White 7-2, Schlichting, L&R, etc.

F. Illingworth-Stewartson … Transformations for Compressible B.L. : (steady flow)

- Continuity eqn:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad \Rightarrow \text{let } \frac{\partial \psi}{\partial y} = \rho u$$

$$\partial \psi / \partial x = -\rho v$$

to eliminate continuity. Try to use similarity variables on DE's – one for stream fn & one for compressibility effects.

Let $\psi(\xi,\eta) = \int \rho u\, dy \equiv G(\xi) f(\eta)$

$\Rightarrow u(\xi,\eta) = U_e(\xi) f'(\eta)$ ('split' off density effects)

Illingworth Transformation:

$$\xi = \int_0^x \rho_e(x) U_e(x) \mu_e(x)\, dx = \xi(x) \text{ only (simplifies)}$$

$$\eta = \frac{U_e}{\sqrt{2\xi}} \int_0^y \rho\, dy = \eta(x,y)$$

Substitute into steady-flow momentum eqn:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p_e}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$ ($p = p_e$ because $\partial p/\partial y = 0$.)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\cancel{\frac{\partial \xi}{\partial y}}^{\,0}$$

$$\frac{\partial \eta}{\partial x} = \frac{U_e}{\sqrt{2\xi}}\left[\int_0^y \frac{\partial \rho}{\partial x} dy\right] + \int_0^y \rho\, dy \left[\frac{\sqrt{2\xi}\,\frac{\partial U_e}{\partial x} + U_e \frac{1}{2}\sqrt{2}\,\xi^{-1/2}\frac{\partial \xi}{\partial x}}{2\xi}\right]$$

$$\frac{\partial \eta}{\partial y} = \frac{u_e}{\sqrt{2\xi}}\rho \qquad \frac{\partial \xi}{\partial x} = \rho_e u_e \mu_e$$

$$u(\xi,\eta) = u_e(\xi) f'(\eta) \qquad \frac{\partial u}{\partial \eta} = u_e f'' \quad \frac{\partial u}{\partial \xi} = u_e' f'$$

Substitute:

$$\rho u_e f' \left[u_e f'' \frac{\partial \eta}{\partial x} + u_e' f' \rho_e u_e \mu_e \right]$$

$$- \frac{\partial \psi}{\partial x}\left[u_e f'' \frac{u_e}{\sqrt{2\xi}} \rho \right] = -\frac{\partial p_e}{\partial x} + \frac{\partial}{\partial y}\left(\mu u_e f'' \frac{u_e \rho}{\sqrt{2\xi}} \right)$$

$$\cancel{\frac{\partial p_e}{\partial x} = \frac{\partial p_e}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial p_e}{\partial \xi}\frac{\partial \xi}{\partial x}} \text{? no, use } u_e$$

$$\frac{\partial p_e}{\partial x} = -\rho_e u_e \frac{\partial u_e}{\partial x} \qquad \text{(inviscid outer flow)}$$

$\boxed{u_e = u_e(\xi) \text{ only}}$

$$\frac{\partial u_e}{\partial x} = \underbrace{\cancel{\frac{\partial u_e}{\partial \eta}}}_{0}\frac{\partial \eta}{\partial x} + \frac{\partial u_e}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{d u_e}{d\xi} \rho_e u_e \mu_e$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial x} \; ; \; \psi = G(\xi) f(\eta)$$

$$= G f' \frac{\partial \eta}{\partial x} + G' f \rho_e u_e \mu_e$$

back into mom:

$$\rho \cancel{u_e} f' \left[\cancel{u_e} f'' \frac{\partial \eta}{\partial x} + u_e' f' \rho_e \cancel{u_e} \mu_e \right]$$

$$- \left[G f' \frac{\partial \eta}{\partial x} + G' f \rho_e u_e \mu_e \right] \left[\cancel{u_e}^{2} \frac{f'' \rho}{\sqrt{2\xi}} \right]$$

