I. Derivation of the Equations of Motion.

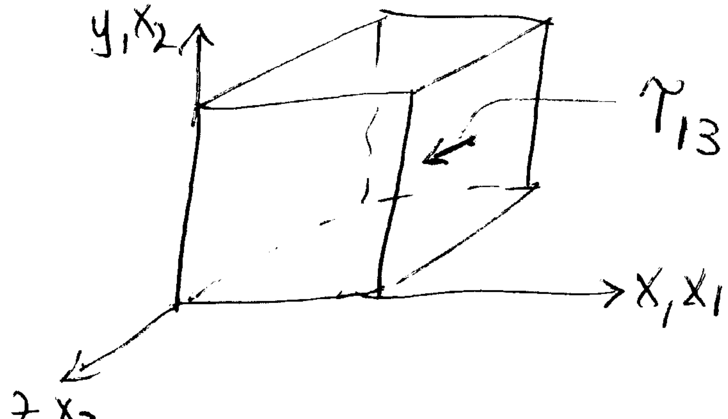
A. Review of Cartesian Tensor Notation

References for Cartesian Tensors:

Butkov, Chap 16; Jeffreys and Jeffreys, Chapter 3 (2nd); Fung, Foundations of Solid Mechanics, Chapter 2; etc, etc.

1. Introduction: Shear Stress: (Tensor)

Use the symbol τ_{ij} for the shear stress on the i face of a box, in the j'th direction:



Sign convention

$\tau_{ij} > 0$ if pointing in $+j$ direction on $+i$ face, or in $-j$ direction on $-i$ face.

Note: for an infinitesimal cube (shear stress at a point), $\tau_{ij} = \tau_{ji}$

$$\underset{\sim}{\tau} = \tau_{ij} = \begin{bmatrix} \tau_{11} & \tau & \tau \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix}$$ — (check if transposed)

Shear is a tensor, not just a collection of 9 numbers; it must be invariant under changes in the coordinate system (rotations, translations, etc).

2. The Summation Convention.

The divergence of the velocity $\underset{\sim}{u}$ can be written as

$$\nabla \cdot \underset{\sim}{u} = \operatorname{div} \underset{\sim}{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \sum_j \frac{\partial u_j}{\partial x_j} = \frac{\partial u_j}{\partial x_j}$$

, with the Σ understood, whenever an index is repeated (Einstein summ. conv.)

$$\underset{\sim}{u} \equiv (u, v, w) = u\hat{e}_x + v\hat{e}_y + w\hat{e}_z = u\hat{i} + v\hat{j} + w\hat{k}$$

Note also that $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_j b_j$

where $\underset{\sim}{a} = a_1 \hat{e}_x + a_2 \hat{e}_y + a_3 \hat{e}_z = (a_1, a_2, a_3)$

3. Some special tensors:

- δ_{ij} – the Kronecker Delta function. (or identity matrix) $= 1$ if $i = j$, $= 0$ if $i \neq j$

- ε_{ijk} – the permutation tensor $= 1$ if ijk follow in cyclical order ($\cdots 12312 \cdots$)
  $= -1$ if out of order by 1 permutation ($\cdots 21321 0 \cdots$)
  $= 0$ otherwise. (any repeated #)

See J&J p. 69ff

③

Note: $\delta_{ij}\delta_{jk} = \delta_{ik}$, since $\delta_{ij}a_j = a_k$

why? ① if $i \neq j$ or $j \neq k$, $= 0$; if $i = j$ and $j = k$, $= 1$ but $i = k$.

② $\delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k} = \delta_{ij}\delta_{jk}$; if $i \neq k$ all $3 = 0$
if $i = k$, one is $+1$

Note: $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ — very useful

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why? terms with $i = j$ or $i = k = 0$, $i = l$ or $i = m = 0$.
① if $j = k$ or $l = m$, $= 0$.

if j is one above i, then l must be above or below, otherwise term is zero. If j is above i, then k is above j. (only possibility). If l is below i, then m must be below l (only possibility).

Thus, if in same order, get $+1 \cdot +1$ or $-1 \cdot -1 = +1$.
if in different order, get $-1 \cdot +1$ or $+1 \cdot -1 = -1$.

same order means when $j = l$ and $k = m$; different order means when $j = m$ and $k = l$.

explanation is hard. use i same order or opposite order argument.

(2) other way: $\varepsilon_{ijk} = 1$ if $i=1$, $j=2$ and $k=3$ or $j=3$ and $k=2$, otherwise $\varepsilon_{ijk} = 0$. similiar with 2d term.

4. Vector Calculus in Index Notation.

Note $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_j b_j$ as before.

Note $\underset{\sim}{a} \times \underset{\sim}{b} = \varepsilon_{ijk} a_j b_k = (a_2 b_3 - a_3 b_2)\hat{e}_1 + (a_3 b_1 - a_1 b_3)\hat{e}_2 + (a_1 b_2 - a_2 b_1)\hat{e}_3$

$$= \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Thus, $\nabla \times \underset{\sim}{u} = \text{curl}\, \underset{\sim}{u} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$

Note: Compute $\nabla \times (\nabla \times \underset{\sim}{u}) = \nabla \times \underset{\sim}{w}$:

$$\nabla \times \nabla \times \underset{\sim}{u} = \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left[\varepsilon_{k\ell m} \frac{\partial}{\partial x_\ell} u_m\right] = \varepsilon_{ijk}\varepsilon_{k\ell m} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_\ell} u_m$$

$$= \varepsilon_{kij}\, \varepsilon_{k\ell m} \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_\ell} u_m\right) = (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) \frac{\partial^2 u_m}{\partial x_j \partial x_\ell}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \frac{\partial^2}{\partial x_j \partial x_j}(u_i) = \nabla(\nabla \cdot \underset{\sim}{u}) - \nabla^2(\underset{\sim}{u})$$

$$= \nabla(\nabla \cdot \underset{\sim}{u}) - \nabla \cdot (\nabla \underset{\sim}{u})$$

to here 10-10

B. Some Theorems from Vector Calculus

(1) Stokes Theorem:

$$\Gamma = \int_C \underset{\sim}{u} \cdot \underset{\sim}{dx} = \iint_S (\text{curl } \underset{\sim}{u}) \cdot \hat{n}\, dS$$

(curl is local 'circulation')

The line integral around a curve is equal to the surface integral of the curl. (remember right-hand rule for calculating line integral, and for curl)

(2) Gauss' Theorem:

$$\iint_S \underset{\sim}{u} \cdot \hat{n}\, dS = \text{flux out} = \iiint_V \text{div } \underset{\sim}{u}\, dV$$

(divergence is the local 'flux out')

Note: There is a form of Gauss' theorem which is sometimes useful when working with the curl.

$$\iiint (\nabla \times \underset{\sim}{f})\, dV = \iint (\hat{n} \times \underset{\sim}{f})\, dS$$

This derivation is left for a homework problem. (a lemma to Gauss' theorem)

Note also: These theorems are also valid for tensors, not just for vectors:

$$\iint_S \underset{\approx}{\tau} \cdot \hat{n}\, dS = \iiint_V \operatorname{div} \underset{\approx}{\tau}\, dV$$

$$\underset{\approx}{\tau} \cdot \hat{n} = \tau_{ij} n_i \,, \quad \operatorname{div} \underset{\approx}{\tau} = \frac{\partial}{\partial x_i}(\tau_{ij})$$

C. Review of Thermodynamics

① First Law of Thermodynamics – Conservation of Energy.

If ΔQ is the heat added to a system, and ΔW is the work done on a system, and if E is the internal energy of a system,

$$\Delta W + \Delta Q = \Delta E, \text{ or}$$

$$\boxed{đW + đQ = dE}$$

($đ$ – not a perfect differential)

Cannot be integrated without knowing path

② Second Law of Thermodynamics – Entropy

Entropy is a variable of state, and for a closed system, S increases in any spontaneous process. Also, for a reversible process,

L&R p.20

$$S_B - S_A = \int_A^B \frac{dQ}{T}$$, where $A \to B$ is a reversible process.

then $$dS\Big|_{rev.} = \frac{dQ}{T}\Big|_{rev.}$$

or $$\boxed{TdS = dE + PdV}$$, where we have set $dW = PdV$, assuming all work done through pressure.

<u>Thermodynamic Identity</u>

* (~~I'm not sure if this is~~ Not ~~still~~ ~~also~~ true ~~when not~~ unless all of the work is done through pressure forces; but shear usually produces heat, not work — see Schilling - Eqnderiv.)

This equation involves only variables of state, so it can be integrated, and the result is <u>path-independent</u>

$$\int_1^2 dS = S_2 - S_1 = \int_1^2 \frac{dE}{T} + \int_1^2 \frac{P}{T} dV$$

Since the integral of S is path independent, dS is a perfect differential, and

$$dS = \frac{1}{T} dE + \frac{P}{T} dV = \frac{\partial S}{\partial E}\Big|_V dE + \frac{\partial S}{\partial V}\Big|_E dV$$

thus, $\frac{1}{T} = \frac{\partial S}{\partial E}\Big|_V$, $\frac{P}{T} = \frac{\partial S}{\partial V}\Big|_E$, etc.

