr>
<td>0.350</td>
<td>1.690</td>
<td>1.690</td>
</tr>
<tr>
<td>0.400</td>
<td>1.581</td>
<td>1.581</td>
</tr>
<tr>
<td>0.450</td>
<td>1.491</td>
<td>1.491</td>
</tr>
<tr>
<td>0.500</td>
<td>1.414</td>
<td>1.414</td>
</tr>
<tr>
<td>0.550</td>
<td>1.313</td>
<td>1.246</td>
</tr>
<tr>
<td>0.600</td>
<td>1.221</td>
<td>1.092</td>
</tr>
<tr>
<td>0.650</td>
<td>1.135</td>
<td>0.949</td>
</tr>
<tr>
<td>0.700</td>
<td>1.052</td>
<td>0.815</td>
</tr>
<tr>
<td>0.750</td>
<td>0.971</td>
<td>0.687</td>
</tr>
<tr>
<td>0.800</td>
<td>0.899</td>
<td>0.562</td>
</tr>
<tr>
<td>0.850</td>
<td>0.820</td>
<td>0.440</td>
</tr>
<tr>
<td>0.900</td>
<td>0.705</td>
<td>0.315</td>
</tr>
<tr>
<td>0.950</td>
<td>0.577</td>
<td>0.182</td>
</tr>
<tr>
<td>0.995</td>
<td>0.317</td>
<td>0.032</td>
</tr>
</tbody>
</table>

Fig. 4.54. Target modulus buckling curves for strong and weak axis buckling of intermediate columns.
Tangent modulus buckling - Numerical

1. Discretize the cross-section into fibers
   Think about the discretization. Do you need the flange
   To be discretized along the length and width?

2. For each fiber, save the area of fiber ($A_{fib}$), the
   distances from the centroid ($y_{fib}$ and $x_{fib}$)
   $I_{x,fb}$ and $I_{y,fb}$ the fiber number in the matrix.

3. Discretize residual stress distribution

4. Calculate residual stress ($\sigma_{fib}$)
   each fiber

5. Check that sum($\sigma_{fib}A_{fib}$) for
   Section = zero

Tangent Modulus Buckling - Numerical

6. Calculate effective residual strain ($\varepsilon_r$) for each fiber
   $\varepsilon_r = \sigma_r/E$

7. Assume centroidal strain $\varepsilon$

8. Calculate total strain for each fiber
   $\varepsilon_{tot} = \varepsilon + \varepsilon_r$

9. Assume a material stress-strain curve for each fiber

10. Calculate stress in each fiber $\sigma_{fib}$

11. Calculate Axial Force = $P$
    Sum ($\sigma_{fib}A_{fib}$)

12. Calculate average stress $\sigma = P/A$

13. Calculate the tangent $(EI)_{TX}$ and $(EI)_{TY}$ for the $\sigma$
    $(EI)_{TX} = \sum (E_{T,fb}A_{fib}^2 y_{fib}^2 + I_{x,fb})$
    $(EI)_{TY} = \sum (E_{T,fb}A_{fib}^2 x_{fib}^2 + I_{y,fb})$

14. Calculate the critical $(KL)_{Xcr}$ and $(KL)_{Ycr}$ for the $\sigma$
    $(KL)_{Xcr} = \pi \sqrt{[(EI)_{TX}/P]}
    $(KL)_{Ycr} = \pi \sqrt{[(EI)_{TY}/P]}

15. Calculate the critical $(KL)_{Xcr}$ and $(KL)_{Ycr}$ for the $\sigma$
    $(KL)_{Xcr} = \pi \sqrt{[(EI)_{TX}/P]}
    $(KL)_{Ycr} = \pi \sqrt{[(EI)_{TY}/P]}$

16. Calculate the tangent $(EI)_{TX}$ and $(EI)_{TY}$ for the $\sigma$
    $(EI)_{TX} = \sum (E_{T,fb}A_{fib}^2 y_{fib}^2 + I_{x,fb})$
    $(EI)_{TY} = \sum (E_{T,fb}A_{fib}^2 x_{fib}^2 + I_{y,fb})$

