A&AE 613
Spring 2000
Professor Steve Schneider
Problem Set 1
Handed Out: Friday, 21 January
Due: Friday, 28 January

1. Using index notation, and no other formulas from vector calculus, derive the following formulas from the vector calculus:

   (a)
   $$\nabla \cdot \left(\phi \vec{f}\right) = \phi \left(\nabla \cdot \vec{f}\right) + \vec{f} \cdot \nabla \phi.$$

   (b)
   $$\nabla \times \left(\vec{f} \times \vec{g}\right) = \vec{f} \left(\nabla \cdot \vec{g}\right) - \vec{g} \left(\nabla \cdot \vec{f}\right) + \left(\vec{g} \cdot \nabla\right) \vec{f} - \left(\vec{f} \cdot \nabla\right) \vec{g}.$$

   Here, ϕ is a scalar.

2. A perfectly insulated cylinder of volume V and cross-sectional area A has a perfectly insulated piston in one side of it. The gas is initially at temperature T_0 and pressure p_0. Work the following problems for the case of a perfect gas with constant specific heats.

   (a) The gas is compressed to a volume $V/2$ by a very slow steady action of the piston. Find the final temperature and pressure of the gas (T_1 and p_1) in terms of the initial conditions. Determine the work done by the piston, W_1. Give numerical results for the case of air.

   (b) A second copy of the original system in the original state is compressed by the sudden movement of the piston, creating shock waves and turbulence in the gas. This sudden movement is created by placing a weight $p_2 A$ on the piston, and suddenly releasing it, causing the final pressure in the cylinder to be p_2. The weight $p_2 A$ is selected so that the work done is the same as that done in part (a). Find the final temperature T_2 and the final pressure p_2, in terms of the initial conditions and the work done. Give numberical results for the case of air. Discuss any assumptions you may find it necessary to make. Hint: simplify the analysis of the complex 3D unsteady flow by assuming that p instantly reaches p_2 when the piston is released.

(c) Compare and contrast the two cases, for air with $\gamma = 1.4$. In which case is there more ability to do useful work, after the compression?

3. Work the following problem, taken from Liepmann and Roshko, *Elements of Gasdynamics*, Wiley, 1957, p. 384, problem 1.9. Brackets have been added to include the λ in the argument of the entropy expression. Check this correction of a semi-typographical error.

   A perfect gas, enclosed by an insulated (upright) cylinder and piston, is at equilibrium at conditions p_1, v_1, T_1. A weight is placed on the piston. After a number of oscillations, the motion subsides and the gas reaches a new equilibrium at conditions p_2, v_2, T_2. Find the temperature ratio T_2/T_1 in terms of the pressure ratio $\lambda = p_2/p_1$. Show that the change of entropy is given by

$$s_2 - s_1 = R \log \left[\left[\frac{1 + (\gamma - 1)\lambda}{\gamma} \right]^{\gamma/(\gamma-1)} \frac{1}{\lambda} \right].$$

   Show that, if the initial disturbance is small, so that $\lambda = 1 + \epsilon, \epsilon \ll 1$, then

$$(s_2 - s_1)/R \simeq \epsilon^2/(2\gamma).$$