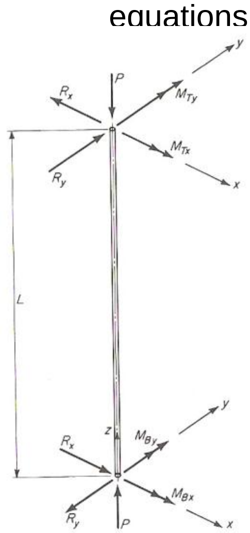
Elastic Buckling Behavior of Beams

CE579 - Structural Stability and Design

ELASTIC BUCKLING OF BEAMS

- Going back to the original three second-order differential equations:



Therefore,

1 $$E\,I_x\,v'' + P\,v - \phi\left(P\,x_0 + M_{BY} - \frac{z}{L}(M_{TY} + M_{BY})\right) = M_{BX} - \frac{z}{L}(M_{TX} + M_{BX})$$

2 $$E\,I_y\,u'' + P\,u - \phi\left(-P\,y_0 + M_{BX} - \frac{z}{L}(M_{TX} + M_{BX})\right) = -M_{BY} + \frac{z}{L}(M_{TY} + M_{BY})$$

3 $$E\,I_w\,\phi''' - (G\,K_T + \bar{K})\,\phi' + u'\,(-M_{BX} - \frac{z}{L}(M_{BX} + M_{TX}) + P\,y_0)$$
$$- v'\,(M_{BY} + \frac{z}{L}(M_{BY} + M_{TY}) + P\,x_0) - \frac{v}{L}(M_{TY} + M_{BY}) - \frac{u}{L}(M_{TX} + M_{BX}) = 0$$

ELASTIC BUCKLING OF BEAMS

- Consider the case of a beam subjected to uniaxial bending only:
  - because most steel structures have beams in uniaxial bending
  - Beams under biaxial bending do not undergo elastic buckling
- P=0; M_{TY}=M_{BY}=0
- The three equations simplify to:

1 $$E\,I_x\,v'' = M_{BX} - \frac{z}{L}\left(M_{TX} + M_{BX}\right)$$

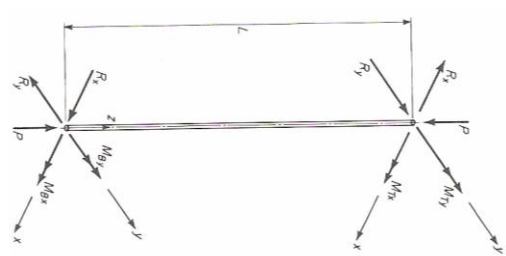
2 $$E\,I_y\,u'' - \phi\,M_{BX} = \frac{z}{L}\left(M_{TX} + M_{BX}\right)(-\phi)$$

3 $$E\,I_w\,\phi''' - (G\,K_T + \bar{K})\,\phi' + u'\left(-M_{BX} - \frac{z}{L}(M_{BX} + M_{TX})\right) - \frac{u}{L}(M_{TX} + M_{BX}) = 0$$

- Equation (1) is an uncoupled differential equation describing in-plane bending behavior caused by M_{TX} and M_{BX}

ELASTIC BUCKLING OF BEAMS

- Equations (2) and (3) are coupled equations in u and ϕ – that describe the lateral bending and torsional behavior of the beam. In fact they define the lateral torsional buckling of the beam.
- The beam must satisfy all three equations (1, 2, and 3). Hence, beam in-plane bending will occur UNTIL the lateral torsional buckling moment is reached, when it will take over.
- Consider the case of uniform moment (M_o) causing compression in the top flange. This will mean that
  - $-M_{BX} = M_{TX} = M_o$



Uniform Moment Case

- For this case, the differential equations (2 and 3) will become:

$$E\, I_y\, u'' + \phi\, M_o = 0$$

$$E\, I_w\, \phi''' - (G\, K_T + \bar{K})\, \phi' + u' \left(M_o\right) = 0$$

where:

$\bar{K}$ = *Wagner's effect due to warping caused by torsion*

$$\bar{K} = \int_A \sigma\, a^2\, dA$$

$$But, \sigma = \frac{M_o}{I_x} y \quad \Rightarrow \text{neglecting higher order terms}$$

$$\therefore \bar{K} = \int_A \frac{M_o}{I_x} y \left[(x_o - x)^2 + (y_o - y)^2 \right] dA$$

$$\therefore \bar{K} = \frac{M_o}{I_x} \int_A y \left[x_o^2 + x^2 - 2xx_0 + y_o^2 + y^2 - 2yy_0 \right] dA$$

$$\therefore \bar{K} = \frac{M_o}{I_x} \left[x_o^2 \int_A y\, dA + \int_A y \left[x^2 + y^2 \right] dA - x_0 \int_A 2xy\, dA + y_o^2 \int_A y\, dA - 2y_o \int_A y^2 dA \right]$$

ELASTIC BUCKLING OF BEAMS

$$\therefore \bar{K} = \frac{M_o}{I_x} \left[\int_A y \left[x^2 + y^2 \right] dA - 2y_o I_x \right]$$

$$\therefore \bar{K} = M_o \left[\frac{\int_A y \left[x^2 + y^2 \right] dA}{I_x} - 2y_o \right]$$

$$\therefore \bar{K} = M_o \beta_x \qquad \Rightarrow \text{where, } \beta_x = \frac{\int_A y \left[x^2 + y^2 \right] dA}{I_x} - 2y_o$$

β_x *is a new sectional property*

The beam buckling differential equations become:

$$(2) \quad E\, I_y\, u'' + \phi\, M_o = 0$$

$$(3) \quad E\, I_w\, \phi''' - (G\, K_T + M_o \beta_x)\, \phi' + u' \left(M_o\right) = 0$$

