4. Beam - Columns

- Members subjected to bending & axial compression are called beam - columns.

- Beam - Columns in frames are usually subjected to end forces only. However, beam - columns may also be subjected to transverse forces in addition to end-forces.

- Behavior of beam - columns is similar somewhat to beams & columns.

4.1 Force - Deformation Behavior

\[ \text{Let, } \quad M_0 = -MBz \]

\[ \kappa M_0 = M_{tx} \]

\[ -1 \leq \kappa \leq 1 \]

\[ \text{change of variable} \quad \int_{0}^{1} \kappa = 1 \rightarrow \]

\[ \kappa = -1 \rightarrow \]
Fig. 5.1. Beam-columns and beam-column loadings. (a) Beam-columns in a frame, (b) typical beam-columns in frames, (c) typical beam-columns with transverse loads.

Fig. 5.2. End forces, dimensions, and sign conventions. (a) Beam-columns with end forces, (b) beam-column sign conventions.
Consider, the experimental behavior of a wide-flange beam-column subjected to $P = 0.49 P_Y$ (concentric) & $M_0$ (increasing) with $\kappa = 0$.

![Diagram showing in-plane behavior and out-of-plane deformation with bracing.]

- Behavior of beam-columns is different than the behavior of beams or columns.

  - $P < P_{\text{max}}$ (column) → Therefore, some reserve capacity to carry M
  - $M < M_p$ → moment carried is less than $M_p$ because $P \neq 0$

  * $M_0|_{\text{max}}$ for the beam-column is not maintained through indefinite rotation.
Note that for beams, moment is reduced due to LTB or LB buckling. → out-of-plane

But, for the beam-column → Mo/\text{max} is reached due to in-plane behavior only → no LTB or LB before Mo/\text{max} is reached.

\[ M(\zeta) = Mo \left(1 - \frac{\zeta}{L}\right) + P\zeta \]

secondary moment

As Mo increases, \( \zeta \) increases

increasing \( P\zeta \)

When member yields, \( \zeta \) increases more rapidly

\( P\zeta \) increases at such a rate that it dominates

Consider, overall behavior

\[ P \zeta \text{ are constant} \]
Fig. 5.4. Effect of lateral-torsional and local buckling
4.2 Elastic Behavior

\[ M_{By} = M_{Ty} = 0 \quad , \quad M_{Rx} = -M_o \quad & \quad M_{Tx} = K \cdot M_o \quad , \quad P = P \]

- 2nd order differential equations are:

\[ EI_x v'' + P_x - P_{xo} \phi = M_o \left[ -1 + \frac{E}{L} (1-K) \right] \quad \rightarrow (1) \]

\[ EI_y u'' + P_u + M_o \phi \left[ 1 - \frac{E}{L} (1-K) \right] + P_{yo} \phi = 0 \quad \rightarrow (2) \]

\[ EI_0 \phi''' - (G_0 K_T + K) \phi' + M_0 u' \left[ 1 - \frac{E}{L} (1-K) \right] + P_{yo} u' - P_{xo} v' + \frac{M_0}{L} (1-K) u = 0 \quad \rightarrow (3) \]

- Eq. (1), (2) & (3) are coupled. Eq (1) is also coupled through

This means that for most general cross-sections with \( x_0 \neq 0 \)
& \( y_0 \neq 0 \) → applying \( P \& M_o \rightarrow u, v, \phi \)

- For a singly symmetric cross-section with \( x_0 = 0 \)
& moments acting in the plane of symmetry (\( y-z \) plane)

\[ EI_x v'' + P_x = M_o \left[ -1 + \frac{E}{L} (1-K) \right] \quad \rightarrow (4) \]

\[ EI_y u'' + P_u + M_o \phi \left[ 1 - \frac{E}{L} (1-K) \right] + P_{yo} \phi = 0 \quad \rightarrow (5) \]

\[ EI_0 \phi''' - (G_0 K_T + K) \phi' + M_0 u' \left[ 1 - \frac{E}{L} (1-K) \right] + P_{yo} u' - P_{xo} v' + \frac{M_0}{L} (1-K) u = 0 \quad \rightarrow (6) \]
\[ (7) \quad EI_x \varphi^iv'' + P \varphi'' = 0 \]
\[ (8) \quad EI_y \varphi^iv'' + Pu'' + M_0 \left[ 1 - \frac{E}{L} (1-K) \right] \varphi'' - \frac{2 M_0}{L} (1-K) \varphi' + Pyo \varphi'' = 0 \]
\[ (9) \quad EI_{yo} \varphi^iv'' - (G Kr + Ko) \varphi'' - Kr \varphi' + M_0 \left[ 1 - \frac{E}{L} (1-K) \right] u'' + Pyo u'' = 0 \]

