Taylor-Maccoll Flow


Consider a perfect sharp cone with an attached shock, perfectly conical. By symmetry, expect a flow with conical symmetry, where properties remain unchanged along rays from the cone vertex. Assume flow is steady and inviscid.

A perfect conical shock is always at the same angle to the uniform freestream, so the flow immediately behind the shock is at uniform conditions. If the flow is inviscid and any body forces are conservative, then the initially irrotational flow will remain irrotational. This means

\[ \nabla \times \mathbf{u} = 0 \quad (T \, 10.1) \]
Continuity for steady compressible flow is
\[ \nabla \cdot (\rho \mathbf{v}) = 0 \quad (T \ 10.2) \]

Finally, the energy equation for steady flow without viscosity or heat transfer, and without body forces, is
\[ h = 0 = h + \frac{1}{2} u^2 \quad (cp. \ LR \ 7.25, \ T \ 10.3) \]

Consider the flow field as follows:

![Diagram of conical shock wave and streamlines](image)

Figure 16.36  Flow model for Taylor-Maccoll flow over a cone.
Use spherical coordinates, with $\phi$ the azimuthal angle, as shown as follows:

![Diagram of spherical coordinates]

Figure 16.37 The spherical coordinate system.

In polar coordinates,

$$ u = u_r e_r + u_\theta e_\theta + u_\phi e_\phi $$

Note $\frac{\partial}{\partial r} = 0$ by assumption, as is $\frac{\partial}{\partial \phi}$ (conical flow).

$$ \nabla \cdot \mathbf{f} \text{ is then } \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 f_r) + \frac{1}{rs \sin \psi} \frac{\partial}{\partial \theta} (f_s \sin \psi) $$

(cp. Gradshteyn & Ryzhik p. 1086)
and \( \nabla \times \mathbf{v} = \frac{1}{r^2 \sin \psi} \begin{vmatrix} \hat{e}_r & \hat{e}_\psi & r \sin \psi \hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \psi} & \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial r} & r \hat{e}_\psi & 0 \end{vmatrix} \)

\((G & R \ p.1086)\)

**Irrotationality gives:**

\[ 0 = \hat{e}_r \left( \frac{\partial}{\partial \phi} (r \ u_\psi) \right) \]

\[ -r \hat{e}_\psi \left( \frac{\partial}{\partial \phi} (u_r) \right) \]

\[ + r \sin \psi \ e_\phi \left( \frac{\partial}{\partial r} (r \ u_\psi) - \frac{\partial}{\partial \psi} (u_r) \right) = \nabla \times \mathbf{v} \]

**Continuity gives**

\[ \nabla \cdot \mathbf{v} = \frac{1}{r^2 \sin \psi} \left( \frac{\partial}{\partial r} \left( r^2 \ u_r \right) + \frac{1}{r \sin \psi} \frac{\partial}{\partial \psi} \left( r \ u_\psi \sin \psi \right) \right) \]

\[ = \frac{\sin \psi}{r} \left[ r^2 \frac{\partial}{\partial r} (\epsilon u_r) + \epsilon u_r \sin \psi \sin \psi \right] + \frac{\partial}{\partial \psi} (\epsilon u_\psi \sin \psi) \]
\[ 0 = 2\sin \psi \cos \psi + \frac{\partial}{\partial \psi} (e\psi \sin \psi) \] (T_910.14)

Now note that \( \frac{\partial}{\partial \psi} \) is really \( \frac{d}{d\psi} \), all variables depend on \( \psi \) alone, by symmetry.