17. Calculate average stress $\sigma = P/A$

18. Calculate Axial Force = $P$
    Sum ($\sigma_{fib}A_{fib}$)

19. Calculate stress in each fiber $\sigma_{fib}$

20. Calculate the tangent $(EI)_{TX}$ and $(EI)_{TY}$ for the $\sigma$
    $(EI)_{TX} = \sum (E_{T,fb}A_{fib}^2 y_{fib}^2 + I_{x,fb})$
    $(EI)_{TY} = \sum (E_{T,fb}A_{fib}^2 x_{fib}^2 + I_{y,fb})$

21. Calculate average stress $\sigma = P/A$

22. Calculate Axial Force = $P$
    Sum ($\sigma_{fib}A_{fib}$)

23. Calculate stress in each fiber $\sigma_{fib}$

24. Calculate the tangent $(EI)_{TX}$ and $(EI)_{TY}$ for the $\sigma$
    $(EI)_{TX} = \sum (E_{T,fb}A_{fib}^2 y_{fib}^2 + I_{x,fb})$
    $(EI)_{TY} = \sum (E_{T,fb}A_{fib}^2 x_{fib}^2 + I_{y,fb})$

25. Calculate average stress $\sigma = P/A$

26. Calculate Axial Force = $P$
    Sum ($\sigma_{fib}A_{fib}$)

27. Calculate stress in each fiber $\sigma_{fib}$
### Tangent Modulus Buckling - numerical

**Section Dimension**

<table>
<thead>
<tr>
<th>d</th>
<th>b</th>
<th>No. of fibers</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>50</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>d</th>
<th>b</th>
<th>No. of fibers</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>fiber no.</th>
<th>Xfib</th>
<th>Yfib</th>
<th>Xr-fib</th>
<th>Yr-fib</th>
<th>Ir-fib</th>
<th>Iy-fib</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.4</td>
<td>-5.7</td>
<td>0</td>
<td>-17.5</td>
<td>6.034E-04</td>
<td>3.2</td>
</tr>
<tr>
<td>2</td>
<td>2.4</td>
<td>-4.5</td>
<td>0</td>
<td>-12.5</td>
<td>4.310E-04</td>
<td>3.2</td>
</tr>
<tr>
<td>3</td>
<td>2.4</td>
<td>-3.9</td>
<td>0</td>
<td>-7.5</td>
<td>2.586E-04</td>
<td>3.2</td>
</tr>
<tr>
<td>4</td>
<td>2.4</td>
<td>-3.3</td>
<td>0</td>
<td>-2.5</td>
<td>8.621E-05</td>
<td>3.2</td>
</tr>
<tr>
<td>5</td>
<td>2.4</td>
<td>-2.7</td>
<td>0</td>
<td>2.5</td>
<td>8.621E-05</td>
<td>3.2</td>
</tr>
<tr>
<td>6</td>
<td>2.4</td>
<td>-2.1</td>
<td>0</td>
<td>7.5</td>
<td>2.586E-04</td>
<td>3.2</td>
</tr>
<tr>
<td>7</td>
<td>2.4</td>
<td>-1.5</td>
<td>0</td>
<td>12.5</td>
<td>4.310E-04</td>
<td>3.2</td>
</tr>
<tr>
<td>8</td>
<td>2.4</td>
<td>-0.9</td>
<td>0</td>
<td>17.5</td>
<td>6.034E-04</td>
<td>3.2</td>
</tr>
<tr>
<td>9</td>
<td>2.4</td>
<td>-0.3</td>
<td>0</td>
<td>22.5</td>
<td>7.759E-04</td>
<td>3.2</td>
</tr>
<tr>
<td>10</td>
<td>2.4</td>
<td>0</td>
<td>0</td>
<td>27.5</td>
<td>9.460E-04</td>
<td>3.2</td>
</tr>
<tr>
<td>11</td>
<td>2.4</td>
<td>0.3</td>
<td>0</td>
<td>32.5</td>
<td>1.10E+00</td>
<td>3.2</td>
</tr>
<tr>
<td>12</td>
<td>2.4</td>
<td>0.9</td>
<td>0</td>
<td>37.5</td>
<td>1.24E+00</td>
<td>3.2</td>
</tr>
<tr>
<td>13</td>
<td>2.4</td>
<td>1.6</td>
<td>0</td>
<td>42.5</td>
<td>1.38E+00</td>
<td>3.2</td>
</tr>
<tr>
<td>14</td>
<td>2.4</td>
<td>2.1</td>
<td>0</td>
<td>47.5</td>
<td>1.52E+00</td>
<td>3.2</td>
</tr>
<tr>
<td>15</td>
<td>2.4</td>
<td>2.7</td>
<td>0</td>
<td>52.5</td>
<td>1.66E+00</td>
<td>3.2</td>
</tr>
<tr>
<td>16</td>
<td>2.4</td>
<td>3.3</td>
<td>0</td>
<td>57.5</td>
<td>1.80E+00</td>
<td>3.2</td>
</tr>
<tr>
<td>17</td>
<td>2.4</td>
<td>3.9</td>
<td>0</td>
<td>62.5</td>
<td>1.94E+00</td>
<td>3.2</td>
</tr>
<tr>
<td>18</td>
<td>2.4</td>
<td>4.5</td>
<td>0</td>
<td>67.5</td>
<td>2.08E+00</td>
<td>3.2</td>
</tr>
<tr>
<td>19</td>
<td>2.4</td>
<td>5.1</td>
<td>0</td>
<td>72.5</td>
<td>2.22E+00</td>
<td>3.2</td>
</tr>
<tr>
<td>20</td>
<td>2.4</td>
<td>5.7</td>
<td>0</td>
<td>77.5</td>
<td>2.36E+00</td>
<td>3.2</td>
</tr>
</tbody>
</table>