The value of \( K = \int \sigma \, dA \)

\[ \begin{cases} 
\sigma_z = -\frac{P}{A} + \frac{M_z \beta_x}{I_x} \\
\end{cases} \]

(10) \quad \therefore \quad \overline{K} = -P \frac{r_0^2}{K} + M_z \beta_x \]

\[ \text{column} \quad \text{beam} \]

where, \( M_z = M_0 \left( 1 - \frac{E}{L} (1-K) \right) \rightarrow \text{moment within the span of b.c.} \)

\[ \therefore \quad \text{Eq. (9) can be written as} \]

\[ EI_{yo} \varphi^iv'' - \left[ G Kr - P \frac{r_0^2}{K} + M_z \beta_x \right] \varphi'' + M_0 \beta_x \varphi' \frac{(1-K)}{L} + M_z u'' + Pyo u'' = 0 \]

\[ \rightarrow (11) \]

- Now the final equations are (7), (8) & (11)

where, Eq. (7) is uncoupled from Eq. (8) & (11)

Eq. (7) defines the in-plane bending deformations

Eq. (8) & (11) correspond to the occurrence of LTB.
4.2.1 IN-PLANE behavior & strength

\[ \text{EI}_x \, \nu^{iv} + P \, \nu^{iv} = 0 \]

\[ \nu^{iv} + \frac{P}{\text{EI}_x} \, \nu^{iv} = 0 \]

Let \( \frac{P}{\text{EI}_x} = F_{v^2} \)

\[ \nu^{iv} + F_{v^2} \, \nu^{iv} = 0 \quad \to \quad (12) \]

Solution is: \( \nu = C_1 \sin(F_{v^2}z) + C_2 \cos(F_{v^2}z) + C_3 z + C_4 \quad \to \quad (13) \)

Use boundary conditions: \( \nu(0) = \nu(L) = 0 \)
\[ \nu''(0) = -\frac{M_0}{\text{EI}_x} \quad \& \quad \nu''(L) = -\frac{\kappa \, M_0}{\text{EI}_x} \]

solve for constants of integration:

\[ (14) \quad \nu = \frac{M_0}{P} \left[ \left( \frac{\kappa - \cos F_{v^2} L}{\sin F_{v^2} L} \right) \sin F_{v^2} z + \cos F_{v^2} z + \frac{z}{L} (1 - \kappa) - 1 \right] \]

bending moment at any point \( z \) is computed as:

\[ (15) \quad M = -\text{EI}_x \, \nu'' = M_0 \left[ \left( \frac{\kappa - \cos F_{v^2} L}{\sin F_{v^2} L} \right) \sin(F_{v^2}z) + \cos(F_{v^2}z) \right] \]

max. moment location & value

\[ \frac{dM}{dz} = 0 \quad \therefore \quad M_0 \left[ \left( \frac{\kappa - \cos F_{v^2} L}{\sin F_{v^2} L} \right) \cos(F_{v^2}z) - \sin(F_{v^2}z) \right] = 0 \]
Max. moment occurs when

\[ \tan \left( \frac{F_e}{\varepsilon} \right) = \left( \frac{k - \cos F_e L}{\sin F_e L} \right) \rightarrow (16) \]

corresponding \( M_{\max} = M_0 \left[ \tan \left( \frac{F_e}{\varepsilon} \right) \sin \left( \frac{F_e}{\varepsilon} \right) + \cos \left( \frac{F_e}{\varepsilon} \right) \right] \)

which can be simplified to \( M_{\max} = M_0 \varphi \rightarrow (17) \)

where, \( \varphi = \frac{1}{\cos \left( \frac{F_e}{\varepsilon} \right)} \)

