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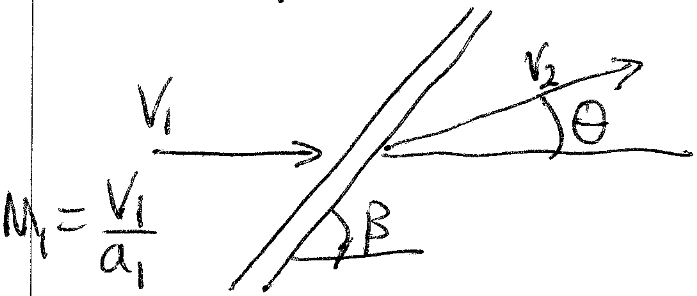
AAE 519

Hypersonic Shock Relations

Chapter 2 of Anderson 1989 contains the ~~normal~~ shock relations worked out for the case of high Mach number (hypersonic). As an example of these relations, here we will work out eqn 2.28, the pressure ratio across an oblique shock for hypersonic conditions.

We begin with (2.1), the exact oblique shock relation for pressure ratio:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 \sin^2\beta - 1\right) \qquad \begin{array}{l}(\text{cp. LR } 4.3)\\ (2.1)\end{array}$$



like wedge of half-angle θ

To obtain (2.2), Anderson looks at $M_1 \to \infty$, $M_1^2 \sin^2\beta \gg 1$, so both the -1 & 1 drop out.

Does ~~Is~~ β really remain finite (non-zero) as $M_1 \to \infty$? Yes, since $\beta > \theta$. ~~(but need to check attached shock limit.)~~

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\left(M_1^2 \sin^2\beta\right) \qquad (2.2)$$

Now look at the relation between θ, β, and M_1. The exact perfect-gas result is

$$\tan\theta = 2\cot\beta\left[\frac{M_1^2\sin^2\beta-1}{M_1^2(\gamma+\cos 2\beta)+2}\right]$$ Anderson 2.16, LR 4.10

For small β, $\sin\beta \cong \beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \cdots \cong \beta$

(Taylor series) $\cos\beta \cong 1 - \frac{\beta^2}{2!} + \cdots \cong 1$

If β is small, θ is small. For $\theta \ll 1$,

$$\tan\theta \cong \frac{\sin\theta}{\cos\theta} \cong \frac{\theta}{1} \cong \theta$$

(A 2.16) reduces to

(2.17 l.h.s.) $$\theta \cong 2\frac{1}{\beta}\left[\frac{M_1^2\beta^2-1}{M_1^2(\gamma+1)+2}\right] \cong \frac{2}{\beta}\left[\frac{M_1^2\beta^2-1}{M_1^2(\gamma+1)}\right]$$

for $M_1 \gg 1$

So far we assumed $M_1 \gg 1$, $\beta \ll 1$, $\theta \ll 1$. We have said nothing about $M_1\beta$ & $M_1\theta$. It is very important to use a consistent set of approximations.

If $M_1\beta >> 1$ also, then

$$\theta \cong \frac{2}{\beta}\left[\frac{M_1^2\beta^2}{M_1^2(\gamma+1)}\right], \text{ or}$$

$$\frac{\theta}{\beta} \cong \frac{2}{\gamma+1}, \quad \boxed{\frac{\beta}{\theta} \cong \frac{\gamma+1}{2}} \cong 1.2 \text{ for } \gamma = 1.4 \quad (A2.19)$$

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Now go back to (2.17), which assumes $\beta << 1$, $\theta << 1$. Rewrite

$$\frac{\beta\theta}{2}\left(M_1^2(\gamma+1)+2\right) = M_1^2\beta^2-1$$

If we assume that $M_1^2 >> 1$, we have

(2.22) $$\frac{\beta\theta}{2}M_1^2(\gamma+1) = M_1^2\beta^2-1$$

Here we do not assume $M_1\beta >> 1$. a choice

rearrange

(2.23) $$0 = \left(\frac{\beta}{\theta}\right)^2 - \left(\frac{\beta}{\theta}\right)\frac{\gamma+1}{2} - \frac{1}{M_1^2\theta^2}$$