$$= \rho_e^2 \cancel{u_e}^{2} \mu_e \frac{du_e}{d\xi} + \frac{\partial}{\partial y}\left(\mu \frac{\cancel{u_e} \rho f''}{\sqrt{2\xi}} \right)$$

put into form of White (7-20) $\mu_e = \mu_e(x)$, $\rho_e = \rho_e(x)$, $u_e = u_e(x)$
so can pass these thru $\partial/\partial y$:

$$\frac{\rho f' f''}{\rho_e \mu_e}\frac{\partial \eta}{\partial x} + \rho f' f' u_e' - \left[G f' \frac{\partial \eta}{\partial x} \frac{1}{\rho_e \mu_e} + G' f u_e \right] \frac{f'' \rho}{\sqrt{2\xi}}$$

$$= \rho_e \frac{du_e}{d\xi} + \frac{\partial}{\partial y}\left(\frac{\rho \mu}{\rho_e \mu_e} \frac{f''}{\sqrt{2\xi}} \right)$$

$$C \equiv \frac{\rho \mu}{\rho_e \mu_e}$$ – Chapman-Rubesin parameter — skip from here.

Now $$\frac{\partial}{\partial y}(\,) = \frac{\partial(\,)}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial(\,)}{\partial \xi}\frac{\partial \xi}{\partial y}^{\to 0} = \frac{\partial(\,)}{\partial \eta}\frac{u_e \rho}{\sqrt{2\xi}}$$

$$= \frac{u_e \rho}{\sqrt{2\xi}} \frac{\partial}{\partial \eta}\left(C \frac{f''}{\sqrt{2\xi}} \right) = \frac{\rho u_e}{2\xi} \frac{\partial}{\partial \eta}\left(C f'' \right)$$

$$\frac{\rho f' f''}{\rho_e \mu_e}\frac{\partial \eta}{\partial x} + \rho f'^2 u_e' - \frac{f''\rho}{\sqrt{2\xi}}\left[\frac{G f'}{\rho_e \mu_e}\frac{\partial \eta}{\partial x} + G' f u_e\right]$$

$$= \rho_e \frac{du_e}{d\xi} + \frac{\rho u_e}{2\xi}\frac{\partial}{\partial \eta}(Cf'')$$

put leading term and left & get rid of factor on it:

$$\frac{\partial}{\partial \eta}(Cf'') + \frac{\rho_e}{\rho}\frac{2\xi}{u_e}\frac{du_e}{d\xi} - f'f''\frac{\partial \eta}{\partial x}\frac{\rho}{\rho_e\mu_e}\frac{2\xi}{\rho u_e}$$

$$-\cancel{\rho} f'^2 u_e' \frac{2\xi}{\cancel{\rho} u_e} + \frac{f''\cancel{\rho}}{\sqrt{2\xi}}\frac{2\xi}{\cancel{\rho} u_e}\left[\frac{G f'}{\rho_e \mu_e}\frac{\partial \eta}{\partial x} + G' f u_e\right] = 0$$

Now

$$\frac{\partial \eta}{\partial x} = \eta + \frac{\sqrt{2\xi}}{u_e}\eta\left[\frac{\partial u_e}{\partial x}\frac{1}{\sqrt{2\xi}} + \frac{u_e\sqrt{2}}{4}\xi^{-3/2}\frac{\partial \xi}{\partial x}\right]$$

$$= \eta + \frac{\eta}{u_e}\left[\frac{\partial u_e}{\partial x} + \frac{u_e}{2}\frac{1}{\xi}\frac{\partial \xi}{\partial x}\right]$$

but $\frac{\partial u_e}{\partial x} = \frac{du_e}{d\xi}\rho_e u_e \mu_e$; $\frac{\partial \xi}{\partial x} = \rho_e \mu_e u_e$, so

$$= \eta + \eta\frac{1}{\cancel{u_e}}\left[\frac{du_e}{d\xi}\rho_e\mu_e\cancel{u_e} + \frac{u_e}{2\xi}\rho_e\mu_e\cancel{u_e}\right]$$

$$\frac{\partial \eta}{\partial x} = \eta + \eta\rho_e\mu_e\left(\frac{du_e}{d\xi} + \frac{u_e}{2\xi}\right)$$

