-2-12-01-

SPS
2/2/99

Viscous Flow Issues for 519 Scope

416 is not a prerequisite.

Some have taken 416, some are taking 416

Don't want to duplicate 416

613 covers in more depth. Don't want to duplicate 613. Some of 416 & 519 can be covered in more depth in 613, some overlap OK.

511 Covers a lot of viscous flow, in more depth than in here. 519 is more of a survey course, and covers many more topics. 511 is mostly grads, 519 has more undergrads than grads. Few in 519 have taken 511 (the grads).

How many have taken 416 or 511, or are taking one of these?

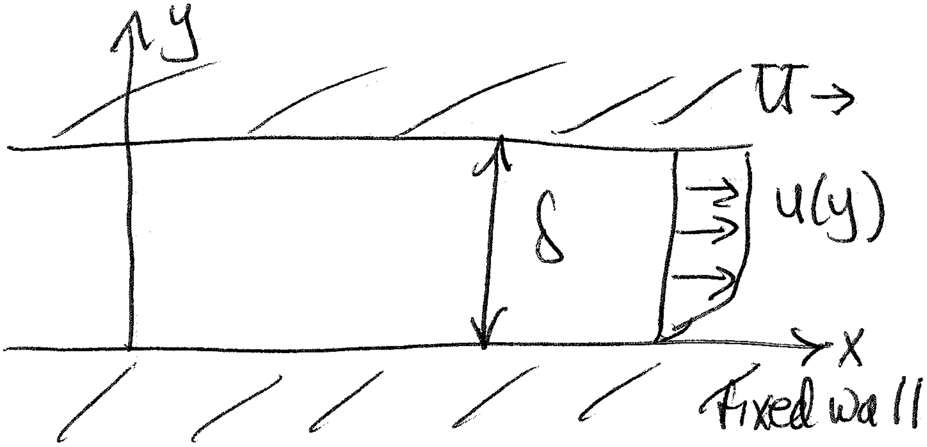
2-15-99

Laminar Compressible Viscous Flow

Couette Flow – An Exact Solution

Introduce using in 519 u Se fall 99. No 416/613 people

Liepmann & Roshko p 306 – 313

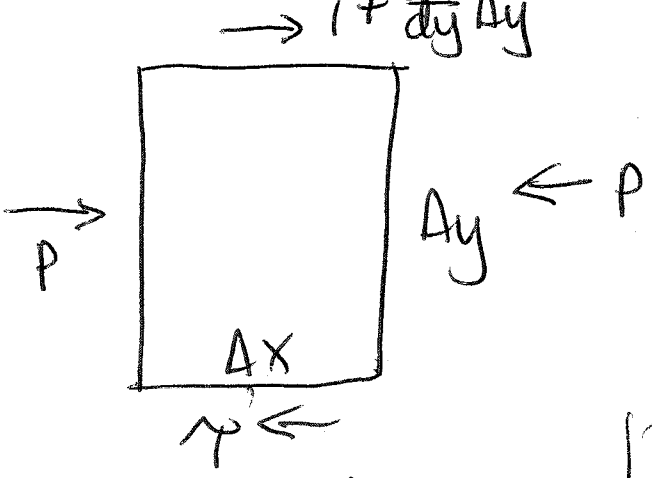


eqn 3-21 Schetz [illegible]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \tau}{\partial y}+\frac{f_x}{\rho}$$

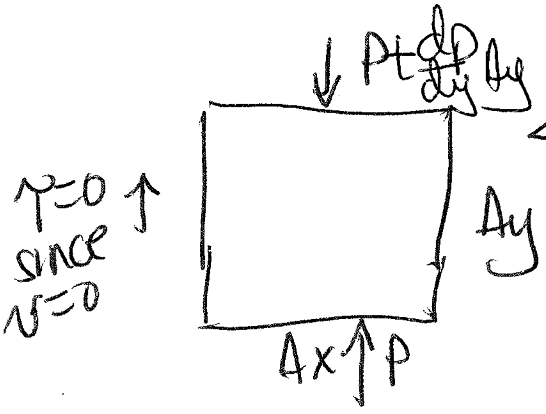
$$3\text{-}23 \quad \tau = \mu\frac{\partial u}{\partial y}$$

Assume 2-D, steady, neglect gravity, flow independent of x.



Equilibrium in x-direction shows $\boxed{\frac{d\tau}{dy}=0}$

(a bit high-handed)



$$\cancel{\frac{\partial \rho u}{\partial x}}^{0}+\frac{\partial \rho v}{\partial y}=0$$

$\Rightarrow v=0$ so no shear on vertical sides

→ since $v=0$ $\boxed{\frac{\partial p}{\partial y}=0 \text{ also}}$

$\Rightarrow \tau = \mu\frac{\partial u}{\partial y} = \text{const} = \tau_w$ (a bit high-handed too)

Schetz 2, 3-4 f

~~$$\rho\left[\frac{\partial}{\partial t}\left(h+\frac{u^2}{2}\right)+u\frac{\partial}{\partial x}\left(h+\frac{u^2}{2}\right)+v\frac{\partial}{\partial y}\left(h+\frac{u^2}{2}\right)\right]=-\frac{\partial q_y}{\partial y}+\frac{\partial}{\partial y}(\tau u)+\frac{\partial p}{\partial t}$$~~

But $u \neq \frac{\hat{\tau}}{\mu} y + \text{const}$, for $\mu \neq \text{const}$!!

μ depends on T. (molecular argument).

(also true for heat-transfer if ρ = const)

02-14-01

2-4-97

temperature varies because viscous work → heat.

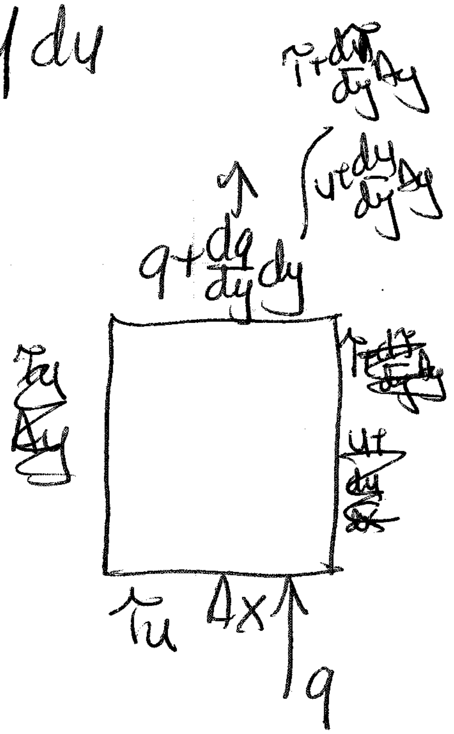
$$\frac{\mu}{\tau}du = dy \quad \text{so} \quad y = \frac{1}{\tau_w}\int \mu \, du$$

Energy Equation needed (steady)

$$\left[q-\left(q+\frac{dq}{dy}\Delta y\right)\right]\Delta x +$$

— work done by shear

$$\left[\left(\tau+\frac{d\tau}{dy}\Delta y\right)\left(u+\frac{du}{dy}\Delta y\right)-\tau u\right]\Delta x = 0$$



$$-\frac{dq}{dy}\Delta y\,\Delta x + \tau\frac{du}{dy}\Delta y\Delta x + u\frac{d\tau}{dy}\Delta y\,\Delta x + \frac{d\tau}{dy}\frac{du}{dy}(\Delta y)^2\Delta x = 0$$

divide by $\Delta y \Delta x$, take limit, last term drops out.

$$-\frac{dq}{dy}+\frac{d}{dy}(\tau u)=0 \quad \text{or} \quad \underline{\tau u - q = \text{const}}$$

~~2-4-96.~~

2-20-98 - but need to draw free-body, only did from Schetz.

9-31-99

Evaluate at wall: (lower wall, $u=0$, $q=q_w$)

$\tau u - q \equiv q_w$ heat transfer at fixed wall.

-2-10-95

Note 1: $\tau u - q = \mu u \frac{\partial u}{\partial y} + k \frac{\partial T}{\partial y}$

$$= \mu \left(\frac{\partial}{\partial y} (u^2/2) + \frac{k}{\mu c_p} \frac{\partial h}{\partial y} \right)$$

$Pr \equiv \frac{c_p \mu}{k}$ = Prandtl number.

ratio of diffusion of momentum to diffusion of heat.

for many substances, Pr is a const, since μ & k vary in the same way.

~~if we take Pr = 1, then~~

~~$\tau u + q = \mu \frac{\partial}{\partial y}\left(h + \frac{1}{2}u^2\right) \Rightarrow h + \frac{1}{2}u^2 = H = \text{const.}$~~

~~a great simplification.~~

$\mu \neq \text{const.}$

9-3-99 did this

Note 2

$q - u\tau = \text{const}$ is the balance btwn heat transfer and viscous dissipation which is required in order to have steady 2-D flow

---

here fr

using note I,

$$-q_w = \mu\left[\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) + \frac{1}{Pr}\frac{\partial h}{\partial y}\right]$$

$$= \mu \frac{\partial}{\partial y}\left(\frac{u^2}{2} + \frac{h}{Pr}\right)$$

Assume constant Prandtl #

"near const. even for dissociation since Cp & k peak at same place"

Integrate:

$$\frac{u^2}{2} + \frac{h}{Pr} = \int \frac{q_w}{\mu}\,dy + \text{const.}$$

$$\frac{u^2}{2} + \frac{h}{Pr} = -\int_0^y \frac{q_w}{\mu}\,dy + \frac{h_{wall}}{Pr} = -q_w \int_0^y \frac{dy}{\mu} + \frac{h_{wall}}{Pr}$$

$$= -q_w \int_0^y \frac{du}{\tau} + \frac{h_{wall}}{Pr}$$

(using mom. $\tau = \mu \frac{\partial u}{\partial y}$)

$$\frac{u^2}{2}+\frac{h}{Pr} = -\frac{q_w}{\tau_w}\int_0^y du + \frac{h_{wall}}{Pr}$$

$$= -\frac{q_w}{\tau_w}u + \frac{h_{wall}}{Pr}$$

So $\boxed{\frac{u^2}{2}+\frac{h-h_{wall}}{Pr} = -\frac{q_w}{\tau_w}u}$ Energy eqn. for Couette flow: (const Pr.)

at upper wall, $y=\delta$,

$\boxed{\frac{U^2}{2}+\frac{h_\delta - h_w}{Pr} = -\frac{q_w}{\tau_w}U}$ q_w = lower wall heat transfer

Bring in a new concept: Recovery temperature or enthalpy:

Recovery temperature is the wall temperature, if the wall is insulated, or adiabatic.

$h_w(q_w=0) = h_r$ at lower wall

so $\boxed{h_{r,w} = \frac{U^2}{2}(Pr) + h_\delta} = h_\delta\left(1+Pr\frac{U^2}{2h_\delta}\right)$

so recovery enthalpy at the lower wall is

$$\frac{h_r}{h_\delta} = 1 + \frac{Pr}{2}\frac{U^2}{h_\delta}$$

$$\frac{U^2}{h_\delta} = \frac{U^2}{c_p T_\delta \gamma R \frac{1}{\gamma R}} = \frac{U^2}{a_\delta^2}\,\frac{1}{\cancel{c_p}\,\frac{1}{\cancel{c_p}/c_v(c_p-c_v)}} = (\gamma-1)M_\delta^2$$

so
$$\boxed{\frac{h_{r,w}}{h_\delta} = 1 + \frac{\gamma-1}{2} Pr\, M_\delta^2}$$

$h_r \sim M_\delta^2$

Look at numbers for transonic, supersonic, hypersonic.

not just U_δ^2 or M_δ^2 but has Pr # factor

Note: since $\frac{h_\delta}{Pr} + \frac{U^2}{2} = \frac{h_r}{Pr}$, and $\frac{h_\delta - h_w}{Pr} = -\frac{q_w}{\tau_w}U - \frac{U^2}{2}$,

$$\frac{h_r}{Pr} - \cancel{\frac{U^2}{2}} = \frac{h_w}{Pr} - \frac{q_w}{\tau_w}U - \cancel{\frac{U^2}{2}}$$

$$\frac{q_w}{\tau_w}U = \frac{h_w - h_r}{Pr}, \text{ or } \boxed{q_w = \frac{\tau_w}{U\,Pr}(h_w - h_r)}$$

113B

For positive heat transfer, $\underline{h_w > h_r}$.

wall has a higher natural temperature than the ambient stream. To make heat flow from the wall, wall temp. must be >> ambient at high speed. Very difficult to cool a wall in high-speed flow – rather, the flow heats the wall.