ELASTIC BUCKLING OF BEAMS

$$Equation\ (2)\ gives\ \ u'' = -\frac{M_o}{E\,I_y}\phi$$

$$Substituting\ u''\ from\ Equation\ (2)\ in\ (3)\ gives:$$

$$E\,I_w\,\phi^{iv} - (G\,K_T + M_o\beta_x)\,\phi'' - \frac{M_o^{\,2}}{E\,I_y}\phi = 0$$

$$For\ doubly\ symmetric\ section: \beta_x = 0$$

$$\therefore \phi^{iv} - \frac{G\,K_T}{E\,I_w}\phi'' - \frac{M_o^{\,2}}{E^2\,I_y I_w}\phi = 0$$

$$Let,\ \lambda_1 = \frac{G\,K_T}{E\,I_w} \quad and \quad \lambda_2 = \frac{M_o^{\,2}}{E^2\,I_y I_w}$$

$$\therefore \phi^{iv} - \lambda_1\,\phi'' - \lambda_2\phi = 0 \quad \Rightarrow becomes\ the\ combined\ d.e.\ of\ LTB$$

ELASTIC BUCKLING OF BEAMS

$$Assume\ solution\ is\ of\ the\ form \quad \phi = e^{\lambda z}$$

$$\therefore \left(\lambda^4 - \lambda_1\,\lambda^2 - \lambda_2\right)e^{\lambda z} = 0$$

$$\therefore \lambda^4 - \lambda_1\,\lambda^2 - \lambda_2 = 0$$

$$\therefore \lambda^2 = \frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2} \quad , \quad -\frac{\sqrt{\lambda_1^2 + 4\lambda_2} - \lambda_1}{2}$$

$$\therefore \lambda = \pm\sqrt{\frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}} \quad , \quad \pm i\sqrt{\frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}}$$

$$\therefore Let,\ \lambda = \pm\alpha_1 \quad , \quad and \quad \pm i\,\alpha_2$$

$$Above\ are\ the\ four\ roots\ for\ \lambda$$

$$\therefore \phi = C_1 e^{\alpha_1 z} + C_2 e^{-\alpha_1 z} + C_3 e^{i\alpha_2 z} + C_4 e^{-i\alpha_2 z}$$

$$\therefore collecting\ real\ and\ imaginary\ terms$$

$$\therefore \phi = G_1\cosh(\alpha_1 z) + G_2\sinh(\alpha_1 z) + G_3\sin(\alpha_2 z) + G_4\cos(\alpha_2 z)$$

ELASTIC BUCKLING OF BEAMS

- Assume simply supported boundary conditions for the beam:

$$\therefore \phi(0) = \phi''(0) = \phi(L) = \phi''(L) = 0$$

Solution for ϕ must satisfy all four b.c.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ \alpha_1^2 & 0 & 0 & -\alpha_2^2 \\ \cosh(\alpha_1 L) & \sinh(\alpha_1 L) & \sin(\alpha_2 L) & \cos(\alpha_2 L) \\ \alpha_1^2\cosh(\alpha_1 L) & \alpha_1^2\sinh(\alpha_1 L) & -\alpha_2^2\sin(\alpha_2 L) & -\alpha_2^2\cos(\alpha_2 L) \end{bmatrix} \times \begin{Bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{Bmatrix} = 0$$

For buckling coefficient matrix must be singular:

$$\therefore \text{determinant of matrix} = 0$$

$$\therefore \left(\alpha_1^2 + \alpha_2^2\right) \times \sinh(\alpha_1 L) \times \sin(\alpha_2 L) = 0$$

Of these:

only $\sin(\alpha_2 L) = 0$

$$\therefore \alpha_2 L = n\pi$$

ELASTIC BUCKLING OF BEAMS

$$\therefore \alpha_2 = \frac{n\pi}{L}$$

$$\therefore \sqrt{\frac{\sqrt{\lambda_1^2 + 4\lambda_2} - \lambda_1}{2}} = \frac{\pi}{L}$$

$$\therefore \sqrt{\lambda_1^2 + 4\lambda_2} - \lambda_1 = \frac{2\pi^2}{L^2}$$

$$\therefore \lambda_2 = \frac{\left(\frac{2\pi^2}{L^2} + \lambda_1\right)^2 - \lambda_1^2}{4} = \frac{\left(\frac{2\pi^2}{L^2} + 2\lambda_1\right)\left(\frac{2\pi^2}{L^2}\right)}{4}$$

$$\therefore \lambda_2 = \left(\frac{\pi^2}{L^2} + \lambda_1\right)\left(\frac{\pi^2}{L^2}\right)$$

$$\therefore \lambda_2 = \frac{M_o^2}{E^2 I_y I_w} = \left(\frac{\pi^2}{L^2} + \frac{G K_T}{E I_w}\right)\left(\frac{\pi^2}{L^2}\right)$$

$$\therefore M_o = \sqrt{\left(E^2 I_y I_w\right)\left(\frac{\pi^2}{L^2} + \frac{G K_T}{E I_w}\right)\left(\frac{\pi^2}{L^2}\right)}$$

$$\therefore M_o = \sqrt{\frac{\pi^2 E I_y}{L^2}\left(\frac{\pi^2 E I_w}{L^2} + G K_T\right)}$$

Uniform Moment Case

- The critical moment for the uniform moment case is given by the simple equations shown below.

$$M_{cr}^{o} = \sqrt{\frac{\pi^2 E I_y}{L^2} \times \left(\frac{\pi^2 E I_w}{L^2} + G K_T \right)}$$

$$M_{cr}^{o} = \sqrt{P_y \times P_\phi \times \bar{r}_o^2}$$

- The AISC code massages these equations into different forms, which just look different. Fundamentally the equations are the same.
  - The critical moment for a span with distance L_b between lateral - torsional braces.
  - P_y is the column buckling load about the minor axis.
  - P_ϕ is the column buckling load about the torsional z- axis.