3rd term

So

$$\frac{\partial}{\partial \eta}(Cf'') + \frac{\rho_e}{\rho}\frac{2\xi}{u_e}\frac{du_e}{d\xi} - \frac{2\xi}{\rho_e \mu_e u_e} f' f'' \left[\eta + \eta \rho_e \mu_e \left(\frac{du_e}{d\xi} + \frac{u_e}{2\xi}\right)\right]$$

$$- \underset{\uparrow}{f'^2}\, 2\xi \frac{du_e}{d\xi}\frac{1}{u_e} + \sqrt{2\xi}\, f'' \frac{1}{u_e}\left[\frac{G f'}{\rho_e \mu_e}\left(\eta + \eta \rho_e \mu_e \left(\frac{du_e}{d\xi} + \frac{u_e}{2\xi}\right)\right) + G' f u_e\right] = 0$$

4th term

$$\frac{\partial}{\partial \eta}(Cf'') + \left(\frac{\rho_e}{\rho} - f'^2\right)\frac{2\xi}{u_e}\frac{du_e}{d\xi}$$

$$- \frac{2\xi \eta}{u_e} f' f'' \left[\frac{1}{\rho_e \mu_e} + \frac{du_e}{d\xi} + \frac{u_e}{2\xi}\right]$$

$$+ \sqrt{2\xi}\, f'' \left[\eta \frac{G f'}{u_e}\left(\frac{1}{\rho_e \mu_e} + \frac{du_e}{d\xi} + \frac{u_e}{2\xi}\right) + G' f\right] = 0$$

$\longrightarrow$ if $G' = \frac{1}{\sqrt{2\xi}}$, may march: $\Rightarrow G = \frac{1}{\sqrt{2}}\xi^{1/2} \cdot 2 + \cancel{\text{const.}}^{\,C_0}$

remaining terms:

$$0 \overset{?}{=} -2\xi \cancel{\eta} \frac{\cancel{f' f''}}{u_e}\left[\cancel{\frac{1}{\rho_e \mu_e} + \frac{du_e}{d\xi}} + \frac{u_e}{2\xi}\right] + \sqrt{2\xi}\, \cancel{f''}\, \cancel{\eta} \frac{G f'}{u_e}\left[\cancel{\frac{1}{\rho_e \mu_e} + \frac{du_e}{d\xi}} + \frac{u_e}{2\xi}\right]$$

$$0 \overset{?}{=} -\frac{2\xi}{\cancel{u_e}} + \sqrt{2\xi}\,\frac{G}{\cancel{u_e}} = -2\xi + \sqrt{2\xi}\left(\sqrt{2\xi} + C_0\right) = 0 \text{ if } C_0 = 0 \checkmark$$

$$\boxed{G(\xi) = \sqrt{2\xi}}$$

eqn becomes

$$\boxed{\frac{\partial}{\partial \eta}\left(C f''\right) + \frac{2\xi}{u_e}\frac{du_e}{d\xi}\left(\frac{\rho_e}{\rho} - f'^2\right) + f f'' = 0}$$

c.f. White

(7-20)

B.C. (solid wall):

$$\frac{\partial \psi}{\partial x} = -\rho v = \frac{\partial}{\partial x}\left(G f\right) = \frac{\partial()}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial()}{\partial \xi}\frac{\partial \xi}{\partial x}$$

$$= G f' \frac{\partial \eta}{\partial x} + G' f \frac{\partial \xi}{\partial x}$$ but $G = G(\xi)$ indep. of y.

$$\Rightarrow \boxed{f(\eta=0) = f'(\eta=0) = 0 \text{ on wall, } \eta = 0}$$

$u(\xi,\eta) = u_e f'(\eta) = 0$ if $f'(\eta=0)=0$, OK.