Note: Can get various relations by equating the mixed partials, e.g.

$$\frac{\partial^2 S}{\partial E \partial V} = \frac{\partial^2 S}{\partial V \partial E} \Rightarrow \frac{\partial}{\partial V}\left(\frac{1}{T}\right) = \frac{\partial}{\partial E}\left(\frac{P}{T}\right)$$

$$-\frac{1}{T^2}\frac{\partial T}{\partial V}\bigg|_E = \frac{T\frac{\partial P}{\partial E} - P\frac{\partial T}{\partial E}}{T^2} \qquad (E, V \text{ space})$$

$$\frac{\partial T}{\partial V}\bigg|_E = P\frac{\partial T}{\partial E}\bigg|_{\cancel{E}V} - T\frac{\partial P}{\partial E}\bigg|_{\cancel{E}V} \qquad \text{(get Maxwell relations this way)}$$

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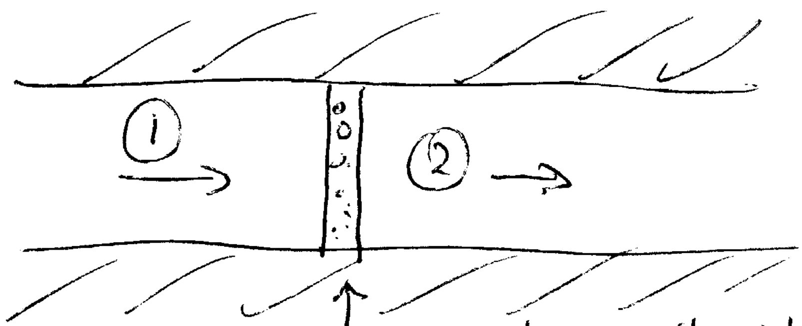
(3) Example - Joule-Thomson Throttling Process

See L&L.



porous plug, or throttling valve, or screen.

Steady flow through a heat-insulating pipe with a porous plug.

Assume fluid flows slowly, so that its kinetic energy is small compared to its enthalpy ($\frac{1}{2}\rho u^2 \ll h$)

Consider passage of a unit mass through the plug:

8/25

Change in internal energy is

$$\Delta E = e_2 - e_1$$

work done (as fluid leaves left and pushes out on right) is

$$W = \int_1 p\,dv - \int_2 p\,dv$$

p_1 and p_2 are constant in this steady process, so

work done on gas $= W = p_1 v_1 - p_2 v_2$ (goes out left, comes in on right)

where v_1 is volume of gas when all on l.h.s.

v_2 is volume of gas when all on r.h.s.

By the first law, since the heat flux out is zero, (assume no conduction in fluid or to walls)

$$\Delta E = W$$

$$e_2 - e_1 = p_1 v_1 - p_2 v_2$$

$$h_1 = e_1 + p_1 v_1 = e_2 + p_2 v_2 = h_2 \quad (h = e + pv)$$

<u>In adiabatic flow through a resistance, the enthalpy per unit mass is the same upstream and downstream.</u>

"adiabatic" - a system without heat conduction to the outside.

8/20/96. - Gollete - Cp blunt nose in S.S. flow →

Fri 1-2 - dont skip eqns.

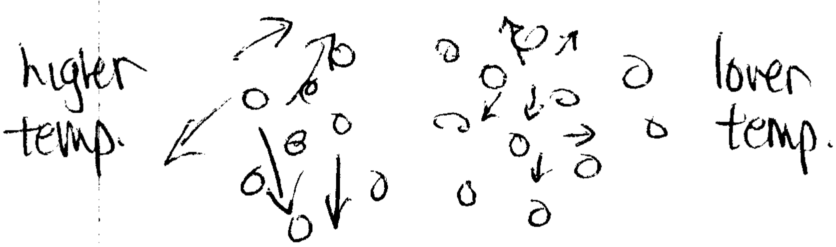
(4) Perfect gas: $pv = RT$ R is the specific gas constant

calorically perfect: $h = C_p T$, $e = C_v T$, C_p & C_v are constant, T absolute.

will use to save on laborious calculations.

D. Heat Transfer (Conduction) (Diffusion)

Diffusion is transport by molecular motion. The simplest example is conductive heat transfer (on the molecular scale, only conduction and radiation exist as forms of heat transfer).



(diffusion of mass and momentum work by same process)

more molecules from higher temp randomly move to lower temp than vice versa. Thus, high-speed molecular motion region diffuses into low-speed region. Heat diffuses. Diffusion is approximated as a linear process - heat diffuses down the temperature gradient.

For a discussion of the limitations on Fourier's Law for heat conduction, see "Heat Under the Microscope", by Maasilta and Minnich, Physics Today, August 2014, pp. 27-32. When the gradients in temperature become significant on the scale of the mean free path of the phonons, Fourier's law breaks down.

thus, say

$$\underset{\sim}{q} = \underset{\sim}{q}(\nabla T, \text{second deriv. of } T)$$

ignore

assume ∇T small (compared to what?)

assume that $\underset{\sim}{q}$ is a linear vector function of ∇T.

$$q_i = a_{ij} \frac{\partial T}{\partial x_j} = \underset{\approx}{a} \cdot \nabla T$$

$\underset{\approx}{a}$ is the heat conductivity tensor

for an isotropic substance, all directions are the same,

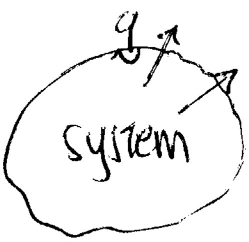
$$a_{ij} = -\lambda \delta_{ij}$$

thus, $\underset{\sim}{q} = -\lambda \nabla T$ where λ is some constant.

usually use $\boxed{\underset{\sim}{q} = -k \nabla T}$ k is the thermal conductivity

(see White p. 30)

Now consider the flow of heat in some substance.



$$\Delta(\text{Total heat enclosed}) = \int \underset{\sim}{q} \cdot \hat{n} \, \Delta t$$

$$\frac{\partial}{\partial t}\left(\iiint_V \rho e \, dV \right) = -\iint_S \underset{\sim}{q} \cdot \hat{n} \, dS$$

for perfect gas, $e = c_v T$

$$\iiint_V \frac{\partial}{\partial t}(\rho c_v T)\,dV + \iiint_V \operatorname{div} \underset{\sim}{q}\,dV = 0$$

using gauss' law, and interchange $\int$ & $\partial/\partial t$

true for any V, so

$$\frac{\partial}{\partial t}(\rho c_v T) + \operatorname{div} \underset{\sim}{q} = 0 \quad \text{everywhere}$$

assume ρ constant, then, using $\underset{\sim}{q} = -k\nabla T$, k constant,

$$\rho c_v \frac{\partial T}{\partial t} - \nabla \cdot k \nabla T = 0$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_v}\nabla^2 T \qquad (\nabla^2 T \equiv \nabla \cdot \nabla T)$$

$k/\rho c_v = \alpha$ = thermal diffusivity - White p.83 - (but see p.34 ~~different~~ typo).

See Thompson problem 6.13

thus, $\boxed{\frac{\partial T}{\partial t} = \alpha \nabla^2 T}$ – the heat equation.

for the conduction of heat in a stationary fluid.

- mom. diffusion more complex eqn, since mom. a vector; but same process.
- same equation for mass diffusion

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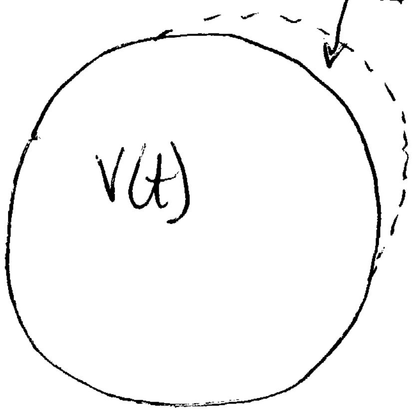
E. The Reynolds Transport Theorem

Want to write laws for conservation of mass, momentum, and energy. But these laws are usually written for a fixed mass; need to rephrase them for volumes fixed in space (Eulerian description)

8/26/96.

$$\Delta V = \iint_S \underset{\sim}{v} \cdot \hat{n} \, \Delta t \, dS$$



Consider any quantity ϕ — could be scalar, vector, tensor (2nd order tensor). Let Φ be the total amount of ϕ in some volume,

$$\Phi = \iiint_{V(t)} \phi \, dV$$ where V moves with the fluid

What is the rate of change of Φ?

$$\frac{d\Phi}{dt} = \lim_{\Delta t \to 0} \frac{\Phi(t+\Delta t) - \Phi(t)}{\Delta t}$$

Φ changes for 2 reasons: ① volume enclosed by moving fluid changes w/ time ② fluid inside the volume changes even if remains inside

$$\Phi(t+\Delta t) = \iiint_{V(t+\Delta t)} \phi(t+\Delta t)\,dV$$

$\phi(x,y,z,t+\Delta t)$

$$\Phi(t) = \iiint_{V(t)} \phi(t)\,dV$$

$$\Phi(t+\Delta t) - \Phi(t) = \iiint_{V(t+\Delta t)} \phi(t+\Delta t)\,dV - \iiint_{V(t)} \phi(t)\,dV$$

Now, add & subtract $\iiint_{V} \phi(t+\Delta t)\,dV$

$$\Phi(t+\Delta t) - \Phi(t) = \underbrace{\iiint_{V(t+\Delta t)} \phi(t+\Delta t)\,dV - \iiint_{V(t)} \phi(t+\Delta t)\,dV}_{\mathrm{I}}$$

$$+ \iiint_{V(t)} \underbrace{\left[\phi(t+\Delta t) - \phi(t)\right]}_{\mathrm{II}}\,dV$$

What is I? $V(t+\Delta t) = V(t) + \Delta V \Rightarrow$

$$I = \iiint_{(V(t+\Delta t) - V(t))} \phi(t+\Delta t)\, dV = \iint_S \phi \underbrace{\underset{\sim}{u} \cdot \hat{n}\, \Delta t}_{\text{distance} \perp \text{ to surface}}\, dS$$

stuff carried w/ fluid which moved out.

dividing both sides of equation by Δt, we have that

$$\frac{\Phi(t+\Delta t) - \Phi(t)}{\Delta t} = \iint_S \phi \underset{\sim}{u} \cdot \hat{n}\, dS + \iiint_V \frac{\phi(t+\Delta t) - \phi(t)}{\Delta t}\, dV$$

as $\Delta t \to 0$, we get, in the limit,

$$\frac{d}{dt}\int \phi\, dV \Big|_{\text{following fluid}} = \boxed{\left.\frac{d\Phi}{dt}\right|_{\text{following the fluid}} = \iint_{S} \phi \underset{\sim}{u} \cdot \hat{n}\, dS + \iiint_{V} \frac{\partial \phi}{\partial t}\, dV}$$

S, V fixed in space

First Form of Reynolds Transport Theorem.