\[ \varphi = \frac{\sqrt{1 + \frac{k^2}{\sin^2 \left( \frac{F_e}{\varepsilon} \right)}}}{\sin \left( \frac{F_e}{\varepsilon} \right)} \]

\[ = \frac{k - \cos F_e L}{\sqrt{\sin^2 \left( \frac{F_e}{\varepsilon} \right) + \left( k - \cos F_e L \right)^2}} \]

\[ = \frac{k - \cos F_e L}{\sqrt{1 + k^2 - 2k \cos F_e L}} \sin F_e L \]

\[ \therefore M_{\max} = M_0 \times \frac{\sqrt{1 + \frac{k^2}{\sin^2 \left( \frac{F_e}{\varepsilon} \right)}}}{\sin \left( \frac{F_e}{\varepsilon} \right)} \rightarrow (18) \]

Thus, for a beam-column subjected to \( P \) & \( M \)

\[ \text{Moment magnification} \]

\[ \begin{array}{c}
\begin{align*}
M_0 & \rightarrow M_{\max} \\
\end{align*}
\end{array} \]

\[ \begin{array}{c}
\begin{align*}
P & \rightarrow K M_0 \\
\end{align*}
\end{array} \]
Location $z$ from: \[ \cos(Fz - \bar{z}) = \frac{\sin FzL}{\sqrt{1 + \kappa^2 - 2\kappa \cos(FzL)}} \tag{19} \]

Eq. (19) may result in values of $\bar{z}$ that are negative.

- This means that $M_{\text{max}}$ does not occur in $0 \leq E \leq L$ and $M_0$ is the max. moment for the beam-column.

- When does $M_{\text{max}}$ occur within the span:
  \[
  \begin{align*}
  \text{If} \quad \kappa & \geq \cos(FzL) \quad \text{then} \quad \varphi = 1.0 \quad (M_{\text{max}} \text{ out span}) \\
  \text{If} \quad \kappa & \leq \cos(FzL) \quad \text{then} \quad \varphi = \text{Eq. (18)} \quad \left[ M_{\text{max}} \text{ in span} \right]
  \end{align*}
  \]

Another limiting situation, when $P = \frac{\pi^2 E I x}{L^2}$

then, no moment can be applied

\[ M_0 = \frac{M_{\text{max}}}{\varphi} = 0 \]

\[ \therefore \varphi = \infty \]

\[ \therefore \sin(FzL) = 0 \]

\[ \therefore (FzL)_{\text{max}} = \pi \quad \rightarrow (20) \]
4.2.2. IN-PLANE STRENGTH

Limit of applicability of the equations is when \( \sigma_Y \) is the max. stress

\[
\sigma_{\text{max}} = \sigma_Y = \frac{P}{A} + \frac{M_{\text{max}}}{S_x}
\]

\[
\frac{P}{A \sigma_Y} + \frac{M_{\text{max}}}{S_x \sigma_Y} = 1.0
\]

\[
\frac{P}{P_Y} + \frac{M_{\text{max}}}{M_Y} = 1.0
\]

\[
\frac{P}{P_Y} + \varphi \frac{M_0}{M_Y} = 1.0 \quad \text{Re expressed limiting condition}
\]

amplification factor.

- Eq. (21) is also called an interaction equation
  Thus for a given \( P \), one can determine \( M_0 \) for the beam-column to limit \( \sigma \) to \( \sigma_Y \).

- Eq. (21) applies for \( 0 \leq F + L \leq \pi \)
  i.e. \( 0 \leq \sqrt[4]{\frac{P}{EI_z}} \leq \pi \)
In Eq. (21)

\[ \begin{align*}
\text{I: } & \quad \kappa \geq \cos (F_{rL}) \text{ then } \varphi = \frac{\sqrt{1 + \kappa^2 - 2\kappa \cos (F_{rL})}}{\sin (F_{rL})} \\
\text{II: } & \quad \kappa \leq \cos (F_{rL}) \text{ then } \varphi = 1.0
\end{align*} \]