Use the energy equation to eliminate the density:
\[ h + \frac{1}{2}u^2 = \text{const.} \]

So \( dh + u du = 0 \)

To simplify the following, assume a perfect gas with constant specific heats. Then
\[ p = eRT \]
\[ dp = eRdT + RT \frac{de}{e} \]
\[ \frac{1}{e} dp = RdT + RT \frac{de}{e} \]

Now use \( dh = Tds + \frac{1}{e} dp \), noting \( s = 0 \), so \( dh = \frac{1}{e} dp \)
\[ dh = RdT + RT \frac{de}{e} \]
\[ C_p dT - R dT = \kappa R T \frac{d\rho}{\rho} \frac{1}{\kappa} \]

\[ \kappa (C_p - (C_p + u))^2 dT = a^2 \frac{d\rho}{\rho} \]

\[ \frac{C_p}{C_v} C_v dT = a^2 \frac{d\rho}{\rho} = dh \]

So \[ a^2 \frac{d\rho}{\rho} + u d\rho = 0 \Rightarrow \frac{d\rho}{\rho} = -\frac{u}{a^2} d\rho \] \((*)\)

Now, differentiate (10.14)

\[ 0 = 2 \sin \psi \psi_r \rho + \rho \frac{\partial}{\partial \psi} (u \psi \sin \psi) + i \psi \sin \psi \frac{d\rho}{d\psi} \]

or \[ 0 = 2 \sin \psi \psi_r \rho + u \psi \cos \psi + \sin \psi \frac{du}{d\psi} + u \psi \sin \psi \frac{d\rho}{d\psi} \]

Now, (*) implies that \[ \frac{1}{\rho} \frac{d\rho}{d\psi} = -\frac{u}{a^2} \frac{du}{d\psi} \]

or

\[ \frac{1}{\rho} \frac{d\rho}{d\psi} = -\frac{1}{a^2} \frac{d}{d\psi} \left( \frac{u^2}{2} \right) = -\frac{1}{2a^2} \frac{d}{d\psi} (u^2 + u \psi) \]
So

\[ 0 = 2 \sin \psi \cos \psi \frac{d\psi}{d\chi} + 2 \sin \psi \cos \psi \frac{d\psi}{d\psi} \]

\[ + \sin \psi \cos \psi \left( \frac{1}{2\alpha^2} \right) \left( 2 \frac{d\gamma}{d\psi} \frac{d\gamma}{d\psi} + 2 \frac{d\chi}{d\psi} \frac{d\chi}{d\psi} \right) \]

\[ \frac{\sin \psi}{\alpha^2} \left( \frac{d\gamma}{d\psi} \frac{d\gamma}{d\psi} + \frac{d\chi}{d\psi} \frac{d\chi}{d\psi} \right) = 2 \sin \psi \cos \psi \frac{d\psi}{d\psi} + \sin \psi \cos \psi \frac{d\psi}{d\psi} \]

\[ \psi \left( \frac{d\gamma}{d\psi} \frac{d\gamma}{d\psi} + \frac{d\chi}{d\psi} \frac{d\chi}{d\psi} \right) = \alpha^2 \left[ \frac{d\psi}{d\psi} + \psi \cot \psi \frac{d\psi}{d\psi} \right] \]

\[ \psi \left( \frac{d\gamma}{d\psi} \frac{d\gamma}{d\psi} + \frac{d\chi}{d\psi} \frac{d\chi}{d\psi} \right) = \alpha^2 \left[ \frac{d\psi}{d\psi} + 2 \frac{d\gamma}{d\psi} \frac{d\gamma}{d\psi} + \psi \cot \psi \frac{d\psi}{d\psi} \right] = 0 \]

(2.81 16.52, or \( T_1 \) line above 10.16)

Solve this, together with (10.13)

\[ 0 = \psi \frac{d\gamma}{d\psi} \]
28H now use \( \bar{u} \) for \( u \) and \( \bar{v} \) for \( v \).