Use Gauss' Theorem to change the surface integral to a volume $\int$.

Second Form.

$$\boxed{\left.\frac{d\Phi}{dt}\right|_{\text{following the fluid}} = \iiint_{V} \left(\operatorname{div}(\phi \underset{\sim}{u}) + \frac{\partial \phi}{\partial t} \right) dV = \left.\frac{d}{dt}\right|_{\text{following fluid}} \iiint_V \phi\, dV}$$

V fixed in space

Remember, ϕ can be anything which is carried with the fluid.

F. The Continuity Equation

The simplest equation of motion expresses mass conservation. Since mass is neither created nor destroyed (if we don't consider nuclear reactions) then the amount of mass within a volume of fluid that stays with the fluid particles must remain the same. That is,

$$\underbrace{\frac{d}{dt}\iiint_{v(t)} \rho \, dv}_{\text{following fluid}} = 0 = \underbrace{\iiint_{V} \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underset{\sim}{u})\right] dV}_{\text{fixed in space}}$$

where I have used Reynolds transport theorem.

Since this ~~must be~~ is true for an arbitrary volume V, it must be true at every point (from 'weak' to 'strong' forms - actually 'weak', averaging form, is more physically true - important for math. proofs). Thus

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underset{\sim}{u}) = 0}$$

Mass conservation or Continuity Equation

Of course, we can express this same equation in various 'control volume' or integral forms, using Gauss' theorem; e.g.

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV + \iint_S \rho\, \underset{\sim}{u} \cdot \hat{n}\, dS = 0$$

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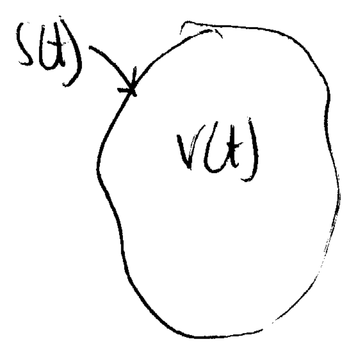
6. Momentum Equation

We know that

$$\underset{\sim}{F} = \frac{d}{dt}(\text{momentum})$$

Consider the fluid within some volume V, which moves with the fluid.



We must have that

$$\sum \underset{\sim}{F} = \frac{d}{dt} \iiint_{V(t)} \rho\, \underset{\sim}{u}\, dV$$

moving with the fluid.

Or, using Reynolds transport theorem,

$$\sum \underset{\sim}{F} = \iiint_V \frac{\partial}{\partial t}(\rho \underset{\sim}{u})dV + \iint_S \rho \underset{\sim}{u}(\underset{\sim}{u}\cdot\hat{n})dS$$

with the volume and S now fixed in space.

Use the extended form of Gauss Law,

$$\sum \underset{\sim}{F} = \iiint_V \frac{\partial}{\partial t}(\rho \underset{\sim}{u})dV + \iiint_V \operatorname{div}(\rho \underset{\sim}{u}\underset{\sim}{u})d$$

$$\left(\operatorname{div}(\rho \underset{\sim}{u}\underset{\sim}{u}) = \frac{\partial}{\partial x_j}(\rho u_i u_j)\right)$$

or $$\sum \underset{\sim}{F} = \iiint_V \left[\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \operatorname{div}(\rho \underset{\sim}{u}\underset{\sim}{u})\right]dV$$

What are the forces $\sum \underset{\sim}{F}$? Two types:

(1) Surface forces: act on contact, on the surface (pressure, shear)

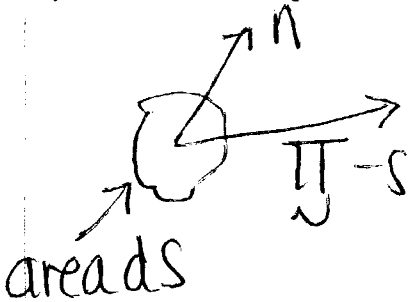
(2) Body forces: Act at a distance, on fluid in whole volume. (gravity, electromagnetic forces, etc)

If we assume that the body forces are independent of the velocity, and so on (like gravity), then

$$\sum \underset{\sim}{F}_{body} = \iiint_V \underset{\sim}{f}\, dv$$ where $\underset{\sim}{f}$ is the body force per unit volume.

I won't do E&M forces here. (does anybody want them?)

The surface forces usually considered are the pressure and the shear (are there others? – surface tension?) Combine these into one symbol, let $\underset{\sim}{\Pi}$ be the force per unit area on an element of surface $d\underset{\sim}{a} = \hat{n}\, dS$



$\underset{\sim}{\Pi}$ – surface force per unit area. ; $\sum \underset{\sim}{F}_{surface} = \iint_S \underset{\sim}{\Pi}\, dS$

$\underset{\sim}{\Pi}$ depends on the orientation of the surface dS (depends on $\hat{n}$) as well as on its position; it is thus more complex than $\underset{\sim}{f}$.

We will return to this. For now, add these

$$\sum \underset{\sim}{F} = \sum \underset{\sim}{F}_{body} + \sum \underset{\sim}{F}_{surface} = \iiint_V \underset{\sim}{f}\, dv + \iint_S \underset{\sim}{\Pi}\, dS$$

So that the momentum conservation equation is

$$\boxed{\iiint_V \left[\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \mathrm{div}(\rho \underset{\sim}{u}\underset{\sim}{u})\right] dV = \iiint_V \underset{\sim}{f}\, dV + \iint_S \underset{\sim}{\Pi}\, dS}$$

where all integrals are now fixed in space

Need to simplify this equation to use it. What is $\underset{\sim}{\Pi}$?
Clearly, $\underset{\sim}{\Pi}$ can depend on the surface orientation (think of how pressure force does). $\underset{\sim}{\Pi}$ is thus a function of $\hat{n}$. What kind of function?

To avoid infinite forces on a closed volume as the volume shrinks to zero (no ∞ volume forces at a point), must have $\underset{\sim}{\Pi}$ a linear function of $\hat{n}$.
(see Batchelor p. 9 & 10, etc) Proof left as exercise.
—hand out—

8-29-94

If Π_i is to be a linear function of n_j, the most complex possible is

$$\Pi_i = \sigma_{ij} n_j$$

$$\underset{\sim}{\Pi} = \underset{\approx}{\sigma} \cdot \hat{n}$$

σ_{ij} the stress <u>tensor</u>
(because has to be invariant under rotations of coords, etc)

Now this σ_{ij} includes pressure forces and shear forces, for the moment — it represents all the surface forces.

Then we can rewrite momentum equation as:

$$\iiint_V \left[\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \operatorname{div}(\rho \underset{\sim}{u}\underset{\sim}{u})\right] dV = \iiint_V \underset{\sim}{f}\, dV + \iint_S \underset{\approx}{\sigma} \circ \hat{n}\, dS$$

use Gauss' law on last term:

$$\iiint_V \left[\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \operatorname{div}(\rho \underset{\sim}{u}\underset{\sim}{u})\right] dV = \iiint_V \underset{\sim}{f}\, dV + \iiint_V \operatorname{div} \underset{\approx}{\sigma}\, dV$$

or, combining all volume $\int$,

$$\iiint_V \left[\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \operatorname{div}(\rho \underset{\sim}{u}\underset{\sim}{u}) - \underset{\sim}{f} - \operatorname{div} \underset{\approx}{\sigma}\right] dV = 0$$

Since this is true for an arbitrary volume V, it must be true everywhere (weak soln → strong soln)

$$\boxed{\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \operatorname{div}(\rho \underset{\sim}{u}\underset{\sim}{u}) - \underset{\sim}{f} - \operatorname{div} \underset{\approx}{\sigma} = 0}$$

or

$$\boxed{\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \operatorname{div}(\rho \underset{\sim}{u}\underset{\sim}{u} - \underset{\approx}{\sigma}) = \underset{\sim}{f}}$$

Now, separate out the thermodynamic pressure from the rest of the surface stress:

$$\underset{\sim}{T} = T_{ij} \equiv -\delta_{ij}\, p + \tau_{ij} = -\underset{\sim}{I}\, p + \underset{\sim}{\tau}$$

where $\underset{\sim}{\tau}$ is the viscous shear part (note $\tau_{ii} \neq 0$ in general)
(p is pressure inside)(was force on element)

Then we have

$$\boxed{\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \operatorname{div}(\rho \underset{\sim}{u}\underset{\sim}{u} + p\underset{\sim}{I} - \underset{\sim}{\tau}) = \underset{\sim}{f}}$$

hard term to deal with.

here 8/26/96.