Non-uniform moment

- The only case for which the differential equations can be solved analytically is the uniform moment.
- For almost all other cases, we will have to resort to numerical methods to solve the differential equations.
- Of course, you can also solve the uniform moment case using numerical methods

$$E\, I_x\, v'' = M_{BX} - \frac{z}{L}(M_{TX} + M_{BX})$$

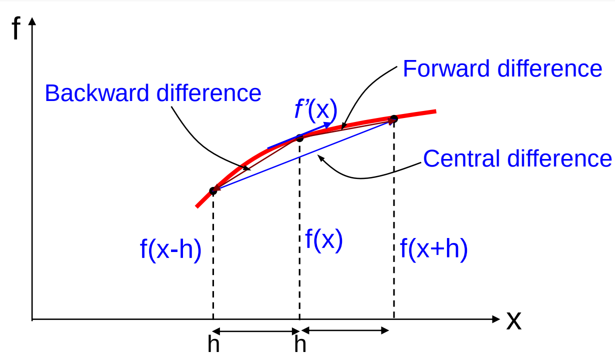
$$E\, I_y\, u'' - \phi\, M_{BX} = \frac{z}{L}(M_{TX} + M_{BX})\ \phi$$

$$E\, I_w\, \phi''' - (G\, K_T + \bar{K})\, \phi' + u' \left(-M_{BX} - \frac{z}{L}(M_{BX} + M_{TX}) \right) - \frac{u}{L}(M_{TX} + M_{BX}) = 0$$

What numerical method to use

- What we have is a problem where the governing differential equations are known.
  - The solution and some of its derivatives are known at the boundary.
  - This is an ordinary differential equation and a boundary value problem.
- We will solve it using the finite difference method.
  - The FDM converts the differential equation into algebraic equations.
  - Develop an FDM mesh or grid (as it is more correctly called) in the structure.
  - Write the algebraic form of the d.e. at each point within the grid.
  - Write the algebraic form of the boundary conditions.
  - Solve all the algebraic equations simultaneously.

Finite Difference Method



$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{iv}(x) + \ldots$$

$$\therefore f'(x) = \frac{f(x+h) - f(x)}{h} + \frac{h}{2!} f''(x) + \frac{h^2}{3!} f'''(x) + \frac{h^3}{4!} f^{iv}(x) + \ldots$$

$$\therefore f'(x) = \frac{f(x+h) - f(x)}{h} + O(h) \qquad \Rightarrow Forward\,difference\,equation$$

Finite Difference Method

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \ldots$$

$$\therefore f'(x) = \frac{f(x) - f(x-h)}{h} + \frac{h}{2!}f''(x) - \frac{h^2}{3!}f'''(x) + \frac{h^3}{4!}f^{iv}(x) + \ldots$$

$$\therefore f'(x) = \frac{f(x) - f(x-h)}{h} + O(h) \quad \Rightarrow \text{Backward difference equation}$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \ldots$$

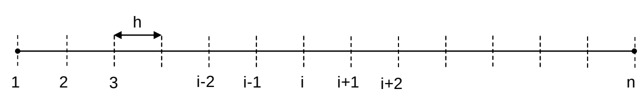
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \ldots$$

$$\therefore f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \frac{2h^2}{3!}f'''(x) + \ldots$$

$$\therefore f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \quad \Rightarrow \text{Central difference equation}$$

Finite Difference Method

- The central difference equations are better than the forward or backward difference because the error will be of the order of h-square rather than h.
- Similar equations can be derived for higher order derivatives of the function f(x).
- If the domain x is divided into several equal parts, each of length h.



- At each of the ‘nodes’ or ‘section points’ or ‘domain points’ the differential equations are still valid.

Finite Difference Method

- Central difference approximations for higher order derivatives:

$$y_i' = \frac{1}{2h}\left(y_{i+1} - y_{i-1}\right)$$

$$y_i'' = \frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right)$$

$$y_i''' = \frac{1}{2h^3}\left(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right)$$

$$y_i^{iv} = \frac{1}{h^4}\left(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right)$$

Notation

$y = f(x)$

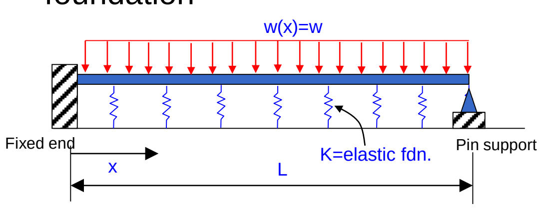
$y_i = f(x = i)$

$y_i' = f'(x = i)$

$y_i'' = f''(x = i) \quad and\ so\ on \cdots$

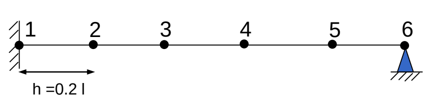
FDM - Beam on Elastic Foundation

- Consider an interesting problemn --> beam on elastic foundation



$$EI\,y^{iv} + k\,y(x) = w(x)$$

- Convert the problem into a finite difference problem.



$$EI\,y_i^{iv} + k\,y_i = w$$

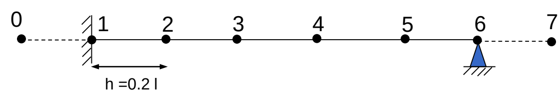
Discrete form of differential equation

FDM - Beam on Elastic Foundation

$$EI\,y_i^{iv} + k\,y_i = w$$

$$\therefore \frac{EI}{h^4}\left(y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}\right) + ky_i = w$$

$$write\,4\ equations for\ i = 2,\ 3,\ 4,\ 5$$



Need two imaginary nodes that lie within the boundary
Hmm…. These are needed to only solve the problem
They don't mean anything.