note typo in Anderson: β not B

Typo fixed in AIAA printings

Solve w/ quadratic eqn

$$\frac{\beta}{\theta} = \frac{\frac{\gamma+1}{2} \pm \sqrt{\left(\frac{\gamma+1}{2}\right)^2 + 4\frac{1}{M_1^2\theta^2}}}{2}$$

$$\frac{\beta}{\theta} = \frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}} \qquad (2.24)$$

still have not assumed anything about $M_1\theta$

using the pressure ratio eqn (2.1), but assuming only that $\beta \ll 1$,

$$\frac{p_2}{p_1} \cong 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2\beta^2 - 1\right) \qquad (2.25)$$

combine 2.23 with 2.24 (not 2.22, typo) typo fixed in AIAA printings

$$\left(\frac{\beta}{\theta}\right)^2 = \frac{1}{M_1^2\theta^2} + \frac{\gamma+1}{2}\left[\frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}\right]$$

$$\beta^2 = \frac{1}{M_1^2} + \left[\frac{(\gamma+1)^2}{8} + \frac{\gamma+1}{2}\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}\right]\theta^2 \qquad (2.26)$$

sub. into 2.25

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left[\cancel{1} + M_1^2\theta^2\left[\frac{(\gamma+1)^2}{8} + \frac{\gamma+1}{2}\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}\right] \cancel{-1}\right]$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma(\gamma+1)}{8}M_1^2\theta^2 + \gamma\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}\,M_1^2\theta^2 \qquad (2.27)$$

let $K \equiv M_1\theta$, hypersonic similarity param, then

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{\gamma(\gamma+1)}{4}K^2 + \gamma K^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}}} \qquad 2.28$$

The pressure coefficient is

$$C_p = \frac{p_2 - p_1}{q_1} = \frac{p_2 - p_1}{\frac{\gamma}{2}p_1 M_1^2} = \frac{2}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right)$$

since $q_1 = \frac{1}{2}\rho_1 V_1^2 = \frac{1}{2}\cancel{\rho_1} V_1^2 \cdot \frac{p_1 \gamma}{\cancel{\rho_1} R T_1 \gamma} = \frac{\frac{1}{2}p_1\gamma}{a_1^2}V_1^2$

$q_1 = \frac{\gamma}{2}p_1 M_1^2$

so

$$C_p = \frac{2}{\cancel{\gamma} M_1^2}\left[\frac{\cancel{\gamma}(\gamma+1)}{4}K^2 + \cancel{\gamma}K^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}}\right]$$

$$\boxed{C_p = 2\theta^2\left[\frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}}\right]} \qquad \text{Eqn 2.29, LR 10.38}$$

if $M_1\theta \gg 1$, $M_1\beta \gg 1$, then $K \gg 1$,

$$C_p = 2\theta^2\,\frac{\gamma+1}{2} = \theta^2(\gamma+1)$$

Similar to Newtonian. recovers N. as $\gamma \to 1$. $\beta \to \theta$

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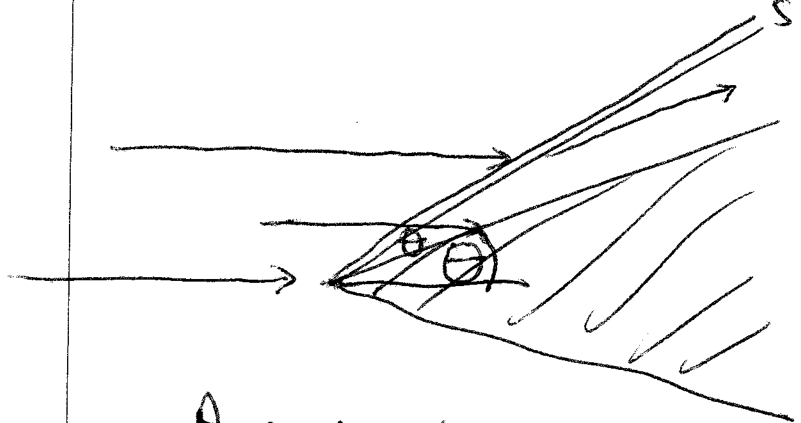
Newtonian Approx. for Pressure Coefficient

