SOLUTION MANUAL FOR
WAVES AND FIELD IN INHOMOGENEOUS MEDIA

EDITED BY W. C. CHEW

WITH CONTRIBUTIONS FROM:

I. AKSUN, J. H. LIN, C. C. LU, G. OTTO
Y. M. WANG, R. WAGNER AND W. WEEDON

PUBLISHED BY
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
UNIVERSITY OF ILLINOIS
URBANA-CHAMPAIGN, IL 61801

CHAPTER 1

EXERCISE SOLUTIONS

By W.C. Chew

§1.1

Using

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{A}(\mathbf{r},\omega)e^{-i\omega t}d\omega,$$

where $\mathbf{A}$ can be $\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}$, or $\mathbf{J}$, and letting

$$\rho(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(\mathbf{r},\omega)e^{-i\omega t}d\omega,$$

it follows that

$$\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)\mathbf{A}(\mathbf{r},\omega)e^{-i\omega t}.$$

Then, Fourier transforming (1.1.1) to (1.1.4) yields (1.1.12) to (1.1.15).

§1.2

(a) 1 volt = 1 watt/amp. 1 watt = 1 joule/sec. 1 joule has the dimension of force $\times$ distance. Force has the dimension of mass $\times$ acceleration = kg $\times$ m/s. Therefore, dim (watt) (which stands for dimension of watt) is kg $\cdot$ m^2/s^3. But dim (amp) = coulomb/sec = C s^{-1}. Therefore dim(watt) = kg $\cdot$ m^2/(C s^2).

(b) Since $\nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H}$, the LHS has a dimension of V/m^2, while $\omega\mathbf{H}$ has a dimension of A/(m $\cdot$ s) where V = volt and A = Amp. Therefore, μ_0 has a dimension of (V/m^2) $\cdot$ m $\cdot$ s/A = V $\cdot$ s/ (m $\cdot$ A). Putting volt and amp in fundamental units, dim (μ_0) = kg $\cdot$ m^2/(C s^2) $\cdot$ s/ (m $\cdot$ C/s) = kg$\cdot$ m/ C^2.

(c) If the size of μ_0 is altered, then the size of coulomb has to be altered, since μ_0 is tied to the definition of kg $\cdot$ m/C^2. Therefore, in order to make μ_0 from $4\pi \times 10^{-7}$ to 4π or 10^7 times bigger, the size of coulomb will have to increase by $10^{3.5}$. Then, 1 coulomb, instead of being 6.25×10^{18} electrons is now 1.98×10^{22} electrons. 1 new volt = $10^{-3.5}$ volt, and 1 new amp = $10^{3.5}$ amp. These units of voltage and amperes are too small and too large respectively. Hence, μ_0 is defined to be $4\pi \times 10^{-7}$.

§1.3

$$A(\mathbf{r},t) = a(\mathbf{r})\cos(\omega t + \phi_1)$$
$$B(\mathbf{r},t) = b(\mathbf{r})\cos(\omega t + \phi_2).$$

$$\begin{aligned}\langle A(\mathbf{r},t), B(\mathbf{r},t)\rangle &= \lim_{T\to\infty} \frac{1}{T} a(\mathbf{r})b(\mathbf{r}) \int_0^T \cos(\omega t + \phi_1)\cos(\omega t + \phi_2)dt \\ &= \lim_{T\to\infty} \frac{1}{2T} a(\mathbf{r})b(\mathbf{r}) \int_0^T [\cos(\phi_1 - \phi_2) - \cos(2\omega t + \phi_1 + \phi_2)]dt \\ &= \frac{1}{2} a(\mathbf{r})b(\mathbf{r})\cos(\phi_1 - \phi_2).\end{aligned}$$