The stagnation enthalpy $H = h + \frac{1}{2}u^2$. This differs from the recovery enthalpy by a Pr # factor. Why? – heat transfer.

[H_δ could be $h_\delta + \frac{1}{2}U^2$]

2-16-96
10-4-99

Note that

$$\frac{h_r - h_\delta}{c_p} = T_r - T_\delta = \frac{U^2}{2} Pr \frac{1}{c_p}$$

Note also that $\frac{T_0}{T_\delta} = 1 + \frac{\gamma-1}{2} M_\delta^2$ from isentropic relations.

$$\frac{T_r - T_\delta}{T_0 - T_\delta} = r = \text{recovery factor} = \frac{\frac{U^2}{2c_p} Pr}{T_\delta \frac{\gamma-1}{2} M_\delta^2}$$

$$\frac{U^2}{2h_\delta} = \frac{U^2}{2c_pT_\delta} = \frac{U^2}{2c_p\frac{1}{\gamma R}(\gamma R T_\delta)} = M_\delta^2 \frac{1}{2}\frac{c_p - c_v}{c_p/\gamma} = M_\delta^2 \frac{\gamma-1}{2\cancel{\gamma/\gamma}}$$

$$\text{so } r = \frac{M_\delta^2 \frac{\gamma-1}{2} Pr}{\frac{\gamma-1}{2} M_\delta^2} = \boxed{Pr = r}$$

For flat plate
lam B.L., Text
$r \cong \sqrt{Pr}$ eqn (6.90)
p. 249

if $Pr = 1$, stagnation ~~enthalp~~ temp = recovery temp.
Different because $Pr \neq 1$, momentum & heat don't diffuse at same rate.

(p. 310 L&R).

So what is the velocity profile?

$$\tau = \tau_w = \mu \frac{du}{dy}$$

$$dy = \frac{\mu}{\tau_w} du \Rightarrow y = \frac{1}{\tau_w} \int_0^u \mu \, du \quad \text{(velocity profile)}$$

what is $\mu(T)$ or $\mu(h)$? (see White p28ff)

can get a good approx. for dilute gases using a power-law:

$$\frac{\mu}{\mu_\delta} = \left(\frac{T}{T_\delta}\right)^\omega \qquad \frac{1}{2} \le \omega < 1$$

so

$$y = \frac{\mu_\delta \mathcal{U}}{\tau_w} \int_0^{u/\mathcal{U}} \frac{\mu}{\mu_\delta} \, d\left(\frac{u}{\mathcal{U}}\right)$$

(u depends on $\mathcal{U}$, h, h_δ, etc, known; $\mu/\mu_\delta = f(h/h_\delta)$ too. So this can be found, maybe numerically)

evaluate at upper wall:

$$\delta = \frac{\mu_\delta}{\tau_w} \mathcal{U} \int_0^1 \frac{\mu}{\mu_\delta} \, d\left(\frac{u}{\mathcal{U}}\right)$$

$$\Rightarrow \boxed{\tau_w = \frac{\mu_\delta}{\delta} \mathcal{U} \int_0^1 \frac{\mu}{\mu_\delta} \, d\left(\frac{u}{\mathcal{U}}\right)}$$

Now remember

$$h = h_w - \frac{q_w u}{\tau_w} P_r - \frac{P_r}{2} u^2$$

and

$$\frac{U^2}{2} + \frac{h_\delta - h_w}{P_r} = -\frac{q_w}{\tau_w} U$$

$$\text{so } h = h_w + P_r \left(\frac{U^2}{2} + \frac{h_\delta - h_w}{P_r} \right) \frac{u}{U} - \frac{P_r}{2} u^2$$

$$h = h_w - \frac{P_r}{2} u^2 + \frac{P_r}{2} u U + (h_\delta - h_w) \frac{u}{U}$$

$$\text{or } \boxed{h = h_w \left(1 - \frac{u}{U}\right) + \frac{P_r}{2} u (U - u) + \frac{u}{U} h_\delta}$$

at $u = 0$, $h = h_w$ ok; at $u = U$, $h = h_\delta$ ok ✓

$$\boxed{\frac{h}{h_\delta} = \frac{h_w}{h_\delta}\left(1 - \frac{u}{U}\right) + \frac{P_r u}{2 h_\delta}(U - u) + \frac{u}{U}}$$

2-25-98 but reduced to ms form

2-19-97

use this and power law for μ / μ_δ in integral eqn also for τ_w to get τ_w: $\left(\mu / \mu_\delta = f(h / h_\delta) \right) \left(\xi \equiv \frac{u}{U} \right)$

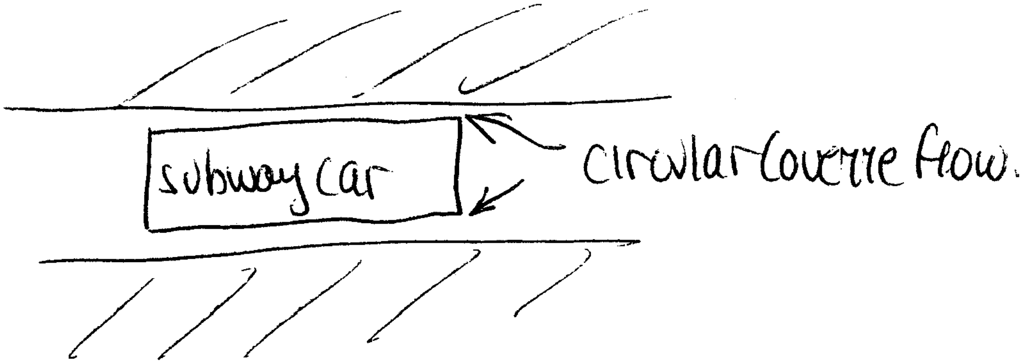
$$\tau_w = \frac{\mu_\delta}{\delta} U \int_0^1 f\left[\frac{h_w}{h_\delta}(1 - \xi) + \frac{P_r U^2}{2 h_\delta}(\xi)(1 - \xi) + \xi \right] d\xi$$

2-19-99
~~2-19-01~~
[illegible]
Mon
3-18

11-5

1st term is heat transfer from wall, rest is Mach # effects.

M is measure of energy in velocity vs. energy in enthalpy.



for a perfect gas, $h = C_pT = \frac{C_p}{\gamma R}\gamma RT = \frac{C_p}{\gamma(C_p - C_\gamma)}a^2 = \frac{a^2}{\gamma-1}$

eqn for h/h_δ becomes $\qquad h_\delta = \frac{a_\delta^2}{\gamma-1}$

$$\frac{T}{T_\delta} = \frac{T_w}{T_\delta}\left(1-\frac{u}{U}\right) + \frac{Pr}{2}\frac{U^2}{(\gamma-1)^{-1}a_\delta^2}\frac{u}{U}\left(1-\frac{u}{U}\right) + \frac{u}{U}$$

$$= \frac{T_w}{T_\delta}\left(1-\frac{u}{U}\right) + \frac{Pr}{2}(\gamma-1)M^2\left(\frac{u}{U}\right)\left(1-\frac{u}{U}\right) + \frac{u}{U}$$

but integral of power-law still hard (not analytic)

HwL "In most cases where M large, have gases, and $Pr \cong 1$
then $\mu(T)$ is the most important tricky part"

for a perfect gas, then

$$\tau_w = \frac{\mu_\delta}{\delta} U \int_0^1 f\left[\frac{T_w}{T_\delta}(1-\xi) + \frac{Pr}{2}(\gamma-1)M^2\xi(1-\xi) + \xi\right] d\xi$$

if $f(x)=x$, or linear, then can decouple the terms in the integrals

where ~~[illegible]~~, $\frac{\mu}{\mu_0} \cong \left(\frac{T}{T_0}\right)^n$

~~$\frac{\mu}{\mu_\delta} = \frac{\mu_0 \mu_0}{\mu_0 \mu_\delta} = \frac{\mu_0}{\mu_\delta}\left(\frac{T}{T_0}\right)^n = \frac{\mu_0}{\mu_\delta}\left(\frac{T_\delta}{T_0}\right)^n\left(\frac{T}{T_\delta}\right)^n$~~

~~so $f(x) = \frac{\mu_0}{\mu_\delta}\left(\frac{T_\delta}{T_0}\right)^n x^n$~~ ~~$n \cong 0.7,\ T_0 = 492°R$~~

~~$\mu_0 = .1716\ mP$~~

~~$f(x) = C_1 x^n$~~ ~~$\left(C_1 = \left(\frac{T_0}{T_\delta}\right)^n\left(\frac{T_\delta}{T_0}\right)^n = 1\right)$~~

~~$f(x) = x^n$~~

so

$$\tau_w = \frac{\mu_\delta}{\delta} U \int_0^1 \left[\frac{T_w}{T_\delta}(1-\xi) + \frac{Pr}{2}(\gamma-1)M^2\xi(1-\xi) + \xi\right]^n d\xi$$

$$[\tau_w] = \left[\mu \frac{du}{dy}\right] = \left[\mu \frac{u}{y}\right]$$

$$\bar{\tau}_w = \frac{\tau_w}{\frac{\mu_\delta U}{\delta}} = \int_0^1 \left[\frac{T_w}{T_\delta}(1-\xi) + \frac{Pr}{2}(\gamma-1)M^2\xi(1-\xi) + \xi\right]^n d\xi$$

2-15-95

$\frac{T_w}{T_\delta}$ - reg 1

~~Re - reg 2~~

~~M - reg 3~~

n - reg 4

ξ - reg 0

$\frac{Pr}{2}(\gamma-1)M^2$ - reg 2

ck prgm:
all 1's
(reg 1, 2, 4)
✓ ✓
7/16

Example 1: Does the Prandtl # affect result much? (often "no")

What are T_w & T_δ? say, heat or cool to keep $T_w = T_\delta$.

let $\gamma = 1.4$, $M = 3$, $Pr = ?$, $n = 0.7$

if $Pr = 0.7$, $\bar{\tau}_w = 1.142$ } not big
$= 1.0$, $\bar{\tau}_w = 1.20$ } effect

Obviously, as $M \to \infty$, $\bar{\tau}_w \sim (Pr)^n$ with Pr, nearly linear.

for $M \to 0$, Pr doesn't matter at all.

at $M = 10$, $\gamma = 1.4$, $n = 0.7$ $\{Pr = 0.7,\ \bar{\tau}_w = 2.29;\ Pr = 1,\ 2.75\}$

Ex2: does the power law exponent affect the result much?

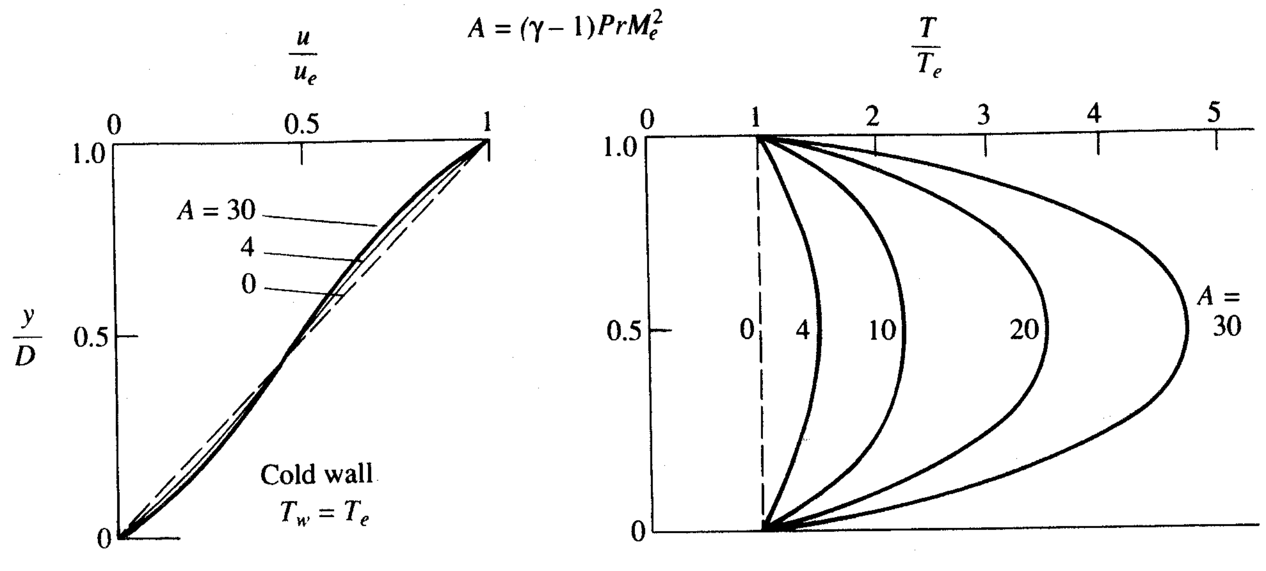
$$\cancel{\bar{\tau}_w = \int_0^1 (f(\xi))^n \, d\xi = \int_0^1 e^{n \ln f(\xi)} \, d\xi}$$

$$\cancel{\bar{\tau}_w(n \approx n_0) = \bar{\tau}_w(n_0) + \frac{d\bar{\tau}_w}{dn_0}(n - n_0) + \dots}$$

$$\cancel{\frac{\partial \bar{\tau}_w}{\partial n} = \int_0^1 f(\xi) e^{n \ln f(\xi)} \, d\xi \quad \text{hard.}}$$

if $\gamma = 1.4$, $M = 3$, $Pr = 0.7$, $n = 0.7$, $\bar{\tau}_w = 1.14$ } not a big
$n = 1.0$, $\bar{\tau}_w = 1.21$ } effect either.