FDM - Beam on Elastic Foundation

$$At\ i = 2:\quad \frac{625\,EI}{L^4}\left(y_0 - 4y_1 + 6y_2 - 4y_3 + y_4\right) + ky_2 = w$$

$$At\ i = 3:\quad \frac{625\,EI}{L^4}\left(y_1 - 4y_2 + 6y_3 - 4y_4 + y_5\right) + ky_3 = w$$

$$At\ i = 4:\quad \frac{625\,EI}{L^4}\left(y_2 - 4y_3 + 6y_4 - 4y_5 + y_6\right) + ky_4 = w$$

$$At\ i = 5:\quad \frac{625\,EI}{L^4}\left(y_3 - 4y4_1 + 6y_5 - 4y_6 + y_7\right) + ky_5 = w$$

- Lets consider the boundary conditions:

$$y(0) = 0 \quad \Rightarrow \quad y_1 = 0 \qquad (1)$$

$$y(L) = 0 \quad \Rightarrow \quad y_6 = 0 \qquad (2)$$

$$M(L) = 0 \qquad (3)$$

$$\theta(0) = 0 \quad \Rightarrow \quad y'(0) = 0 \qquad (4)$$

FDM - Beam on Elastic Foundation

$$y(0)=0 \quad \Rightarrow \quad y_1=0 \qquad (1)$$

$$y(L)=0 \quad \Rightarrow \quad y_6=0 \qquad (2)$$

$$M(L)=0 \qquad (3)$$

$$\therefore EI\, y''(L)=0 \quad \Rightarrow \quad y''_6=0$$

$$\therefore y''_6=\frac{1}{h^2}\left(y_5-2y_6+y_7\right)=0$$

$$\therefore y_7=-y_5 \qquad (3+)$$

$$\theta(0)=0 \quad \Rightarrow \quad y'(0)=0 \qquad (4)$$

$$\therefore y'_1=\frac{1}{2h}\left(y_2-y_0\right)=0$$

$$\therefore y_2=y_0 \qquad (4+)$$

FDM - Beam on Elastic Foundation

- Substituting the boundary conditions:

$$At\ i=2: \quad \left(7y_2-4y_3+y_4\right)+\frac{kL^4}{625\,EI}y_2=\frac{wL^4}{625\,EI}$$

$$At\ i=3: \quad \left(-4y_2+6y_3-4y_4+y_5\right)+\frac{kL^4}{625\,EI}y_3=\frac{wL^4}{625\,EI}$$

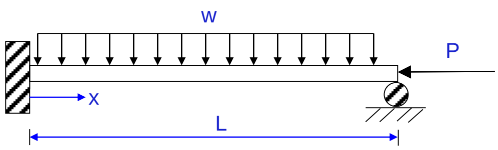
$$At\ i=4: \quad \left(y_2-4y_3+6y_4-4y_5\right)+\frac{kL^4}{625\,EI}y_4=\frac{wL^4}{625\,EI}$$

$$At\ i=5: \quad \left(y_3-4y_1+5y_5\right)+\frac{kL^4}{625\,EI}y_5=\frac{wL^4}{625\,EI}$$

Let a = $kl^4/625EI$

$$\begin{bmatrix} 7+a & -4 & 1 & 0 \\ -4 & 6+a & -4 & 1 \\ 1 & -4 & 6+a & -4 \\ 0 & 1 & -4 & 5+a \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \frac{wL^4}{625\,EI}$$

FDM - Column Euler Buckling

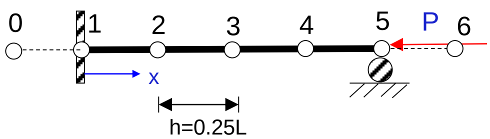


Buckling problem: Find axial load P for which the nontrivial Solution exists.

Ordinary Differential Equation

$$y^{iv}(x) + \frac{P}{EI} y''(x) = \frac{w}{EI}$$

Finite difference solution. Consider case Where w=0, and there are 5 stations



FDM - Euler Column Buckling

Finite difference method

$$y_i^{iv} + \frac{P}{EI} y_i'' = 0$$

At stations $i = 2, 3, 4$

$$\frac{1}{h^4}\left(y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}\right) + \frac{P}{EI} \bullet \frac{1}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) = 0$$

Boundary conditions

$$y_1 = 0 \qquad (1)$$

$$y_5 = 0 \qquad (2)$$

$$y_1' = 0$$

$$\therefore \frac{1}{2h}\left(y_2 - y_0\right) = 0$$

$$\therefore y_0 = y_2 \qquad (3)$$

$$M_5 = 0$$

$$\therefore EI \bullet y_5'' = 0$$

$$\therefore (y_6 - 2y_5 + y_4) = 0$$

$$\therefore y_6 = -y_4 \qquad (4)$$

FDM - Column Euler Buckling

- Final Equations

$$\frac{1}{h^4}(7y_2-4y_3+y_4)+\frac{P}{EI}\bullet\frac{1}{h^2}(-2y_2+y_3)=0$$
$$\frac{1}{h^4}(-4y_2+6y_3-4y_4)+\frac{P}{EI}\bullet\frac{1}{h^2}(y_2-2y_3+y_4)=0$$
$$\frac{1}{h^4}(y_2-4y_3+5y_4)+\frac{P}{EI}\bullet\frac{1}{h^2}(y_3-2y_4)=0$$

$$\therefore Matrix\,Form$$
$$\begin{bmatrix}7 & -4 & 1\\ -4 & 6 & -4\\ 1 & -4 & 5\end{bmatrix}\begin{Bmatrix}y_2\\ y_3\\ y_4\end{Bmatrix}+\frac{PL^2}{16EI}\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{Bmatrix}y_2\\ y_3\\ y_4\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$$

FDM - Euler Buckling Problem

- [A]{y}+λ[B]{y}={0}
  - How to find P? Solve the eigenvalue problem.
- Standard Eigenvalue Problem
  - [A]{y}=λ{y}
    - Where, λ = eigenvalue and {y} = eigenvector
  - Can be simplified to [A-λI]{y}={0}
  - Nontrivial solution for {y} exists if and only if | A-λI|=0
  - One way to solve the problem is to obtain the characteristic polynomial from expanding | A-λI|=0
  - Solving the polynomial will give the value of λ
  - Substitute the value of λ to get the eigenvector {y}
  - This is not the best way to solve the problem, and will not work for more than 4or 5th order polynomial

