SPS
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Compressible BL Similarity Solution Code

See Anderson, HHTG or White

Look at flat plate case $\frac{dh_e}{d\xi}=0$, $\frac{du_e}{d\xi}=0$.

Have

$$(Cf'')' + ff'' = 0 \quad (A6.68)$$

$$\left(\frac{C}{Pr}g'\right)' + fg' + C\frac{u_e^2}{h_e}(f'')^2 = 0 \quad (A6.69)$$

Assume perfect gas with constant specific heats.

$$\frac{u_e^2}{h_e} = \frac{u_e^2}{c_p T_e}\frac{\gamma R}{\gamma R} = \frac{u_e^2}{a_e^2}\frac{\frac{c_p}{c_v}(c_p-c_v)}{c_p} = (\gamma-1)M_e^2$$

so energy becomes

$$\left(\frac{C}{Pr}g'\right)' + fg' + C(\gamma-1)M_e^2(f'')^2 = 0$$

B.C. are $f(0)=0$, $f'(0)=0$ (no slip, no flow through)

and $f'(\infty)=1$ (match at freestream)

$g(\infty)=1$ (match at freestream)

Also need B.C. on wall temp. ($g(0)$) or on wall heat transfer.

$$-q_w = k_w \frac{\partial T}{\partial y}\bigg|_w = +\frac{k_w}{c_{p_w}} \frac{\partial h}{\partial y}\bigg|_w$$

$$-q_w = \frac{k_w}{c_{p_w}} \frac{h_e \rho_e u_e}{\sqrt{2\xi}} g'\Big|_{\eta=0}$$ (cp. A p. 237) (sign error there)

if $q_w = 0$, $g'(0) = 0$

Need an equation for $C(h)$. Use Sutherland viscosity law

$$C = \frac{\rho \mu}{\rho_e \mu_e}; \quad \frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + S}{T + S}$$

$$P = \rho R T = \text{const across layer} \Rightarrow \frac{\rho}{\rho_e} = \frac{T_e}{T}$$

$$\frac{\mu}{\mu_e} = \frac{\mu/\mu_0}{\mu_e/\mu_0} = \frac{(T/\cancel{T_0})^{3/2} \frac{\cancel{T_0 + S}}{T+S}}{(T_e/\cancel{T_0})^{3/2} \frac{\cancel{T_0+S}}{T_e+S}} = \left(\frac{T}{T_e}\right)^{3/2} \frac{T_e + S}{T + S}$$

$$\text{So } C = \frac{T_e}{T}\left(\frac{T}{T_e}\right)^{3/2} \frac{T_e + S}{T + S} = \left(\frac{T}{T_e}\right)^{1/2} \frac{T_e + S}{T + S}$$

$$C = \left(\frac{h}{h_e}\right)^{1/2}\left(\frac{h_e + c_p S}{h + c_p S}\right)$$ (assume perf. gas with const. c_p)

but $g = h/h_e$

$$\boxed{C = g^{1/2}\left(\frac{h_e + c_p S}{g h_e + c_p S}\right)}$$

Have to write DEls in terms of f, f', etc.

A6.68 is $$C f''' + C' f'' + f f'' = 0$$

what is C'?

$$C' = g^{1/2}\left(h_e + c_p S\right)(-1)\left(g h_e + c_p S\right)^{-2} h_e g'$$

$$+\left(\frac{h_e + c_p S}{g h_e + c_p S}\right)\frac{1}{2} g^{-1/2} g'$$

$$\boxed{C' = \frac{h_e + c_p S}{g h_e + c_p S}\left[g^{1/2}\frac{(-h_e g')}{g h_e + c_p S} + \frac{1}{2}\frac{g'}{g^{1/2}}\right]}$$

$$\boxed{f''' = \frac{1}{C}\left(-C' f'' - f f''\right)}$$

A 6.691s (assuming Pr = const)

$$\frac{1}{Pr}\left(C'g' + Cg''\right) + fg' + C(\gamma-1)M_e^2 (f'')^2 = 0$$

$$\boxed{Cg'' = -Pr\, fg' - Pr\, C(\gamma-1)M_e^2 (f'')^2 - C'g'}$$

Simplify C' further:

$$C' = \left(\frac{1+\frac{c_p S}{h_e}}{g+\frac{c_p S}{h_e}}\right)\left[\frac{g'}{g^{1/2}}\right]\left[\frac{1}{2} - \frac{\frac{h_e}{c_p S}\, g}{g\frac{h_e}{c_p S}+1}\right]$$

$$C' = \left[\frac{\frac{h_e}{c_p S}+1}{g\frac{h_e}{c_p S}+1}\right]\frac{g'}{g^{1/2}}\left[\frac{1}{2}\right]\cdot\left[\frac{g\frac{h_e}{c_p S}+1-2\frac{h_e}{c_p S}\, g}{g\frac{h_e}{c_p S}+1}\right]$$

$$h_e/c_p S = T_e/S \equiv C_1$$

$$\boxed{C' = \left(\frac{C_1+1}{(gC_1+1)^2}\right)\frac{g'}{2\sqrt{g}}\left(gC_1+1-2C_1 g\right)}$$

or,

$$\boxed{C' = \frac{g'}{2\sqrt{g}} \frac{c_1+1}{(gc_1+1)^2}(1-c_1 g)}$$

Now, find Reynolds analogy

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \mu_w u_e f'' \frac{\partial \eta}{\partial y}\Big|_{y=0}$$

$$= \mu_w u_e f''(0) \frac{u_e}{\sqrt{2\xi}} \rho\big|_{y=0}$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} = \frac{2}{\rho_e \cancel{u_e^2}} \frac{\mu_w \cancel{u_e^2} f''(0)\rho(0)}{\sqrt{2}\sqrt{\rho_e \mu_e u_e x}}$$

$$c_f = \frac{\sqrt{2}\,\mu_w f''(0)\rho_w}{\rho_e\sqrt{\rho_e \mu_e u_e x}} = \sqrt{2}\frac{\rho_w \mu_w}{\rho_e \mu_e} f''(0)\sqrt{\frac{\mu_e}{\rho_e u_e x}}$$

$$\boxed{c_f = \sqrt{2}\frac{\rho_w \mu_w}{\rho_e \mu_e}\frac{f''(0)}{\sqrt{Re_{x,e}}}} \quad (A6.71)$$

$$C_H = \frac{|q_w|}{\rho_e u_e (h_{aw} - h_w)} \quad (A.\ p.\ 237)$$

$$C_H = \frac{k_w}{c_{p_w}} \frac{h_e \rho_w u_e}{\sqrt{2 \rho_e \mu_e u_e x}}\, g'(0)\, \frac{1}{\rho_e u_e (h_{aw} - h_w)} \quad (cp.\ 6.65)$$

$$C_H = \frac{1}{\sqrt{2}} \frac{k_w}{\mu_e c_{p_w}} \frac{\rho_w h_e\, g'(0)}{\sqrt{\rho_e u_e x / \mu_e}} \frac{1}{\rho_e (h_{aw} - h_w)}$$

$$\boxed{C_H = \frac{1}{\sqrt{2}} \frac{k_w}{\mu_e c_{p_w}} \frac{\rho_w}{\rho_e} \frac{h_e}{h_{aw} - h_w} \frac{g'(0)}{\sqrt{Re_{x,e}}}} \quad (A6.77)$$

$$\frac{c_f}{C_H} = \frac{\sqrt{2}\, \frac{\cancel{\rho_w} \mu_w}{\cancel{\rho_e} \cancel{\mu_e}} \frac{f''(0)}{\cancel{\sqrt{Re_{x,e}}}}}{\frac{1}{\sqrt{2}} \frac{k_w}{\cancel{\mu_e} c_{p_w}} \frac{\cancel{\rho_w}}{\cancel{\rho_e}} \frac{h_e}{h_{aw} - h_w} \frac{g'(0)}{\cancel{\sqrt{Re_{x,e}}}}}$$

$$\frac{c_f}{C_H} = 2 \frac{\mu_w f''(0) c_{p_w}}{k_w\, g'(0)} \frac{h_{aw} - h_w}{h_e} = 2 \frac{Pr_w f''(0)}{g'(0)} \frac{h_{aw} - h_w}{h_e}$$

$$\boxed{\left(\frac{c_f}{C_H}\right)\left(\frac{h_e}{h_{aw} - h_w}\right) = \frac{2 Pr_w f''(0)}{g'(0)}}$$