From Anderson, Fundamentals of Aerodynamics,
McGraw-Hill, 2nd. ed.

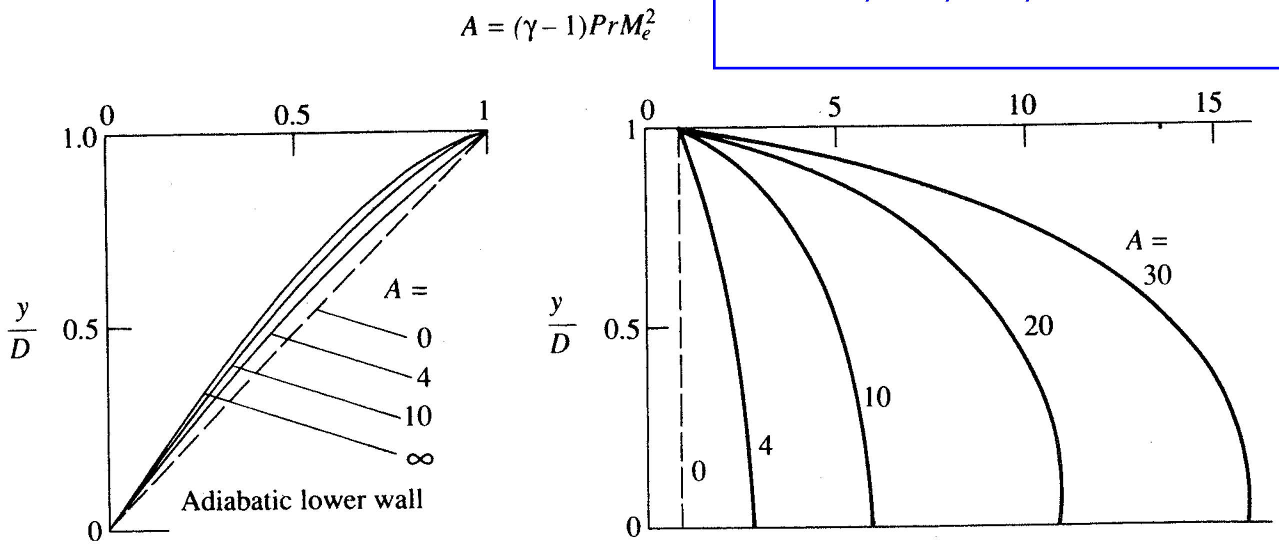


(a)

(b)

FIGURE 16.9
Velocity and temperature profiles for compressible Couette flow. Cold wall cases. (*From White, Ref. 43.*)

For gamma=1/4, Pr=0.7, this is M=0,3.8,6.0,8.5 & 10.4



(a)

(b)

FIGURE 16.10
Velocity and temperature profiles for compressible Couette flow. Adiabatic lower wall. (*From Ref. 43.*)

Power ~~Sutherland~~ law for μ.

2-2-96 2-5-989

2-5-97

1-30-95

82B

The BL equations can be solved for Blasius flow, flat plate, $\frac{\partial U}{\partial x} = 0$.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}\ ;\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Dim. analysis: $\underset{U}{u}, v, x, y, \nu$. 6 var, 2 dimensions, 4 π's

$\frac{u}{U}, \frac{v}{U}$, 2 more. $\frac{x}{y}$? no change w/ Re. $\left[\frac{\nu}{U}\right] = m$

2-26-01

Try a similiarrey solotion.

Let $\eta = \frac{y}{\sqrt{\frac{x\nu}{U}}}$. $\eta = \frac{y}{x}$ clearly won't wor

Like $\frac{y}{\sqrt{\nu t'}}$, $t' = \frac{x}{U}$

Let $u = U f(\eta)$ U = const for Blasius, but not F-S

$$\frac{\partial u}{\partial x} = f'(\eta) U\, y\sqrt{\frac{U}{\nu}}\left(-\frac{1}{2}\right)x^{-3/2} = U f'\left(-\frac{1}{2}\right)\frac{y}{\sqrt{\frac{x\nu}{U}}}\,\frac{1}{x}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\eta f' U \frac{1}{x}$$

2-9-98

$$\frac{\partial u}{\partial y} = U f' \frac{1}{\sqrt{\frac{\nu x}{U}}}$$

Note $\frac{\partial u}{\partial y} \sim \frac{1}{\sqrt{\nu}} \to \infty$ as $\nu \to 0$.

$\frac{\partial u}{\partial x}$ stays finite, <u>smaller</u>

$$\frac{\partial^2 u}{\partial y^2} = U f'' \frac{1}{\frac{\nu x}{U}} = \frac{U^2}{\nu x} f''$$

'partial integ' – <u>note</u> – along const x

$$v = -\int \frac{\partial u}{\partial x}\, dy \quad \text{from continuity.}$$

$$= -\int \left(-\tfrac{1}{2}\right) \eta f' U \tfrac{1}{x}\, dy \qquad d\eta = \frac{dy}{\sqrt{\frac{\nu x}{U}}}$$

$$= \frac{1}{2} U \frac{1}{x} \sqrt{\frac{\nu x}{U}} \int \eta f'\, d\eta \; \left[+\ \cancelto{0}{f(x)} \text{ only}\right]$$

$$= \frac{1}{2} \sqrt{\frac{U\nu}{x}} \int \eta f'\, d\eta$$

Integrate by parts $\int u\, dv = uv - \int v\, du$

$dv = f'\, d\eta \qquad u = \eta$

$v = f \qquad du = d\eta$

$$v = \frac{1}{2} \sqrt{\frac{U\nu}{x}} \left[\eta f - \int f\, d\eta \right]$$

Note that $\frac{v}{U} \sim \sqrt{\frac{\nu}{Ux}} \sim \frac{1}{Re_x^{1/2}}$ as $Re_x \to \infty$, $v \to 0$. OK.

Assemble:

$$U f\left(-\frac{1}{2}\right)\eta f' U \frac{1}{x} + \frac{1}{2}\sqrt{\frac{U\nu}{x}}\left[\eta f - \int f d\eta\right] U f' \frac{1}{\sqrt{\frac{\nu x}{U}}}$$

$$= \cancel{\nu}\frac{U^2}{\cancel{\nu} x} f''$$

Simplify.

here
2195

$$-\frac{1}{2} f\eta f' \frac{U^2}{x} + \frac{1}{2}\sqrt{\frac{U^2}{x^2}}\, U f'\left[\eta f - \int f d\eta\right] = \frac{U^2}{x} f''$$

divide out U^2/x.

$$\cancel{-\frac{1}{2} f\eta f'} + \cancel{\frac{1}{2} f'\eta f} - \frac{1}{2}\left[f'\int f d\eta\right] = f''$$

$$f'' + \frac{1}{2} f' \int f d\eta = 0$$

let $g = \int f d\eta$. $g' = f$, $g'' = f'$, $g''' = f''$

$$\boxed{g''' + \frac{1}{2} g'' g = 0}$$ Blasius Equation.

3 BC needed.

82E

B.C. For flat plate, $u=v=0$ at $\eta=0$ $(=y=0)$
(except where $x=0$)

$u(y=0)=0$

$\Rightarrow f(0)=0 \Rightarrow \boxed{g'(0)=0}$

$$v=\frac{1}{2}\sqrt{\frac{U\nu}{x}}\left(\eta g'-g\right)$$

$v(0)=0 \Rightarrow \boxed{g(0)=0}$

as $y\to\infty$, $\eta\to\infty$, $u\to U \Rightarrow f(\infty)=1$ or $\boxed{g'(\infty)=1}$

(See solution plot)

Note: $v(\infty)\neq 0$.

as $\eta\to\infty$, $g'\to 1$, $\eta\to\infty$, $g\to\infty$.

$\eta g'-g \to 1.7208$

$$v(y\to\infty)=\frac{1}{2}\sqrt{\frac{U\nu}{x}}(1.7208)$$

$$\frac{v}{U}=\frac{1.7208}{2}\sqrt{\frac{\nu}{Ux}}=\frac{1.7208}{2}\frac{1}{Re_x^{1/2}}$$

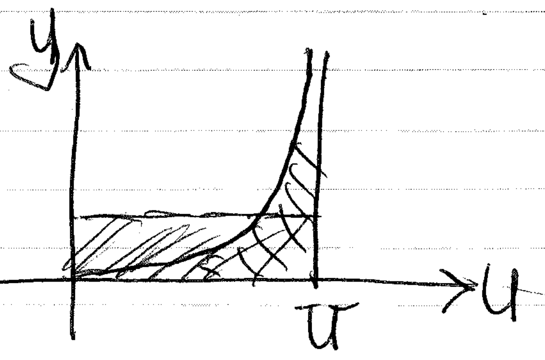
Finite movenent out.

2-28-01
3 1/2 lbt
did solns

1-7-97 show solns.
2-11-98 showed solns already.

82F

This can be accounted for by the displacement thickness:



2-5-99

$$\int_0^\infty (U-u)\,dy \equiv \delta^* U$$ - equivalent displacement of outer flow.

$$\delta^* = \int_0^\infty \left(1-\frac{u}{U}\right)dy = \sqrt{\frac{x\nu}{U}}\int_0^\infty (1-g')\,d\eta$$

$$\frac{\delta^*}{x} = \frac{1}{Re_x^{1/2}}\int_0^\infty (1-g')\,d\eta = \frac{1}{\sqrt{Re_x}}\Big[\eta - g\Big]_0^\infty$$

since $g' \to 1$ as $\eta \to \infty$, this is

$$\boxed{\frac{\delta^*}{x} = \frac{1}{\sqrt{Re_x}}\,1.7208}$$

Leads to higher order correction to potential flow - look at inviscid flow over body + displacement thickness.

2-5-96

Heat Transfer in Falkner-Skan Flows

Incompressible 2-D B.L.

Need the energy equation. Here, assume low speed, so $\frac{1}{2}u^2 << h$, and neglect viscous dissipation into heat.