B.C. (free stream):

$u(\xi,\eta) \to u_e(\xi)$ as $\eta \to \infty$ $\Rightarrow \boxed{f'(\eta \to \infty) \to 1}$

Notes: if $M \ll 1$, $u_e, \rho, \mu = \text{const}$, $C \cong 1$, $\frac{du_e}{d\xi} = 0$, get

$f''' + f f'' = 0$, Blasius.

$$\xi = \rho \mu u_e x$$

$$\eta = \frac{\rho u_e y}{\sqrt{2\rho\mu u_e x}} = \sqrt{\frac{\rho u}{2\mu x}}\, y$$

do this over and look for errors

here wed 3-28

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(144)

Energy Equation: try $h(x,y) = h_e(\xi)\, g(\eta)$ (steady)

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u \frac{\partial p}{\partial x} + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right)$$

but $dh = c_p dT$, so

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u \frac{\partial p_e}{\partial x} + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\frac{k\mu}{c_p \mu} \frac{\partial h}{\partial y}\right)$$

$$\boxed{\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u \frac{\partial p_e}{\partial x} + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\frac{\mu}{Pr} \frac{\partial h}{\partial y}\right)}$$

again, $\frac{\partial p_e}{\partial x} = -\rho_e u_e \frac{\partial u_e}{\partial x} = -(\rho_e u_e)^2 \mu_e \frac{d u_e}{d\xi}$

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial h}{\partial \xi}\frac{\partial \xi}{\partial x} = h_e g' \frac{\partial \eta}{\partial x} + h_e' g \frac{\partial \xi}{\partial x}$$

$$= h_e g' \frac{\partial \eta}{\partial x} + h_e' g \rho_e u_e \mu_e$$

$u = u_e f'$

$\rho v = -\frac{\partial \psi}{\partial x} = -G f' \frac{\partial \eta}{\partial x} - G' f \rho_e u_e \mu_e$ as before.

$$\frac{\partial h}{\partial y} = \frac{\partial h}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial h}{\partial \xi}\cancel{\frac{\partial \xi}{\partial y}}^{\,0} = \frac{\partial h}{\partial \eta}\frac{\rho u_e}{\sqrt{2\xi}} = h_e g' \frac{\rho u_e}{\sqrt{2\xi}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\cancel{\frac{\partial \xi}{\partial y}}^{\,0} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \eta}\frac{u_e \rho}{\sqrt{2\xi}}$$

$$= \frac{u_e \rho}{\sqrt{2\xi}} u_e f''(\eta)$$

so we get:

$$\rho u_e f'\left[h_e g' \frac{\partial \eta}{\partial x} + h_e' g \rho_e u_e \mu_e\right] - \left[G f' \frac{\partial \eta}{\partial x} + G' f \rho_e u_e \mu_e\right] h_e g' \frac{\rho u_e}{\sqrt{2\xi}}$$

$$= u_e f'(-)(\rho_e u_e)^2 \mu_e \frac{du_e}{d\xi} + \mu\left(\frac{\rho u_e^2}{\sqrt{2\xi}} f''\right)^2 + \frac{\partial}{\partial y}\left(\frac{\mu h_e g'}{P_r}\frac{\rho u_e}{\sqrt{2\xi}}\right)$$

but $G = \sqrt{2\xi}$, $G' = \frac{1}{\sqrt{2\xi}}$

$$\frac{\partial}{\partial y}(\ ) = \frac{\partial(\ )}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial(\ )}{\partial \xi}\cancel{\frac{\partial \xi}{\partial y}}^{\,0} = \frac{\partial(\ )}{\partial \eta}\frac{\rho u_e}{\sqrt{2\xi}}$$

First, put into form of White (7-25)