### Strain Increment

<table>
<thead>
<tr>
<th>Fiber no.</th>
<th>Xfib</th>
<th>Yfib</th>
<th>Xr-fib</th>
<th>Yr-fib</th>
<th>Ir-fib</th>
<th>Iy-fib</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.0003</td>
<td>0</td>
<td>-11.2</td>
<td>92800</td>
<td>23900</td>
<td>2.26E+06</td>
</tr>
<tr>
<td>2</td>
<td>-0.0003</td>
<td>0</td>
<td>-21.2</td>
<td>92800</td>
<td>23900</td>
<td>6.034E+04</td>
</tr>
<tr>
<td>3</td>
<td>-0.0003</td>
<td>0</td>
<td>-31.2</td>
<td>92800</td>
<td>23900</td>
<td>2.26E+06</td>
</tr>
<tr>
<td>4</td>
<td>-0.0003</td>
<td>0</td>
<td>-41.2</td>
<td>92800</td>
<td>23900</td>
<td>6.034E+04</td>
</tr>
<tr>
<td>5</td>
<td>-0.0003</td>
<td>0</td>
<td>-51.2</td>
<td>92800</td>
<td>23900</td>
<td>2.26E+06</td>
</tr>
<tr>
<td>6</td>
<td>-0.0003</td>
<td>0</td>
<td>-61.2</td>
<td>92800</td>
<td>23900</td>
<td>6.034E+04</td>
</tr>
<tr>
<td>7</td>
<td>-0.0003</td>
<td>0</td>
<td>-71.2</td>
<td>92800</td>
<td>23900</td>
<td>2.26E+06</td>
</tr>
<tr>
<td>8</td>
<td>-0.0003</td>
<td>0</td>
<td>-81.2</td>
<td>92800</td>
<td>23900</td>
<td>6.034E+04</td>
</tr>
<tr>
<td>9</td>
<td>-0.0003</td>
<td>0</td>
<td>-91.2</td>
<td>92800</td>
<td>23900</td>
<td>2.26E+06</td>
</tr>
<tr>
<td>10</td>
<td>-0.0003</td>
<td>0</td>
<td>-101.2</td>
<td>92800</td>
<td>23900</td>
<td>6.034E+04</td>
</tr>
<tr>
<td>11</td>
<td>-0.0003</td>
<td>0</td>
<td>-111.2</td>
<td>92800</td>
<td>23900</td>
<td>2.26E+06</td>
</tr>
<tr>
<td>12</td>
<td>-0.0003</td>
<td>0</td>
<td>-121.2</td>
<td>92800</td>
<td>23900</td>
<td>6.034E+04</td>
</tr>
<tr>
<td>13</td>
<td>-0.0003</td>
<td>0</td>
<td>-131.2</td>
<td>92800</td>
<td>23900</td>
<td>2.26E+06</td>
</tr>
<tr>
<td>14</td>
<td>-0.0003</td>
<td>0</td>
<td>-141.2</td>
<td>92800</td>
<td>23900</td>
<td>6.034E+04</td>
</tr>
<tr>
<td>15</td>
<td>-0.0003</td>
<td>0</td>
<td>-151.2</td>
<td>92800</td>
<td>23900</td>
<td>2.26E+06</td>
</tr>
<tr>
<td>16</td>
<td>-0.0003</td>
<td>0</td>
<td>-161.2</td>
<td>92800</td>
<td>23900</td>
<td>6.034E+04</td>
</tr>
<tr>
<td>17</td>
<td>-0.0003</td>
<td>0</td>
<td>-171.2</td>
<td>92800</td>
<td>23900</td>
<td>2.26E+06</td>
</tr>
<tr>
<td>18</td>
<td>-0.0003</td>
<td>0</td>
<td>-181.2</td>
<td>92800</td>
<td>23900</td>
<td>6.034E+04</td>
</tr>
<tr>
<td>19</td>
<td>-0.0003</td>
<td>0</td>
<td>-191.2</td>
<td>92800</td>
<td>23900</td>
<td>2.26E+06</td>
</tr>
<tr>
<td>20</td>
<td>-0.0003</td>
<td>0</td>
<td>-201.2</td>
<td>92800</td>
<td>23900</td>
<td>6.034E+04</td>
</tr>
</tbody>
</table>
### Tangent Modulus Buckling - Numerical