Anderson HHTG (6.30) gives (6.30 in 2e and 3e also)

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + u\frac{\partial p_e}{\partial x} + \mu\left(\frac{\partial u}{\partial y}\right)^2$$

derivation of this equation is tricky, and given in 615. here, note that $\mu\left(\frac{\partial u}{\partial y}\right)^2$ is the viscous dissipation into heat (neglect here), and that

$$u\frac{\partial p_e}{\partial x} = u\frac{\partial}{\partial x}\left(-\frac{1}{2}\rho U^2\right) = -uU\frac{\partial U}{\partial x}\rho$$

$$= -\rho u\frac{\partial}{\partial x}\left(\frac{1}{2}U^2\right) << \rho u\frac{\partial h}{\partial x} \quad \text{(in magnitude)}$$

for low speed, so neglect it. (see HW1, Ae101b Final, 1984)

we now have, assuming $h = c_p T$, $c_p = \text{const}$,

$$\rho c_p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) = k \frac{\partial^2 T}{\partial y^2}$$

which is consistent with White (4-35c)

From the momentum-equation solution, we have u & v. Here, we assume that the changes in T do not affect u & v – a poor assumption in many cases, since $\mu = \mu(T)$. Here we assume μ = const, etc.

we have $u = U f'(\eta)$, $\eta = y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}$, and

$$v = -\sqrt{\frac{2\nu K}{m+1}}\left[\frac{m+1}{2} x^{\frac{m-1}{2}} f + x^{\frac{m+1}{2}} \frac{\partial \eta}{\partial x} f'\right]$$

following White, let $\theta = \frac{T - T_e}{T_w - T_e}$, so $\frac{\partial \theta}{\partial x} = \frac{1}{T_w - T_e}\frac{\partial T}{\partial x}$

ρ & c_p are constant. So

$$\rho c_p U f' \cancel{(T_w - T_e)} \frac{\partial \theta}{\partial x}$$

$$+ \rho c_p \cancel{(T_w - T_e)} \frac{\partial \theta}{\partial y}\left[(-)\sqrt{\frac{2\nu K}{m+1}}\right]\left[\frac{m+1}{2} x^{\frac{m-1}{2}} f + x^{\frac{m+1}{2}} \frac{\partial \eta}{\partial x} f'\right]$$

$$= \cancel{(T_w - T_e)} \frac{\partial^2 \theta}{\partial y^2} k$$

now $\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\theta(\eta(x,y))\right) = \frac{d\theta}{d\eta}\frac{\partial \eta}{\partial x}$, etc,

so we have

$$\rho c_p \bar{U} f' \theta' \frac{\partial \eta}{\partial x}$$

$$- \rho c_p \theta' \frac{\partial \eta}{\partial y}\left[\sqrt{\frac{2\nu k}{m+1}}\right]\left[\frac{m+1}{2} x^{\frac{m-1}{2}} f + x^{\frac{m+1}{2}} \frac{\partial \eta}{\partial x} f'\right]$$

$$= \frac{d^2\theta}{d\eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 k = \theta''\left(\frac{m+1}{2}\frac{\bar{U}}{\nu x}\right) k$$

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$$\bar{U} f' \theta' \frac{\partial \eta}{\partial x} - \theta' \sqrt{\frac{\cancel{m+1}}{\cancel{2}}\frac{\bar{U}}{\cancel{\nu} x}}\sqrt{\frac{\cancel{2\nu} k}{\cancel{m+1}}}\;\frac{m+1}{2} x^{\frac{m-1}{2}} f$$

$$- \theta' \sqrt{\frac{\cancel{m+1}}{\cancel{2}}\frac{\bar{U}}{\cancel{\nu} x}}\sqrt{\frac{\cancel{2\nu} k}{\cancel{m+1}}}\; x^{\frac{m+1}{2}} \frac{\partial \eta}{\partial x} f' = \theta'' \frac{m+1}{2}\frac{\bar{U}}{\nu x}\frac{k}{c_p \rho}$$

$$\bar{U} f' \theta' \frac{\partial \eta}{\partial x} - \theta' f \sqrt{\frac{\bar{U}}{x}}\sqrt{k}\, x^{m/2}\, \frac{m+1}{2\sqrt{x}}$$

$$- \theta' \sqrt{\frac{U}{\cancel{x}}}\sqrt{k}\, x^{m/2} \cancel{\sqrt{x}} \frac{\partial \eta}{\partial x} f' = \theta'' \frac{m+1}{2}\frac{\bar{U}}{\nu x}\frac{k}{\rho c_p}$$

$$\cancel{U f'\theta' \frac{\partial\eta}{\partial x}} - f\theta'\sqrt{\frac{u}{x}}\sqrt{U}\frac{1}{\sqrt{x}}\frac{m+1}{2} f$$

$$-\cancel{f'\theta'\sqrt{U}\sqrt{U}\frac{\partial\eta}{\partial x}} = \frac{k}{\rho c_p}\frac{m+1}{2}\frac{U}{\nu x}\theta''$$

$$0 = \frac{k}{\rho c_p}\cancel{\frac{m+1}{2}}\cancel{\frac{u}{\nu x}}\theta'' + \theta' f\cancel{\frac{u}{x}}\cancel{\frac{m+1}{2}}$$

$$0 = \frac{k}{\cancel{\rho} c_p}\frac{\cancel{\rho}}{\mu}\theta'' + f\theta'$$

$$0 = \theta'' + \frac{\mu c_p}{k} f\theta' \; ; \; Pr \equiv \frac{\mu c_p}{k}$$

$$\boxed{0 = \theta'' + Pr\, f\theta'}$$ agrees with White (4-76)

note that $u = U(x) f'(\eta) \equiv \frac{\partial\psi}{\partial y}$, ψ = stream fn. (e.g. White 4-44)

so $\psi = \int U(x) f'(\eta)\, dy \Big|_{x \text{ held const.}}$

$$= U(x)\frac{1}{\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}}\int f'(\eta)\, d\eta$$

or $\psi = \sqrt{\frac{2}{m+1} \nu x U(x)} \, f$ (agrees with White 4-77)

Now f is assumed constant, known from F-S soln.
so

$$\frac{d^2\theta}{d\eta^2} + Pr\, f(\eta) \frac{d\theta}{d\eta} = 0 \; ; \text{ let } \alpha = \frac{d\theta}{d\eta}$$

$$\frac{d\alpha}{d\eta} = -Pr f \alpha \Rightarrow \frac{d\alpha}{\alpha} = -Pr f d\eta$$

$$\ln \alpha = -Pr \int f d\eta + \text{const}$$

$$\alpha = \text{const}\; e^{-Pr\int f d\eta} = \frac{d\theta}{d\eta}$$

$$\theta = \text{const} \int e^{-Pr\int f d\eta} d\eta + \text{const.}$$

at $\eta = 0$, $\theta = 1$, since $T = T_w$

$\eta = \infty$ $\theta = 0$, since $T = T_e$

let

$$\theta = \text{const} \int e^{-Pr\int_0^\eta f d\eta} d\eta + \text{const}$$

by inspection, White (4-58) fits the B.C.,

$$\boxed{\theta = \frac{\int_\eta^\infty \exp\left(-Pr\int_0^\eta f d\eta\right) d\eta}{\int_0^\infty \exp\left(-Pr\int_0^\eta f d\eta\right) d\eta}}$$

$$q_w = -k\frac{\partial T}{\partial y}\Big|_{y=0} = -k(T_w - T_e)\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial y}\Big|_{y=0}$$

$$q_w = -k(T_w - T_e)\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}\;\frac{d\theta}{d\eta}\Big|_{\eta=0} \quad \sim \underline{\underline{\frac{1}{\sqrt{x}}}}$$

$$Nu_x \equiv \frac{q_w x}{k\cancel{x}(T_w - T_e)} = -\theta'(0)\sqrt{\frac{m+1}{2}}\sqrt{\frac{Ux}{\nu}}$$

$$Nu_x = -\theta'(0)\sqrt{\frac{m+1}{2}}\,Re_x^{1/2}$$

$$\frac{d\theta}{d\eta}\Big|_{\eta=0} = \theta'(0) = \frac{1}{\int_0^\infty ()d\eta}\frac{d}{d\eta}\left[\int_\eta^\infty \exp\left(-Pr\int_0^\eta f(s)ds\right)d\eta\right]$$

$$\left.\frac{d\theta}{d\eta}\right|_{\eta=0} = \frac{1}{\int_0^\infty e^{-Pr\int_0^\eta f(s)ds}\, d\eta} \,.$$

$$\left[(-)\; e^{-Pr\int_0^\eta f(s)ds}\right]_{\eta=0} = \frac{-1}{\int_0^\infty e^{-Pr\int_0^\eta f(s)ds}\, d\eta}$$

so

$$\boxed{Nu_x = \frac{\sqrt{\frac{m+1}{2}}\; Re_x^{1/2}}{\int_0^\infty e^{-Pr\int_0^\eta f(s)ds}\, d\eta}} \qquad \left(\text{cp. White } 4\text{-}78\right)$$

workout <u>Numerically</u>.

Stanton number

$$\left(\text{White } 4\text{-}22\right) \quad C_h = \frac{q_w}{\rho U c_p (T_w - T_e)} = \frac{Nu_x\, k \cancel{(T_w-T_e)}}{x} \, \frac{1}{\rho U c_p \cancel{(T_w-T_e)}}$$

$$= \frac{Nu_x\, k}{x} \, \frac{\nu}{\rho U c_p \, \mu/\cancel{\rho}} = \frac{Nu_x}{Re_x\, Pr}$$

$$C_h = Nu_x \, Re_x^{-1} \, Pr^{-1}$$

$$C_h = \frac{\sqrt{\frac{m+1}{2}} \, Re_x^{-1/2} \, Pr^{-1}}{\int_0^\infty \exp\left(-Pr \int_0^\eta f(s)\,ds\right) d\eta} \equiv \frac{\sqrt{\frac{m+1}{2}} \, Re_x^{-1/2} Pr^{-1}}{I}$$

both C_h & C_f scale with $Re_x^{-1/2}$

$$\frac{C_f}{C_h} = \frac{f''(0)\sqrt{2(\cancel{m+1})} \; \cancel{Re_x^{-1/2}}}{\sqrt{\frac{\cancel{m+1}}{2}} \; \cancel{Re_x^{-1/2}} \, Pr^{-1}} \, I$$

$$\frac{C_f}{C_h} = f''(0) \, Pr \cdot 2 I$$

$$\boxed{\frac{C_f}{C_h} = 2 f''(0) \, Pr \int_0^\infty \exp\left(-Pr \int_0^\eta f(s)\,ds\right) d\eta}$$

again evaluate numerically. (for PS5)

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Crocco-Busemann Integrals

Derived in 613

The 2-D boundary layer equations for compressible flow can be shown to be (steady flow, Anderson p. 226)
(p. 276 in 2e. eqn no. unchanged)

Continuity $\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0$ (A 6.27)

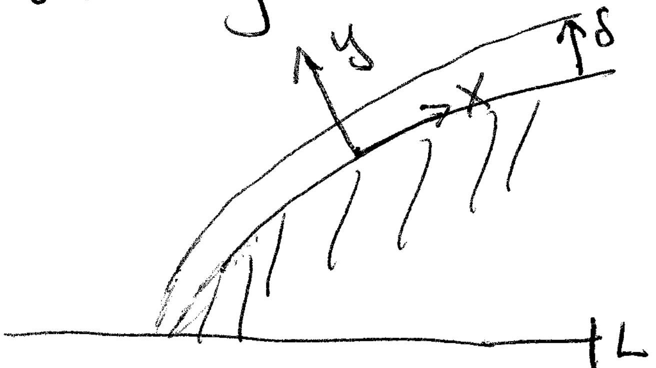
x-momentum $\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$ (A 6.28)

y-momentum $\frac{\partial p}{\partial y}=0$ (A. 6.29) Not always true for hypersonic

Energy: $\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=u\frac{\partial p}{\partial x}+\mu\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$

(A 6.30)

x-momentum is similar to the incompressible equations. 416 covers B.L. theory, incl. some compressible flow, and cannot be reproduced here. B.L. theory assumes
$\frac{\partial}{\partial x}<<\frac{\partial}{\partial y}$



$\delta<<L,\ \frac{\partial}{\partial x}<<\frac{\partial}{\partial y}$

get a form of the energy equation that replaces h with $H = h + \frac{1}{2}u^2$, the total enthelpy.

take momentum, multiply by u, and add to energy

$$\rho u\left(\frac{\partial h}{\partial x} + u\frac{\partial u}{\partial x}\right) + \rho v\left(\frac{\partial h}{\partial y} + u\frac{\partial u}{\partial y}\right) = u\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$

note $u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(u^2/_2\right)$

$$\frac{\partial}{\partial y}\left(\mu u\frac{\partial u}{\partial y}\right) = u\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \mu\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$$, so

$$\boxed{\rho u\frac{\partial H}{\partial x} + \rho v\frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\mu u\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)}$$

$Pr =$ diff. $\left(\frac{\text{mom}}{\text{heat}}\right)$

Note $Pr \equiv \frac{\mu c_p}{k}$, $dh = c_p dT$ perfect gas, c_p not const.

$$\rho u\frac{\partial H}{\partial x} + \rho v\frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left[\mu u\frac{\partial u}{\partial y} + \frac{\mu c_p}{Pr}\frac{\partial T}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}\left[\mu u\frac{\partial u}{\partial y} + \frac{\mu}{Pr}\frac{\partial h}{\partial y}\right]$$

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Now $\frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(h + \frac{1}{2}u^2\right) = \frac{\partial h}{\partial y} + u\frac{\partial u}{\partial y}$, so

$$\rho u\frac{\partial H}{\partial x} + \rho v\frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\mu}{Pr}\left(\frac{\partial H}{\partial y} - u\frac{\partial u}{\partial y}\right) + \mu u\frac{\partial u}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{\mu}{Pr}\frac{\partial H}{\partial y} + u\frac{\partial u}{\partial y}(\mu)\left(1 - \frac{1}{Pr}\right)\right]$$

$$\boxed{\rho u\frac{\partial H}{\partial x} + \rho v\frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left[\left(1 - \frac{1}{Pr}\right)\mu u\frac{\partial u}{\partial y}\right]}$$

White, Viscous Fluid Flow, 2e, (7-8)

If $Pr = 1$, 1st Crocco-Busemann integral:

H = const is a solution of this equation.

$$\Rightarrow \frac{\partial}{\partial y}(H) = 0 = \frac{\partial}{\partial y}\left(h + \frac{1}{2}u^2\right) = \frac{\partial h}{\partial y} + u\frac{\partial u}{\partial y}$$

at wall, $u = 0$, so $\left.\frac{\partial h}{\partial y}\right|_w = 0$, $\Rightarrow$ adiabatic wall.

$$\boxed{\text{so } Pr = 1, \text{ adiabatic wall}, \Rightarrow H = h + \tfrac{1}{2}u^2 = \text{const.}}$$

For gases, $Pr \cong 0.7$, this is a fair approx.