FDM - Euler Buckling Problem

- For solving Buckling Eigenvalue Problem
- [A]{y} + λ[B]{y}={0}
- [A+ λ B]{y}={0}
- Therefore, det |A+ λ B|=0 can be used to solve for λ

$$A = \begin{bmatrix} 7 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$and\ \lambda = \frac{PL^2}{16EI}$$

$$\begin{vmatrix} 7-2\lambda & -4+\lambda & 1 \\ -4+\lambda & 6-2\lambda & -4+\lambda \\ 1 & -4+\lambda & 5-2\lambda \end{vmatrix} = 0$$

$$\therefore \lambda = 1.11075$$

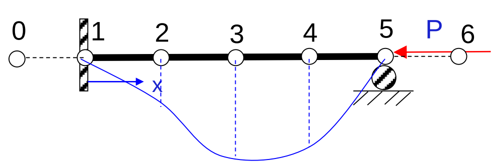
$$\therefore \frac{PL^2}{16EI} = 1.11075$$

$$\therefore P_{cr} = 17.772\frac{EI}{L^2}$$

$$Exact\ solution is\ 20.14\frac{EI}{L^2}$$

FDM - Euler Buckling Problem

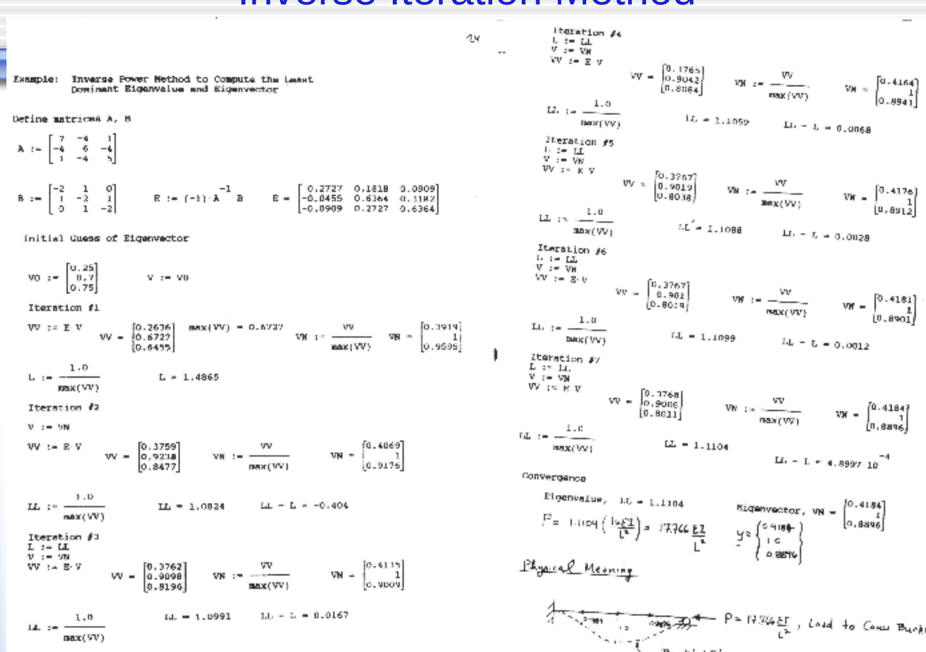
- 11% error in solution from FDM
- {y}= {0.4184 1.0   0.8896}^{T}



FDM Euler Buckling Problem

- Inverse Power Method: Numerical Technique to Find Least Dominant Eigenvalue and its Eigenvector
  - Based on an initial guess for eigenvector and iterations
- Algorithm
  - 1) Compute $[E]=-[A]^{-1}[B]$
  - 2) Assume initial eigenvector guess $\{y\}^0$
  - 3) Set iteration counter i=0
  - 4) Solve for new eigenvector $\{\underline{y}\}^{i+1}=[E]\{y\}^i$
  - 5) Normalize new eigenvector $\{y\}^{i+1}=\{\underline{y}\}^{i+1}/\max(\underline{y}_j^{i+1})$
  - 6) Calculate eigenvalue = $1/\max(\underline{y}_j^{i+1})$
  - 7) Evaluate convergence: $\lambda^{i+1}-\lambda^i < tol$
  - 8) If no convergence, then go to step 4
  - 9) If yes convergence, then $\lambda=\lambda^{i+1}$ and $\{y\}=\{y\}^{i+1}$

Inverse Iteration Method



Example: Inverse Power Method to Compute the Least Dominant Eigenvalue and Eigenvector

Define matrices A, B

$$A := \begin{bmatrix} 7 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix}$$

$$B := \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad E := (-1)\cdot A^{-1}\cdot B \quad E = \begin{bmatrix} 0.2727 & 0.1818 & 0.0909 \\ -0.0455 & 0.6364 & 0.3182 \\ -0.0909 & 0.2727 & 0.6364 \end{bmatrix}$$

Initial Guess of Eigenvector

$$V0 := \begin{bmatrix} 0.25 \\ 0.7 \\ 0.75 \end{bmatrix} \quad V := V0$$

Iteration #1

$$VV := E\cdot V \quad VV = \begin{bmatrix} 0.2636 \\ 0.6727 \\ 0.6455 \end{bmatrix} \quad \max(VV) = 0.6727 \quad VN := \frac{VV}{\max(VV)} \quad VN = \begin{bmatrix} 0.3919 \\ 1 \\ 0.9595 \end{bmatrix}$$

$$L := \frac{1.0}{\max(VV)} \quad L = 1.4865$$

Iteration #2

$V := VN$

$$VV := E\cdot V \quad VV = \begin{bmatrix} 0.3759 \\ 0.9238 \\ 0.8477 \end{bmatrix} \quad VN := \frac{VV}{\max(VV)} \quad VN = \begin{bmatrix} 0.4069 \\ 1 \\ 0.9176 \end{bmatrix}$$

$$LL := \frac{1.0}{\max(VV)} \quad LL = 1.0824 \quad LL - L = -0.404$$

Iteration #3

$L := LL$

$V := VN$

$$VV := E\cdot V \quad VV = \begin{bmatrix} 0.3762 \\ 0.9098 \\ 0.8196 \end{bmatrix} \quad VN := \frac{VV}{\max(VV)} \quad VN = \begin{bmatrix} 0.4115 \\ 1 \\ 0.9009 \end{bmatrix}$$