Plot relationship between \( \varphi \) & \( F_{rL} \) for diff. \( \kappa \)

\[ \text{Fig. 5.7. Charts for determining } \varphi \]
\[ \frac{P}{P_Y} + \varphi \frac{M_o}{M_Y} = 1.0 \rightarrow \text{Eq. (2a)} \]

\[ \varphi \rightarrow f \left( \kappa, \frac{F_o L}{P} \right) \]

\[ 2 F_o L = \frac{L}{\sqrt{\frac{E}{I_x}}} = \frac{\pi L}{\sqrt{\pi^2 E I_x}} = \pi \sqrt{\frac{P}{P_E}} \]

\[ \therefore F_o L = \frac{L}{\frac{r}{E}} \sqrt{\frac{E}{I_x}} \times \frac{P}{P_Y} \]

- The elastic in-plane strength depends on four non-dimensionalized parameters \( \frac{P}{P_Y}, \frac{M_o}{M_Y}, \kappa, \frac{L}{r_x} \sqrt{E} \)

- **Interaction curves**: Slenderness effects \( \left( \frac{L}{r_x} \right) \)

\[ \frac{P}{P_Y} - \frac{M_o}{M_Y} \text{ interaction curves.} \]

- Consider the effect of slenderness \( \frac{L}{r_x} \times \sqrt{E} \) for beam-columns bent into single curvature by equal end moments \( \left( \kappa = 1.0 \right) \).

- Slender columns deflect more \( \rightarrow \ldots \text{more } P_x \text{ effects} \)
Fig. 5.8. Elastic limit interaction curves for $\kappa = 1.0$

Fig. 5.9. Exact and approximate elastic-limit interaction curves
- Introd equation: effect of moment gradient $K$

\[- \frac{L}{Kx} \sqrt{E} = 4.0 \quad \text{single curvature } K = 1.0\]

\[- \frac{L}{Kx} \sqrt{E} = 1.0 \quad \text{double curvature } K = -1.0\]

for double curvature case: max. moment at member ends.

- Other cases & APPROXIMATION

\[- \frac{M_0}{M_y} = 1.0 \quad \rightarrow \text{Eq. (21) Interaction eqn}\]

The Eq. (21) was derived for linear moment gradient

with loads. However, this form holds for other loading cases (pg. 245 of book).

The solution for $\varphi$ depends on $K$ & (FwL). It has to be \( \varphi \) for each problem. Therefore, an approximation for $\varphi$ developed

\[
\varphi = \varphi \bigg|_{K=1} \quad \text{(kind of a separation of variables)}
\]

$\alpha$ depends on FwL
Now: \( \varphi \bigg|_{k=1} = \frac{\sqrt{\kappa - 2 \cos \theta \cdot L}}{\sin \theta \cdot L} \)

\[ \approx \frac{1}{1 - \frac{P}{P_e}} \]

\[ \approx \frac{1}{1 - \left( \frac{F\cdot L}{\pi} \right)^2} \]

Fig. 5.7. Charts for determining \( \varphi \)

And: \( C_m = \sqrt{0.3 \kappa^2 + 0.4 \kappa + 0.3} \) \hspace{1cm} \text{Massonet (Europe)}

or \( C_m = 0.6 + 0.4 \kappa \) \hspace{1cm} \text{Anotia (USA)}

Values are reasonable for \(-0.5 \leq \kappa \leq 1.0\)
\[ \frac{P}{P_Y} + \frac{C_m}{1 - \frac{P}{P_E}} \frac{M_o}{M_Y} = 1.0 \rightarrow \text{Eq. (22)} \]

where, \[ C_m = 0.6 + 0.4 K \]

\[ \frac{P}{P_Y} + \frac{M_o}{M_Y} = 1.0 \rightarrow \text{Eq. (24)} \]

The approximate Eq. (22) does not a/c for the case where \( M_{max} = M_o \) at member end. Therefore need additional condition Eq. (24) must also be satisfied.

See above figure.
4.3 ELASTIC IN-PLANE BEHAVIOR OF BEAM-COLUMNS

BLE 4.1 Moment Amplification Factors

<table>
<thead>
<tr>
<th>Case</th>
<th>Maximum First-order Moment</th>
<th>Moment Amplification Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>εL</td>
<td>M₀ ≤ cos kL</td>
<td>φ = 1.0 if ε ≤ cos kL</td>
</tr>
<tr>
<td>εL</td>
<td>M₀ ≥ sin kL</td>
<td>φ = \sqrt{1 + \epsilon^2 - 2\epsilon \cos kL} if ε ≥ cos kL</td>
</tr>
</tbody>
</table>