Since \( \frac{\partial}{\partial y} \) is really now \( \frac{d}{dy} \), rewrite as

\[
\bar{u} \left( \bar{u} \frac{d\bar{u}}{dy} + \bar{v} \frac{d\bar{v}}{dy} \right) - a^2 \left( \frac{d\bar{v}}{dy} + 2\bar{u} + \bar{v} \cot y \right) = 0
\]

(16.56)

\[
\frac{d\bar{u}}{dy} = \bar{u}
\]

(16.57)

Substitute

\[
\bar{v} \left( \bar{u} \bar{v} + \bar{v} \frac{d\bar{v}}{dy} \right) - a^2 \left( \frac{d\bar{v}}{dy} + 2\bar{u} + \bar{v} \cot y \right) = 0
\]

\[
(\bar{u}^2 - a^2) \frac{d\bar{v}}{dy} + \bar{u} \bar{v}^2 - a^2 2\bar{u} - a^2 \bar{v} \cot y = 0
\]

\[
(\bar{u}^2 - a^2) \frac{d\bar{v}}{dy} = -\bar{u} (\bar{v}^2) + 2\bar{u} a^2 + a^2 \bar{v} \cot y
\]

\[
= -\bar{u} (\bar{v}^2 - a^2) + a^2 (\bar{u} + \bar{v} \cot y)
\]

\[
\frac{d\bar{v}}{dy} = -\bar{u} + \frac{a^2 (\bar{u} + \bar{v} \cot y)}{\bar{u}^2 - a^2}
\]

(16.58)
Put in non-dimensional form using the critical speed of the freestream gas, \( a^* \). \( a^* \) is the speed of sound in a place where the freestream gas might be isentropically brought to Mach 1.

\[
\begin{align*}
\bar{U}^* &= \frac{U}{a^*}, \\
\bar{V}^* &= \frac{V}{a^*} \\
\end{align*}
\]

(16.59 in 2nd Ed)

So we now have

\[
\frac{d\bar{U}^*}{dy} = \bar{U}^* \quad (16.60)
\]

\[
\frac{d\bar{V}^*}{dy} = -\bar{U}^* + \frac{(\gamma/\gamma^*)^2(\bar{U}^* + \bar{V}^* + 4)}{\bar{U}^2 - (\gamma/\gamma^*)^2} \quad (16.61)
\]

B.C. - March freestream, no flow through cone surface. What is the equation of state relating \( a^* \) and \( a^* \)?


For a perfect gas without heat conduction,

\[
H = h + \frac{1}{2}u^2 = \text{constant} \quad \text{(also true across shock)}
\]
then let $a^2 = KRT_1$ and $h = c_p T_1$ (calorically perfect) and $c_p T + \frac{1}{2} u^2 = \text{const} = c_p \frac{a^2}{x^*} + \frac{1}{2} u^2$.

$$\text{const} = \frac{c_p}{\rho / c_p} \frac{1(a^2)}{c_p - c_p} + \frac{1}{2} u^2$$

$$\text{const} = \frac{a^2}{x-1} + \frac{1}{2} u^2 \quad (LR2.29)$$

At $u = a$, $u = a = a^*$ by definition, so

$$\frac{a^*}{x-1} + \frac{1}{2} a^*\frac{a^2}{x-1} = \frac{a^2}{x-1} + \frac{1}{2} u^2$$

$$\frac{2(x-1)}{(x-1)2} a^* = \frac{a^2}{x-1} + \frac{1}{2} u^2$$

$$\frac{1}{2(x-1)} a^* = \frac{a^2}{x-1} + \frac{1}{2} u^2 \quad (LR2.33)$$

$$\frac{1}{2} a^* \frac{a^2}{x-1} - \frac{1}{2} (x-1) u^2 \frac{a^2}{x^*} = \frac{a^2}{a^*}$$

$$\frac{a^2}{a^*} = \frac{x+1}{2} - \frac{x-1}{2} M^2$$

\[ \begin{array}{c} 2 \cdot 8 + 16.62 \end{array} \]
Note that here

\[(M^+)^2 = \left(\frac{u}{a^+}\right)^2 = \frac{u^2 + \nu^2}{a^+} = \frac{u_{\infty}^2 + \nu_{\infty}^2}{a^+}\]

(28416 16.63)

Follow 28416 16-56 for the numerical integration.

Have to assume a shock angle, compute prop. behind shock. Integrate to the surface, surface is where \(u^* = 0\). Find accurate value of surface angle. Will not be exactly the angle sought. So make a new guess for the shock angle, and iterate to convergence.