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26 Jan 2001

Taylor–Maccoll Flow

Follow Zucrow and Hoffman, Gas Dynamics, Vol. II, Multidimensional Flows, section 16.5, and Philip A. Thompson, Compressible-Fluid Dynamics, section 10.3.

Consider a perfect sharp cone with an attached shock, perfectly conical. By symmetry, expect a flow with conical symmetry, where properties remain unchanged along rays from the cone vertex.

Assume flow is steady and inviscid.

A perfect conical shock is always at the same angle to the uniform freestream, so the flow immediately behind the shock is at uniform conditions.

If the flow is inviscid and any body forces are conservative, then the initially irrotational flow will remain irrotational. This means

$$\nabla \times \underset{\sim}{u} = 0 \qquad (T\ 10.1)$$

Continuity for steady compressible flow is

$$\nabla \cdot (\rho \underset{\sim}{u}) = 0 \qquad (T\ 10.2)$$

Finally, the energy equation for steady flow without viscosity or heat transfer, and without body forces, is

$$H = 0 = h + \frac{1}{2}u^2 \qquad (\text{cp. L\&R } 7.25;\ T\ 10.3)$$

Consider the flow field as follows:

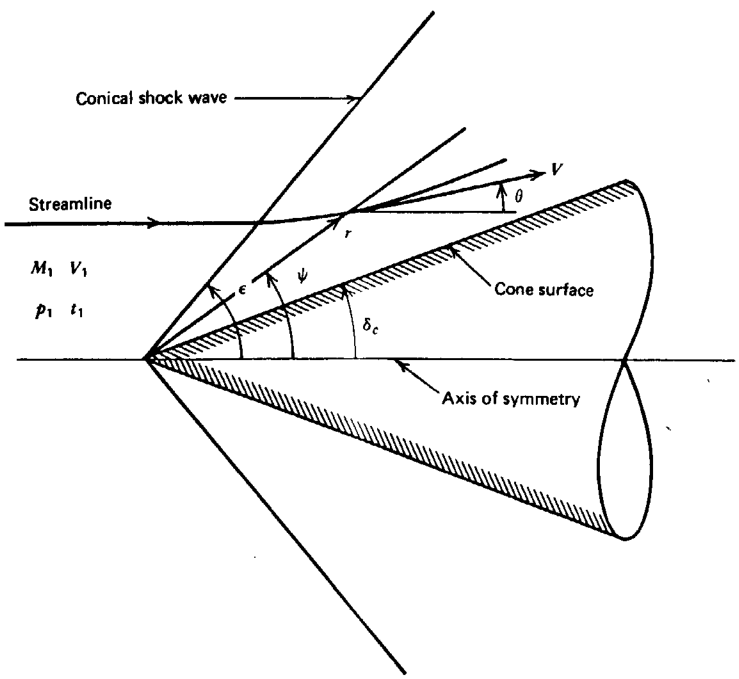

Figure 16.36 Flow model for Taylor-Maccoll flow over a cone.

Use spherical coordinates, with ϕ the azimuthal angle, as shown as follows:

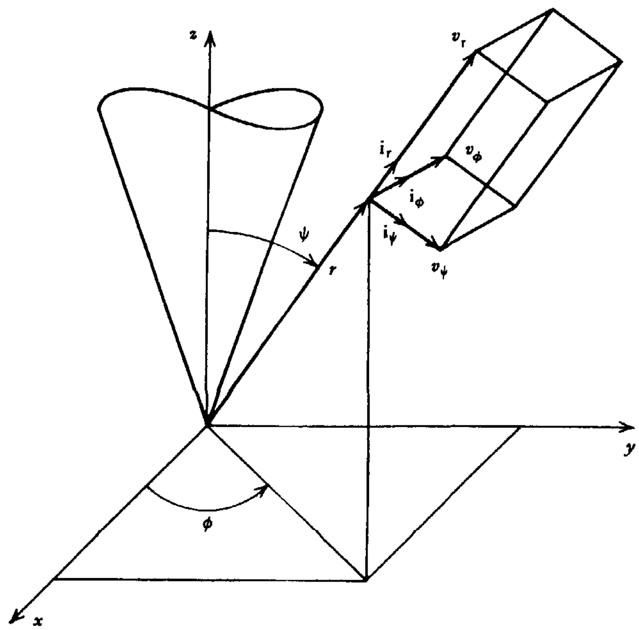

Figure 16.37 The spherical coordinate system.

