Infiltration

Governing laws governing water movement in soil:

Darcy’s law: \( q_s = -K \frac{\partial H}{\partial s} \)

- \( q_s \) = flux in s direction (flow/unit area) (L over T)
- \( K \) = hydraulic conductivity (L/T)
- \( H \) = total potential head = \( h - z \)
- where \( h \) = pressure head
  - \( z \) = distance below the surface

for saturated conditions \( \theta = \partial s \), \( h > 0 \)
for unsaturated conditions \( h < 0 \)

The relationship between \( h \) and \( \theta \) = soil characteristic curve.

The curve is not unique (hysteresis)

For unsaturated conditions
\[ q_s = -K(\theta) \frac{\partial H}{\partial s} \quad \text{or} \quad \text{(Figure 4.4)} \]
\[ q_s = -K(h) \frac{\partial H}{\partial s} - (1) \]

Conservation of mass equation

\[ \frac{\partial \theta}{\partial t} = -\nabla \cdot q = -\left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) \]

where \( \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \)

\( \bar{q} = \text{flux vector} \)

For vertical flow:

\[ \frac{\partial \theta}{\partial t} = -\frac{\partial q_z}{\partial z} \]

(1) and (2) yields Richards equation

\[ C(h) \frac{d}{dh} \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial t} \left[ K(h) \frac{\partial h}{\partial z} \right] - \frac{\partial K(h)}{\partial z} \]

For 2D flow add \( \frac{\partial}{\partial x} (K(h) \frac{\partial h}{\partial x}) \). For \( \theta \) as independent variable:

\[ \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial t} \left[ D(\theta) \frac{\partial \theta}{\partial z} \right] - \frac{\partial K}{\partial z} \]

The three variable parameters \( K(h), C(h), D(\theta) \) make the equation non-linear.

For saturated conditions (h-based) equation

\( C(h) = 0 \) Equation becomes Laplace

\( \theta \)-based becomes invalid for saturated conditions

Numerical Solution for Richards Equation:

B.C., I.C. one for each differential

For ponded conditions:

\[ h=\delta \quad z=0 \quad t>0 \]
\[ h=h_i \quad z\rightarrow\infty \quad t\geq \]
\[ h=h_i \quad z\geq0 \quad t=0 \]

analytical solution is not possible due to non-linearity.

Discritization

1. Explicit scheme
2. Implicit scheme
3. Crank-Nicholson scheme

(Define: explicit, implicit, CN)
Explicit scheme:

\[ h_{i}^{j+1} = h_{i}^{j} + \frac{\Delta t}{C_{i}^{j} \Delta z} \left[ K_{i}^{j+\frac{1}{2}} \left( \frac{h_{i+1}^{j} - h_{i}^{j}}{\Delta Z} - 1 \right) - K_{i}^{\frac{1}{2}} \left( \frac{h_{i-1}^{j} - h_{i}^{j}}{\Delta Z} - 1 \right) \right] \]

\( j = \) time
\( i = \) spatial increment

The method is only stable for
\[ \Delta t < 0.25 \Delta z^2 \frac{C}{K} \]

Implicit scheme:

\[ a_{i} h_{i}^{j+1} + b_{i} h_{i}^{j+1} + c_{i} h_{i}^{j+1} = \text{RHSi} \]

\[ a_{i} = \frac{K_{i}^{j+\frac{1}{2}}}{\Delta X^2} \]
\[ b_{i} = - \left[ \left( K_{i}^{j+\frac{1}{2}} + K_{i}^{\frac{1}{2}} \right) / \Delta Z^2 + C_{i} / \Delta t \right] \]
\[ c_{i} = K_{i}^{\frac{1}{2}} / \Delta Z^2 \]

\[ \text{RHSi} = \left( K_{i}^{j+\frac{1}{2}} - K_{i}^{\frac{1}{2}} \right) / \Delta Z - C_{i} h_{i}^{j} / \Delta t \]

This is not fully implicit since \( C_{i} \) and \( K_{i}^{j+\frac{1}{2}} \) are still represented at \( j^{th} \) time rather than \( J+1 \)