Note that $k \neq 0$! $Pr = 1$ means heat & momentum have same diffusivity.

Cp. to BLUNT Code arper!

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For $Pr=1$, $\frac{\partial p}{\partial x}=0$, NOT adiabatic wall, 2nd Crocco-Busemann integral: (steady)

mom: $\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$

energy: $\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\mu}{\cancelto{1}{Pr}}\frac{\partial h}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2$

Same equation almost, except for dissipation term.

Try $h=h(u)$ as a solution: $\frac{\partial h}{\partial y}=\frac{dh}{du}\frac{\partial u}{\partial y}$

energy becomes:

$$\rho u\frac{dh}{du}\frac{\partial u}{\partial x}+\rho v\frac{dh}{du}\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{dh}{du}\frac{\partial u}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

use momentum on l.h.s.

$$\cancel{\frac{dh}{du}\left[\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right]}=\cancel{\frac{dh}{du}\left[\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right]}+\mu\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(\frac{dh}{du}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

$$0=\mu\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(\frac{dh}{du}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

Note: $\frac{\partial}{\partial y}\left(\frac{dh}{du}(u(x,y))\right)=\frac{d^2h}{du^2}\frac{\partial u}{\partial y}$

$$0 = \mu \frac{\partial u}{\partial y} \frac{d^2 h}{du^2} \frac{\partial u}{\partial y} + \mu \left(\frac{\partial u}{\partial y}\right)^2$$

$$0 = \mu \left(\frac{\partial u}{\partial y}\right)^2 \left[\frac{d^2 h}{du^2} + 1\right]$$

Satisfied iff $\frac{d^2 h}{du^2} = -1 \Rightarrow \frac{dh}{du} = -u + C_1$

$$h = -\frac{u^2}{2} + C_1 u + C_2$$

Since $u = 0$ at wall, $h = h_w = C_2$

Since $h = h_{ext} \equiv h_e$ at $u = U_e$, $h_e = -\frac{u_e^2}{2} + C_1 u_e + h_w$

$$C_1 = \frac{h_e - h_w + u_e^2/2}{u_e} \; ; \quad h + \frac{u^2}{2} = C_1 u + C_2$$

So 2nd Crocco-Busemann integral is

$$\boxed{Pr = 1,\ \frac{dp}{dx} = 0,\ H = h + \frac{u^2}{2} = h_w + \left(\frac{u}{U_e}\right)\left(h_e - h_w + \frac{u_e^2}{2}\right)}$$

or $\boxed{H = h_w + \frac{u}{U_e}\left(H_e - h_w\right)}$

So H varies linearly with u across B.L.

if $H_e = h_w = H = \text{const}$, recover 1st Crocco-Busemann.

Note: recovery factor (empirical)

if $h = c_p T$, $c_p = \text{const}$, $Pr = 1$, $dp/dx = 0$,

$$T = T_w + \left(T_e + \frac{u_e^2}{2c_p} - T_w\right)\frac{u}{u_e} - \frac{u^2}{2c_p}$$

if wall is adiabatic, $H = \text{const} \Rightarrow H_w = H_e \Rightarrow$

$$T_e + \frac{u_e^2}{2c_p} = T_{aw}, \text{ so}$$

$$T = T_w + (T_{aw} - T_w)\frac{u}{u_e} - \frac{u^2}{2c_p}$$

semi-empirical correction factor, for $Pr \neq 1$, etc, is r,

$$\boxed{T = T_w + (T_{aw} - T_w)\frac{u}{u_e} - r\frac{u^2}{2c_p}}$$

if $T_w < T_{aw}$, this term cause T to increase with u away from wall.

If $T_w > T_{aw}$, this term causes T to decrease with u away from wall.

note $+q_w = k_w \frac{\partial T}{\partial y}\Big|_w = \left(-\frac{T_{aw} - T_w}{u_e}\frac{\partial u}{\partial y}\Big|_w + \frac{r}{c_p}2\cancelto{0}{u_w}\frac{\partial u}{\partial y}\Big|_w\right)k$

$$q_w = -\frac{T_{aw} - T_w}{u_e \mu_w} \tilde{\tau}_w K$$

q_w is ~ τ_w, Reynolds analogy.

also, +q_w proportional to $T_w - T_{aw}$ — T_w has to be more than T_{aw} to ~~radia~~ conduct heat into the fluid. Very hard at hypersonic speeds.

Next, do B.L. similarity equations, leading to stagnation-point solution for laminar perfect-gas flow.

Jan 01 – First, do Falkner-Skan. (Do Blasius as warmup.) inclass

Do main solution for τ_w, similar to 416

Do heat transfer solution inclass.

3-7-01

4 Feb 99
SPS

Cf White sec. 7-2, Schlichting,
Anderson sec. 6.5

Illingworth-Stewartson .... Transformations for Compressible B.L.

(steady flow) 2D

modified from 1990 613 notes, to get early G fx, needed for $u=f'(\eta)$ statement but omitted earlier.

Continuity: $\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$

let $\rho u = \frac{\partial \psi}{\partial y}$ and $\rho v = -\frac{\partial \psi}{\partial x}$, to satisfy identically.

Try similarity variables on PDE's — one for streamfunction and one for compressibility effects.

$$\text{let } \psi(\xi,\eta) = \int \rho u\, dy \equiv G(\xi) f(\eta) = \psi(x,y)$$

↑ differs from Anderson, who allows $f(\xi,\eta)$

use Illingworth transformation,

$$\xi \equiv \int_0^x \rho_e(x) u_e(x) \mu_e(x)\, dx = \xi(x) \text{ only (simplifies)}$$

$$\eta \equiv \frac{u_e}{\sqrt{2\xi}} \int_0^y \rho\, dy = \eta(x,y)$$

(typo in Anderson 6.34!)

First, get $u(\xi,\eta) = \frac{1}{\rho}\frac{\partial \psi}{\partial y}$

$$\frac{\partial \psi}{\partial y} = G(\xi)\frac{df}{d\eta}\frac{\partial \eta}{\partial y} = G(\xi)\frac{df}{d\eta}\frac{u_e}{\sqrt{2\xi}}\rho$$

so $u(\xi,\eta) = \frac{1}{\cancel{\rho}} G(\xi)\, f'(\eta) \frac{u_e}{\sqrt{2\xi}} \cancel{\rho}$

let $G(\xi)=\sqrt{2\xi}$ so this simplifies, and because it works. (see Anderson 'Step III')

the $u(\xi,\eta) = u_e(\xi)\, f'(\eta)$

Now substitute into steady-flow momentum eqn. (2D)

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p_e}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \qquad \left(p = p_e \text{ because } \frac{\partial}{\partial y} p = 0\right)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} \quad \text{since } u = u(\eta(x,y), \xi(x))$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \quad \text{since } \frac{\partial \xi}{\partial y} = 0$$

$\frac{\partial \eta}{\partial x}$ will drop out (see Anderson, and watch), so don't evoluate, messy.

$\frac{\partial \eta}{\partial y} = \frac{u_e}{\sqrt{2\xi}}\rho$, $\frac{\partial \xi}{\partial x} = \rho_e u_e \mu_e$, and

$$\frac{\partial u}{\partial \eta} = u_e f''(\eta), \quad \frac{\partial u}{\partial \xi} = u_e'(\xi)\, f'(\eta)$$

substitute

$$\rho u_e f' \left[u_e f'' \frac{\partial \eta}{\partial x} + u_e' f' \rho_e u_e \mu_e \right]$$

$$- \frac{\partial \psi}{\partial x} \left[u_e f'' \frac{u_e}{\sqrt{2\xi}} \rho \right] = - \frac{\partial p_e}{\partial x} + \frac{\partial}{\partial y} \left(\mu u_e f'' \frac{u_e}{\sqrt{2\xi}} \rho \right)$$

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Now $p_e + \frac{1}{2} \rho_e u_e^2 = \text{const}$, so

but no good for curved shock. ($\rho_e \neq \text{const}$) is ok

~~flat plate only~~

$$\frac{\partial p_e}{\partial x} = - \rho_e u_e \frac{d u_e}{dx}$$

(from x-mom with $\frac{\partial}{\partial y} = 0$ & $\mu = 0$)

Euler along streamline (A., Fund. sec 3.2)

Now $\frac{d u_e(x)}{dx} = \frac{d}{dx}\left(u_e(\xi(x)) \right) = \frac{d u_e}{d\xi} \rho_e u_e \mu_e$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \psi}{\partial \xi} \frac{\partial \xi}{\partial x}, \quad \psi = G(\xi) f(\eta) = \sqrt{2\xi}\, f(\eta)$$

$$\frac{\partial \psi}{\partial x} = \sqrt{2\xi}\, f'(\eta) \frac{\partial \eta}{\partial x} + \frac{1}{2} (2\xi)^{-1/2} 2 f(\eta) \rho_e u_e \mu_e$$

$$\frac{\partial \psi}{\partial x} = \sqrt{2\xi}\, f' \frac{\partial \eta}{\partial x} + \frac{\rho_e u_e \mu_e}{\sqrt{2\xi}} f$$

plug these in to momentum

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$$\cancel{\rho u_e^2 f' f'' \frac{\partial \eta}{\partial x}} + \rho u_e (f')^2 u_e' \rho_e u_e \mu_e$$

$$-\cancel{\sqrt{2\xi}\, f' \frac{\partial \eta}{\partial x} \frac{\rho u_e^2}{\sqrt{2\xi}}} f'' - \frac{\rho_e u_e \mu_e}{\sqrt{2\xi}} f \frac{\rho u_e^2}{\sqrt{2\xi}} f''$$

$$= \rho_e u_e \frac{d u_e}{dx} + \frac{u_e^2}{\sqrt{2\xi}} \frac{\partial}{\partial y}\left(\mu \rho f''\right)$$

$\frac{\partial \eta}{\partial x}$ terms cancel! Define $\frac{\rho \mu}{\rho_e \mu_e} \equiv C$, Chapman-Rubesin parameter

$$\rho u_e u_e' \rho_e u_e \mu_e (f')^2 - \frac{\rho u_e^2 \rho_e u_e \mu_e}{2\xi} f f''$$

$$= \rho_e u_e^2 \frac{1}{u_e} \frac{d u_e}{dx} + \rho_e \mu_e \frac{u_e^2}{\sqrt{2\xi}} \frac{\partial}{\partial y}\left(\frac{\rho \mu}{\rho_e \mu_e} f''\right)$$

for last term, note

$$\frac{\partial J(\eta,\xi)}{\partial y} = \frac{\partial J}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial J}{\partial \xi}\cancelto{0}{\frac{\partial \xi}{\partial y}} = \frac{\partial J}{\partial \eta}\frac{\rho u_e}{\sqrt{2\xi}}, \text{ so}$$

$$\frac{\partial}{\partial y}\left(\frac{\rho \mu}{\rho_e \mu_e} f''\right) = \frac{\partial}{\partial \eta}\left(C f''\right)\frac{\rho u_e}{\sqrt{2\xi}}$$