\[
M₀ = \frac{qL^2}{8} \quad \phi = \frac{8}{\left(\frac{1 - \cos kL}{2}\right)^{\frac{1}{3}}} \\
M₀ = \frac{qL^2}{9\sqrt{3}} \quad \phi = \frac{9\sqrt{3}}{(kL)^2} \left[ \sqrt{(kL)^2 - \sin^2 kL} \right] \left[ \frac{\arccos \left(\frac{\sin kL}{kL}\right)}{kL} \right] \\
M₀ = \frac{QL}{4} \quad \phi = \frac{2\tan \frac{\epsilon L}{L}}{kL}
\]

It is obvious that the four amplification factors do not differ from each other a great deal. For practical purposes, the four amplification factors are adequately represented by the much simpler formula

\[
\phi = \frac{1}{1 - P/P_E}
\]  

This formula appears to be intuitive, and it was probably so conceived in the first beam-column interaction equations used in design specifications, but it also has a mathematical significance, as discussed next.

Assume that the deflected shape of the beam-column is represented by the series of \( n \) sine shapes with amplitudes \( a_n \), where \( n = 1, 2, 3, \ldots \)

\[
\nu = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{L}
\]  

The first and second derivatives of \( \nu \) are equal to

\[
\nu' = \sum_{n=1}^{\infty} \frac{n\pi a_n}{L} \cos \frac{n\pi z}{L} \\
-\nu'' = \sum_{n=1}^{\infty} \frac{n^2\pi^2 a_n}{L^2} \sin \frac{n\pi z}{L}
\]  

The strain energy of bending is equal to

\[
U = \frac{1}{2} \int_0^L \left[ EI\left(\frac{n\pi z}{L}\right)^2 dz = \frac{EL\pi^4}{2L^2} \cdot \frac{L}{2} \sum_{n=1}^{\infty} a_n^2 n^4 \right]
\]

because

\[
\int_0^L \sin^2 \frac{n\pi z}{L} dz = \frac{L}{2} \\
\int_0^L \sin \frac{n\pi z}{L} \cdot \sin \frac{m\pi z}{L} dz = 0, \quad n \neq m
\]
Moment equilibrium:  \[ M_{pc} = (d - y_p) b F_y \times \left( d - \frac{2(d - y_p)}{2} \right) \]

\[ M_{pc} = (d - y_p) y_p \times b F_y = \frac{bd^2 F_y}{4} \left( 1 - \frac{P}{P_y} \right) \left( 1 + \frac{P}{P_y} \right) \]

\[ \frac{M_{pc}}{M_p} = 1 - \left( \frac{P}{P_y} \right)^2 \]  (4.44)

Equation 4.44 is the plastic moment in the presence of axial force for a rectangular cross-section. The formulas for \( M_{pc} \) for other cross-sections are derived in a similar manner, but they are more complicated, of course, depending on the cross-sectional geometry. The equations for the plastic moments for x-axis and y-axis bending of wide-flange shapes are in Table 4.4. Simpler approximate formulas are also given in the table (Plastic Design in Steel ASCE 1971). These approximate equations are compared with the analytically exact formulas in Figure 4.36, where the axial ratio is the ordinate and the bending ratio is the abscissa. The exact formulas were calculated for the geometry of a W14 x 99 rolled wide-flange shape. The curves for the approximate formulas are seen to be close enough for practical purposes.

The approximate formula for a solid circular cross-section of radius \( R \) is given as equation 4.45. This is a very good approximation of a complicated exact equation.

\[ \frac{M_{pc}}{M_p} = 1 + 0.08 \frac{P}{P_y} - 1.08 \left( \frac{P}{P_y} \right)^2 ; \quad P_y = \pi R^2 F_y ; \quad M_p = \frac{4R^3 F_y}{3} \]  (4.45)

The curves from the analytically exact equations are shown in Figure 4.37, starting with the top dashed line, of a W14 x 99 wide-flange section bent about the minor axis, a solid circular section, a rectangular section, and a W14 x 99 wide-flange section bent about the major axis, respectively. The solid line represents the lower-bound interaction equation that is the basis of the AISC Specification for the design of beam-columns. As can be seen, the AISC equation closely replicates the x-axis interaction strength of the wide-flange shape. Since the AISC equation is a lower bound to the most frequent practical situation, it was adopted for use in the design standard. It should be realized, however, that for other shapes it can be very conservative. Equation 4.46 is the AISC basic interaction equation.
### Table 4.4 Plastic Moments for Wide-flange Shapes