In polar coordinates,

$$\underset{\sim}{u} = u_r \hat{e}_r + \overset{0 \text{ by symmetry}}{\cancel{u_\phi}} \hat{e}_\phi + u_\psi \hat{e}_\psi$$

note $\frac{\partial}{\partial r} = 0$ by assumption, as is $\frac{\partial}{\partial \phi}$ (conical flow)

$\nabla \cdot \underset{\sim}{f}$ is then $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\psi}\frac{\partial}{\partial \psi}\left(f_\psi \sin\psi\right)$

(cp. Gradshteyn & Ryzhik p. 1086)

and $\nabla \times \underset{\sim}{f} = \dfrac{1}{r^2 \sin\psi} \begin{vmatrix} \hat{e}_r & r\hat{e}_\psi & r\sin\psi\, \hat{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \psi} & \dfrac{\partial}{\partial \phi} \\ f_r & r f_\psi & 0 \end{vmatrix}$

(G&R p.1086)

Irrotationality gives:

$$0 = \hat{e}_r (-) \overbrace{\cancel{\frac{\partial}{\partial \phi}}}^{\text{conical flow}} (r\, u_\psi)$$

$$- r\hat{e}_\psi \left(\overbrace{\cancel{\frac{\partial}{\partial \phi}}}^{0 \text{ conical flow}} (u_r) \right)$$

$$+ r\sin\psi\, e_\phi \left(\frac{\partial}{\partial r}(r\, u_\psi) - \frac{\partial}{\partial \psi}(u_r) \right) = \nabla \times \underset{\sim}{u}$$

$$0 = r \overbrace{\cancel{\frac{\partial u_\psi}{\partial r}}}^{0,\text{ conical flow}} + u_\psi - \frac{\partial u_r}{\partial \psi}$$

$$\boxed{0 = u_\psi - \frac{\partial u_r}{\partial \psi}} \qquad (T\ 10.13) \text{ or } (Z\&H\ 16.53)$$

Continuity gives

$$\nabla \cdot (\rho \underset{\sim}{u}) = 0 = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \rho u_r) + \frac{1}{r\sin\psi} \frac{\partial}{\partial \psi}(\rho u_\psi \sin\psi)$$

$$0 = \frac{\sin\psi}{r} \left[r^2 \overbrace{\cancel{\frac{\partial}{\partial r}}}^{0,\text{ conical flow}} (\rho u_r) + \rho u_r 2r \right] + \frac{\partial}{\partial \psi}(\rho u_\psi \sin\psi)$$

$$\boxed{0 = 2\sin\phi\, \rho u_r + \frac{\partial}{\partial\phi}(\rho u_\phi \sin\phi)} \quad (T, 10.14)$$

Now note that $\frac{\partial}{\partial\phi}$ is really $\frac{d}{d\phi}$, all variables depend on ϕ alone, by symmetry.

Use the energy equation to eliminate the density:

$$h + \tfrac{1}{2}u^2 = \text{const.}$$

So $$dh + u\,du = 0$$

to simplify the following, assume a perfect gas with constant specific heats. Then

$$p = \rho RT$$

$$dp = \rho R\,dT + RT\,d\rho$$

$$\frac{1}{\rho}dp = R\,dT + RT\frac{d\rho}{\rho}$$

Now use $dh = T\,ds + \frac{1}{\rho}dp$, note $ds = 0$, so $dh = \frac{1}{\rho}dp$

$$dh = R\,dT + RT\frac{d\rho}{\rho}$$

$$C_p dT - R\,dT = \gamma R T \frac{d\rho}{\rho}\frac{1}{\gamma}$$

$$\gamma\left(C_p - (C_p - C_v)\right)dT = a^2 \frac{d\rho}{\rho}$$

$$\frac{C_p}{C_v} C_v\, dT = a^2 \frac{d\rho}{\rho} = dh$$

So $a^2 \frac{d\rho}{\rho} + u\,du = 0 \Rightarrow \frac{d\rho}{\rho} = -\frac{u}{a^2}\,du \quad (*)$