τ can depend only on the derivatives of $\underset{\sim}{u}$, and not on $\underset{\sim}{u}$ (must be Galilean invariant). We will approximate that τ is linear in the first derivatives of $\underset{\sim}{u}$ (Newtonian fluid). Just like approximating $\underset{\sim}{q}$ linear in 1st derivatives of T (Fourier heat conduction).

here Fri H9

H. Energy Equation

Look at energy content of fluid inside a volume which moves with the fluid.

actually first term in expansion of $\dfrac{E_0}{(1-v^2/c^2)^{1/2}}$ French p. 22

non-relativistic. $\dfrac{d}{dt}\displaystyle\int_{V(t)} \left(\rho e + \tfrac{1}{2}\rho u^2\right) dV$ = Rate of (work done + heat added)

$V(t)$ ← moving w/ fluid

ρe ← internal energy

$\tfrac{1}{2}\rho u^2$ ← KE

(e = internal energy per unit mass)

use Reynolds transport theorem: ($u^2 \equiv \underset{\sim}{u}\cdot\underset{\sim}{u}$)

$$\iiint_{V \text{ fixed}} \left[\frac{\partial}{\partial t}\left(\rho e + \tfrac{1}{2}\rho u^2\right) + \operatorname{div}\left[\underset{\sim}{u}\left(\rho e + \tfrac{1}{2}\rho u^2\right)\right]\right] dV = \text{Rate of (work + heat)}$$

Forces: $\underset{\sim}{\Pi}$ on S, $\underset{\sim}{f}$ on V (for work done)

heat added: $-\underset{\sim}{q}\cdot d\underset{\sim}{A}$ $\left(\underset{\sim}{u} = \lim \frac{\Delta x}{\Delta t},\ \Delta t \to 0\right)$

$$\text{rate of work done by } \underset{\sim}{\Pi} = \iint_S \underset{\sim}{u}\cdot\underset{\sim}{\Pi}\, dS = \iint_S \underset{\sim}{u}\cdot\underset{\approx}{\sigma}\cdot\hat{n}\, dS = \iiint_V \operatorname{div}\left(\underset{\sim}{u}\cdot\underset{\approx}{\sigma}\right) dV$$

$$\text{rate of work done by } \underset{\sim}{f} = \iiint_V \underset{\sim}{u}\cdot\underset{\sim}{f}\, dV$$

$$\text{rate of heat added} = -\iint_S \underset{\sim}{q}\cdot\hat{n}\, dS = -\iiint_V \operatorname{div}\underset{\sim}{q}\, dV$$

so whole equation becomes:

$$\iiint_{V\,\text{fixed}} \left[\frac{\partial}{\partial t}\left(\rho e + \tfrac{1}{2}\rho u^2\right) + \operatorname{div}\left(\underset{\sim}{u}\left(\rho e + \tfrac{1}{2}\rho u^2\right)\right)\right] dV$$

$$= \iiint_V \left(\operatorname{div}\left(\underset{\sim}{u}\cdot\underset{\approx}{\sigma} - \underset{\sim}{q}\right) + \underset{\sim}{u}\cdot\underset{\sim}{f}\right) dV$$

need $\underset{\approx}{\sigma}$ symm for this

$\left(\underset{\sim}{u}\cdot\underset{\approx}{\sigma} = \underset{\approx}{\sigma}\cdot\underset{\sim}{u}\right)$

Since this is true for arbitrary volume, true everywhere:

$$\frac{\partial}{\partial t}\left[\rho\left(e + \tfrac{1}{2}u^2\right)\right] + \operatorname{div}\left(\underset{\sim}{u}\left(\rho e + \tfrac{1}{2}\rho u^2\right) + \underset{\sim}{q} - \underset{\approx}{\sigma}\cdot\underset{\sim}{u}\right) - \underset{\sim}{f}\cdot\underset{\sim}{u} = 0$$

but $\underset{\approx}{\sigma} = -p\underset{\approx}{I} + \underset{\approx}{\tau}$; $\underset{\approx}{\sigma}\cdot\underset{\sim}{u} = -p\underset{\approx}{I}\cdot\underset{\sim}{u} + \underset{\approx}{\tau}\cdot\underset{\sim}{u}$

$$= -p\underset{\sim}{u} + \underset{\approx}{\tau}\cdot\underset{\sim}{u}$$

so

$$\frac{\partial}{\partial t}\left[\rho\left(e + \tfrac{1}{2}u^2\right)\right] + \operatorname{div}\left[\rho\underset{\sim}{u}\left(e + \tfrac{1}{2}u^2\right) + \underset{\sim}{q} + p\underset{\sim}{u} - \underset{\approx}{\tau}\cdot\underset{\sim}{u}\right] - \underset{\sim}{f}\cdot\underset{\sim}{u} = 0$$

$$\frac{\partial}{\partial t}\left[\rho\left(e + \tfrac{1}{2}u^2\right)\right] + \operatorname{div}\left[\rho\underset{\sim}{u}\left(e + \tfrac{1}{2}u^2 + \frac{p}{\rho}\right) + \underset{\sim}{q} - \underset{\approx}{\tau}\cdot\underset{\sim}{u}\right] - \underset{\sim}{f}\cdot\underset{\sim}{u} = 0$$

but $e + p/\rho = e + pv = h$

$$\frac{\partial}{\partial t}\left[\rho\left(e + \tfrac{1}{2}u^2\right)\right] + \operatorname{div}\left[\rho\underset{\sim}{u}\left(h + \tfrac{1}{2}u^2\right) + \underset{\sim}{q} - \underset{\approx}{\tau}\cdot\underset{\sim}{u}\right] - \underset{\sim}{f}\cdot\underset{\sim}{u} = 0$$

$$\boxed{\frac{\partial}{\partial t}\left[\rho\left(e + \tfrac{1}{2}u^2\right)\right] + \operatorname{div}\left[\rho\underset{\sim}{u}\left(h + \tfrac{1}{2}u^2\right) + \underset{\sim}{q} - \underset{\approx}{\tau}\cdot\underset{\sim}{u}\right] = \underset{\sim}{f}\cdot\underset{\sim}{u}}$$

Energy equation (scalar)

— can be simplified to heat conduction eqn. —

8-31-94 —8-27— Eqns so for general, no E&M, relativity...

I. Fluid Characteristics

Equations of Motion are:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0 \quad \text{(continuity)}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \mathrm{div}(\rho \mathbf{u}\mathbf{u} + p\mathbf{I} - \boldsymbol{\tau}) = \mathbf{f} \quad \text{(momentum)}$$

$$\frac{\partial\, \rho(e+\frac{1}{2}u^2)}{\partial t} + \mathrm{div}\left[\rho \mathbf{u}(h+\tfrac{1}{2}u^2) - \boldsymbol{\tau}\cdot\mathbf{u} + \mathbf{q}\right] = \mathbf{f}\cdot\mathbf{u} \quad \text{(energy)}$$

Too general — Need equations for fluid characteristics (eqns of state) for $\boldsymbol{\tau}$, $\mathbf{q}$, etc.

For gases, some of this can be derived from kinetic theory. For liquids, this has not been successful (long-short range coupling, multi-body interactions, same problem as in turbulence). Use phenomenological models for liquids; might as well use for gases here too.

① $\mathbf{q}$, the heat flux vector, depends on spatial derivatives of T. Assume depends on 1st derivatives only —

Fourier's Law: $\mathbf{q}$ is a linear function of 1st deriv. of T.

Thus, $\underset{\sim}{q} = \underset{\approx}{k} \cdot \nabla T$ or $q_i = k_{ij} \frac{\partial T}{\partial x_j}$

Specify fluid - assume isotropic, with no preferred axes. ∴ then $\underset{\sim}{q}$ must be parallel to ∇T. Thus,

$$\underset{\approx}{K} = k_{ij} = -k\,\delta_{ij} = -k\,\underset{\approx}{I}$$

Minus sign convention so heat flux is positive in direction of decreasing T.

$$q_i = -k\,\delta_{ij} \frac{\partial T}{\partial x_j} = -k \nabla T$$

k is the thermal conductivity, can depend on local thermodynamic state (this is often neglected).

(2) $\underset{\approx}{\tau} = \tau_{ij}$, the viscous stress tensor (Newtonian fluid)

$\underset{\approx}{\tau}$ cannot depend on $\underset{\sim}{x}$ (translation of coords) or on $\underset{\sim}{u}$ (Galilean invariance). Must depend on spatial derivatives of $\underset{\sim}{u}$. Assume depends on 1st derivatives only. (Like a Taylor series ....). Assume a linear relation between τ_{ij} and $\frac{\partial u_i}{\partial x_j}$.

Further assume that a rigid-body rotation does not produce viscous stresses. Thus,

decompose $\frac{\partial u_i}{\partial x_j}$ into symmetric and anti-symmetric parts:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

but $\varepsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ rate of strain tensor

here wed 1-24.

$$\Omega_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \Rightarrow \Omega_{ii}\,(\text{no sum}) = 0$$

$$\Omega_{ij} = -\Omega_{ji}$$

only 3 independent components $\Rightarrow$ can associate a pseudo-vector with this tensor $\rightarrow$ $\begin{pmatrix} 0 & & \Omega_{ij} \\ & 0 & \\ -\Omega_{ij} & & 0 \end{pmatrix}$

$$\Omega_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k, \quad \underset{\sim}{\omega} = \nabla \times \underset{\sim}{u}$$

Ω_{ij} associated with rotation of fluid element. – rigid body rotations $\Rightarrow$ no $\underset{\sim}{\tau}$

τ_{ij} depends only on ε_{ij} – rate of strain tensor.