Substitute & divide through by u_e^2

$$\rho u_e' \cancel{\rho_e} \cancel{\mu_e}(f')^2 - \frac{\rho \cancel{\rho_e} u_e \cancel{\mu_e}}{2\xi} f f''$$

$$= \cancel{\rho_e} \frac{1}{\cancel{u_e}} \frac{du_e}{d\xi} \rho_e \cancel{u_e} \cancel{\mu_e} + \frac{\cancel{\rho_e \mu_e}}{\sqrt{2\xi}} \frac{\rho u_e}{\sqrt{2\xi}} \frac{\partial}{\partial \eta}(C f'')$$

(divide by $\rho_e \mu_e$), so

$$\rho u_e' (f')^2 - \frac{\rho u_e}{2\xi} f f'' = \rho_e \frac{du_e}{d\xi} + \frac{\rho u_e}{2\xi}(C f'')'$$

multiply by $\frac{2\xi}{\rho u_e}$ & switch sides to put leading term on left.

$$(C f'')' + \rho_e \frac{du_e}{d\xi} \frac{2\xi}{\rho u_e} - \rho \frac{du_e}{d\xi}(f')^2 \frac{2\xi}{\rho u_e} + \frac{\cancel{\rho u_e}}{\cancel{2\xi}} f f'' \cdot \frac{\cancel{2\xi}}{\cancel{\rho u_e}} = 0$$

$$(C f'')' + \frac{2\xi \rho_e}{u_e \rho} \frac{du_e}{d\xi} - \frac{2\xi}{u_e} \frac{du_e}{d\xi}(f')^2 + f f'' = 0$$

$$\boxed{(C f'')' + \frac{2\xi}{u_e} \frac{du_e}{d\xi}\left(\frac{\rho_e}{\rho} - (f')^2\right) + f f'' = 0}$$

Ck. White^{2e} eqn. 7-20, Anderson 6.55 (here $f = f(\eta)$ only is already assumed)

B.C. (solid wall) $v(y=0)=0$

$$\rho v = -\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} \quad -\left[\sqrt{2\xi}\, f' \frac{\partial \eta}{\partial x} + \frac{\rho_e u_e \mu_e}{\sqrt{2\xi}} f\right]$$

recall $\eta = \frac{u_e}{\sqrt{2\xi}} \int_0^y \rho\, dy$ (typo in Anderson 6.34!)

so at $y=0$, $\eta=0$

in general, for $v(\eta=0)=0$, $\boxed{f'(0)=f(0)=0}$

no-slip: $u(\eta=0)=0=u_e f'(\eta=0)$, also works with the above.

Freestream match:

$$u(\xi,\eta) \to u_e(\xi) \text{ as } \eta \to \infty, \quad u = u_e f'$$

$$\Rightarrow \boxed{f'(\eta \to \infty) \to 1}$$

Notes: if $M \to 0$, μ, ρ, u_e const, $C=1$, Blasius, get

(White 4-45) $$f''' + f f'' = 0 \quad \left(\frac{du_e}{d\xi}=0\right)$$

$$\xi = \rho \mu u_e x, \quad \eta = \frac{u_e}{\sqrt{2\rho\mu u_e x}}\, \rho y = y\sqrt{\frac{\rho u_e}{2\mu x}} = y\sqrt{\frac{u}{2\nu x}}$$

reduces to Blasius

(White 442

Energy Equation: try $h(x,y) = h_e(\xi)g(\eta)$

again differs from Anderson, $g = g(\eta)$ only.

steady, 2D,

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u \frac{\partial p_e}{\partial x} + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right)$$

but $dh = c_p dT$ for perfect gas, and $\frac{k}{c_p \mu} = \frac{1}{Pr}$, so $\frac{k}{c_p} = \frac{\mu}{Pr}$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u \frac{\partial p_e}{\partial x} + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\frac{\mu}{Pr} \frac{\partial h}{\partial y}\right)$$

again, $\frac{\partial p_e}{\partial x} = -\rho_e u_e \frac{du_e}{dx} = -\rho_e u_e \frac{du_e}{d\xi} \rho_e u_e \mu_e$

and $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial h}{\partial \xi}\frac{\partial \xi}{\partial x} = h_e g'(\eta)\frac{\partial \eta}{\partial x} + \underbrace{h_e' g \rho_e u_e \mu_e}_{\text{Anderson ~~drops this term. Not clear why.~~ not in 5th printing}}$

$u = u_e f'$ from before

$\rho v = -\frac{\partial \psi}{\partial x} = -\left[\sqrt{2\xi} f' \frac{\partial \eta}{\partial x} + \frac{\rho_e u_e \mu_e}{\sqrt{2\xi}} f\right]$ from before

$\frac{\partial h}{\partial y} = \frac{\partial h}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial h}{\partial \xi}\cancel{\frac{\partial \xi}{\partial y}}^{\to 0} = h_e g' \frac{u_e}{\sqrt{2\xi}} \rho$ $\boxed{\partial\eta/\partial y \text{ from before}}$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial \xi} + \frac{\partial u}{\partial \xi}\cancel{\frac{\partial \xi}{\partial y}}^{\to 0} = u_e f'' \frac{u_e}{\sqrt{2\xi}} \rho$

-3-19-01-

plug in

$$\rho u_e f' \left[h_e g' \frac{\partial \eta}{\partial x} + h_e' g \rho_e u_e \mu_e \right]$$

$$- \left[\sqrt{2\xi} f' \frac{\partial \eta}{\partial x} + \frac{\rho_e u_e \mu_e}{\sqrt{2\xi}} f \right] h_e g' \frac{u_e}{\sqrt{2\xi}} \rho$$

$$= u_e f' \left[-\rho_e^2 u_e^2 \mu_e \frac{d u_e}{d \xi} \right] + \mu u_e^2 (f'')^2 \frac{u_e^2}{2\xi} \rho^2$$

$$+ \frac{\partial}{\partial y} \left(\frac{\mu}{Pr} h_e g' \frac{u_e}{\sqrt{2\xi}} \rho \right)$$

2-999 — w/ $\frac{\partial \eta}{\partial x}$ dropped (canceled)

remember $\frac{\partial J}{\partial y} = \frac{\partial J}{\partial \eta} \frac{\rho u_e}{\sqrt{2\xi}}$, so

$$\cancel{\rho u_e f' h_e g' \frac{\partial \eta}{\partial x}} + \rho u_e^2 f' h_e' g \rho_e \mu_e$$

$$- \cancel{f' \frac{\partial \eta}{\partial x} h_e g' u_e \rho} - \frac{\rho_e u_e^2 \mu_e}{2\xi} f h_e g' \rho$$

$$= - \rho_e^2 u_e^3 \mu_e f' \frac{d u_e}{d \xi} + \frac{\mu u_e^4 \rho^2 (f'')^2}{2 \xi}$$

$$+ \frac{\partial}{\partial \eta} \left(\frac{\mu \rho}{Pr} g' \right) \frac{h_e u_e}{\sqrt{2\xi}} \frac{\rho u_e}{\sqrt{2\xi}}$$

$\frac{\partial\eta}{\partial x}$ terms again cancel out (part of reason for $\sqrt{2\xi}$ defn.)

again define $C \equiv \frac{\rho\mu}{\rho_e \mu_e}$

$$\rho \cancel{u_e^2} f' h_e' g \rho_e \mu_e - \rho \cancel{u_e^2} \rho_e \mu_e h_e \frac{fg'}{2\xi}$$

$$= -\rho_e^2 u_e^{\cancel{3}\,1} \mu_e f' \frac{du_e}{d\xi} + \frac{\mu u_e^{\cancel{4}\,2} \rho^2 (f'')^2}{2\xi} + \left(\frac{C}{Pr} g'\right)' \frac{\rho \cancel{u_e^2} h_e \rho_e \mu_e}{2\xi}$$

divide by u_e^2. multiply by 2ξ

$$\rho h_e' \rho_e \mu_e (f'g) 2\xi - \rho \rho_e \mu_e h_e f g'$$

$$= -\rho_e^2 u_e \mu_e \frac{du_e}{d\xi} f' 2\xi + \mu u_e^2 \rho^2 (f'')^2 + \left(\frac{C}{Pr} g'\right)' \rho h_e \mu_e \rho_e$$

Switch sides & move terms

$$\left(\frac{C}{Pr} g'\right)' \rho_e \rho h_e \mu_e + \rho \rho_e \mu_e h_e f g'$$

$$= 2\xi \left[\rho \rho_e h_e' \mu_e f' g + \rho_e^2 u_e \mu_e \frac{du_e}{d\xi} f'\right] - \rho^2 \mu u_e^2 (f'')^2$$

divide by $\rho_e \rho h_e \mu_e$

$$\left(\frac{C}{Pr}g'\right)' + fg' = 2\xi\left[\frac{h_e' \mu_e f' g}{h_e \mu_e} + \frac{\rho_e u_e \frac{du_e}{d\xi} f'}{\rho h_e}\right] - \frac{\rho}{\rho_e}\frac{\mu}{\mu_e}\frac{u_e^2}{h_e}(f'')^2$$

$$\boxed{\left(\frac{C}{Pr}g'\right)' + fg' = 2\xi\left[\underbrace{\frac{h_e'}{h_e}f'g}_{} + \frac{\rho_e u_e}{\rho h_e}\frac{du_e}{d\xi}f'\right] - C\frac{u_e^2}{h_e}(f'')^2}$$

missing 2nd printing: present 5th printing: Anderson ~~6.58 is missing (this term)~~ (2nd printing) (~~but~~ present in 5th printing)

he has $\frac{\partial g}{\partial \xi}$ & $\frac{\partial f}{\partial \xi}$ terms that are both zero by definition/assumption here.

In general, there is no reason why $\frac{dh_e}{d\xi} = 0$.

Lees (1956) assumes $\frac{dH_e}{d\xi} = 0$, which is reasonable, although not true where heat conduction in the entropy layer is significant.

also differs from White (7-25)

(6.54), we finally obtain

$$\boxed{(Cf'')' + ff'' = \frac{2\xi}{u_e}\left[(f')^2 - \frac{\rho_e}{\rho}\right]\frac{du_e}{d\xi} + 2\xi\left(f'\frac{\partial f'}{\partial \xi} - \frac{\partial f}{\partial \xi}f''\right)} \tag{6.55}$$

Eq. (6.55) is the transformed boundary layer x-momentum equation for a two-dimensional, compressible flow.

The boundary layer y-momentum equation, namely Eq. (6.29) stating that $\partial p/\partial y = 0$ becomes in the transformed space

$$\boxed{\frac{\partial p}{\partial \eta} = 0} \tag{6.56}$$

The boundary layer energy equation given by Eq. (6.30) can also be transformed. Defining a nondimensional static enthalpy as

$$g = g(\xi, \eta) = \frac{h}{h_e} \tag{6.57}$$

where h_e is the static enthalpy at the boundary layer edge, and utilizing the same transformation as before, Eq. (6.30) becomes

$$\boxed{\left(\frac{C}{\mathrm{Pr}}g'\right)' + fg' = 2\xi\left[f'\frac{\partial g}{\partial \xi} + \frac{f'g}{h_e}\frac{\partial h_e}{\partial \xi} - g'\frac{\partial f}{\partial \xi} + \frac{\rho_e u_e}{\rho h_e}f'\frac{du_e}{d\xi}\right] - C\frac{u_e^2}{h_e}(f'')^2} \tag{6.58}$$

Anderson, Hypersonic and High-Temp. Gasdynamics, 5th printing

— here fri 3-30-90 —

~~$$\frac{\partial}{\partial\eta}\left(\frac{Cg'}{Pr}\right) + fg' = \frac{2\xi f'}{h_e}\left[\frac{\rho_e u_e}{\rho}\frac{du_e}{d\xi} + g\frac{dh_e}{d\xi}\right] - \frac{C}{h_e}u_e^2(f'')^2$$~~