$h_e = h_e(\xi) = h_e(x)$; $u_e = u_e(\xi) = u_e(x)$, same with ρ_e, μ_e

$$\frac{1}{\rho_e u_e \mu_e}\ \rho u_e f'\left[h_e g' \frac{\partial \eta}{\partial x} + h_e' g \rho_e \mu_e u_e\right] - \left[\frac{G f' \frac{\partial \eta}{\partial x}}{\rho_e u_e \mu_e} + G' f\right] h_e g' \frac{\rho u_e}{\sqrt{2\xi}}$$

$$= -u_e f' \rho_e u_e \frac{du_e}{d\xi} + \frac{\mu}{\mu_e}\frac{\rho^2}{\rho_e} u_e^3 \frac{(f'')^2}{2\xi} + \frac{h_e \cancel{u_e}}{\sqrt{2\xi}\cancel{u_e}}\frac{\partial}{\partial y}\left(\frac{\rho \mu\, g'}{\rho_e \mu_e P_r}\right)$$

so

$$\rho u_e f'\left[\frac{g' h_e \frac{\partial\eta}{\partial x}}{\rho_e \mu_e u_e} + h_e' g\right] - \left[\frac{\sqrt{2\xi}\, f' \frac{\partial\eta}{\partial x}}{\rho_e \mu_e u_e} + \frac{f}{\sqrt{2\xi}}\right]\frac{h_e \rho u_e g'}{\sqrt{2\xi}}$$

$$= -\rho_e u_e^2 f' \frac{du_e}{d\xi} + \rho \frac{\mu \rho}{\rho_e \mu_e} u_e^3 \frac{(f'')^2}{2\xi} + \frac{h_e}{\sqrt{2\xi}} \frac{\rho u_e}{\sqrt{2\xi}} \frac{\partial}{\partial\eta}\left(C \frac{g'}{P_r}\right)$$

put last term on LHS as first, divide by $\frac{h_e u_e \rho}{2\xi}$

$$0 = \frac{\partial}{\partial\eta}\left(\frac{C}{Pr} g'\right) + \frac{\cancel{2\xi}}{h_e u_e \cancel{\rho}} \cancel{\rho} \frac{\rho\mu}{\rho_e \mu_e} u_e^3 \frac{(f'')^2}{\cancel{2\xi}} - \rho_e u_e^2 f' \frac{du_e}{d\xi} \frac{2\xi}{h_e u_e \rho}$$

$$- \frac{\cancel{\rho} \cancel{u_e} f' 2\xi}{h_e \cancel{\rho} \cancel{u_e}}\left[\frac{g' h_e \frac{\partial\eta}{\partial x}}{\rho_e \mu_e u_e} + h_e' g\right] + \frac{\cancel{h_e} \cancel{\rho} \cancel{u_e} g'}{\sqrt{2\xi}} \frac{2\xi}{\cancel{h_e u_e \rho}}\left[\frac{f}{\sqrt{2\xi}} + \frac{\sqrt{2\xi}\, f' \frac{\partial\eta}{\partial x}}{\rho_e \mu_e u_e}\right]$$

but $\frac{\partial\eta}{\partial x} = \eta + \eta \rho_e \mu_e \left(\frac{du_e}{d\xi} + \frac{u_e}{2\xi}\right)$

$$0 = \frac{\partial}{\partial\eta}\left(\frac{C}{Pr} g'\right) + \frac{C}{h_e u_e} u_e^3 (f'')^2 - \frac{\rho_e}{\rho} u_e f' 2\xi \frac{du_e}{d\xi} \frac{1}{h_e}$$

$$- \frac{2\xi f'}{h_e}\left[g h_e' + \frac{g' h_e \eta}{\rho_e \mu_e u_e} + \frac{g' h_e}{\cancel{\rho_e \mu_e} u_e} \eta \cancel{\rho_e \mu_e}\left(\frac{du_e}{d\xi} + \frac{u_e}{2\xi}\right)\right]$$

$$+ \left[f + 2\xi f' \frac{1}{\rho_e \mu_e u_e} \eta + 2\xi \frac{f'}{\cancel{\rho_e \mu_e} u_e} \eta \cancel{\rho_e \mu_e}\left(\frac{du_e}{d\xi} + \frac{u_e}{2\xi}\right)\right] g'$$