Now, differentiate (10.14)

$$0 = 2\sin\psi\, u_r\, \rho + \rho \frac{\partial}{\partial\psi}\left(u_\psi \sin\psi\right) + u_\psi \sin\psi \frac{\partial\rho}{\partial\psi}$$

or $0 = 2\sin\psi\, u_r + u_\psi \cos\psi + \sin\psi \frac{du_\psi}{d\psi} + u_\psi \sin\psi \frac{d\rho}{\rho\, d\psi}$

Now, (*) implies that $\frac{1}{\rho}\frac{d\rho}{d\psi} = -\frac{u}{a^2}\frac{du}{d\psi}$, or

$$\frac{1}{\rho}\frac{d\rho}{d\psi} = -\frac{1}{a^2}\frac{d}{d\psi}\left(\frac{u^2}{2}\right) = -\frac{1}{2a^2}\frac{d}{d\psi}\left(u_r^2 + u_\psi^2\right)$$

So

$$0 = 2\sin\psi\, u_r + u_\psi \cos\psi + \sin\psi \frac{du_\psi}{d\psi} + u_\psi \sin\psi \left(-\frac{1}{2a^2}\right)\left(2u_r \frac{du_r}{d\psi} + 2u_\psi \frac{du_\psi}{d\psi}\right)$$

$$\frac{u_\psi \sin\psi}{a^2}\left(u_r \frac{du_r}{d\psi} + u_\psi \frac{du_\psi}{d\psi}\right) = 2\sin\psi\, u_r + u_\psi \cos\psi + \sin\psi \frac{du_\psi}{d\psi}$$

$$u_\psi\left(u_r \frac{du_r}{d\psi} + u_\psi \frac{du_\psi}{d\psi}\right) = a^2\left[2u_r + u_\psi \cot\psi + \frac{du_\psi}{d\psi}\right]$$

$$\boxed{u_\psi\left(u_r \frac{du_r}{d\psi} + u_\psi \frac{du_\psi}{d\psi}\right) - a^2\left[\frac{du_\psi}{d\psi} + 2u_r + u_\psi \cot\psi\right] = 0}$$

(Z&H 16.52, or T_1 line above 10.16)

Solve this, together with (10.13)

$$\boxed{0 = u_\psi - \frac{du_r}{d\psi}}$$

Z&H now use $\bar{u}$ for u_r & $\bar{v}$ for v_ψ.
Since $\frac{\partial}{\partial\psi}$ is really now $\frac{d}{d\psi}$, rewrite as

$$\bar{v}\left(\bar{u}\frac{d\bar{u}}{d\psi}+\bar{v}\frac{d\bar{v}}{d\psi}\right)-a^2\left(\frac{d\bar{v}}{d\psi}+2\bar{u}+\bar{v}\cot\psi\right)=0 \quad (16.56)$$

$$\frac{d\bar{u}}{d\psi}=\bar{v} \quad (16.57)$$

substitute

$$\bar{v}\left(\bar{u}\bar{v}+\bar{v}\frac{d\bar{v}}{d\psi}\right)-a^2\left(\frac{d\bar{v}}{d\psi}+2\bar{u}+\bar{v}\cot\psi\right)=0$$

$$(\bar{v}^2-a^2)\frac{d\bar{v}}{d\psi}+\bar{u}\bar{v}^2-a^2 2\bar{u}-a^2\bar{v}\cot\psi=0$$

$$(\bar{v}^2-a^2)\frac{d\bar{v}}{d\psi}=-\bar{u}(\bar{v}^2)+2\bar{u}a^2+a^2\bar{v}\cot\psi$$

$$=-\bar{u}(\bar{v}^2-a^2)+a^2(\bar{u}+\bar{v}\cot\psi)$$

$$\boxed{\frac{d\bar{v}}{d\psi}=-\bar{u}+\frac{a^2(\bar{u}+\bar{v}\cot\psi)}{\bar{v}^2-a^2}} \quad \begin{pmatrix}\text{Z\&H}\\ 16.58\end{pmatrix}$$