$$LL := \frac{1.0}{\max(VV)} \quad LL = 1.0991 \quad LL - L = 0.0167$$

Iteration #4

$L := LL$

$V := VN$

$$VV := E\cdot V \quad VV = \begin{bmatrix} 0.3765 \\ 0.9042 \\ 0.8084 \end{bmatrix} \quad VN := \frac{VV}{\max(VV)} \quad VN = \begin{bmatrix} 0.4164 \\ 1 \\ 0.8941 \end{bmatrix}$$

$$LL := \frac{1.0}{\max(VV)} \quad LL = 1.1059 \quad LL - L = 0.0068$$

Iteration #5

$L := LL$

$V := VN$

$$VV := E\cdot V \quad VV = \begin{bmatrix} 0.3767 \\ 0.9019 \\ 0.8038 \end{bmatrix} \quad VN := \frac{VV}{\max(VV)} \quad VN = \begin{bmatrix} 0.4176 \\ 1 \\ 0.8912 \end{bmatrix}$$

$$LL := \frac{1.0}{\max(VV)} \quad LL = 1.1088 \quad LL - L = 0.0028$$

Iteration #6

$L := LL$

$V := VN$

$$VV := E\cdot V \quad VV = \begin{bmatrix} 0.3767 \\ 0.902 \\ 0.8019 \end{bmatrix} \quad VN := \frac{VV}{\max(VV)} \quad VN = \begin{bmatrix} 0.4181 \\ 1 \\ 0.8901 \end{bmatrix}$$

$$LL := \frac{1.0}{\max(VV)} \quad LL = 1.1099 \quad LL - L = 0.0012$$

Iteration #7

$L := LL$

$V := VN$

$$VV := E\cdot V \quad VV = \begin{bmatrix} 0.3768 \\ 0.9006 \\ 0.8011 \end{bmatrix} \quad VN := \frac{VV}{\max(VV)} \quad VN = \begin{bmatrix} 0.4184 \\ 1 \\ 0.8896 \end{bmatrix}$$

$$LL := \frac{1.0}{\max(VV)} \quad LL = 1.1104 \quad LL - L = 4.8997\cdot 10^{-4}$$

Convergence

Eigenvalue, $LL = 1.1104$ Eigenvector, $VN = \begin{bmatrix} 0.4184 \\ 1 \\ 0.8896 \end{bmatrix}$

$P = 1.1104\left(\frac{16EI}{L^2}\right) = 17.766\frac{EI}{L^2}$ $\quad y = \begin{Bmatrix} 0.4184 \\ 1.0 \\ 0.8896 \end{Bmatrix}$

Physical Meaning

$P = 17.766\frac{EI}{L^2}$, Load to Cause Buck

Buckled Shape

The differential equation for the problem is:

$$\phi^{IV} - \lambda_1 \phi'' - \lambda_2 \phi = 0$$

where $\lambda_1 = \dfrac{G K_T}{E I_w}$ $\qquad \lambda_2 = \dfrac{M_0^2}{E^2 I_y I_w}$

λ_1 = constant & $\lambda_2 \rightarrow$ eigenvalue or unknown we are looking

For the W27×94 section:

$K_T = 4.03\ in^4$ $\qquad r_y = 2.12\ in$ $\qquad G = 11200\ ksi$

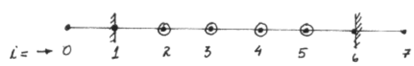
$I_w = 21300\ in^6$ $\qquad I_y = 124\ in^4$ $\qquad E = 29000\ ksi$

Length of beam = $200\ r_y$ = 424 in.

Divide into ⑤ segments for this solution:

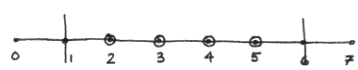
$\therefore h = \dfrac{424}{5} = 84.8$ in.

The original differential equation is valid at each station



$$\therefore \phi_i^{IV} - \lambda_1 \phi_i'' - \lambda_2 \phi_i = 0 \longrightarrow \text{at each station } i$$

=0



@ i = 2

$$\frac{1}{h^4}\phi_0 + \phi_1\left[-\frac{4}{h^4} - \frac{\lambda_1}{h^2}\right]^{\to 0} + \phi_2\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_3\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_4 \cdot \frac{1}{h^4} = 0$$

@ i = 3

$$\frac{1}{h^4}\phi_1^{\to 0} + \phi_2\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_3\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_4\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_5 \cdot \frac{1}{h^4} = 0$$

@ i = 4

$$\frac{1}{h^4}\phi_2 + \phi_3\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_4\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_5\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_6 \cdot \frac{1}{h^4}^{\to 0} = 0$$

@ i = 5

$$\frac{1}{h^4}\phi_3 + \phi_4\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_5\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_6\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right]^{\to 0} + \phi_7 \cdot \frac{1}{h^4} = 0$$

$\phi_1 = \phi_6 = 0$

$\phi_0 = \phi_2$ & $\phi_5 = \phi_7$

$$\phi_2\left[\frac{7}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_3\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_4 \cdot \frac{1}{h^4} = 0$$

$$\phi_2\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_3\left[\frac{6}{h^4} + \frac{2\lambda_1}{h^2} - \lambda_2\right] + \phi_4\left[\frac{-4}{h^4} - \frac{\lambda_1}{h^2}\right] + \phi_5 \cdot \frac{1}{h^4} = 0$$

$$
\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots \\
-\frac{4}{h^4}-\frac{\lambda_1}{h^2} & \frac{6}{h^4}+\frac{2\lambda_1}{h^2}-\lambda_2 & -\frac{4}{h^4}-\frac{\lambda_1}{h^2} & \frac{1}{h^4} \\
\frac{1}{h^4} & -\frac{4}{h^4}-\frac{\lambda_1}{h^2} & \frac{6}{h^4}+\frac{2\lambda_1}{h^2}-\lambda_2 & -\frac{4}{h^4}-\frac{\lambda_1}{h^2} \\
0 & \frac{1}{h^4} & -\frac{4}{h^4}-\frac{\lambda_1}{h^2} & \frac{7}{h^4}+\frac{2\lambda_1}{h^2}-\lambda_2
\end{bmatrix}
\begin{bmatrix} \cdots \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = 0
$$