1st term, last term

$$0 = \frac{\partial}{\partial \eta}\left(\frac{C}{Pr} g'\right) + \frac{C}{h_e} u_e^2 (f'')^2 - \frac{\rho_e u_e}{\rho h_e} f' \, 2\xi \frac{du_e}{d\xi}$$

$$- 2\xi f' g \frac{h_e'}{h_e} \cancel{- \frac{2\xi f' g' \eta}{\rho_e \mu_e u_e}} \cancel{- \frac{2\xi f' g' \eta}{u_e}\left(\frac{du_e}{d\xi} + \frac{u_e}{2\xi}\right)}$$

2nd term

$$+ f g' \cancel{+ 2\xi f' \eta g' \frac{1}{\rho_e \mu_e u_e}} \cancel{+ \frac{2\xi f' \eta g'}{u_e}\left(\frac{du_e}{d\xi} + \frac{u_e}{2\xi}\right)}$$

$$0 = \frac{\partial}{\partial \eta}\left(\frac{C}{Pr} g'\right) + \frac{C}{h_e} u_e^2 (f'')^2 + f g'$$

$$+ 2\xi f' \left\{ -\frac{\rho_e u_e}{\rho h_e} \frac{du_e}{d\xi} - g \frac{h_e'}{h_e} \right\}$$

different from White, at least apparently; Anderson almost same as if $\frac{\partial g}{\partial \xi}$ & $\frac{\partial f}{\partial \xi} = 0$

Anderson 6.58

$$\boxed{\frac{\partial}{\partial \eta}\left(\frac{C}{Pr} g'\right) + f g' = 2\xi f' \left[\frac{\rho_e u_e}{\rho h_e} \frac{du_e}{d\xi} + g \frac{h_e'}{h_e}\right] - \frac{C}{h_e} u_e^2 (f'')^2}$$

$$P \equiv 2\xi f' \left[\frac{\rho_e u_e}{\rho h_e} \frac{du_e}{d\xi} + g \frac{h_e'}{h_e}\right] = ? \qquad H_e = h_e + \frac{1}{2} u_e^2$$

$$\frac{dH_e}{d\xi} = \frac{dh_e}{d\xi} + u_e \frac{du_e}{d\xi}$$

$$= \frac{2\xi f'}{h_e}\left[\frac{\rho_e u_e}{\rho} \frac{du_e}{d\xi} + g \frac{dh_e}{d\xi}\right]$$

Need to eliminate ρ, ρ_e — Cannot get White eqn. 7-25 — see notes — can get Lee's formulation.

Alternate similarity form for energy eqn.

Energy Equation: try $H(x,y) = H_e(\xi)\, g(\eta)$ (steady)

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu u \frac{\partial u}{\partial y}\right)$$ (alternate form)

$dh = c_p dT$, so

$$= \frac{\partial}{\partial y}\left[\frac{k\mu}{c_p \mu}\frac{\partial h}{\partial y} + \mu u \frac{\partial u}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}\left[\mu\left(\frac{1}{Pr}\frac{\partial h}{\partial y} + u\frac{\partial u}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial y}\left[\mu\left(\frac{1}{Pr}\frac{\partial H}{\partial y} - \frac{1}{Pr}\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) + u\frac{\partial u}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial y}\left[\mu\left(\frac{1}{Pr}\frac{\partial H}{\partial y} + u\frac{\partial u}{\partial y}\left(1 - \frac{1}{Pr}\right)\right)\right]$$

$$\boxed{\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\mu}{Pr}\frac{\partial H}{\partial y} + \mu\left(1 - \frac{1}{Pr}\right) u \frac{\partial u}{\partial y}\right]}$$

$\rho v = -G f' \frac{\partial \eta}{\partial x} - G' f \rho_e u_e \mu_e$ as before.