So $\underset{\sim}{\tau}$ is a linear function of $\underset{\sim}{\varepsilon}$ - implies

$\underset{\sim}{\varepsilon}$ not permutation tensor now (only 2 indices)

$$\tau_{ij} = \beta_{ijkl}\varepsilon_{kl}$$

↑ 81 components - and a fluid crystal can have many of them.

Each τ_{ij} in general depends on every ε_{kl}.

To simplify, assume fluid is isotropic. (water, air) (no preferred direction). Most general isotropic tensor

see Batchelor p143ff

$$\beta_{ijkl} = \mu\,\delta_{ik}\delta_{jl} + \mu'\,\delta_{il}\delta_{jk} + \mu''\,\delta_{ij}\delta_{kl}$$

(δ clearly isotropic) (can be shown, in tensor books)

But since $\tau_{ij} = \tau_{ji}$ *, $\beta_{ijkl} = \beta_{jikl} \Rightarrow \mu' = \mu$ (or get using symm of $\underset{\sim}{\varepsilon}$)

*otherwise ∞ rotational moment at a point.

$$\tau_{ij} = \left[\mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \mu''\delta_{ij}\delta_{kl}\right]\varepsilon_{kl}$$

$$\tau_{ij} = \mu(\varepsilon_{ij} + \varepsilon_{ji}) + \mu''\delta_{ij}\varepsilon_{kk}$$

$$\boxed{\tau_{ij} = 2\mu\varepsilon_{ij} + \mu''\delta_{ij}\varepsilon_{kk}}$$

μ, μ'' arbitrary constants.

but $\varepsilon_{kk} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_k} + \frac{\partial u_k}{\partial x_k}\right) = \frac{1}{2} 2\, \nabla \cdot \underset{\sim}{u}$

$$\varepsilon_{kk} = \nabla \cdot \underset{\sim}{u}$$

so $\tau_{ij} = 2\mu\varepsilon_{ij} + \mu'' \delta_{ij} \nabla \cdot \underset{\sim}{u}$

$$= 2\mu \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \mu'' \delta_{ij} \nabla \cdot \underset{\sim}{u}$$

(index must be repeated in a single term for summ. conv.)

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij} \mu'' \nabla \cdot \underset{\sim}{u}$$

Note trace$(\tau_{ij}) = \tau_{ii} = \mu \cdot 2 \cdot \frac{\partial u_i}{\partial x_i} + \underbrace{\delta_{ii}}_{3} \mu'' \frac{\partial u_i}{\partial x_i}$

$$= (2\mu + 3\mu'')\frac{\partial u_i}{\partial x_i} \equiv 3\lambda \frac{\partial u_i}{\partial x_i}$$

$$3\mu'' = 3\lambda - 2\mu$$

so $\mu'' = \lambda - 2/3\mu$

$$\boxed{\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\left(\lambda - 2/3\mu\right)\frac{\partial u_k}{\partial x_k}}$$

See Rosenhead et al. in Proc. Roy Soc. A, 1954, pp. 1-6, re 2nd viscosity

μ is ordinary viscosity.

see PFA5 1267, 1993

λ is the 'second viscosity' (often neglected) (important in shocks, etc)

-9-2-94-

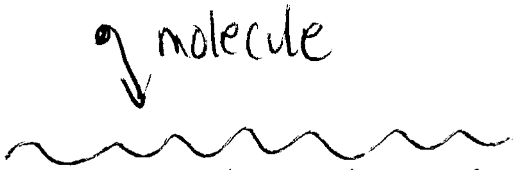
for second viscosity, see also notes from Erickson via Steinle, June 2014, plus AIAA paper 2004-0017 by Pan et al., etc. See also Meador et al., Phys. Fluids, v. 8, pp. 258-261, 1996, and Vincenti and Kruger, Intro. to Physical Gas Dynamics, 1965, pp. 407-412. Also the 23-page article by M.S. Cramer in Physics of Fluids, v. 24, 2012, article 066102. Remains controversial with many unresolved issues. Seem important for certain problems in ultrasound and perhaps second-mode instability waves in hypersonics.

(3) Also note that if we give the equation of state in canonical form, we get the caloric equation at the same time — give

$$h(p, s) \qquad (dh = T\,ds - v\,dp)$$

J. Boundary Conditions

No-slip → $\underset{\sim}{u}_{\text{fluid at surface}} = \underset{\sim}{u}_{\text{boundary}}$
(both parallel and ⊥ components)



on molecular scale, surface is not flat or hard. Bounces back randomly

really: particle is (HUL) usually accepted and then reevaporated. Bit of a monolayer of adsorbed fluid.

If we neglect viscosity can't enforce both B.C. → because viscosity multiplies highest order derivative, if $\mu = 0$ lose one order in the equations.

at HUL 1900T p. 2 notes.

K. Further Simplification of Momentum Equation

Insert the formula for the Newtonian shear into the momentum equation:

$$\frac{\partial(\rho \underset{\sim}{u})}{\partial t} + \operatorname{div}\left(\rho \underset{\sim}{u}\,\underset{\sim}{u} + p\underset{\sim}{I} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \delta_{ij}\left(\lambda - \tfrac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\right) = \underset{\sim}{f}$$

or,

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}\left[\rho u_i u_j + p\delta_{ij} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \delta_{ij}\left(\lambda - \tfrac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\right] = f_i$$

$$\frac{\partial(\rho u_i)}{\partial t} + u_i\frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j\frac{\partial}{\partial x_j}(u_i) + \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)$$

subtract div (continuity)

$$- \frac{\partial}{\partial x_i}\left[\left(\lambda - \tfrac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\right] = f_i$$

If fluid is not incompressible, there is very little which can be done to simplify this equation.

If $\mu = \text{const}$, $\frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) = \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu\frac{\partial^2 u_j}{\partial x_j \partial x_i}$

$$= \mu\nabla^2\underset{\sim}{u} + \mu\nabla(\nabla\cdot\underset{\sim}{u})$$

(three terms total, in $\nabla\cdot\underset{\sim}{u}$ or $\nabla\cdot(\rho\underset{\sim}{u})$)

Continuity is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{u}) = 0$$

$\boxed{\text{if } \rho = \text{const}}$ (incompressible, not stratified fluid) then $\nabla \cdot \underset{\sim}{u} = 0$

then, momentum equation becomes

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial}{\partial x_j}(u_i) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \underset{\sim}{f}$$

or,

$$\boxed{\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho(\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u} = -\nabla p + \mu \nabla^2 \underset{\sim}{u} + \underset{\sim}{f}}$$

'Navier-Stokes' equation.

But this is true only if $\underline{\rho = \text{const}}$ (perhaps this can be relaxed a bit...)

9-7-94. L. <u>Dissipation into Heat, and the Entropy Equation</u>

Energy Equation:

$$\frac{\partial\left(\rho\left(e + \overset{3}{\tfrac{1}{2}u^2}\right)\right)}{\partial t} + \text{div}\left[\overbrace{\rho \underset{\sim}{u}\left(h + \tfrac{1}{2}u^2\right) + \underset{\sim}{q} - \underset{\approx}{\tau} \cdot \underset{\sim}{u}}^{4}\right] = \underset{\sim}{f} \cdot \underset{\sim}{u}$$

Momentum eqn:

$$\frac{\partial \rho \underset{\sim}{u}}{\partial t} + \text{div}\left(\rho \underset{\sim}{u}\underset{\sim}{u} + p \underset{\approx}{I} - \underset{\approx}{\tau}\right) = \underset{\sim}{f}$$

dot momentum eqn. into $\underset{\sim}{u}$ to get mechanical energy eqn:

$$\underset{\sim}{u}\cdot \overset{(1)}{\frac{\partial \rho \underset{\sim}{u}}{\partial t}} + \underset{\sim}{u}\cdot \overset{(2)}{\mathrm{div}}\left[\rho \underset{\sim}{u}\underset{\sim}{u} + p\underset{\sim}{I} - \underset{\sim}{\tau}\right] = \underset{\sim}{u}\cdot\underset{\sim}{f}$$

want to subtract this from energy: Reformulate

(1) $\underset{\sim}{u}\cdot \frac{\partial \rho \underset{\sim}{u}}{\partial t} = u_j \frac{\partial (\rho u_j)}{\partial t}$; $\frac{\partial}{\partial t}(\rho u_j u_j) = \rho u_j \frac{\partial u_j}{\partial t} + u_j \frac{\partial}{\partial t}(\rho u_j)$

so $\underset{\sim}{u}\cdot \frac{\partial \rho \underset{\sim}{u}}{\partial t} = \frac{\partial}{\partial t}(\rho u^2) - \rho u_j \frac{\partial u_j}{\partial t} = \frac{\partial}{\partial t}(\rho u^2) - \rho \frac{\partial}{\partial t}\left(\frac{u^2}{2}\right)$

(2) $\underset{\sim}{u}\cdot \mathrm{div}(\rho \underset{\sim}{u}\underset{\sim}{u}) = u_i \frac{\partial}{\partial x_j}(\rho u_i u_j) = u_i u_j \cancel{\frac{\partial}{\partial x_j}(\rho u_i) + u_i \rho u_i} \frac{\partial u_j}{\partial x_j}$

$=$

$$= u_i u_i \frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j u_i \frac{\partial}{\partial x_j}(u_i)$$

~~energy eqns~~

(3) $\frac{\partial}{\partial t}\left(\rho e + \frac{1}{2}\rho u^2\right) = \frac{\partial}{\partial t}(\rho e) + \frac{1}{2}\frac{\partial}{\partial t}(\rho u^2)$