See Anderson section 3.2 to 3.5

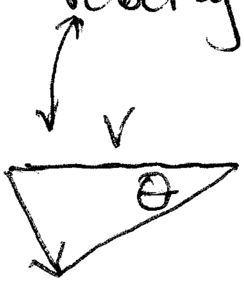


Flow behind shock almost parallel to wall. Since shock angle close to wall angle. ($\beta = 1.2\theta$)

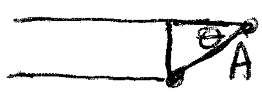
beta = 1.2* theta

Approximation: All velocity normal to the wall goes into the pressure coefficient.

velocity $\perp$ to wall is $V\sin\theta$



Mass flow rate for area A of surface is



$\rho_\infty V_\infty \sin\theta\, A$

so $$p' = \frac{F}{A} = \frac{1}{A}\frac{dp}{dt} = \frac{1}{A}(\dot{m}V)$$

$$(3.2) \qquad p' = \frac{1}{A}\rho V_\infty \sin\theta\, A\, V_\infty \sin\theta = \rho_\infty V_\infty^2 \sin^2\theta$$

here $p' = p - p_\infty$. Anderson argues that p' is pressure due to non-random motion.

then $$\boxed{C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\sin^2\theta}$$ Newtonian law.

Lees approximated better as

$$\boxed{C_p = C_{p_{max}} \sin^2\theta}$$

Modified Newtonian law.

This is ad hoc empirical.

- Surfaces which are in the shadow of the vehicle have $C_p = 0$. (approx.)

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- Fails in some cases. Better for blunt shapes, even at relatively low M. Does not capture expansion/recompression at sphere-cone junction, for example.

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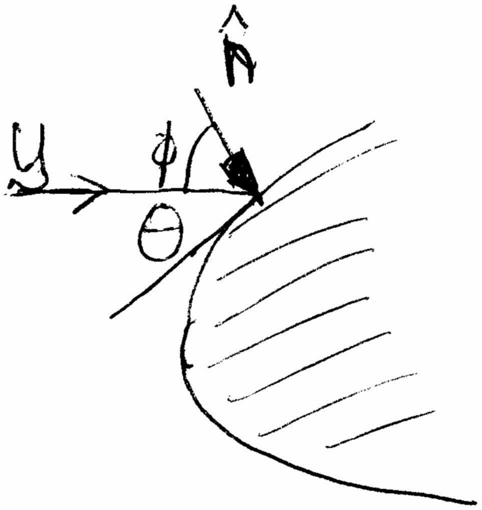
Application of Newtonian Approx.



$\hat{n}$ = UNIT <u>inward</u> normal!

$$C_p = K \sin^2\theta$$

$K = C_{p_{max}}$ or 2 or other empirical.

And. p. 49

For general geometries, and in 3D, easier to work with surface normal.

$$\underset{\sim}{U} \circ \hat{n} = |U| \cos\phi = |U| \sin\left(\frac{\pi}{2} - \phi\right) = |U| \sin\theta.$$

So $\sin\theta = \dfrac{\underset{\sim}{U} \circ \hat{n}}{|U|}$

$$\boxed{C_p = K \left[\frac{\underset{\sim}{U} \circ \hat{n}}{|U|}\right]^2}$$

Applied to flat plate to get L & D by Anderson p. 51
Integrate pressure over surface

For homework, integrate over some surfaces and compare results to Clark & Trimmer AEDC-TR-64-25.
Equation above same as C&T eqn (5).

Note C&T work in body-fixed coordinates, axial & normal force, not in coords relative to $\underset{\sim}{U}$ (L&D).

this page replaced by slides. See handouts-section1ab-2013.pdf or similar. Cover part of the slides, then go back to the blackboard lectures from the notes.

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Other Approximate Methods for Solving Inviscid Flows

It's difficult to decide what to cover. Anderson covers the shock-expansion method, shows Mach number independence in the inviscid Euler eqns and B.C., looks at the small-disturbance eqns (valid for sharp slender bodies), Goes through hypersonic small-disturbance theory, blast-wave theory, and thin-shock-layer theory.

Bertin skips most of this and shows expt. data.

I have the impression that the theoretical methods were dominant from the 50's to the 70's. Some of the simpler approx. methods are incorporated into computer programs (SHABP) and are used in preliminary design (tangent-wedge & cone, shock-exp., Newtonian, ...?)