Put in nondimensional form using the critical speed of the freestream gas, a^*. a^* is the speed of sound in a place where the freestream gas might be isentropically brought to Mach 1.

$$\bar{u}^* = \frac{\bar{u}}{a^*}, \quad \bar{v}^* = \frac{\bar{v}}{a^*} \qquad (16.59 \text{ in Z\&H}).$$

So we now have

$$\boxed{\frac{d\bar{u}^*}{d\psi} = \bar{v}^* \qquad \left(\begin{array}{c}\text{Z\&H}\\ 16.60\end{array}\right)}$$

$$\boxed{\frac{d\bar{v}^*}{d\psi} = -\bar{u}^* + \frac{(a/a^*)^2(\bar{u}^* + \bar{v}^*\cot\psi)}{\bar{v}^{*2} - (a/a^*)^2} \qquad (16.61)}$$

B.C. - March freestream, no flow through cone surface.
What is the equation of state relating a & a^*?
Follow L&R section 2.10. Cp. Anderson, "Fundamentals ...", 2e, section 8.4 (eqn. 8.35).
For a perfect gas without heat conduction,

$$H = h + \tfrac{1}{2}u^2 = \text{constant} \qquad \text{(also true across shock)}$$

then let $a^2 = \gamma RT$, and $h = c_p T$, (calorically perfect),

and $c_p T + \frac{1}{2}u^2 = \text{const} = c_p \frac{a^2}{\gamma R} + \frac{1}{2}u^2$

$$\text{const} = \frac{\cancel{c_p}}{\cancel{c_p}/c_v} \frac{1(a^2)}{c_p - c_v} + \frac{1}{2}u^2$$

$$\text{const} = \frac{a^2}{\gamma-1} + \frac{1}{2}u^2 \qquad (\text{L\&R } 2.29)$$

at $u = a$, $u = a = a^*$ by definition, so

$$\frac{a^{*2}}{\gamma-1} + \frac{1}{2}a^{*2} = \frac{a^2}{\gamma-1} + \frac{1}{2}u^2$$

$$\frac{2+\gamma-1}{(\gamma-1)2} a^{*2} = \frac{a^2}{\gamma-1} + \frac{1}{2}u^2$$

$$\frac{\gamma+1}{2(\gamma-1)} a^{*2} = \frac{a^2}{\gamma-1} + \frac{1}{2}u^2 \qquad (\text{L\&R } 2.33)$$

$$\frac{\gamma+1}{2}\cancel{a^{*2}}\frac{1}{\cancel{a^{*2}}} - \frac{1}{2}(\gamma-1)\frac{u^2}{a^{*2}} = \frac{a^2}{a^{*2}}$$

$$\boxed{\frac{a^2}{a^{*2}} = \frac{\gamma+1}{2} - \frac{\gamma-1}{2}M^{*2}} \qquad \boxed{\text{Z\&H } 16.62}$$

note that here

$$\boxed{(M^*)^2 = \left(\frac{u}{a^*}\right)^2 = \frac{\bar{u}^2 + \bar{v}^2}{a^{*2}} = \bar{u}^{*2} + \bar{v}^{*2}}$$

(Z&H 16.63)

Follow Z&H 16-5b for the numerical integration.
Have to assume a shock angle, compute prop. behind shock.
Integrate to the surface, surface is where
$\bar{v}^* = 0$. Find accurate value of surface angle.
will not be exactly the angle sought.
So make a new guess for the shock angle, and
Iterate to convergence.