~~energy eqns~~

(4) $\mathrm{div}\left(\rho \underset{\sim}{u}\left(h + \frac{1}{2}u^2\right)\right) = \frac{\partial}{\partial x_j}\left[\rho u_j\left(h + \frac{1}{2}u^2\right)\right] = \mathrm{div}(\rho \underset{\sim}{u} h)$

$$+ \rho u_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}u^2\right) + \frac{1}{2}u^2 \frac{\partial}{\partial x_j}(\rho u_j)$$

Subtract ① & ② from ③ & ④:

$$\frac{\partial}{\partial t}(\rho e) + \frac{1}{2}\frac{\partial}{\partial t}(\rho u^2) + \operatorname{div}(\rho \underset{\sim}{u} h) + \cancel{(\rho \underset{\sim}{u} \cdot \nabla)\left(\tfrac{1}{2}u^2\right)} + \frac{1}{2}u^2 \nabla\cdot(\rho \underset{\sim}{u})$$
$$- \frac{\partial}{\partial t}(\rho u^2) + \rho\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) - u^2 \nabla\cdot(\rho \underset{\sim}{u}) - \cancel{(\rho \underset{\sim}{u} \cdot \nabla)\left(\frac{u^2}{2}\right)}$$

$$= \frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho \underset{\sim}{u} h) + \frac{1}{2}\frac{\partial}{\partial t}(\rho u^2) - \frac{1}{2}u^2 \nabla\cdot(\rho \underset{\sim}{u}) - \frac{\partial}{\partial t}(\rho u^2)$$
$$+ \rho\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right)$$

$$= \frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho \underset{\sim}{u} h) - \frac{1}{2}\left[\frac{\partial}{\partial t}(\rho u^2) + u^2 \nabla\cdot(\rho \underset{\sim}{u})\right] + \rho\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right)$$

but $\frac{\partial}{\partial t}(\rho u^2) = u^2\frac{\partial \rho}{\partial t} + \rho\frac{\partial u^2}{\partial t}$

$-\frac{1}{2}\frac{\partial}{\partial t}(\rho u^2) = -\frac{1}{2}u^2\frac{\partial \rho}{\partial t} - \frac{1}{2}\rho\frac{\partial u^2}{\partial t}$

$$= \frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho \underset{\sim}{u} h) - \frac{1}{2}\left[u^2\frac{\partial \rho}{\partial t} + u^2 \nabla\cdot(\rho \underset{\sim}{u})\right] - \cancel{\frac{1}{2}\rho\frac{\partial u^2}{\partial t}} + \cancel{\frac{1}{2}\rho\frac{\partial u^2}{\partial t}}$$

$= u^2(\text{continuity}) = 0$

$$= \frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho \underset{\sim}{u} h)$$

(8) of 16 Jan 86 notes

so, energy eqn − mechanical energy eqn is:

$$\frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho \underset{\sim}{u} h) + \operatorname{div}\left[\underset{\sim}{q} - \underset{\approx}{\tau}\cdot \underset{\sim}{u}\right] - \underset{\sim}{u}\cdot \operatorname{div}\left[p\underset{\approx}{I} - \underset{\approx}{\tau}\right]$$

$$= \underset{\sim}{f}\cdot \underset{\sim}{u} - \underset{\sim}{u}\cdot \underset{\sim}{f} = 0$$

$$\frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho \underset{\sim}{u} h + \underset{\sim}{q} - \underset{\approx}{\tau}\cdot \underset{\sim}{u}) - \underset{\sim}{u}\cdot \operatorname{div}(p\underset{\approx}{I} - \underset{\approx}{\tau}) = 0$$

$$-\operatorname{div}(\underset{\approx}{\tau}\cdot \underset{\sim}{u}) + \underset{\sim}{u}\cdot \operatorname{div}\underset{\approx}{\tau} = -\frac{\partial}{\partial x_j}(\tau_{ij} u_i) + u_i \frac{\partial}{\partial x_j}(\tau_{ij})$$

$$= -\tau_{ij}\frac{\partial u_i}{\partial x_j} - u_i \frac{\partial \tau_{ij}}{\partial x_j} + u_i \frac{\partial \tau_{ij}}{\partial x_j} = -\tau_{ij}\frac{\partial u_i}{\partial x_j}$$

$$\underset{\sim}{u}\cdot \operatorname{div}(p\underset{\approx}{I}) = u_i \frac{\partial}{\partial x_j}(p\,\delta_{ij}) = u_i \frac{\partial p}{\partial x_i} = \underset{\sim}{u}\cdot \nabla p$$

$$\frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho \underset{\sim}{u} h + \underset{\sim}{q}) = \underset{\sim}{u}\cdot \nabla p + \tau_{ij}\frac{\partial u_i}{\partial x_j}$$

$\phi = \tau_{ij}\dfrac{\partial u_i}{\partial x_j}$, the dissipation of mechanical energy into heat through viscous shear.

$\underset{\sim}{u}\cdot \nabla p$ = pressure work term.

$\operatorname{div}\underset{\sim}{q}$ = heat flux term.

here Fri. 1-26.

What is ϕ? (for a Newtonian fluid)

$$\phi = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} + \delta_{ij}\left(\lambda - \tfrac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\frac{\partial u_i}{\partial x_j}$$

$$= \mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} + \left(\lambda - \tfrac{2}{3}\mu\right)\cancel{\frac{\partial u_k}{\partial x_k}\frac{\partial u_i}{\partial x_i}} \rightarrow \left(\frac{\partial u_k}{\partial x_k}\right)^2$$

switch the dummy indices i, j

$$\phi = \mu\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\frac{\partial u_j}{\partial x_i} + \left(\lambda - \tfrac{2}{3}\mu\right)\left(\frac{\partial u_k}{\partial x_k}\right)^2, \text{ add}$$

$$2\phi = \mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) + 2\left(\lambda - \tfrac{2}{3}\mu\right)\left(\frac{\partial u_k}{\partial x_k}\right)^2$$

$$\boxed{\phi = \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2 + \left(\lambda - \tfrac{2}{3}\mu\right)\left(\frac{\partial u_k}{\partial x_k}\right)^2}$$

Positive definite (unless $\lambda - \tfrac{2}{3}\mu$ wierd...?)

Note: Batchelor p.153 seems to assume $\lambda = \tfrac{2}{3}\mu$ → this is associated with his assumption that $\tau_{ii} = 0$ (sum of mean normal viscous stress $= 0$).

Not clear issue (hard to do expts, not important?)

Now, continue w/ derivation of entropy eqn.

$$\frac{\partial}{\partial t}(\rho e) + \operatorname{div}(\rho \mathbf{v} h + q) = \mathbf{v}\cdot\nabla p + \phi$$

want $\frac{\partial}{\partial t}(\rho h) = \frac{\partial}{\partial t}\left(\rho\left(e + \frac{p}{\rho}\right)\right) = \frac{\partial}{\partial t}(\rho e + p)$

$$\frac{\partial}{\partial t}(\rho h) - \frac{\partial p}{\partial t} + \operatorname{div}(\rho \mathbf{v} h) + \operatorname{div} q = \mathbf{v}\cdot\nabla p + \phi$$

$$\frac{\partial}{\partial t}(\rho h) + \operatorname{div}(\rho \mathbf{v} h) - \left(\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p\right) = \phi - \operatorname{div} q$$

$$\rho\frac{\partial h}{\partial t} + h\frac{\partial \rho}{\partial t} + \left(\rho u_j \frac{\partial}{\partial x_j}\right)h + h\frac{\partial}{\partial x_j}(\rho u_j) - \left(\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p\right) = \phi - \operatorname{div} q$$

$$\rho\left[\frac{\partial h}{\partial t} + (\mathbf{v}\cdot\nabla)h\right] + h\underbrace{\cancel{\left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{v})\right]}}_{\text{(continuity)}}^{0} - \left(\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p\right) = \phi - \operatorname{div} q$$

Now derivative following the fluid is

$$\frac{d}{dt} = \frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)$$

(to be discussed later)

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so $\rho\frac{dh}{dt} - \frac{dp}{dt} = \phi - \operatorname{div} q$

but $T\,ds = dh - v\,dp = dh - \frac{1}{\rho}dp$

$$\rho T\,ds = \rho\,dh - dp$$

neglected diffusive effects here $dw = -p\,dV$.

diffusive effects here

so $$\rho T \frac{ds}{dt} = \rho \frac{dh}{dt} - \frac{dp}{dt} \quad \text{(tricky)}$$

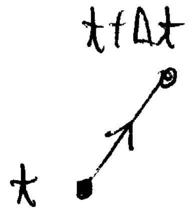
$$\boxed{\rho T \frac{ds}{dt} = \phi - \operatorname{div} q} \qquad \text{(cf. Batchelor 3.4.11)}$$

$$\boxed{\rho T \frac{ds}{dt} = \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 + \left(\lambda - \tfrac{2}{3}\mu\right)\left(\frac{\partial u_k}{\partial x_k}\right)^2 - \operatorname{div} q}$$

Note Derivative following the fluid:

1

'Particle' at (x, y, z) at time Δt moves to $(x+\Delta x, y+\Delta y, z+\Delta z)$ at time $t+\Delta t$



(ϕ any property carried w/ fluid)

$$\left.\frac{d\phi}{dt}\right|_{\text{following fluid}} = \frac{\phi(t+\Delta t) - \phi(t)}{\Delta t}, \quad \Delta t \to 0$$

$$\left.\frac{d\phi}{dt}\right|_{f.f.} = \frac{\phi(t+\Delta t, x+\Delta x, y+\Delta y, z+\Delta z) - \phi(t)}{\Delta t}, \quad \Delta t \to 0$$

but $$\phi(t+\Delta t, \underset{\sim}{x}+\Delta \underset{\sim}{x}) \cong \phi(t, \underset{\sim}{x}) + \frac{\partial \phi}{\partial t}\Delta t + \frac{\partial \phi}{\partial x}\Delta x + \frac{\partial \phi}{\partial y}\Delta y + \frac{\partial \phi}{\partial z}\Delta z + \cdots \text{ h.o.t.}$$

so $$\left.\frac{d\phi}{dt}\right|_{ff} = \lim_{\Delta t \to 0} \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial \phi}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial \phi}{\partial z}\frac{\Delta z}{\Delta t} = \frac{\partial \phi}{\partial t} + (\underset{\sim}{u} \cdot \nabla)\phi$$

Note 2: s is the entropy per unit mass, so ρs is the entropy per unit volume. ϕ is the increase of entropy due to diffusion of momentum (randomness increases on molecular reduction of gradients).