$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial y} = H_e g' \frac{\rho u_e}{\sqrt{2\xi}}$; $u = u_e f'$

A2

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \cancel{\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y}}^{\,0} = u_e f'' \frac{\rho u_e}{\sqrt{2\xi}}$$

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial H}{\partial \xi}\frac{\partial \xi}{\partial x} = H_e g' \frac{\partial \eta}{\partial x} + H_e' g \rho_e \mu_e u_e$$

$$\rho u_e f'\left[H_e g' \frac{\partial \eta}{\partial x} + H_e' g \rho_e \mu_e u_e\right] - \left[G f' \frac{\partial \eta}{\partial x} - G' f \rho_e u_e \mu_e\right] H_e g' \frac{\rho u_e}{\sqrt{2\xi}}$$

$$= \frac{\partial}{\partial y}\left[\frac{\mu}{Pr} H_e g' \frac{\rho u_e}{\sqrt{2\xi}} + \mu\left(1-\frac{1}{Pr}\right) u_e f' u_e f'' \frac{\rho u_e}{\sqrt{2\xi}}\right]$$

$$G = \sqrt{2\xi} \qquad G' = \frac{1}{\sqrt{2\xi}}$$

get

$$\rho u_e f'\left[H_e g' \frac{\partial \eta}{\partial x} + H_e' g \rho_e \mu_e u_e\right] - \left[\sqrt{2\xi} f' \frac{\partial \eta}{\partial x} - \frac{1}{\sqrt{2\xi}} f \rho_e \mu_e u_e\right] \frac{H_e g' \rho u_e}{\sqrt{2\xi}}$$

$$= \frac{\rho u_e}{\sqrt{2\xi}} \frac{\partial}{\partial \eta}\left[\frac{\mu H_e g' \rho u_e}{Pr\sqrt{2\xi}} + \mu\left(1-\frac{1}{Pr}\right)\frac{u_e^3 f' f'' \rho}{\sqrt{2\xi}}\right]$$

$$\cancel{\rho u_e f' g' H_e \frac{\partial \eta}{\partial x}} + \cancel{\rho} u_e^2 \rho_e \mu_e H_e' f' g - \cancel{H_e f' g' \rho u_e \frac{\partial \eta}{\partial x}} - \frac{f \cancel{\rho} \rho_e u_e^2 \mu_e g' H_e}{2\xi}$$

$$= \frac{\cancel{\rho} u_e}{\sqrt{2\xi}} \frac{1}{\sqrt{2\xi}} u_e \frac{\partial}{\partial \eta}\left[\frac{\rho \mu H_e g'}{Pr} + \mu\left(1-\frac{1}{Pr}\right)\rho u_e^2 f' f''\right]$$

divide by $\rho_e \mu_e u_e$: (indep. of η)

$$u_e H_e' f' g - \frac{f g' u_e H_e}{2\xi}$$

$$= \frac{u_e}{2\xi}\frac{\partial}{\partial\eta}\left[\frac{\rho\mu}{\rho_e\mu_e}\frac{H_e g'}{Pr} + \frac{\rho\mu}{\rho_e\mu_e}\left(1-\frac{1}{Pr}\right)u_e^2 f' f''\right]$$

$$2\xi \cancel{u_e} H_e' f' g - f g_1' \overset{H_e}{\cancel{u_e}} = \cancel{u_e}\frac{\partial}{\partial\eta}\left[C\frac{H_e g'}{Pr} + C\left(1-\frac{1}{Pr}\right)u_e^2 f' f''\right]$$

$$2\xi\frac{dH_e}{d\xi} f' g - f g_1' H_e = \frac{\partial}{\partial\eta}\left[C\frac{g'}{Pr}\right]H_e + u_e^2\frac{\partial}{\partial\eta}\left[C\left(1-\frac{1}{Pr}\right)f' f''\right]$$

$$\frac{\partial}{\partial\eta}\left[C\frac{g'}{Pr}\right] + \frac{u_e^2}{H_e}\frac{\partial}{\partial\eta}\left[C\left(1-\frac{1}{Pr}\right)f' f''\right] + f g' - \frac{2\xi}{H_e}\frac{dH_e}{d\xi} f' g = 0$$

$$\boxed{\left(C\frac{g'}{Pr}\right)' + f g' + \frac{u_e^2}{H_e}\left[C\left(1-\frac{1}{Pr}\right)f' f''\right]' - \frac{2\xi}{H_e}\frac{dH_e}{d\xi} f' g = 0}$$

for "isoenergetic flow" $\frac{dH_e}{d\xi} = 0$, got result of Lees (10)

agrees ~~exa~~ completely with Dorrance book* (2-102) (without reactions)