~~energy equation (does not check White 7-25)~~

G1. Flat Plate in Compressible Flow

for flat plate (or cone) $\frac{dh_e}{d\xi} = 0$, $\frac{du_e}{d\xi} = 0$ (constant outer conditions)

eqns become:

$$\boxed{(Cf'')' + ff'' = 0}$$ (momentum BL eqn)

$$\left(\frac{Cg'}{Pr}\right)' + fg' = -\frac{C}{h_e}u_e^2(f'')^2$$ (energy BL eqn)

for a perfect gas with constant specific heats,

$$\frac{u_e^2}{h_e} = \frac{u_e^2}{c_p T_e} = \frac{u_e^2}{c_p}\frac{\gamma R}{\gamma R T_e} = \frac{u_e^2\gamma(c_p - c_v)}{c_p a_e^2} = M_e^2\frac{\gamma(\gamma-1)}{\gamma}$$

$$= M_e^2(\gamma-1)$$

so $$\boxed{\left(\frac{Cg'}{Pr}\right)' + fg' = -CM_e^2(\gamma-1)(f'')^2}$$

-here 11-3

B.C. $$\boxed{\begin{array}{l} f(0) = f'(0) = 0, \quad f'(\infty) = 1; \\ g(\infty) = 1 \ (\text{so } h = h_e) \end{array}}$$ (+)

heat transfer: $-q_w = k_w \frac{\partial T}{\partial y}\Big|_w = \frac{k_w}{c_{p_w}} \frac{\partial h}{\partial y}\Big|_w$

$$-q_w = \frac{k_w}{c_{p_w}} \frac{h_e \rho_e u_e}{\sqrt{2\xi}} g'\Big|_{\eta=0}$$

for adiabatic wall, $q_w = 0 \Rightarrow \boxed{g'(0) = 0}$ (adiabatic)

or $\boxed{g(0) = 1 \ (\text{cold wall}, \ h_w = h_e}$

$\boxed{+ \text{ need an eqn for } C(h)}$ (if C = const, change variables & use Blasius soln)

~~Compressible displacement thickness:~~

$$\cancel{\delta^* = \int_0^\infty \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy = \int_0^\infty \left(1 - \frac{\rho}{\rho_e} f'\right) dy}$$

$C = \frac{\rho \mu}{\rho_e \mu_e}$; $\frac{\mu}{\mu_0} \cong \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + S}{T + S}$

Air: $T_0 = 491.6R$, $S = 199°R$

$\mu_0 = .1716$ mP

OK for 300-3420R (White)

$p = \rho R T \Rightarrow \frac{p}{\cancel{RT}} \cdot \frac{\cancel{R}T_e}{\cancel{p_e}} = \frac{\rho}{\rho_e} = \frac{T_e}{T}$ perf gas w/ const c_p

$$\frac{\mu}{\mu_e} = \frac{\mu/\mu_0}{\mu_e/\mu_0} = \frac{\left(T/\cancel{T_0}\right)^{3/2}\frac{\cancel{T_0+S}}{T+S}}{\left(T_e/\cancel{T_0}\right)^{3/2}\frac{\cancel{T_0+S}}{T_e+S}} = \left(\frac{T}{T_e}\right)^{3/2}\frac{T_e+S}{T+S}$$

so $C \cong \frac{T_e}{T}\left(\frac{T}{T_e}\right)^{3/2}\frac{T_e+S}{T+S} \cong \left(\frac{T}{T_e}\right)^{1/2}\frac{T_e+S}{T+S}$ for range given.

$C \cong \left(\frac{h}{h_e}\right)^{1/2}\left(\frac{h_e+c_pS}{h+c_pS}\right)$ but $g = h/h_e$

$$\boxed{C \cong \sqrt{g}\left(\frac{h_e+c_pS}{gh_e+c_pS}\right)}$$ h_e known, c_p, S known

have to solve numerically. — see plots —

(Beware of dissociation effects, nonequilibrium chemistry effects)

Work heat transfer & wall temp. for this

102099

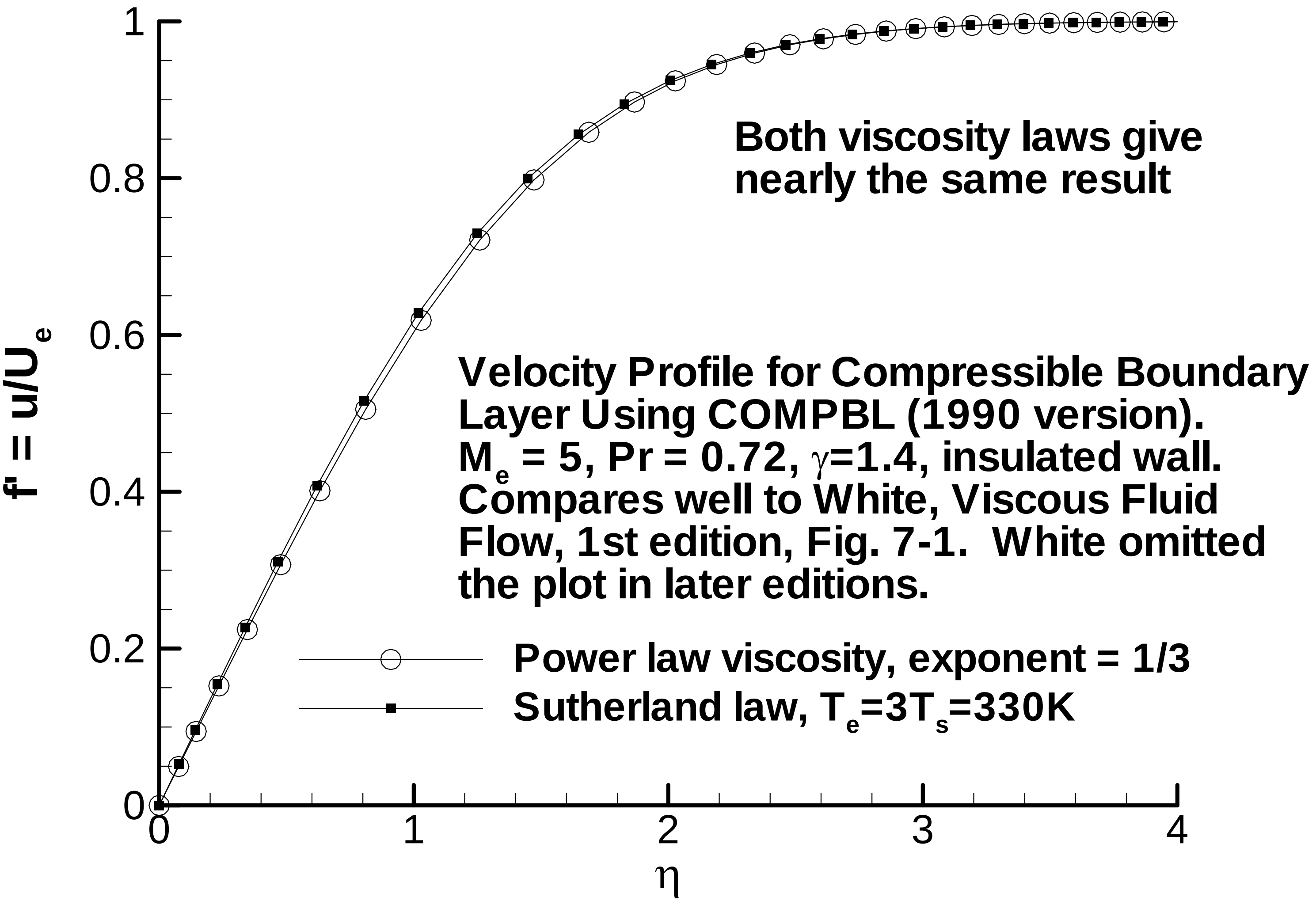
3-21-01

H. Approximations for Flat Plate in Compressible Flow

- Reference Temperature

Assume that C is a constant, so that can solve the momentum equation using the Blasius solns. Pick a "reference temperature" T^* to be an appropriate characteristic temp. at which to evaluate C (and

S.P. Schneider, Purdue AAE | 24 Oct 2005 | COMPBL data from March 1990, mach5.p1 and mach5.s1

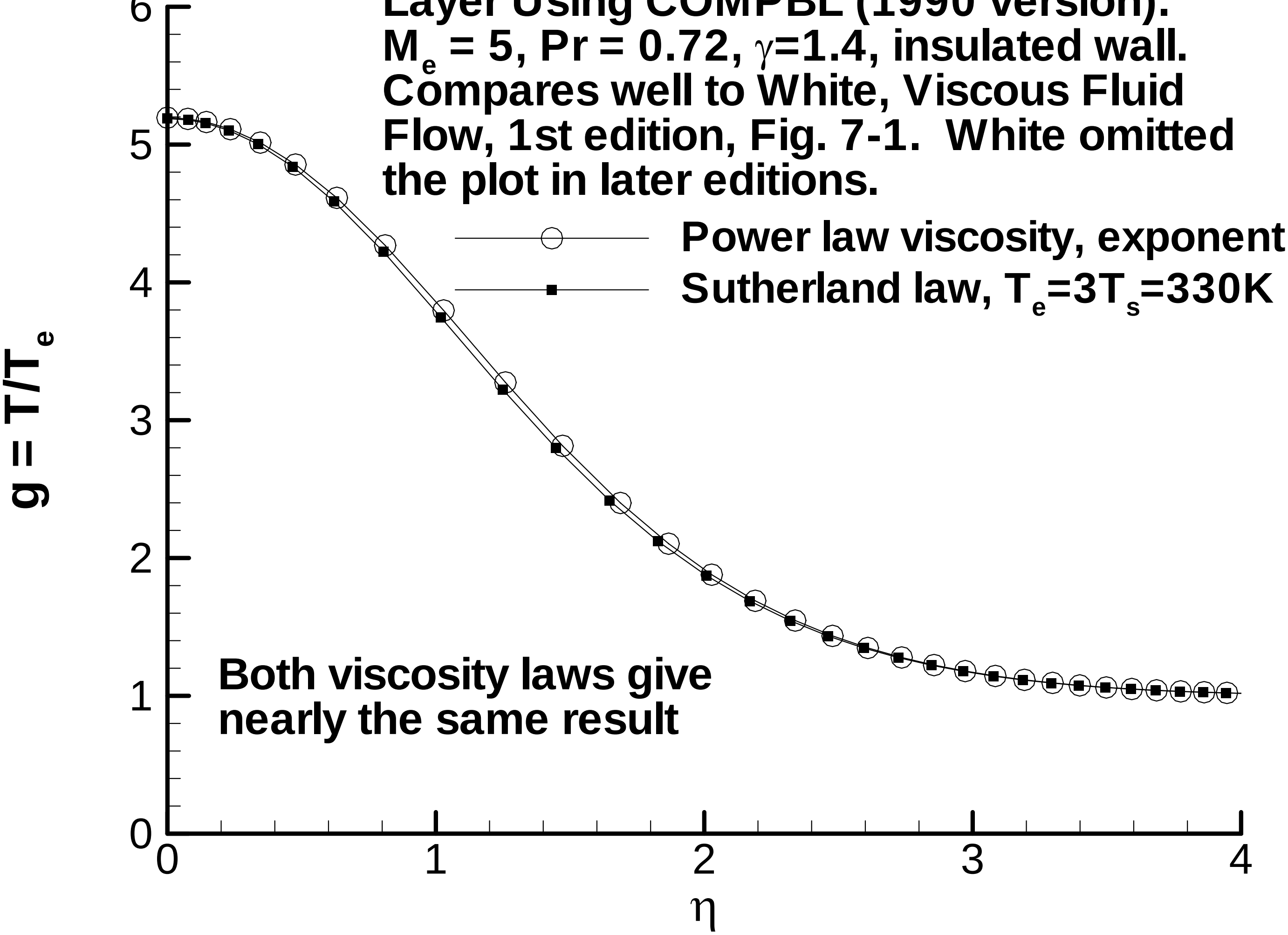


Both viscosity laws give nearly the same result

Velocity Profile for Compressible Boundary Layer Using COMPBL (1990 version). M_e = 5, Pr = 0.72, γ=1.4, insulated wall. Compares well to White, Viscous Fluid Flow, 1st edition, Fig. 7-1. White omitted the plot in later editions.