Why div q? Must be similar; if $\underset{\sim}{q} = -k\nabla T$, k=const,

$\mathrm{div}\, \underset{\sim}{q} = -k\nabla^2 T$, $-\mathrm{div}\, \underset{\sim}{q} = k\nabla^2 T$, so

$$\boxed{\rho T \frac{ds}{dt} = \phi + k\nabla^2 T}$$

& the similiarity is apparent. $k\nabla^2 T$ is reduction of gradients in T due to molecular diffusion.

Hard to see much more.

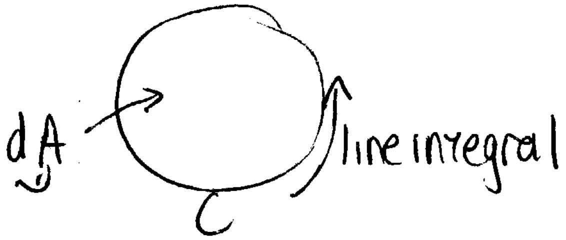
-9-3 M. Vorticity (see, e.g. pp ff in Rosenhead, by Lighthill)

see Saffman, in HWL festschrift '85 → "the sinews and bones of fluid motion" (Küchemann)

$$\underset{\sim}{\omega} \equiv \nabla \times \underset{\sim}{v} = \mathrm{curl}\, \underset{\sim}{v}$$

$$\int_S \underset{\sim}{\omega} \cdot dA = \int_C \underset{\sim}{v} \cdot d\underset{\sim}{l}$$ by Greens theorem, (or Stokes),

So vorticity associated with the rotation of a fluid element.



We have also seen this from the rotation tensor Ω_{ij}

Some theorems about vorticity:

(1) Helmholtz Law's: Vortex lines cannot end in the fluid:

since $\nabla\cdot(\nabla\times \underset{\sim}{v})=0$ for any vector field $\underset{\sim}{v}$,

$$\nabla\cdot\underset{\sim}{\omega}=0$$

A vortex line is a line everywhere tangent to the local vorticity vector (similar to a streamline). Since there is no divergence of $\underset{\sim}{\omega}$, these lines cannot end in the fluid. In a viscous flow, they cannot end on the wall either; or else the no-slip condition would not be satisfied.

(2) Helmholtz Law #2: Consider a vortex tube, which is a tube with sides tangent to the vorticity (sides made up of vortex lines) so that $\underset{\sim}{\omega}\cdot\hat{n}=0$ on the sides. What is the circulation around the ends?

Look at $\int_A \underset{\sim}{\omega}\cdot d\underset{\sim}{A}=\int_A \underset{\sim}{\omega}\cdot\hat{n}\,dS$

(A is a closed surface.)

$\int_A \underset{\sim}{\omega}\cdot\hat{n}\,dS=\iiint_V \operatorname{div}\underset{\sim}{\omega}\,dV$ by divergence thm.

$$\operatorname{div}\underset{\sim}{\omega}=\nabla\cdot(\nabla\times\underset{\sim}{u})=\frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}\frac{\partial}{\partial x_j}u_k\right)=0$$

$$=\varepsilon_{ijk}\frac{\partial^2 u_k}{\partial x_i\partial x_j}$$

zero because j, k pairs cancel out, for each i.

thus, $\int_A \underset{\sim}{\omega} \cdot d\underset{\sim}{A} = 0$, for any flow.

(3) Kelvin's Theorem: In an inviscid fluid in which $\rho = \text{const}$ or $p = p(\rho)$ only, and in which any body forces are conservative, vorticity moves with the fluid. (Action of viscosity pretty much restricted to diffusion).

Prove that $\left.\dfrac{d\Gamma}{dt}\right|_{\text{moving w/ the fluid}} = 0$ with conditions above

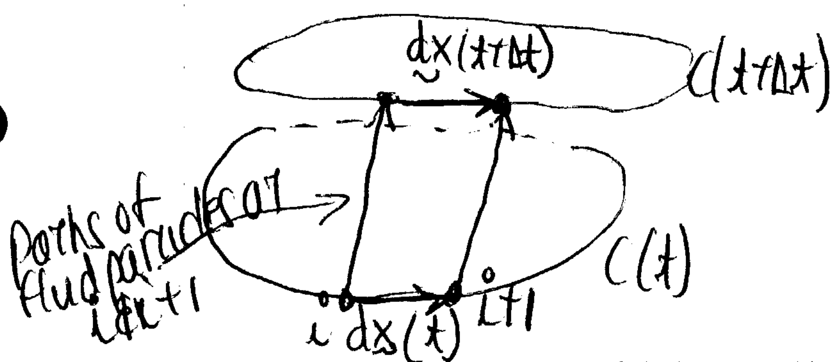
see AAE333 notes p.128ff

$$\Gamma \equiv \int_C \underset{\sim}{u} \cdot d\underset{\sim}{l} = \iint_A \underset{\sim}{\omega} \cdot d\underset{\sim}{A}$$

See Sommerfeld, Mechanics of Deformable Bodies, p.132ff.

$$\Gamma(t+\Delta t) - \Gamma(t) = \oint_{C(t+\Delta t)} \underset{\sim}{u} \cdot d\underset{\sim}{x} - \oint_{C(t)} \underset{\sim}{u} \cdot d\underset{\sim}{x}$$

where C marks the location of the same set of fluid particles, in both cases.



Note: $\underset{\sim}{u}_i \Delta t + d\underset{\sim}{x}(t+\Delta t) = d\underset{\sim}{x}(t) + \underset{\sim}{u}_{i+1} \Delta t$

(2 paths around 'square')

$$\left.\frac{d\Gamma}{dt}\right|_{\text{moving w/ fluid}} = \lim_{\Delta t\to 0} \frac{\Gamma(t+\Delta t)-\Gamma(t)}{\Delta t}$$

$$= \left.\frac{d}{dt}\right|_{m.w.f.} \oint \underset{\sim}{u}\cdot d\underset{\sim}{x} = \oint \left.\frac{d}{dt}\right|_{mwf} (\underset{\sim}{u}\cdot d\underset{\sim}{x}) \qquad \left(\Sigma \frac{d}{dt} = \frac{d}{dt}\Sigma\right)$$

here 9-29-94.

$$= \oint \left(\left.\frac{d\underset{\sim}{u}}{dt}\right|_{mwf} \cdot d\underset{\sim}{x} + \underset{\sim}{u}\cdot \left.\frac{d(d\underset{\sim}{x})}{dt}\right|_{mwf} \right) = \oint \left.\frac{d\underset{\sim}{u}}{dt}\right|_{mwf} \cdot d\underset{\sim}{x} + \oint \underset{\sim}{u}\cdot \left.\frac{d(d\underset{\sim}{x})}{dt}\right|_{mwf}$$

but $\displaystyle \left.\frac{d(d\underset{\sim}{x})}{dt}\right|_{mwf} = \frac{d\underset{\sim}{x}(t+\Delta t) - d\underset{\sim}{x}(t)}{\Delta t}, \ \Delta t \to 0$

$$= \frac{\underset{\sim}{u}_{i+1}\Delta t - \underset{\sim}{u}_i \Delta t}{\Delta t} \ (\text{from } \square) = \underset{\sim}{u}_{i+1} - \underset{\sim}{u}_i$$

so $\displaystyle \oint \underset{\sim}{u}\cdot \left.\frac{d(d\underset{\sim}{x})}{dt}\right|_{mwf} \cong \sum_{i \text{ around loop}} \underset{\sim}{u}_i \cdot (\underset{\sim}{u}_{i+1} - \underset{\sim}{u}_i)$

but $\displaystyle \sum_i \underset{\sim}{u}_i \cdot (\underset{\sim}{u}_{i+1} - \underset{\sim}{u}_i) \cong \oint \underset{\sim}{u}\cdot d\underset{\sim}{u}$; $\left(\begin{array}{l} \text{as } i\to\infty,\ \Delta \underset{\sim}{u} \to 0, \\ \underset{\sim}{u}_{i+1} - \underset{\sim}{u}_i = \Delta\underset{\sim}{u} \to d\underset{\sim}{u} \\ \quad \Sigma \to \int \end{array} \right)$

$$\oint \underset{\sim}{u}\cdot d\underset{\sim}{u} = \frac{1}{2}(\underset{\sim}{u}\cdot\underset{\sim}{u})\Big|_{1}^{2}$$

but for closed curve $1 = 2$, $\oint \underset{\sim}{u}\cdot d\underset{\sim}{u} = 0$.

$$\left(\oint (u\,du + v\,dv + w\,dw) = \oint u\,du + \cdots = \frac{1}{2}\oint d(u^2) + \cdots = \frac{1}{2}\underset{\sim}{u}\cdot\underset{\sim}{u} \right)$$

$$\text{so } \left.\frac{d\Gamma}{dt}\right|_{mwf} = \oint \left.\frac{d\underset{\sim}{u}}{dt}\right|_{mwf} \cdot d\underset{\sim}{x}$$

← gortere 1-31

but $\left.\frac{d\underset{\sim}{u}}{dt}\right|_{mwf}$ is the 'material derivative'; use momentum eqn:

$$\left.\frac{d}{dt}\right|_{mwf} \underset{\sim}{u} = \frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} = -\frac{\nabla p}{\rho} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right] + \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\lambda - \tfrac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k} + \frac{\underset{\sim}{f}}{\rho}$$

Terms in Integral:

a) if $\underset{\sim}{f}$ is conservative, $\underset{\sim}{f} = \nabla\varphi$, φ some potential.

