A&AE 613, Professor Steve Schneider
Summary Sheet for Index Notation and Cartesian Tensors

The vector $\vec{u}$ can be written as

$$\vec{u} = u_x \hat{e}_x + u_y \hat{e}_y + u_z \hat{e}_z,$$

or as

$$\vec{u} = u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3,$$

or as

$$\vec{u} = \sum_{i=1}^{3} u_i \hat{e}_i.$$

If the summation is implied whenever an index is repeated (Einstein summation convention), then this last is the same as

$$\vec{u} = u_i \hat{e}_i.$$

The unit vectors are often omitted, leaving their presence implied only. We introduce the Kronecker delta or identity tensor δ_{ij}, which is 1 if $i = j$ and zero otherwise. We also introduce the permutation tensor ϵ_{ijk}, which is 1 if i, j, k are $1, 2, 3$ or $2, 3, 1$ or $3, 1, 2$, is -1 if i, j, k are $2, 1, 3$ or $1, 3, 2$ or $3, 2, 1$, and is zero otherwise.

In this notation, we have the following equivalencies:

$$\vec{a} \cdot \vec{b} = a_i b_i$$

$$\vec{a} \times \vec{b} = \epsilon_{ijk} \hat{e}_i a_j b_k$$

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

$$\nabla = \hat{e}_i \frac{\partial}{\partial x_i}$$

$$\nabla \phi = \hat{e}_i \frac{\partial \phi}{\partial x_i}$$

$$\nabla \cdot \vec{u} = \frac{\partial u_i}{\partial x_i}$$

$$\nabla \times \vec{u} = \hat{e}_i \epsilon_{ijk} \frac{\partial}{\partial x_j} u_k$$

All the usual vector identities can be derived using these relations, keeping in mind that we are in general restricted to Cartesian coordinates.

The advantage of this system is that it can also be used for tensor quantities like the shear stress,

$$\boldsymbol{\tau} = \tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}.$$

Here,

$$\nabla \vec{u} = \frac{\partial u_i}{\partial x_j},$$

a tensor, and

$$\nabla \cdot \boldsymbol{\tau} = \hat{e}_i \frac{\partial}{\partial x_j} \tau_{ij},$$

and

$$\nabla \times \boldsymbol{\tau}$$

has no definition. Gauss' theorem, or the divergence theorem, still holds for these tensor quantities,

$$\int_V \nabla \cdot \boldsymbol{\tau} \, dV = \int_S \boldsymbol{\tau} \cdot \vec{n} \, dS = \int_S \tau_{ij} n_j \, dS.$$

Remember that $\tau_{ij} = \tau_{ji}$ in fluids.