**Bending about x-axis**

For $0 \leq \frac{P}{P_y} \leq \frac{t_w(d - 2t_f)}{A}$

$$
\frac{M_{pcx}}{M_{px}} = 1 - \frac{A^2 \left( \frac{P}{P_y} \right)^2}{4t_wZ_x}
$$

For $\frac{t_w(d - 2t_f)}{A} \leq \frac{P}{P_y} \leq 1$

$$
\frac{M_{pcx}}{M_{px}} = 1 - \frac{A \left( 1 - \frac{P}{P_y} \right)}{2Z_x}
$$

**Approximation:**

$$
\frac{M_{pcx}}{M_{px}} = 1.18 \left( 1 - \frac{P}{P_y} \right) \leq 1.0
$$

**Bending about y-axis**

For $0 \leq \frac{P}{P_y} \leq \frac{t_wd}{A}$

$$
\frac{M_{pcy}}{M_{py}} = 1 - \frac{A^2 \left( \frac{P}{P_y} \right)^2}{4dt_fZ_y}
$$

For $\frac{t_wd}{A} \leq \frac{P}{P_y} \leq 1$

$$
\frac{M_{pcy}}{M_{py}} = \frac{A^2 \left( 1 - \frac{P}{P_y} \right) \left[ Abt_f \right]}{8t_fZ_y} - \left( 1 - \frac{P}{P_y} \right)
$$

**Approximation:**

$$
\frac{M_{pcy}}{M_{py}} = 1.19 \left[ 1 - \left( \frac{P}{P_y} \right)^2 \right] \leq 1.0
$$

---

**Fig. 4.36** Exact and approximate $M$-$P$ relations for x-axis bending (lower curves) and y-axis bending (upper curves) of a W14 × 99 wide-flange section.

$$
\begin{align*}
\frac{P}{P_y} + \frac{M_{pc}}{M_p} &= 1 & \text{if } \frac{P}{P_y} \leq 0.2 \\
\frac{P}{P_y} + \frac{8M_{pc}}{9M_p} &= 1 & \text{if } \frac{P}{P_y} \geq 0.2
\end{align*}
$$

The previous discussion above considered the plastic strength of a cross-section that is subjected to an axial force at its geometric centroid and to a

---

**Fig. 4.37** Comparison of cross-section interaction curves with AISC interaction equation.
4.2.3 Elastic Lateral-Torsional Buckling behavior

Most general equations were (8) & (11)

\[ EI_y u'''' + Pu'' + M_0 \left[ 1 - \frac{E}{L} (1-\nu) \right] \phi'' - \frac{E M_0}{L} (1-\nu) \phi' + Py_0 \phi'' = 0 \tag{8} \]

\[ EI_\omega \phi'''' - \left[ G K_t - P I_o^2 + M_0 \beta_\omega \right] \phi'' + M_0 \beta_\omega \phi' \left( \frac{1-\nu}{L} \right) + M_0 u'' + P y_0 u'' = 0 \tag{18} \]

The phenomenon will be identical to the lateral-torsional buckling of beams & columns.

The critical combination of loads producing it are the max. practical load that can be sustained by the member.

Eq. (8) & (11) can be solved for singly symmetric sections ($\kappa = \nu$) & unequal end moments ($\kappa \neq \nu$) by numerical methods.

For doubly symmetric sections ($y_0 = \beta_\omega = 0$) the d.e. become

\[ EI_y u'''' + Pu'' + M_0 \left[ 1 - \frac{E}{L} (1-\nu) \right] \phi'' - \frac{E M_0}{L} (1-\nu) \phi' = 0 \tag{25} \]

\[ EI_\omega \phi'''' - \left[ G K_t - P I_o^2 \right] \phi'' + M_0 \left[ 1 - \frac{E}{L} (1-\nu) \right] u'' = 0 \tag{26} \]
Eq. (25) & (26) are best solved by numerical or energy methods. Salvadossi used the Rayleigh-Ritz method and presented the results.