Multiplying all equations by h^4

$$
\begin{bmatrix}
7+2\lambda_1 h^2-\lambda_2 h^4 & -4-\lambda_1 h^2 & 1 & 0 \\
-4-\lambda_1 h^2 & 6+2\lambda_1 h^2-\lambda_2 h^4 & -4-\lambda_1 h^2 & 1 \\
1 & -4-\lambda_1 h^2 & 6+2\lambda_1 h^2-\lambda_2 h^4 & -4-\lambda_1 h^2 \\
0 & 1 & -4-\lambda_1 h^2 & 7+2\lambda_1 h^2-\lambda_2 h^4
\end{bmatrix}
\begin{Bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix}
$$

Solve using inverse iteration:

$$\lambda_1 = \frac{GK_T}{EI_w} \checkmark \qquad \lambda_2 = \frac{M_o^2}{E^2 I_y I_w}$$

$$[A] - \lambda_2 [B] = \{0\}.$$

$$
\therefore [A] =
\begin{bmatrix}
7+2\lambda_1 h^2 & -4-\lambda_1 h^2 & 1 & 0 \\
-4-\lambda_1 h^2 & 6+2\lambda_1 h^2 & -4-\lambda_1 h^2 & 1 \\
1 & -4-\lambda_1 h^2 & 6+2\lambda_1 h^2 & -4-\lambda_1 h^2 \\
0 & 1 & -4-\lambda_1 h^2 & 7+2\lambda_1 h^2
\end{bmatrix}
$$

$$\lambda_1 = \frac{GK_T}{EI_w} = \frac{11200 \times 0.492}{29000 \times 21200} = 7.342 \times 10^{-5}$$

$$\lambda_2 = \frac{M_o^2}{E^2 I_y I_w}$$

$$h = \frac{L}{5} = \frac{424}{5} = 84.8 \text{ in}$$

$$
\therefore
\begin{bmatrix}
8.055932 & -4.52766 & 1 & 0 \\
-4.52766 & 7.055932 & -4.52766 & 1 \\
1 & -4.52766 & 7.055932 & -4.52766 \\
0 & 1 & -4.52766 & 8.055932
\end{bmatrix}
- h^4 \lambda_2 [I] = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

Using Matlab eigenvalue command

$$h^4 \lambda_2 = \begin{Bmatrix} 0.8107 \\ 4.0175 \\ 9.7735 \\ 15.6220 \end{Bmatrix}$$

0.8107 → main eigenvalue → eigenvector = $\begin{Bmatrix} -0.3095 \\ -0.6358 \\ -0.6358 \\ -0.3095 \end{Bmatrix}$

4.0175, 9.7735, 15.6220 → other eigenvalues

$$h^4 \cdot \frac{M_o^2}{E^2 I_y I_w} = 0.8107$$

$$\therefore M_o^2 = 0.8107 \times E^2 \times 124 \times \frac{21200}{84.8^4}$$

$$\therefore M_o = 5887 \text{ k-in}$$

Different Boundary Conditions

Table 3.2. EFFECTIVE LENGTH FACTORS IN LATERAL-TORSIONAL BUCKLING

Boundary Conditions $z = 0$	Boundary Conditions $z = L$	K_y	K_z	K_{13}
$u = u'' = \phi = \phi'' = 0$	$u = u'' = \phi = \phi'' = 0$	1.000	1.000	1.000
$u = u'' = \phi = \phi'' = 0$	$u = u'' = \phi = \phi' = 0$	0.904	0.693	0.904
$u = u'' = \phi = \phi'' = 0$	$u = u' = \phi = \phi'' = 0$	0.626	1.000	0.904
$u = u'' = \phi = \phi'' = 0$	$u = u' = \phi = \phi' = 0$	0.693	0.693	1.000
$u = u'' = \phi = \phi' = 0$	$u = u'' = \phi = \phi' = 0$	0.883	0.492	0.883
$u = u' = \phi = \phi'' = 0$	$u = u' = \phi = \phi' = 0$	0.431	0.693	0.875
$u = u' = \phi = \phi' = 0$	$u = u' = \phi = \phi' = 0$	0.492	0.492	1.000
$u = u' = \phi = \phi'' = 0$	$u = u' = \phi = \phi'' = 0$	0.434	1.000	0.883
$u = u' = \phi = \phi'' = 0$	$u = u'' = \phi = \phi' = 0$	0.606	0.492	0.875

Beams with Non-Uniform Loading

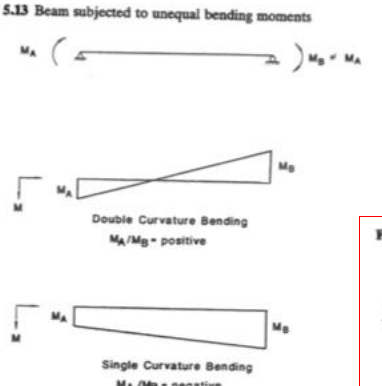
- Let M_o^{cr} be the lateral-torsional buckling moment for the case of uniform moment.
- If the applied moments are non-uniform (but varying linearly, i.e., there are no loads along the length)
  - Numerically solve the differential equation using FDM and the Inverse Iteration process for eigenvalues
  - Alternately, researchers have already done these numerical solution for a variety of linear moment diagrams
  - The results from the numerical analyses were used to develop a simple equation that was calibrated to give reasonable results.

Beams with Non-uniform Loading

- Salvadori in the 1970s developed the equation below based on the regression analysis of numerical results with a simple equation
  - $M^{cr} = C_b\, M_o^{cr}$
  - Critical moment for non-uniform loading = C_b x critical moment for uniform moment.