Boundary conditions

a) Constant flux at surface
b) Ponded surface
c) Constant water content at the surface

\[ q = R = -K \left( \frac{\partial h}{\partial t} - 1 \right) \] at \( z=0; 0 \leq t < t_{p} \)

Factors affecting infiltration
- Soil properties
Initial water content

**Rainfall rate:**

Surface Sealing
- Layered soil
- Air entrapment

**Movement and Entrapment of Soil Air**

Two phase flow air-water

**Water phase continuity**

\[
\frac{\partial q}{\partial z} = -\phi \frac{\partial S_w}{\partial t}
\]

**Air phase continuity**

\[
\frac{\partial}{\partial z} (\rho_a q_a) = -\phi \frac{\partial}{\partial t} (\rho_a S_a)
\]

S = saturation = \(\frac{\phi}{\Phi}\)

\(\Phi\) = porosity

\(\rho\) = fluid density

**Water phase Darcy’s**

\[
q = -\frac{K k_r}{\mu} \left( \frac{\partial P}{\partial z} - \rho_s \right)
\]

**Air phase Darcy’s:**

\[
q_a = \frac{K k_{ra}}{\mu a} \left( \frac{\partial P}{\partial z} - \rho_s g \right)
\]

where \(K\) = absolute or saturated permeability in the z-direction (\(L^2\))

\[
K = \frac{k \rho g}{\mu}
\]

\(k, k_{ra}\) = relative permeabilities of water and air

\(S_w + S_a = 1\)

\(P_c = P_a - P\)

\(\rho_a = \frac{P_a}{R_o T}\) (perfect gas)
\( k_r = K_r(S_w) \)
\( k_{ra} = K_{ra}(S_a) = k_{ra}(1-S_w) \)

Combining the above equations:

**Air pressure**
\[
\frac{\partial}{\partial z} \left[ \frac{K_{kr}}{\mu} \left( \frac{\partial \rho_a}{\partial z} - \frac{\partial P_c}{\partial z} - \rho g \right) \right] + \frac{1}{\rho a} \frac{\partial}{\partial z} \left[ \frac{K_{kr} \rho_a}{Ma} \left( \frac{\partial P_a}{\partial z} - \rho_a g \right) \right] = \phi S_a \frac{\partial P_a}{P_a} \frac{\partial t}{t}
\]

**Water saturated**
\[
\frac{\partial S_w}{\partial t} = \frac{1}{(\phi)} \frac{\partial}{\partial z} \left[ \frac{K_{kr}}{\mu} \left( \frac{\partial P}{\partial z} - \rho g \right) \right]
\]

**Air saturation**
\[
\frac{\partial S_a}{\partial t} = \frac{1}{(\phi P_a)} \frac{\partial}{\partial z} \left[ \frac{K_{kr} \rho_a}{\mu a} \left( \frac{\partial P_a}{\partial z} - \rho_a g \right) \right] - S_a \frac{\partial P_a}{P_a} \frac{\partial t}{t}
\]

Solve these equations for air pressures at \( t+1 \), and air and water saturation

**Approximate equations**

**Horton** (ponded)
\[
f_p = f_c + (f_o - f_c) e^{\beta t}
\]

**Green-Ampt** (ponded)
\[
f_p = K_s (h_o + S_c + L_f) / L_f
\]
from Darcy’s law
\[ H_0 + S_f + L_f = \text{total potential at front} \]

\[ F = \int f_p = (\theta_s - \theta_i) L_f = ML_f \]

If \( H_o > 0 \) the equation is then

\[ f_p = \frac{K_s + K_s M S_f}{F} \]

\[ f_p = \frac{dF}{dt} \]

**Philip Equation**

\[ f_p = \frac{S}{2} t^{1/2} + C_a \]

\( S = \text{sorptivity} \)

\( C_a = \text{final infiltration rate at large } t. \)

\[ C_a = \frac{2}{3} K_s \]

\[ S = \sqrt{2 M K_s S_f} \]

\( M = \text{fillable porosity} \)

\( (\theta_s - \theta_i) \)

\( S_f = \text{effective suction at wetting front.} \)