$A(\mathbf{r},t) = \Re e[A(\mathbf{r})e^{-i\omega t}]$, where $A(\mathbf{r}) = a(\mathbf{r})e^{i\phi_1}$, $B(\mathbf{r},t) = \Re e[B(\mathbf{r})e^{-i\omega t}]$, where $B(\mathbf{r}) = b(\mathbf{r})e^{i\phi_2}$. Therefore,

$$\begin{aligned}\frac{1}{2}\Re e\{A(\mathbf{r})B^*(\mathbf{r})\} &= \frac{1}{2}\Re e\{a(\mathbf{r})b(\mathbf{r})e^{i(\phi_1-\phi_2)}\} \\ &= \frac{1}{2}a(\mathbf{r})b(\mathbf{r})\cos(\phi_1 - \phi_2) \\ &= \langle A(\mathbf{r},t), B(\mathbf{r},t)\rangle.\end{aligned}$$

§1.4

If we have a point charge, it produces an electric field given by $\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$. If we have a DC current in a wire, it produces a magnetic field given by $\mathbf{H} = \frac{I}{2\pi\rho}\hat{\phi}$. If we put the point charge next to the wire, $\mathbf{E} \times \mathbf{H}$ is nonzero locally. But both $\mathbf{E}$ and $\mathbf{H}$ are static fields, and there cannot be power flow.

§1.5

(a)

$$V(t) = V_0 \cos\omega t, I(t) = I_I \cos\omega t + I_Q \sin\omega t.$$

Then,

$$V(t)I(t) = V_0 I_I \cos^2\omega t + \frac{1}{2}V_o I_Q \sin 2\omega t.$$

(b) $V(t) = \Re e\{V_0 e^{-i\omega t}\}$, $I(t) = \Re e\{(I_I + iI_Q)e^{-i\omega t}\}$. The phasors are V_0 and $I_I + iI_q$ for voltage and current respectively.

(c) The complex power is $P = VI^*$ were V and I are phasors. Therefore, $P = V_0 I_I + iV_0 I_Q$. $\frac{1}{2}\Re e\{P\}$ is equal to the time average power from the in-phase component of the current. $\frac{1}{2}\Im m\{P\}$ is equal to the amplitude of the time-varying power from the quadrature component of the current.

(d) The imaginary part of the complex power $\frac{1}{2}\Im m\{P\}$, is also known as the reactive power. This reactive power is equal to the amplitude of the time varying power which has zero time averages.

§1.6

$\overline{\mathbf{a}} = i(\overline{\mu}^+ - \overline{\mu}), \overline{\mathbf{b}} = i(\overline{\epsilon}^+ - \overline{\epsilon})$ are Hermitian because $\overline{\mathbf{a}}^+ = \overline{\mathbf{a}}, \overline{\mathbf{b}}^+ = \overline{\mathbf{b}}$. Therefore, they are either positive, negative definite, or indefinite. They correspond to lossy, active and lossless media respecitively.

§1.7

Another positive set of replacement rule is that

$$\mathbf{E} \to \mathbf{H}, \quad \mathbf{H} \to -\mathbf{E}, \quad \overline{\mu} \to \overline{\epsilon}, \quad \overline{\epsilon} \to \overline{\mu},$$
$$\mathbf{M} \to -\mathbf{J}, \quad \mathbf{J} \to \mathbf{M}, \quad \rho_m \to -\rho, \quad \rho \to \rho_m.$$

§1.8

(a) $\nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \frac{\partial}{\partial t}(\rho \nu) = \mathbf{F}$. Using the chain rule,

$$\nabla \cdot (\rho \mathbf{v} v_i) = (\nabla \cdot \rho \mathbf{v}) v_i + \rho \mathbf{v} \cdot \nabla v_i,$$

or

$$\nabla \cdot (\rho \mathbf{v}\mathbf{v}) = (\nabla \cdot \rho \mathbf{v})\mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v}.$$

But from (1.2.1), $(\nabla \cdot \rho \mathbf{v}) = -\frac{\partial \rho}{\partial t}$, therefore $\nabla \cdot (\rho \mathbf{v}\mathbf{v}) = -\mathbf{v}\frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v}$. Furthermore, $\frac{\partial}{\partial t}(\rho \mathbf{v}) = \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t}$. Consequently, $\nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \frac{\partial}{\partial t}(\rho \mathbf{v}) = \rho[\mathbf{v} \cdot \nabla \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}] = \mathbf{F}$.