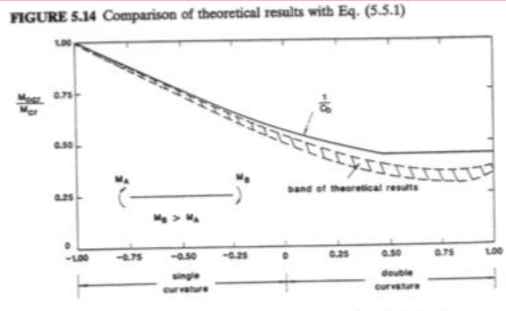
Beams with Non-uniform Loading

IGURE 5.13 Beam subjected to unequal bending moments



$$C_b = 1.75 + 1.05\left(\frac{M_A}{M_B}\right) + 0.3\left(\frac{M_A}{M_B}\right)^2 \le 2.3 \qquad (5.5.2)$$

FIGURE 5.14 Comparison of theoretical results with Eq. (5.5.1)



Beams with Non-uniform Loading

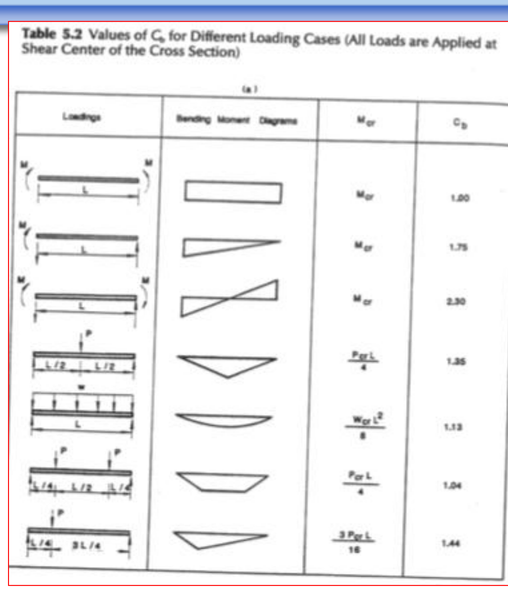


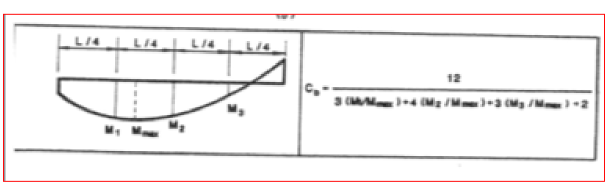
Table 5.2 Values of C_b for Different Loading Cases (All Loads are Applied at Shear Center of the Cross Section)

(a)

Loadings	Bending Moment Diagrams	M_{cr}	C_b
		M_{cr}	1.00
		M_{cr}	1.75
		M_{cr}	2.30
L/2, L/2		$\frac{P_{cr}L}{4}$	1.35
L		$\frac{W_{cr}L^2}{8}$	1.13
L/4, L/2, L/4		$\frac{P_{cr}L}{4}$	1.04
L/4, 3L/4		$\frac{3P_{cr}L}{16}$	1.44

Beams with Non-Uniform Loading

- In case that the moment diagram is not linear over the length of the beam, i.e., there are transverse loads producing a non-linear moment diagram
  - The value of C_b is a little more involved



$$C_b = \frac{12}{3(M_1/M_{max}) + 4(M_2/M_{max}) + 3(M_3/M_{max}) + 2}$$

Beams with non-simple end conditions

- $M_o^{cr} = (P_y P_\phi \underline{r_o}^2)^{0.5}$
    - P_Y with K_b
    - P_ϕ with K_t

Table 5.6 Effective Length Factors for Beams Under Uniform Moment with Various Boundary Conditions (Adapted from Ref. 19)

Boundary Conditions		K_b	K_t
$z = 0$	$z = L$		
$u = u'' = \gamma = \gamma'' = 0$	$u = u'' = \gamma = \gamma'' = 0$	1.000	1.000
$u = u'' = \gamma = \gamma'' = 0$	$u = u'' = \gamma = \gamma' = 0$	0.904	0.693
$u = u'' = \gamma = \gamma'' = 0$	$u = u' = \gamma = \gamma'' = 0$	0.626	1.000
$u = u'' = \gamma = \gamma'' = 0$	$u = u' = \gamma = \gamma' = 0$	0.693	0.693
$u = u'' = \gamma = \gamma' = 0$	$u = u'' = \gamma = \gamma' = 0$	0.883	0.492
$u = u' = \gamma = \gamma'' = 0$	$u = u' = \gamma = \gamma' = 0$	0.431	0.693
$u = u' = \gamma = \gamma' = 0$	$u = u' = \gamma = \gamma' = 0$	0.492	0.492
$u = u' = \gamma = \gamma'' = 0$	$u = u' = \gamma = \gamma'' = 0$	0.434	1.000
$u = u' = \gamma = \gamma'' = 0$	$u = u'' = \gamma = \gamma' = 0$	0.606	0.492

Beam Inelastic Buckling Behavior

- Uniform moment case

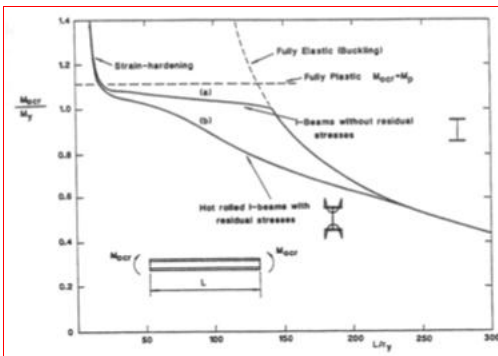
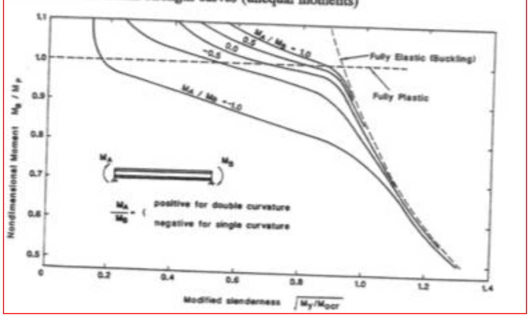


FIGURE 5.27 Beam strength curves (equal and opposite moments)

Beam Inelastic Buckling Behavior

- Non-uniform moment

FIGURE 5.28 Beam strength curves (unequal moments)



Beam In-plane Behavior

- Section capacity M_p
- Section M-ϕ behavior

Beam Design Provisions

CHAPTER F in AISC Specifications