Since the development and ease of use of numerical methods, hypersonic small-disturbance theory seems to be less used. In part because few hypersonic vehicles are slender, ~~and none are~~ ~~sharp~~ or sharp. The Taylor-Maccoll results are much used.

S. P. Schneider
26 Jan 2001

Taylor-Maccoll Flow

Follow Zucrow and Hoffman, Gas Dynamics, Vol. II, Multidimensional Flows, section 16-5, and Philip A. Thompson, Compressible-Fluid Dynamics, section 10.3.

Consider a perfect sharp cone with an attached shock, perfectly conical. By symmetry, expect a flow with conical symmetry, where properties remain unchanged along rays from the cone vertex.

Assume flow is steady and inviscid.

A perfect conical shock is always at the same angle to the uniform freestream, so the flow immediately behind the shock is at uniform conditions

If the flow is inviscid and any body forces are conservative, then the initially irrotational flow will remain irrotational. This means

$$\nabla \times \underset{\sim}{u} = 0 \qquad (T\ 10.1)$$

Continuity for steady compressible flow is

$$\nabla \cdot (\rho \underset{\sim}{u}) = 0 \qquad (T\ 10.2)$$

Finally, the energy equation for steady flow without viscosity or heat transfer, and without body forces, is

$$H = \overset{\text{const}}{\cancel{0}} = h + \tfrac{1}{2}u^2 \qquad (\text{cp. L\&R } 7.25;\ T\ 10.3)$$

Consider the flow field as follows:

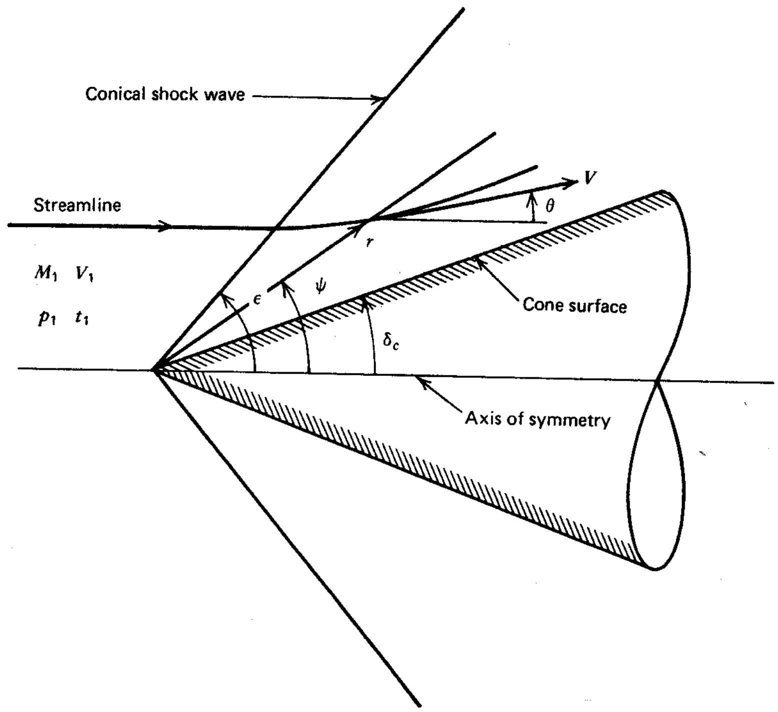

Figure 16.36 Flow model for Taylor-Maccoll flow over a cone.

Use spherical coordinates, with ϕ the azimuthal angle, as shown as follows:

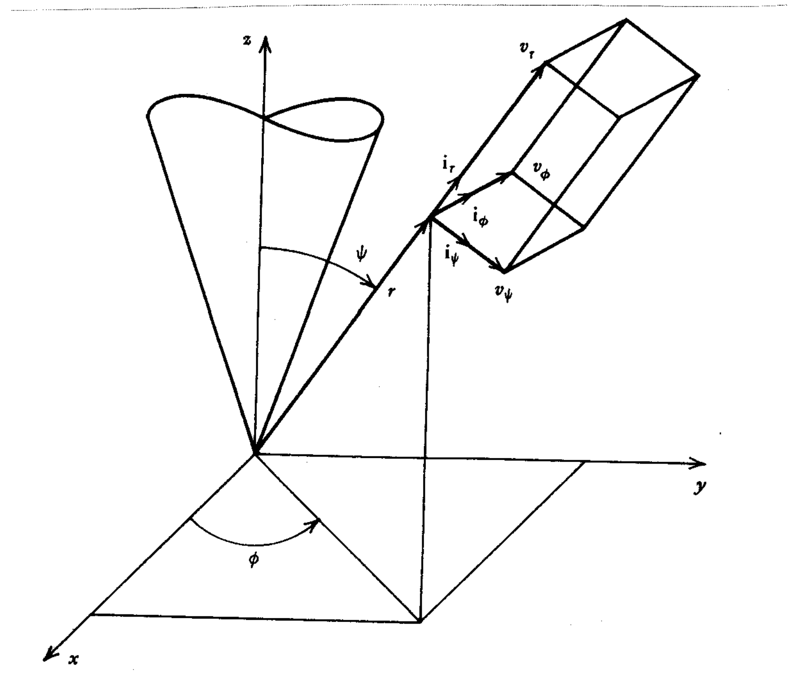

Figure 16.37 The spherical coordinate system.

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In ~~polar~~ spherical coordinates,

$$\underset{\sim}{u} = u_r \hat{e}_r + \overset{0 \text{ by symmetry}}{\cancel{u_\phi}} \hat{e}_\phi + u_\psi \hat{e}_\psi$$

note $\frac{\partial}{\partial r} = 0$ by assumption, as is $\frac{\partial}{\partial \phi}$ (conical flow)

$\nabla \cdot \underset{\sim}{f}$ is then $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\psi}\frac{\partial}{\partial \psi}\left(f_\psi \sin\psi\right)$

(cp. Gradshteyn & Ryzhik p. 1086)

and $$\nabla \times \underset{\sim}{f} = \frac{1}{r^2 \sin\psi} \begin{vmatrix} \hat{e}_r & r\hat{e}_\psi & r\sin\psi\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\psi} & \frac{\partial}{\partial\phi} \\ f_r & r f_\psi & 0 \end{vmatrix}$$

(G&R p. 1086)

Irrotationality gives:

$$0 = \hat{e}_r(-)\cancelto{\text{conical flow}}{\frac{\partial}{\partial\phi}(r u_\psi)}$$

$$-r\hat{e}_\psi\left(\cancelto{0 \text{ conical flow}}{\frac{\partial}{\partial\phi}(u_r)}\right)$$

$$+r\sin\psi\, e_\phi\left(\frac{\partial}{\partial r}(r u_\psi) - \frac{\partial}{\partial\psi}(u_r)\right) = \nabla\times\underset{\sim}{u}$$

$$0 = \cancelto{0,\text{ conical flow}}{r\frac{\partial u_\psi}{\partial r}} + u_\psi - \frac{\partial u_r}{\partial\psi}$$

$$\boxed{0 = u_\psi - \frac{\partial u_r}{\partial \psi}}$$ (T 10.13) or (Z&H 16.53)

Continuity gives

$$\nabla\cdot(\rho \underset{\sim}{u}) = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u_r) + \frac{1}{r\sin\psi}\frac{\partial}{\partial\psi}(\rho u_\psi \sin\psi)$$

$$0 = \frac{\sin\psi}{r}\left[r^2 \cancelto{0,\text{ conical flow}}{\frac{\partial}{\partial r}(\rho u_r)} + \rho u_r 2r\right] + \frac{\partial}{\partial\psi}(\rho u_\psi\sin\psi)$$

$$\boxed{0 = 2\sin\psi\, \rho u_r + \frac{\partial}{\partial\psi}(\rho u_\psi \sin\psi)}\quad (T, 10.14)$$

Now note that $\frac{\partial}{\partial\psi}$ is really $\frac{d}{d\psi}$, all variables depend on ψ alone, by symmetry.