<table>
<thead>
<tr>
<th>$\Delta \varepsilon$</th>
<th>P</th>
<th>$E_1 x$</th>
<th>$E_2 x$</th>
<th>$K_{L x}$</th>
<th>$K_{L y}$</th>
<th>$\sigma_T / \sigma_Y$</th>
<th>$x_{cr}$</th>
<th>$y_{cr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.001</td>
<td>-417.6</td>
<td>185600</td>
<td>1670400</td>
<td>209.4395</td>
<td>628.3185</td>
<td>0.174</td>
<td>181.3799</td>
<td>181.3799</td>
</tr>
<tr>
<td>-0.004</td>
<td>-556.8</td>
<td>185600</td>
<td>1670400</td>
<td>209.4395</td>
<td>628.3185</td>
<td>0.232</td>
<td>157.0796</td>
<td>157.0796</td>
</tr>
<tr>
<td>-0.005</td>
<td>-696</td>
<td>185600</td>
<td>1670400</td>
<td>209.4395</td>
<td>628.3185</td>
<td>0.279</td>
<td>140.4962</td>
<td>140.4962</td>
</tr>
<tr>
<td>-0.006</td>
<td>-835.2</td>
<td>185600</td>
<td>1670400</td>
<td>209.4395</td>
<td>628.3185</td>
<td>0.330</td>
<td>128.2549</td>
<td>128.2549</td>
</tr>
<tr>
<td>-0.007</td>
<td>-974.4</td>
<td>185600</td>
<td>1670400</td>
<td>209.4395</td>
<td>628.3185</td>
<td>0.381</td>
<td>118.7410</td>
<td>118.7410</td>
</tr>
<tr>
<td>-0.008</td>
<td>-1113.6</td>
<td>185600</td>
<td>1670400</td>
<td>209.4395</td>
<td>628.3185</td>
<td>0.433</td>
<td>111.0720</td>
<td>111.0720</td>
</tr>
<tr>
<td>-0.009</td>
<td>-1252.8</td>
<td>185600</td>
<td>1670400</td>
<td>209.4395</td>
<td>628.3185</td>
<td>0.485</td>
<td>104.7198</td>
<td>104.7198</td>
</tr>
<tr>
<td>-0.010</td>
<td>-1384.8</td>
<td>185600</td>
<td>1670400</td>
<td>209.4395</td>
<td>628.3185</td>
<td>0.537</td>
<td>94.4927</td>
<td>94.4927</td>
</tr>
<tr>
<td>-0.011</td>
<td>-1510.08</td>
<td>1670400</td>
<td>15177216</td>
<td>109.1105</td>
<td>294.5984</td>
<td>0.589</td>
<td>85.0432</td>
<td>85.0432</td>
</tr>
<tr>
<td>-0.012</td>
<td>-1624.32</td>
<td>1484800</td>
<td>8552448</td>
<td>94.9835</td>
<td>227.9603</td>
<td>0.641</td>
<td>65.8065</td>
<td>65.8065</td>
</tr>
<tr>
<td>-0.013</td>
<td>-1734.72</td>
<td>1313600</td>
<td>5729472</td>
<td>85.9752</td>
<td>180.5479</td>
<td>0.693</td>
<td>52.1197</td>
<td>52.1197</td>
</tr>
<tr>
<td>-0.014</td>
<td>-1832.16</td>
<td>1113600</td>
<td>360864</td>
<td>75.5861</td>
<td>136.0173</td>
<td>0.745</td>
<td>39.2649</td>
<td>39.2649</td>
</tr>
<tr>
<td>-0.015</td>
<td>-1924.9</td>
<td>928000</td>
<td>1968064</td>
<td>65.2262</td>
<td>103.6811</td>
<td>0.797</td>
<td>28.7118</td>
<td>28.7118</td>
</tr>
<tr>
<td>-0.016</td>
<td>-2008.32</td>
<td>742400</td>
<td>742400</td>
<td>57.5812</td>
<td>87.3926</td>
<td>0.849</td>
<td>19.9467</td>
<td>19.9467</td>
</tr>
<tr>
<td>-0.017</td>
<td>-2083.2</td>
<td>556800</td>
<td>556800</td>
<td>49.2763</td>
<td>69.0974</td>
<td>0.901</td>
<td>12.8023</td>
<td>12.8023</td>
</tr>
<tr>
<td>-0.018</td>
<td>-2152.8</td>
<td>371200</td>
<td>371200</td>
<td>40.5641</td>
<td>52.1197</td>
<td>0.953</td>
<td>6.8527</td>
<td>6.8527</td>
</tr>
<tr>
<td>-0.019</td>
<td>-2205.92</td>
<td>185600</td>
<td>185600</td>
<td>27.7074</td>
<td>23.9953</td>
<td>1.000</td>
<td>2.3935</td>
<td>2.3935</td>
</tr>
</tbody>
</table>