(b) The force acting in the $+x$ direction between x and $x + \Delta x$ is $[p(x) - p(x + \Delta x)]A$. The volume is $A\Delta x$. Therefore, the force/unit volume is $[p(x) - p(x + \Delta x)]/\Delta x$ or $F_x = -\frac{\partial p}{\partial x}$ where F is the force density.

(c) We can apply the above derivative to a cube of volume $\Delta x \Delta y \Delta z$. We will get $F_x = -\frac{\partial p}{\partial x}, F_y = -\frac{\partial p}{\partial y}$, and $F_z = -\frac{\partial p}{\partial z}$. Therefore, $\mathbf{F} = -\nabla p$.

§1.9

(a)

$$\begin{aligned}\frac{D}{Dt}\rho(\mathbf{r}(t), t) &= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho\end{aligned}$$

where $\quad \mathbf{v} = \hat{x}\frac{dx}{dt} + \hat{y}\frac{dy}{dt} + \hat{z}\frac{dz}{dt}.$

(b) $p(\rho, S)$, where $\frac{DS}{Dt} = 0$. Then,

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial \rho}\frac{D\rho}{Dt} + \frac{\partial p}{\partial S}\frac{DS}{Dt} = \frac{\partial p}{\partial \rho}\frac{D\rho}{Dt} = \frac{\partial p}{\partial \rho}\left[\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho\right]$$

But also,

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho$$

Therefore,

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p = \frac{\partial p}{\partial \rho}\left[\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho\right].$$

By using the perturbation expansions of (1.2.6), and collecting the first order terms, we have

$$\frac{\partial p_1}{\partial t} + \mathbf{v}_1\cdot\nabla\rho_0 = \frac{\partial p_0}{\partial \rho_0}\left[\frac{\partial \rho_1}{\partial t} + \mathbf{v}_1\cdot\nabla\rho_0\right].$$

Letting $C^2 = \frac{\partial p_0}{\partial \rho_0}$, we have (1.2.13).

§1.10

If $\mathbf{E} = \hat{r}E_r + \hat{\theta}E_0 + \hat{\phi}E_\phi$ for example, $\hat{r}, \hat{\theta}$, and $\hat{\phi}$ are functions of position (x, y, z). Therefore, they do not commute with ∇^2, or that $\nabla^2\hat{r} \neq \hat{r}\nabla^2, \nabla^2\hat{\theta} \neq \hat{\theta}\nabla^2$, and $\nabla^2\hat{\phi} \neq \hat{\phi}\nabla^2$. Consequently, $(\nabla^2 + k^2)E_r \neq 0, (\nabla^2 + k^2)E_\theta \neq 0$, and $(\nabla^2 + k^2)E_\phi \neq 0$.

§1.11

For $n = 0$, the wavefront of $e^{ik_z z + ik_\rho \rho}$ looks like Figure 1.

Figure 1 for Exercise Solution 1.11.

For $n \neq 0$, for constant z, the wavefront looks like $e^{i\frac{n}{\rho}\rho + ik_\rho\rho}$. The azimuthal wave number is $\frac{n}{\rho}$ which is a function of ρ. So the wave front looks like Figure 2. It spirals in the counterclockwise direction.

Figure 2 for Exercise Solution 1.11.