Use the energy equation to eliminate the density:

$$h + \tfrac{1}{2}u^2 = \text{const.}$$

So $$dh + u\,du = 0$$

to simplify the following, assume a perfect gas with constant specific heats. Then

$$p = \rho R T$$

$$dp = \rho R\,dT + RT\,d\rho$$

$$\frac{1}{\rho}dp = R\,dT + RT\frac{d\rho}{\rho}$$

Now use $dh = T\,ds + \frac{1}{\rho}dp$, note $ds = 0$, so $dh = \frac{1}{\rho}dp$

$$dh = R\,dT + RT\frac{d\rho}{\rho}$$

$$C_p dT - R dT = \gamma R T \frac{d\rho}{\rho} \frac{1}{\gamma}$$

$$\gamma (C_p - (C_p - C_v)) dT = a^2 \frac{d\rho}{\rho}$$

$$\frac{C_p}{C_v} C_v \, dT = a^2 \frac{d\rho}{\rho} = dh$$

So $a^2 \frac{d\rho}{\rho} + u\,du = 0 \Rightarrow \frac{d\rho}{\rho} = -\frac{u}{a^2} du \quad (*)$

Now, differentiate (10.14)

$$0 = 2 \sin\psi \, u_r \, \rho + \rho \frac{\partial}{\partial \psi}(u_\psi \sin\psi) + u_\psi \sin\psi \frac{\partial \rho}{\partial \psi}$$

or $0 = 2 \sin\psi \, u_r + u_\psi \cos\psi + \sin\psi \frac{du_\psi}{d\psi} + u_\psi \sin\psi \frac{d\rho}{\rho \, d\psi}$

Now, (*) implies that $\frac{1}{\rho} \frac{d\rho}{d\psi} = -\frac{u}{a^2} \frac{du}{d\psi}$, or

$$\frac{1}{\rho} \frac{d\rho}{d\psi} = -\frac{1}{a^2} \frac{d}{d\psi}\left(\frac{u^2}{2}\right) = -\frac{1}{2a^2} \frac{d}{d\psi}\left(u_r^2 + u_\psi^2\right)$$

So

$$0 = 2\sin\psi\, u_r + u_\psi \cos\psi + \sin\psi \frac{du_\psi}{d\psi}$$
$$+ u_\psi \sin\psi \left(-\frac{1}{2a^2}\right)\left(2u_r \frac{du_r}{d\psi} + 2u_\psi \frac{du_\psi}{d\psi}\right)$$

$$\frac{u_\psi \sin\psi}{a^2}\left(u_r \frac{du_r}{d\psi} + u_\psi \frac{du_\psi}{d\psi}\right) = 2\sin\psi\, u_r + u_\psi \cos\psi + \sin\psi \frac{du_\psi}{d\psi}$$

$$u_\psi\left(u_r \frac{du_r}{d\psi} + u_\psi \frac{du_\psi}{d\psi}\right) = a^2\left[2u_r + u_\psi \cot\psi + \frac{du_\psi}{d\psi}\right]$$

$$\boxed{u_\psi\left(u_r \frac{du_r}{d\psi} + u_\psi \frac{du_\psi}{d\psi}\right) - a^2\left[\frac{du_\psi}{d\psi} + 2u_r + u_\psi \cot\psi\right] = 0}$$

(Z&H 16.52, or T_1 line above 10.16)

Solve this, together with (10.13)

$$\boxed{0 = u_\psi - \frac{du_r}{d\psi}}$$

Z&H now use $\bar{u}$ for u_r & $\bar{v}$ for v_ψ.
Since $\frac{\partial}{\partial\psi}$ is really now $\frac{d}{d\psi}$, rewrite as

$$\bar{v}\left(\bar{u}\frac{d\bar{u}}{d\psi}+\bar{v}\frac{d\bar{v}}{d\psi}\right)-a^2\left(\frac{d\bar{v}}{d\psi}+2\bar{u}+\bar{v}\cot\psi\right)=0 \quad (16.56)$$

$$\frac{d\bar{u}}{d\psi}=\bar{v} \quad (16.57)$$

substitute

$$\bar{v}\left(\bar{u}\bar{v}+\bar{v}\frac{d\bar{v}}{d\psi}\right)-a^2\left(\frac{d\bar{v}}{d\psi}+2\bar{u}+\bar{v}\cot\psi\right)=0$$

$$\left(\bar{v}^2-a^2\right)\frac{d\bar{v}}{d\psi}+\bar{u}\bar{v}^2-a^2 2\bar{u}-a^2\bar{v}\cot\psi=0$$