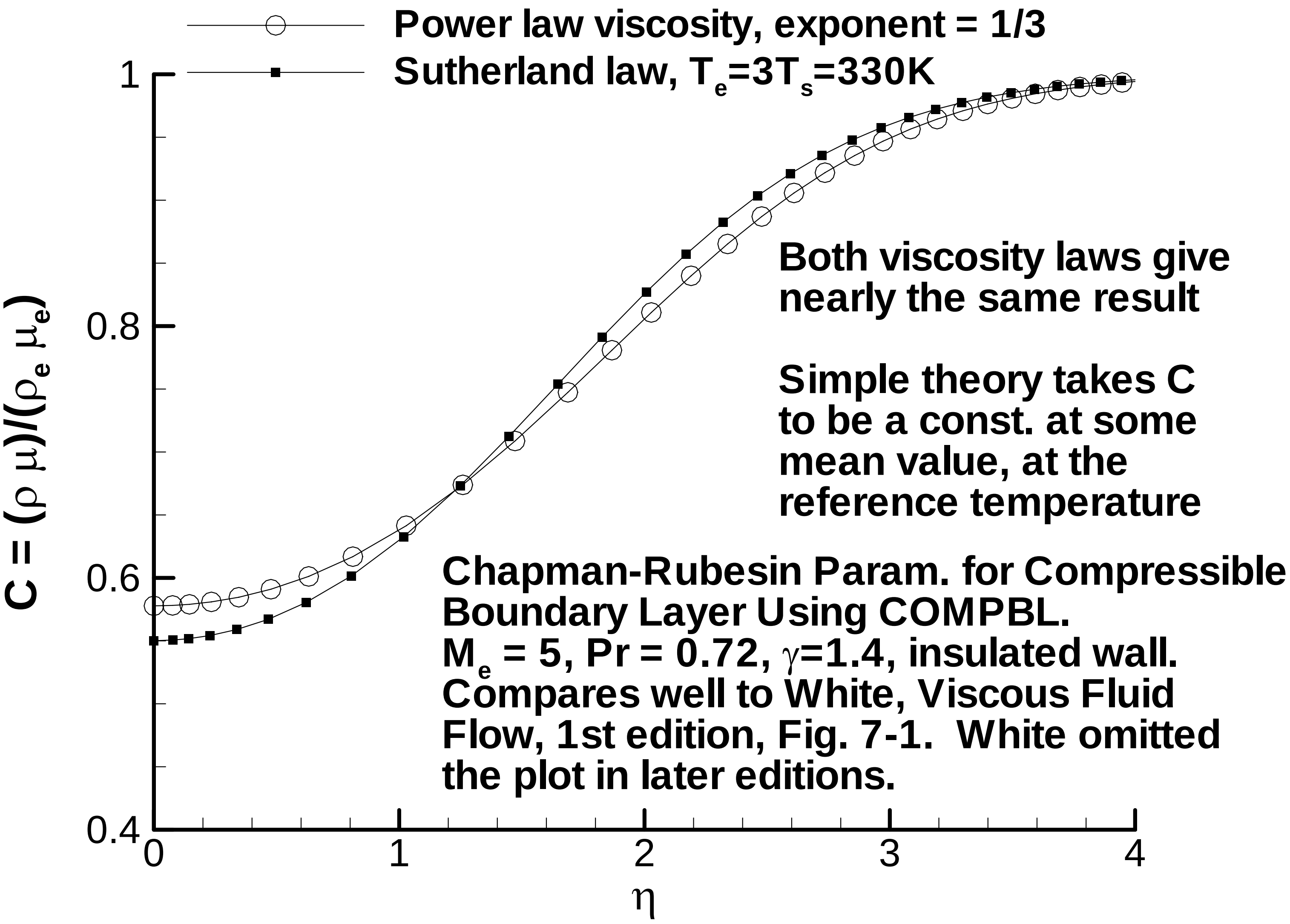
Temperature Profile for Compressible Boundary Layer Using COMPBL (1990 version).

M_e = 5, Pr = 0.72, γ=1.4, insulated wall.
Compares well to White, Viscous Fluid Flow, 1st edition, Fig. 7-1. White omitted the plot in later editions.



Both viscosity laws give nearly the same result

S.P. Schneider, Purdue AAE | 24 Oct 2005 | COMPBL data from March 1990, mach5.p1 and mach5.s1



Both viscosity laws give nearly the same result

Simple theory takes C to be a const. at some mean value, at the reference temperature

Chapman-Rubesin Param. for Compressible Boundary Layer Using COMPBL.
M_e = 5, Pr = 0.72, γ=1.4, insulated wall.
Compares well to White, Viscous Fluid Flow, 1st edition, Fig. 7-1. White omitted the plot in later editions.

y and so on). Then

$$\frac{d}{d\eta}\left(c\frac{d^2f}{d\eta^2}\right)+f\frac{d^2f}{d\eta^2}=0 \qquad f(0)=f'(0)=0,\ f'(\infty)=1$$

let $\eta=c_0\alpha$, $f=c_1\tilde{f}$, c_1 & c_0 = const.

$$\frac{1}{c_0^3}c_1\frac{d}{d\alpha}\left(c\frac{d^2\tilde{f}}{d\alpha^2}\right)+\frac{c_1^2}{c_0^2}\tilde{f}\frac{d^2\tilde{f}}{d\alpha^2}=0;$$

want $\tilde{f}(\alpha=0)=0$, $\tilde{f}'(\alpha=0)=0$, $\tilde{f}'(\alpha=\infty)=1$

so $\tilde{f}$ is $f_{Blasius}$

1st two come automatically from B.C. on f.

3rd means $\frac{d\tilde{f}}{d\alpha}=\frac{c_0}{c_\phi}\frac{df}{d\eta}=\frac{c_0}{c_\phi}$ at $\alpha\to\infty$ $=1$

need $c_1=c_0$.

eqn becomes

$$\frac{1}{c_0c_1}c\frac{d^3\tilde{f}}{d\alpha^3}+\tilde{f}\frac{d^2\tilde{f}}{d\alpha^2}=0$$

need $c=c_0c_1=c_0^2 \Rightarrow c_0=(c)^{1/2}=c_1$

then $\tilde{f}'''+\tilde{f}\tilde{f}''=0$ & $\tilde{f}(\alpha)=f_{Blasius}(\alpha)\equiv f_B(\alpha)$

24199

So $f(\eta) = c_1 \tilde{f}(\eta) = c_1 \tilde{f}(c_0 \alpha) = c_1 \, f_B(c_0 \eta_{Blasius})$

$$\left(f(\eta=1) = c_1 \tilde{f}(\eta=1) = c_1 \tilde{f}(c_0\alpha = 1) = etc \right)$$

$$\frac{df}{d\eta} = f' = \frac{u}{u_e} = \frac{c_1}{c_0}\frac{d\tilde{f}}{d\alpha} = \tilde{f}' = f_B'(c_0 \eta_{Bl})$$

$$\frac{d^2 f}{d\eta^2} = f'' = \frac{c_1}{c_0^2}\frac{d^2 \tilde{f}}{d\alpha^2} = \frac{1}{\sqrt{C}}\tilde{f}'' = \frac{f_B''(c_0 \eta_B)}{\sqrt{C}}$$

So $f''(0) = \frac{f_B''(0)}{\sqrt{C}} = \frac{.4696}{\sqrt{C}}$

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_0 = \mu \frac{\partial}{\partial y}\left(u_e f' \right)\Big|_0 = \mu \frac{u_e^2 \rho}{\sqrt{2\xi}} f''(0)$$

$$\xi = \int^x \rho_e u_e \mu_e \, dx = \rho_e \mu_e u_e x$$

$$\tau_w = \frac{\rho \mu u_e^2}{\sqrt{2}\sqrt{\rho_e \mu_e u_e x}} f''(0) = \frac{\rho\mu}{\rho_e \mu_e}\frac{\sqrt{\rho_e \mu_e}}{\sqrt{u_e x}}\frac{u_e^2}{\sqrt{2}} f''(0)$$

$$\tau_w = C \rho_e \sqrt{\frac{\mu_e}{u_e \rho_e x}} \frac{u_e^2}{\sqrt{2}} f''(0) = C \rho_e \left(Re_{ext,x} \right)^{-1/2} \frac{u_e^2}{\sqrt{2}} f''(0)$$

$$\boxed{\tau_w = C \rho_e \left(Re_{ext,x} \right)^{-1/2} \frac{u_e^2}{2} \frac{.4696}{\sqrt{C}} = \sqrt{\frac{C}{2}} \left(Re_{x,ext} \right)^{-1/2} .4696 \, \rho_e u_e^2}$$

Reference temp. Idea: Use this approx, and pick C as some average, so that profiles are $\cong$ correct. Since C depends on temp. alone, to good approx, evaluate C at some "reference temp" T^*. Can do empirically or using integral form of eqns. (White p.590-91).

$$\boxed{\frac{T^*}{T_e} \cong .5 + .5\frac{T_w}{T_e} + 0.039\,Ma_e^2}$$

White (7-46b)

is one good approx. Numerical solns show $r \cong \sqrt{Pr}$ $= T_{aw} + \frac{r u_e^2}{2c_p}$ White 7-3

— here Mon 2 April (skipped deca. I)

I. B.L. Displacement Thickness for Similarity Solns

flat plate

$$\delta^* \equiv \int_0^\infty \left(1 - \frac{\rho}{\rho_e}\frac{u}{u_e}\right)dy = \int_0^\infty \left(1 - \frac{\rho}{\rho_e}f'\right)dy$$

but $\frac{\rho}{\rho_e} = \frac{T_e}{T} = \frac{h_e}{h}$ since $\frac{dp}{dy} = 0$, and if $h = c_pT$

so $\frac{\rho}{\rho_e} = \frac{1}{g}$ where $g \equiv \frac{h}{h_e}$

$$\Rightarrow \delta^* = \int_0^\infty \left(1 - \frac{f'}{g}\right)dy.$$

5th printing, from Shin. 2-99

Anderson, Hypersonic and High-Temp. Gasd.

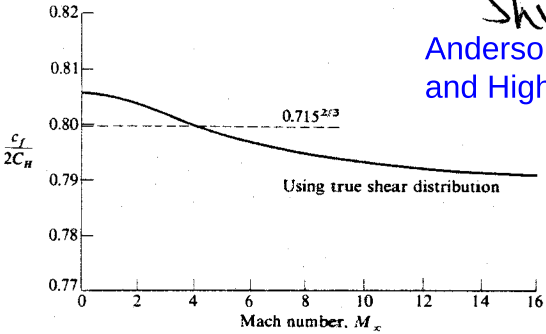


FIGURE 6.13
Comparison of exact and approximate Reynolds analogy factor for laminar flow over a flat plate. (*Ref. 90.*)

is expressed by Eq. (6.83); numerical solutions are given in Fig. 6.13, obtained from Ref. 90. Note that the ratio c_f/C_H decreases as M_e increases across the hypersonic regime. However, again note that the ordinate is an expanded scale, and c_f/C_H decreases by only 2 percent from $M_e = 0$ to 16. Thus, the incompressible result given by Eq. (6.82) is a reasonable approximation at hypersonic speeds, namely

$$\frac{C_H}{c_f} = \frac{1}{2}\mathrm{Pr}^{-2/3} \qquad (6.91)$$

But need $\delta^*(\eta)$ to integrate similarity soln:

$$\eta = \frac{u_e}{\sqrt{2\xi}} \int_0^y \rho \, dy \; ; \; \xi = \int_0^x \rho_e u_e \mu_e \, dx$$

$$\frac{\partial \eta}{\partial y} = \frac{u_e}{\sqrt{2\xi}} \rho \; ; \; dy = \frac{\partial y}{\partial \eta} d\eta + \cancel{\frac{\partial y}{\partial \xi}}^{0} d\xi$$

y indepd $\xi(x)$

so $dy = \frac{\sqrt{2\xi}}{\rho u_e} d\eta$; but <u>for flat plate</u> $\xi = \rho_e u_e \mu_e x$

10-22-99

$$dy = \sqrt{\frac{2\rho_e u_e \mu_e x}{\rho^2 u_e^2}} \, d\eta = \frac{d\eta}{\rho} \sqrt{\frac{2\mu_e x \rho_e}{u_e}}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{f'}{g}\right) \frac{1}{\rho} \sqrt{\frac{2\mu_e x}{u_e \rho_e}} \, d\eta \, (\rho_e)$$

$$\frac{\delta^*}{x} = \int_0^\infty \left(1 - \frac{f'}{g}\right) g \sqrt{\frac{2\mu_e}{\rho_e u_e x}} \, d\eta$$

$$\frac{\delta^*}{x} = \underbrace{\left[\sqrt{2} \int_0^\infty (g - f') \, d\eta\right]}_{=I} (Re_x)^{-1/2}$$

↑ integral depends only on M_e, T_e, Pr.

given a viscosity law, Pr, and gamma (perfect gas).

for Blasius, $M=0$, T doesn't matter, $I = 1.7208$
$(g=1)$

for adiabatic wall conditions, Sutherland law, $Pr = 0.72$ (air), see following plots.

Boundary-Layer Thickness Integral for Flat Plate. Isentropic expansion from 300K without bow shock.

Air, γ = 1.4, Sutherland law, Pr=0.72