§1.12

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{n^2}{\rho^2} + k_\rho^2\right]B_n(k\rho) = 0$$

is the equation for cylindrical Bessel functions. If we let $f_n(\rho) = \frac{1}{\sqrt{\rho}}B_n(k\rho)$ then

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}B_n(\rho) = \left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right]\sqrt{\rho}f_n(\rho)$$
$$= +\frac{1}{4}\rho^{-\frac{3}{2}}f_n(\rho) + 2\rho^{-\frac{1}{2}}f_n'(\rho) + \sqrt{\rho}f_n''(\rho)$$

and (1.2.28) becomes

$$f_n''(\rho) + \frac{2}{\rho}f_n'(\rho) + \frac{\frac{1}{4} - n^2}{\rho^2}f_n(\rho) + k_\rho^2 f_n(\rho) = 0.$$

But $f_n''(\rho) + \frac{2}{\rho}f_n'(\rho) = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho}f_n(\rho)$, and $\frac{1}{4} - n^2 = (\frac{1}{2} - n)(\frac{1}{2} + n)$, therefore, the above can be written as

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho}f_{n'+\frac{1}{2}}(\rho) - \frac{n'(n'+1)}{\rho^2}f_{n'+\frac{1}{2}}(\rho) + k_\rho^2 f_{n'+\frac{1}{2}}(\rho) = 0$$

where $n' = n - \frac{1}{2}$. So if we let $\rho \to r$, $k_\rho^2 \to k^2$, and let $b_{n'}(kr) = \sqrt{\frac{\pi}{2k}}f_{n'+\frac{1}{2}}(\rho)$, then the above becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}b_{n'}(kr) - \frac{n'(n'+1)}{r^2}b_{n'}(kr) + k^2 b_{n'}(kr) = 0$$

where

$$b_{n'}(kr) = \sqrt{\frac{\pi}{2kr}}B_{n'+\frac{1}{2}}(kr),$$

which is precisely (1.2.40a).

§1.13

$$B_n'(k_\rho\rho) = B_{n-1}(k_\rho\rho) - \frac{n}{k_\rho\rho}B_n(k_\rho\rho) = -B_{n+1}(k_\rho\rho) + \frac{n}{k_\rho\rho}B_n(k_\rho\rho).$$

Letting $b_{n-\frac{1}{2}}(kr) = \sqrt{\frac{\pi}{2kr}}B_n(kr)$ and $\rho \to r, k_\rho \to k$, we have

$$\begin{aligned}\frac{\partial}{\partial kr}\sqrt{kr}b_{n-\frac{1}{2}}(kr) &= \sqrt{kr}b_{n-\frac{3}{2}}(kr) - \frac{n}{\sqrt{kr}}b_{n-\frac{1}{2}}(kr)\\ &= -\sqrt{kr}b_{n+\frac{1}{2}}(kr) + \frac{n}{\sqrt{kr}}b_{n-\frac{1}{2}}(kr).\end{aligned}$$

But

$$\frac{\partial}{\partial x}\sqrt{x}b_{n-\frac{1}{2}}(x) = \frac{1}{2\sqrt{x}}b_{n-\frac{1}{2}}(x) + \sqrt{x}b'_{n-\frac{1}{2}}(x).$$

Therefore,

$$\begin{aligned}b'_{n-\frac{1}{2}}(kr) &= b_{n-\frac{3}{2}}(kr) - \frac{n+\frac{1}{2}}{kr}b_{n-\frac{1}{2}}(kr)\\ &= -b_{n+\frac{1}{2}}(kr) + \frac{n-\frac{1}{2}}{kr}b_{n-\frac{1}{2}}(kr).\end{aligned}$$

Letting $n - \frac{1}{2} = n'$, we obtain

$$\begin{aligned}b'_{n'}(kr) &= b_{n'-1}(kr) - \frac{n'+1}{kr}b_{n'}(kr)\\ &= -b_{n'+1}(kr) + \frac{n'}{kr}b_{n'}(kr).\end{aligned}$$

§1.14

$$\nabla \cdot p^{-1}\nabla\phi(\mathbf{r}) + k^2\phi(\mathbf{r}) = s(\mathbf{r}).$$

(a) We can integrate the above about a pill box at an interface where $p(\mathbf{r})$ is discontinuous. Then,

$$\int_S \hat{n} \cdot p^{-1}\nabla\phi(\mathbf{r})d\mathbf{r} = 0$$

assuming that no source resides at the interface. The subsequent boundry condition is that

$$\hat{n} \cdot p_1^{-1}\nabla\phi_1 = \hat{n} \cdot p_2^{-1}\nabla\phi_2$$

at interface. Moreover, we need $\phi_1 = \phi_2$ at interface, otherwise, $\nabla\phi$ will be singular.