$$\left(\bar{v}^2-a^2\right)\frac{d\bar{v}}{d\psi}=-\bar{u}\left(\bar{v}^2\right)+2\bar{u}a^2+a^2\bar{v}\cot\psi$$

$$=-\bar{u}\left(\bar{v}^2-a^2\right)+a^2\left(\bar{u}+\bar{v}\cot\psi\right)$$

$$\boxed{\frac{d\bar{v}}{d\psi}=-\bar{u}+\frac{a^2\left(\bar{u}+\bar{v}\cot\psi\right)}{\bar{v}^2-a^2}} \quad \begin{pmatrix}\text{Z\&H}\\ 16.58\end{pmatrix}$$

1-2-02

Put in nondimensional form using the critical speed of the freestream gas, a^*. a^* is the speed of sound in a place where the freestream gas might be isentropically brought to Mach 1.

$$\bar{u}^* = \frac{\bar{u}}{a^*}, \quad \bar{v}^* = \frac{\bar{v}}{a^*} \qquad (16.59 \text{ in 2eH}).$$

So we now have

$$\frac{d\bar{u}^*}{d\psi} = \bar{v}^* \qquad \left(\begin{matrix}\text{2eH}\\ 16.60\end{matrix}\right)$$

$$\frac{d\bar{v}^*}{d\psi} = -\bar{u}^* + \frac{(a/a^*)^2(\bar{u}^* + \bar{v}^*\cot\psi)}{\bar{v}^{*2} - (a/a^*)^2} \qquad (16.61)$$

B.C. – March freestream, no flow through cone surface.
What is the equation of state relating a & a^*?
Follow L&R section 2.10. Cp. Anderson, "Fundamentals ...",
2e, section 8.4 (eqn. 8.35).
For a perfect gas without heat conduction,

$$H = h + \tfrac{1}{2}u^2 = \text{constant} \qquad (\text{also true across shock})$$

then let $a^2 = \gamma R T$, and $h = c_p T$, (calorically perfect),

and $c_p T + \frac{1}{2}u^2 = \text{const} = c_p \frac{a^2}{\gamma R} + \frac{1}{2}u^2$

$$\text{const} = \frac{\cancel{c_p}}{\cancel{c_p}/c_v} \frac{1(a^2)}{c_p - c_v} + \frac{1}{2}u^2$$

$$\text{const} = \frac{a^2}{\gamma - 1} + \frac{1}{2}u^2 \qquad (\text{L\&R } 2.29)$$

at $u = a$, $u = a = a^*$ by definition, so

$$\frac{a^{*2}}{\gamma - 1} + \frac{1}{2}a^{*2} = \frac{a^2}{\gamma - 1} + \frac{1}{2}u^2$$

$$\frac{2 + \gamma - 1}{(\gamma - 1)2} a^{*2} = \frac{a^2}{\gamma - 1} + \frac{1}{2}u^2$$

$$\frac{\gamma + 1}{2(\gamma - 1)} a^{*2} = \frac{a^2}{\gamma - 1} + \frac{1}{2}u^2 \qquad (\text{L\&R } 2.33)$$

$$\frac{\gamma + 1}{2} \cancel{a^{*2}} \frac{1}{\cancel{a^{*2}}} - \frac{1}{2}(\gamma - 1)\frac{u^2}{a^{*2}} = \frac{a^2}{a^{*2}}$$

$$\boxed{\frac{a^2}{a^{*2}} = \frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} M^{*2}} \qquad \boxed{\text{Z\&H } 16.62}$$

note that here

$$\boxed{(M^*)^2 = \left(\frac{u}{a^*}\right)^2 = \frac{\bar{u}^2+\bar{v}^2}{a^{*2}} = \bar{u}^{*2}+\bar{v}^{*2}}$$

(Z&H 16.63)

Follow Z&H 16-5b for the numerical integration.
Have to assume a shock angle, compute prop. behind shock.
Integrate to the surface, surface is where
$\bar{v}^* = 0$. Find accurate value of surface angle.
will not be exactly the angle sought.
So make a new guess for the shock angle, and
Iterate to convergence.

after this page, go back to the slides in handouts-section1ab-2013 or similar. Cover the rest of those slides. Then go on to Section 2 on viscous flow.