For gravity, $\frac{\underset{\sim}{f}}{\rho} = \nabla\phi$ (gravitational force proportional to mass)
($\underset{\sim}{f}$ is force per unit volume)

but $\oint \nabla\phi \cdot d\underset{\sim}{x} = \phi\Big|_{①}^{②} = 0$ $\left(\phi \text{ single-valued}, \int_1^2 \nabla\phi\cdot d\underset{\sim}{x} = \phi\Big|_1^2\right)$

b) $-\oint \frac{\nabla p}{\rho}\cdot d\underset{\sim}{x}$; if ρ = const, dispose of as above.

if $\rho = \rho(p)$ only, then $\frac{\nabla p}{\rho} = \nabla\left(\int\frac{dp}{\rho}\right)$, and use same trick.

$$\left[\frac{\partial p}{\partial x}\hat{e}_x\frac{1}{\rho} = \hat{e}_x\frac{\partial}{\partial x}\int\frac{dp}{\rho} = \hat{e}_x\frac{(\partial p)}{\rho}\quad (?)\right]$$

loose

When is $\rho=\rho(p)$ only?

$$T\,ds = de + p\,dv = de + p\,d\left(\frac{1}{\rho}\right)$$

ρ depends on s and e & p, but e depends on p & T.

if $s=\text{const}$ (inviscid & non-heat conducting)

then ρ depends on p & T. But we have the eqn of state to give $T=T(\rho \,\&\, p)$ so $\rho=\rho(p)$ only. (isentropic flow $\Leftrightarrow \rho=\rho(p)$)

and if inviscid, the other terms drop out too.

$\boxed{\text{If not inviscid, 'errors' in Kelvin's theorem come from 'overturning' terms caused by density changes due to heating, and diffusion of vorticity caused by viscosity.}}$

c) viscous terms: if $\mu=\text{const}$,

$$\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right) = \frac{\mu}{\rho}\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j}+\frac{\partial^2 u_j}{\partial x_i \partial x_j}\right)$$

this term gives diffusion of vorticity, and stops the vortex tubes from moving only with the flow (think of diffusing line vortex).

$$\frac{1}{\rho}\delta_{ij}\left(\lambda-\tfrac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}.$$

?

So for conservative body forces, inviscid flow, and flow which is incompressible or isentropic, vorticity moves with the fluid (Kelvins Theorem)

4 Vorticity Equation

Start with momentum eqn:

$$\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \mathrm{div}(\rho \underset{\sim}{u}\underset{\sim}{u} + p\underset{\approx}{I} - \underset{\approx}{\tau}) = \underset{\sim}{f}$$

$$\rho\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \underset{\sim}{u}\underset{\sim}{u}) = \underset{\sim}{f} + \mathrm{div}(\underset{\approx}{\tau} - p\underset{\approx}{I})$$

$$\rho\frac{\partial u_i}{\partial t} + u_i\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \quad ''$$

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial}{\partial x_j}(u_i) + \underbrace{u_i\frac{\partial \rho}{\partial t} + u_i\frac{\partial}{\partial x_j}(\rho u_j)}_{\underset{\sim}{u}\cdot(\text{continuity}) = 0} = \quad ''$$

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial}{\partial x_j}(u_i) = \underset{\sim}{f} + \mathrm{div}(\underset{\approx}{\tau} - p\underset{\approx}{I})$$

change form of left hand side:

$$\underset{\sim}{\omega}\times\underset{\sim}{u} = \varepsilon_{\ell i m}\left(\varepsilon_{ijk}\frac{\partial}{\partial x_j}u_k\right)u_m$$

$$= \varepsilon_{im\ell}\,\varepsilon_{ijk}\left(\frac{\partial}{\partial x_j}u_k\right)u_m = \left(\delta_{mj}\delta_{\ell k} - \delta_{mk}\delta_{\ell j}\right)u_m\frac{\partial u_k}{\partial x_j}$$

$$\underset{\sim}{\omega} \times \underset{\sim}{u} = u_j \frac{\partial u_k}{\partial x_j} - u_k \frac{\partial u_k}{\partial x_j} = u_j \frac{\partial u_k}{\partial x_j} - \frac{\partial}{\partial x_j}\left(u_k u_k \tfrac{1}{2}\right)$$

so $$u_j \frac{\partial u_i}{\partial x_j} = \underset{\sim}{\omega} \times \underset{\sim}{u} + \nabla\left(\frac{u^2}{2}\right)$$

$$\boxed{\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho \underset{\sim}{\omega} \times \underset{\sim}{u} + \rho \nabla\left(\frac{u^2}{2}\right) = \underset{\sim}{f} + \operatorname{div} \underset{\approx}{\tau} - \nabla p}$$

alternative form of momentum eqn. actually a correct vector form for l.h.s. -- see Whitham, p 120ff, in Rosenhead (& p.129)

can't make further progress without assuming incompressible. (ρ=const)

then $\operatorname{div} \underset{\approx}{\tau} = \mu \nabla^2 \underset{\sim}{u}$

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{\omega} \times \underset{\sim}{u} + \nabla\left(\frac{u^2}{2} + \frac{p}{\rho}\right) = \frac{1}{\rho}\left(\underset{\sim}{f} + \mu \nabla^2 \underset{\sim}{u}\right)$$

take the curl of this:

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + \nabla \times (\underset{\sim}{\omega} \times \underset{\sim}{u}) + \overbrace{\cancel{\nabla \times \nabla\left(\frac{u^2}{2} + \frac{p}{\rho}\right)}}^{0} = \overbrace{\cancel{\nabla \times \left(\frac{\underset{\sim}{f}}{\rho}\right)}}^{0 \text{ if } \underset{\sim}{f} \text{ conservative}} + \frac{\mu}{\rho} \nabla^2 \underset{\sim}{\omega}$$

$$\nabla \times (\underset{\sim}{\omega} \times \underset{\sim}{u}) = \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left(\varepsilon_{klm}\, \omega_l u_m\right)$$

here fri

9-14-94
Need to do
$\nabla^2\omega$
$\nabla\times(\omega\times u)$

$$= \varepsilon_{kij}\varepsilon_{klm}\frac{\partial}{\partial x_j}(\omega_l u_m)$$

$$= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\left(\frac{\partial}{\partial x_j}(\omega_l u_m)\right)$$

$$= \frac{\partial}{\partial x_j}(\omega_i u_j) - \frac{\partial}{\partial x_j}(\omega_j u_i)$$

$$= \omega_i\frac{\partial u_j}{\partial x_j} + u_j\frac{\partial}{\partial x_j}\omega_i - \omega_j\frac{\partial}{\partial x_j}u_i - u_i\frac{\partial \omega_j}{\partial x_j}$$

$$= \boldsymbol{\omega}\,\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - \mathbf{u}\,\nabla\cdot\boldsymbol{\omega}$$

but $\nabla\cdot\mathbf{u}=0$, incompressible

$$\nabla\cdot\boldsymbol{\omega} = \nabla\cdot(\nabla\times\mathbf{u}) = \frac{\partial}{\partial x_j}\left(\varepsilon_{jlm}\frac{\partial}{\partial x_l}u_m\right) = \underset{=0}{\varepsilon_{jlm}\frac{\partial^2 u_m}{\partial x_j\partial x_l}}$$

$$\nabla\times(\boldsymbol{\omega}\times\mathbf{u}) = (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$$

so

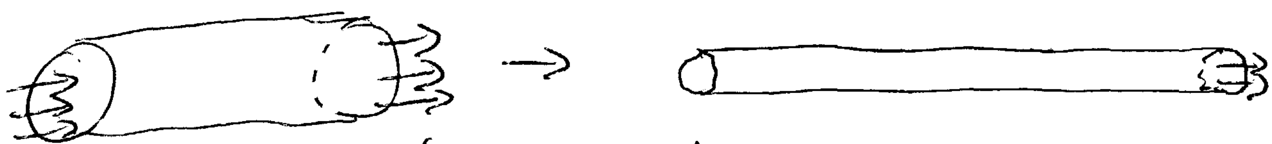
$$\boxed{\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}}$$

Vorticity Eqn. (ρ = const, conservative f)

$$\boxed{\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}}$$

$\nu \nabla^2 \underset{\sim}{\omega}$ is the diffusion of vorticity term.

$(\underset{\sim}{\omega} \cdot \nabla) \underset{\sim}{u}$ is the vortex-stretching term.



if vortex tube lengthens, mass conservation implies that diameter must decrease, Helmholtz law $\Rightarrow \underset{\sim}{\omega}$ increases. Like a spinning figure skater who pulls in her arms.

(see Batchelor or Whitham in Rosenhead for more details)