A&AE 613
Spring 2000
Professor Steve Schneider
Problem Set 2
Handed Out: Friday, 28 January
Due: Friday, 11 February

1. In curvilinear coordinates, the convective part of the momentum equation sometimes contains 'extra' terms, of a type not present in Cartesian coordinates. The following exercise asks you to derive these terms for 2D incompressible flow in polar coordinates, in an elementary way which gives an indication of their physical meaning.

   We need a relation for the term

   $$\mathrm{div}\left(\rho\vec{u}\vec{u}\right). \tag{1}$$

   We know from Gauss' theorem that

   $$\int_V \mathrm{div}\left(\rho\vec{u}\vec{u}\right) dV = \int_S \rho\vec{u}\left(\vec{u}\cdot\hat{n}\right) dS. \tag{2}$$

   Consider 2D flow in polar coordinates. Draw an elementary differential volume element near the point (r,θ), and evaluate the right hand side of equation (2). In this way, give the polar coordinate expression for (1).

   There is a new consideration here, not present in Cartesian coordinates, which gives rise to the extra terms in the expression for (1). What is it?

   Hint: Keep only terms to second order in Δr and $\Delta\theta$. Evaluate the integrals over the various differential surface area elements using the value of the integrand at the midpoint times the surface area (correct to lowest order). Your result should agree with, for example, that listed by Batchelor on p. 603. This problem is fairly laborious but is the simplest elementary way to get the result from first principles (to my knowledge).

(over)

2. The streamlines at an instant t are defined to be the curves everywhere tangent to the velocity vector. They are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

Pathlines are defined to be the lines traced out by a given fluid particle as it flows; they must satisfy the equation

$$\vec{u} = \frac{d\vec{x}}{dt}.$$

(see, e.g. Currie p. 38ff). Show that the streamlines and pathlines coincide if and only if

$$\vec{u}\,(\vec{x}, t) = \vec{u}_a(\vec{x}) f(\vec{x}, t).$$

Results such as these are useful in the interpretation of experimental dye flow visualization information. (See Hama, 1962, *Streaklines in a Perturbed Shear Flow*, Phys. Fluids **5**, 644-650. Problem taken from Ae201a, Caltech, Prof. P.G. Saffman, 1986).

3. For potential flow due to a line vortex the vorticity is concentrated along the axis of the vortex. Assume that viscosity is turned on at time $t = 0$, after which the vortex decays due to viscosity. Assume the flow is incompressible. Let the initial circulation around the line vortex be Γ. Obtain expressions for the velocity $u_\theta(r, t)$ and the pressure $p(r, t)$ in the fluid. Simplify the expressions as much as is feasible. Leave p as an integral if needed. Use a similarity solution. Discuss how Γ changes with time. (Currie, second edition, problem 7.9; similar to White, Viscous Fluid Flow, second edition, problem 3-14. See both listings for useful hints.)