---

### Tangent Modulus Buckling - Numerical

![Inelastic Column Buckling](image)
Figure 6.7.1
Column strength curves for H-shaped sections having compressive residual stress at design tips. (Adapted from Ref. 6.20, p. 39)
Going back to the original three second-order differential equations:

Therefore,

1. $EI_x v'' + P v - \phi \left( P x_0 + M_{by} - \frac{z}{L} (M_{ty} + M_{by}) \right) = M_{bx} - \frac{z}{L} (M_{tx} + M_{bx})$
2. $EI_y u'' + P u - \phi \left( -P y_0 + M_{by} - \frac{z}{L} (M_{tx} + M_{by}) \right) = -M_{by} + \frac{z}{L} (M_{ty} + M_{by})$
3. $EI_w \phi'' + (G K_y + \bar{K}) \phi' + u' \left( -M_{by} - \frac{z}{L} (M_{by} + M_{tx}) + P y_0 \right)$

$$-v' \left( M_{by} + \frac{z}{L} (M_{by} + M_{ty}) + P x_0 \right) - \frac{v}{L} (M_{ty} + M_{by}) - \frac{u}{L} (M_{tx} + M_{bx}) = 0$$
ELASTIC BUCKLING OF BEAMS

Consider the case of a beam subjected to uniaxial bending only:
- because most steel structures have beams in uniaxial bending
- Beams under biaxial bending do not undergo elastic buckling

\[ P=0; \quad M_{TY}=M_{BY}=0 \]

The three equations simplify to:

1. \[ E I_x y'' = M_{BX} - \frac{z}{L} (M_{TX} + M_{BX}) \]
2. \[ E I_y u'' - \phi M_{BX} = \frac{z}{L} (M_{TX} + M_{BX})(-\phi) \]
3. \[ E I_u \phi'' - (G K_f + \bar{K}) \phi' + u' \left( -M_{BX} - \frac{z}{L} (M_{BX} + M_{TX}) \right) - \frac{u}{L} (M_{TX} + M_{BX}) = 0 \]

Equation (1) is an uncoupled differential equation describing in-plane bending behavior caused by \( M_{TX} \) and \( M_{BX} \)

ELASTIC BUCKLING OF BEAMS

Equations (2) and (3) are coupled equations in \( u \) and \( \phi \) – that describe the lateral bending and torsional behavior of the beam. In fact they define the lateral torsional buckling of the beam.

The beam must satisfy all three equations (1, 2, and 3). Hence, beam in-plane bending will occur UNTIL the lateral torsional buckling moment is reached, when it will take over.

Consider the case of uniform moment \( (M_0) \) causing compression in the top flange. This will mean that
- \(-M_{BX} = M_{TX} = M_0\)
ELASTIC BUCKLING OF BEAMS

For this case, the differential equations (2 and 3) will become:

\[ E I_y u'' + \phi M_o = 0 \]
\[ E I_w \phi'' - (G K_T + \bar{K}) \phi' + u' (M_o) = 0 \]

where:
\[ \bar{K} = \text{Wagner's effect due to warping caused by torsion} \]
\[ \bar{K} = \int_A \sigma a^2 \, dA \]

But, \( \sigma = \frac{M_o}{I_x} y \Rightarrow \text{neglecting higher order terms} \)

\[ \therefore \bar{K} = \int_A \frac{M_o}{I_x} y \left[ (x_o - x)^2 + (y_o - y)^2 \right] \, dA \]
\[ \therefore \bar{K} = \frac{M_o}{I_x} \int_A y \left[ x_o^2 + x^2 - 2x x_o + y_o^2 + y^2 - 2y y_o \right] \, dA \]
\[ \therefore \bar{K} = \frac{M_o}{I_x} \left[ \int_A y \, dA + \int_A y \left[ x^2 + y^2 \right] \, dA - x_o \int_A 2x y \, dA + y_o \int_A y^2 \, dA - 2y_o \int_A y^2 \, dA \right] \]

ELASTIC BUCKLING OF BEAMS

\[ \therefore \bar{K} = \frac{M_o}{I_x} \left[ \int_A y \left[ x^2 + y^2 \right] \, dA - 2y_o I_x \right] \]
\[ \therefore \bar{K} = M_o \left[ \frac{\int_A y \left[ x^2 + y^2 \right] \, dA}{I_x} \right] - 2y_o \]
\[ \therefore \bar{K} = M_o \beta_x \Rightarrow \text{where, } \beta_x = \frac{\int_A y \left[ x^2 + y^2 \right] \, dA}{I_x} - 2y_o \]

\( \beta_x \) is a new sectional property

The beam buckling differential equations become:

(2) \[ E I_y u'' + \phi M_o = 0 \]
(3) \[ E I_w \phi'' - (G K_T + M_o \beta_x) \phi' + u' (M_o) = 0 \]
ELASTIC BUCKLING OF BEAMS

Equation (2) gives \( u'' = -\frac{M_o}{EI_y} \phi \)

Substituting \( u'' \) from Equation (2) in (3) gives:

\[ EI_w \phi'' - (GK_T + M_o \beta_z) \phi'' - \frac{M_o^2}{EI_y} \phi = 0 \]

For doubly symmetric section: \( \beta_z = 0 \)

\[ \therefore \phi'' - \frac{GK_T}{EI_w} \phi'' - \frac{M_o^2}{E^2 I_y I_w} \phi = 0 \]

Let, \( \lambda_1 = \frac{GK_T}{EI_w} \) and \( \lambda_2 = \frac{M_o^2}{E^2 I_y I_w} \)

\[ \therefore \phi'' - \lambda_1 \phi'' - \lambda_2 \phi = 0 \Rightarrow \text{becomes the combined d.e. of LTB} \]