*Viscous Hypersonic Flow

B1

$$p = \frac{2\xi f'}{h_e}\left[\frac{\rho_e u_e}{\rho}\frac{du_e}{d\xi} + g\frac{dh_e}{d\xi}\right]$$

but $P_e = \rho_e R T_e = P = \rho R T$ at any given ξ (if perfect gas)

$$\rho_e T_e = \rho T \Rightarrow \frac{\rho_e}{\rho} = \frac{T}{T_e}$$

if a perfect gas with constant specific heats,

$$\frac{\rho_e}{\rho} = \frac{c_p T}{c_p T_e} = \frac{h}{h_e} = g$$

then

$$p = \frac{2\xi f'}{h_e}\left[g u_e \frac{du_e}{d\xi} + g\frac{dh_e}{d\xi}\right] = \frac{2\xi f'}{h_e} g \frac{dH_e}{d\xi}$$

$$= \frac{\xi}{h_e}\frac{dH_e}{d\xi}(f')(2g)\cdot\left(\frac{H_e}{h_e}\frac{h_e}{H_e}\right)$$

$$= \frac{\xi}{H_e}\frac{dH_e}{d\xi} f'\left(2g\cdot\frac{H_e}{h_e}\right) = \frac{\xi}{H_e}\frac{dH_e}{d\xi} f' 2g\cdot\left(\frac{h_e + \frac{1}{2}u_e^2}{h_e}\right)$$

$$= \frac{\xi}{H_e}\frac{dH_e}{d\xi} f' 2g\left(1 + \frac{\frac{1}{2}u_e^2}{h_e}\right) = \frac{\xi}{H_e}\frac{dH_e}{d\xi} f'\left(2g + \frac{g u_e^2}{h_e}\right)$$

let $Q = \frac{g u_e^2}{h_e} = \frac{g h_e u_e^2}{h_e^2} = \frac{h u_e^2}{h_e^2} \Rightarrow$ need $g = f'^2$ to agree with White ...

no way.

— here Fri 3-30-90 —

$$\boxed{\frac{\partial}{\partial\eta}\left(\frac{Cg'}{Pr}\right) + fg' = \frac{2\xi f'}{h_e}\left[\frac{e_\xi u_e}{e}\frac{du_e}{d\xi} + g\frac{dh_e}{d\xi}\right] - \frac{C}{h_e}u_e^2(f'')^2}$$

energy equation (does not check White 7-25)

G. Flat Plate in Compressible Flow

for flat plate (or cone) $\frac{dh_e}{d\xi} = 0$, $\frac{du_e}{d\xi} = 0$ (constant outer conditions)

eqns become:

$$\boxed{(Cf'')' + ff'' = 0} \quad \text{(momentum BL eqn)}$$

$$\left(\frac{Cg'}{Pr}\right)' + fg' = -\frac{C}{h_e}u_e^2(f'')^2 \quad \text{(energy BL eqn)}$$

for a perfect gas with constant specific heats,

$$\frac{u_e^2}{h_e} = \frac{u_e^2}{c_p T_e} = \frac{u_e^2}{c_p}\frac{\gamma R}{\gamma R T_e} = \frac{u_e^2\gamma(c_p - c_v)}{c_p a_e^2} = M_e^2\frac{\gamma(\gamma-1)}{\gamma}$$

$$= M_e^2(\gamma-1)$$

so $$\boxed{\left(\frac{Cg'}{Pr}\right)' + fg' = -CM_e^2(\gamma-1)(f'')^2}$$