Consider Eq. (25) & (26) with $\kappa = 1.0$

$$EI_y U'''' + Pu'' + \frac{1}{3} Mo \phi'' = 0$$

$$EI_0 \phi'''' - (GK_T - P_0^2) \phi'' + Mo u'' = 0$$

Assume s.s. boundary conditions $u(0) = 0, u(L) = 0, u''(0) = 0, u''(L) = 0, \phi(0) = 0, \phi(L) = 0, \phi''(0) = 0, \phi''(L) = 0$

$$u = C_1 \sin \frac{n \pi x}{L} \quad \phi = C_2 \sin \frac{n \pi x}{L}$$

$$\left[ \left( \frac{n^2 EI_y}{L^2} - P \right) C_1 - Mo C_2 \right] \times \frac{n^2 \pi^2}{L^2} \sin \frac{n \pi x}{L} = 0$$

$$\left[ -Mo C_1 + \left( \frac{n^2 EI_0}{L^2} + GK_T - P_0^2 \right) C_2 \right] \times \frac{n^2 \pi^2}{L^2} \sin \frac{n \pi x}{L} = 0$$

$$\left[ \frac{n^2 EI_y}{L^2} - P - Mo \right] \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$\begin{vmatrix} \frac{n^2 EI_0}{L^2} & -P_0^2 + GK_T \\ -Mo & \frac{n^2 EI_0}{L^2} \end{vmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\det = 0$$
\[ (P_y - P) \left( \bar{P}_o^2 P_z - P \bar{P}_o^2 \right) - M_o^2 = 0 \]

\[ M_o = \sqrt{(P_y - P)(P_z - P)\bar{P}_o^2} \quad \longrightarrow \quad (27) \]

Where, \( P_y = \frac{\pi^2 EI_y}{L^2} \) and \( P_z = \frac{\pi^2 EI_z}{\bar{P}_o^2} + G K_T \)

Eq. (23) gives the \( P - M_o \) relationship for a doubly symmetric elastic member with \( \kappa = 0 \)

\[ M_o)_{cr} = \sqrt{(P_y - P)(P_z - P) \times \frac{(I_x + I_y)}{A}} \]

Consider the member with \( r_o = 0; \; y_o \neq 0; \; \beta \neq 0 \) (singly symmetric) with \( \kappa = 1 \)

Then,

\[
\begin{vmatrix}
P_y - P & -(M_0 + Py_o) \\
-(M_0 + Py_o) & (\bar{P}_o^2 P_z - P \bar{P}_o^2 + M_0 \beta z)
\end{vmatrix} = 0
\]

\[ (P_y - P) \left( \bar{P}_o^2 P_z - P \bar{P}_o^2 + M_0 \beta z \right) = (M_0 + Py_o)^2 \quad \longrightarrow \quad (21) \]

Eq. (28) gives the critical relation between \( P - M_o \) leading to elastic lateral-torsional buckling.
- Effect of lateral-torsional buckling on W8 x 31 for

\[ \frac{L}{r_x} x \sqrt{E_T} = 1.99 \quad \text{and} \quad \frac{L}{r_y} x \sqrt{E_T} = 3.98 \]

- Lateral-torsional buckling controls everywhere for

\[ \frac{L}{r_y} \sqrt{E_T} = 8.98 \]

- Lateral-torsional buckling governs for small portion of

\[ \frac{L}{r_y} \sqrt{E_T} = 1.99 \]

Fig. 5.11. Effect of elastic lateral-torsional buckling
Another way of illustrating lateral-torsional buckling

\[ \frac{M_0}{M_Y} \text{ vs. } \frac{L}{f_e} \]

- The curve for elastic strength from \( \frac{P}{P_Y} + \varphi \frac{M_0}{M_Y} = 1.0 \)
- The curve for elastic LTB for W8x31 & N27x94 are also shown.

For efficient use of beam-columns, bracing must be provided between member ends so that LTB strength > in-plane str.

For \( \frac{P}{P_Y} = 0.3, \varepsilon_r = 0.0016, \frac{P}{P_Y} = 2.62 \)

![Graph showing lateral-torsional buckling](image)

**Fig. 5.12.** Cross-section effect on elastic lateral-torsional buckling

- **Boundary Condition effects** see pg. R52 of Galambos