ELASTIC BUCKLING OF BEAMS

Assume solution is of the form \( \phi = e^{i\zeta} \)

\[ \therefore (\lambda^4 - \lambda_1 \lambda^2 - \lambda_2) e^{i\zeta} = 0 \]

\[ \therefore \lambda^4 - \lambda_1 \lambda^2 - \lambda_2 = 0 \]

\[ \therefore \lambda^2 = \frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}, \quad -\frac{\sqrt{\lambda_1^2 + 4\lambda_2} - \lambda_1}{2} \]

\[ \therefore \lambda = \pm \sqrt{\frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}}, \quad \pm i \sqrt{\frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}} \]

\[ \therefore \text{Let, } \lambda = \pm \alpha_1, \quad \text{and} \quad \pm i \alpha_2 \]

Above are the four roots for \( \lambda \)

\[ \therefore \phi = C_1 e^{\alpha_1 z} + C_2 e^{-\alpha_1 z} + C_3 e^{i\alpha_2 z} + C_4 e^{-i\alpha_2 z} \]

\[ \therefore \text{collecting real and imaginary terms} \]

\[ \therefore \phi = G_1 \cosh(\alpha_1 z) + G_2 \sinh(\alpha_1 z) + G_3 \sin(\alpha_2 z) + G_4 \cos(\alpha_2 z) \]
ELASTIC BUCKLING OF BEAMS

Assume simply supported boundary conditions for the beam:

\[
\phi(0) = \phi''(0) = \phi(L) = \phi''(L) = 0
\]

Solution for \( \phi \) must satisfy all four b.c.

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
\alpha_1^2 & \cosh(\alpha_1L) & \sinh(\alpha_1L) & \sin(\alpha_1L) & \cos(\alpha_1L) \\
\alpha_2^2 & \cosh(\alpha_2L) & \alpha_2^2 \sinh(\alpha_2L) & -\alpha_2^2 \sin(\alpha_2L) & -\alpha_2^2 \cos(\alpha_2L)
\end{bmatrix}
\begin{bmatrix}
G_1 \\
G_2 \\
G_3 \\
G_4
\end{bmatrix} = 0
\]

For buckling coefficient matrix must be singular:

\[
\det \text{ant of matrix} = 0
\]

\[
(\alpha_1^2 + \alpha_2^2) \times \sinh(\alpha_1L) \times \sin(\alpha_2L) = 0
\]

Of these:
only \( \sin(\alpha_1L) = 0 \)

\[
\Rightarrow \alpha_1L = n\pi
\]

\[
\Rightarrow \alpha_2 = \frac{n\pi}{L}
\]

\[
\Rightarrow \sqrt{\frac{\alpha_1^2 + 4\alpha_2^2 - \lambda_1}{2}} = \frac{\pi}{L}
\]

\[
\Rightarrow \sqrt{\frac{\lambda_1^2 + 4\lambda_2^2 - \lambda_1}{2}} = \frac{2\pi}{L}
\]

\[
\Rightarrow \lambda_2 = \left(\frac{2\pi^2}{L^2} + \lambda_1\right)^2 - \frac{\lambda_1^2}{4} = \frac{(2\pi^2 + 2\lambda_1)(2\pi^2)}{4}
\]

\[
\Rightarrow \lambda_2 = \left(\frac{\pi^2}{L^2} + \lambda_1\right)
\]

\[
\Rightarrow \frac{M_o^2}{E^2 I_1 I_w} = \left(\frac{\pi^2}{L^2} + \frac{G K_T}{E I_w}\right) \left(\frac{\pi^2}{L^2}\right)
\]

\[
\Rightarrow M_o = \sqrt{\left(\frac{E^2 I_1 I_w}{L^2}\right) \left(\frac{\pi^2}{L^2} + \frac{G K_T}{E I_w}\right) \left(\frac{\pi^2}{L^2}\right)}
\]

\[
\Rightarrow M_o = \sqrt{\frac{\pi^2}{L^2} \left(\frac{\pi^2}{L^2} + \frac{G K_T}{E I_w}\right) \left(\frac{\pi^2}{L^2}\right)}
\]