(b)

$$\nabla \cdot p^{-1}\nabla\phi_1(\mathbf{r}) + k^2\phi_1(\mathbf{r}) = s_1(\mathbf{r}), \tag{1}$$

$$\nabla \cdot p^{-1}\nabla\phi_2(\mathbf{r}) + k^2\phi_2(\mathbf{r}) = s_2(\mathbf{r}), \tag{2}$$

Multiplying (1) by $\phi_2(\mathbf{r})$ and (2) by $\phi_1(\mathbf{r})$, subtracting the two equations and integrate yields

$$\begin{aligned}\langle\phi_2(\mathbf{r}), s_1(\mathbf{r})\rangle &- \langle\phi_1(\mathbf{r}), s_2(\mathbf{r})\rangle \\ &= \int_V dV[\phi_2\nabla\cdot p^{-1}\nabla\phi_1 - \phi_1\nabla\cdot p^{-1}\nabla\phi_2] \\ &= \int_V dV\nabla\cdot[\phi_2 p^{-1}\nabla\phi_1 - \phi_1 p^{-1}\nabla\phi_2] \\ &= \int_S dS\hat{n}\cdot[\phi_2 p^{-1}\nabla\phi_1 - \phi_1 p^{-1}\nabla\phi_2]\end{aligned}$$

We can let $S \to S_{\text{inf}}$. In this case ϕ_1 and ϕ_2 will look like plane waves so that $\nabla\phi_1 = i\mathbf{k}\phi$ and $\nabla\phi_2 = i\mathbf{k}\phi_2$. The above integral is seen to vanish, and we have

$$\langle\phi_2, S_z\rangle = \langle\phi_1, S_2\rangle.$$

§1.15

If $\mathbf{k} = \hat{z}k$, then (1.3.28) becomes

$$k\overline{\mathbf{K}} = k\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

This is a rank two matrix. Since $\overline{\mathbf{F}} = \overline{\mathbf{K}}\cdot\overline{\boldsymbol{\mu}}^{-1}\cdot\overline{\mathbf{K}}\cdot\overline{\boldsymbol{\epsilon}}^{-1}$, then

$$\overline{\mathbf{F}}\cdot\mathbf{D}_0 = \begin{bmatrix}A_x\\ A_y\\ 0\end{bmatrix} \qquad \text{even if} \qquad \mathbf{D}_0 = \begin{bmatrix}D_x\\ D_y\\ D_z\end{bmatrix}.$$

Therefore, for nontrivial $\frac{\omega^2}{k^2}$ in (1.3.29), $D_z = 0$, and an eigenvector

$$\mathbf{D}_0 = \begin{bmatrix}D_x\\ D_y\\ 0\end{bmatrix}.$$

In this space, there are only two independent eigenvectors. Let $\overline{\boldsymbol{\kappa}} = \overline{\boldsymbol{\epsilon}}^{-1}$, and $\overline{\boldsymbol{\nu}} = \overline{\boldsymbol{\mu}}^{-1}$, and we have

$$\overline{\mathbf{K}}\cdot\overline{\boldsymbol{\kappa}} = \begin{bmatrix}-\kappa_{21} & -\kappa_{22} & -\kappa_{23}\\ \kappa_{11} & \kappa_{12} & \kappa_{13}\\ 0 & 0 & 0\end{bmatrix}, \overline{\mathbf{K}}\cdot\overline{\boldsymbol{\nu}} = \begin{bmatrix}-\nu_{21} & -\nu_{22} & -\nu_{23}\\ \nu_{11} & \nu_{12} & \nu_{13}\\ 0 & 0 & 0\end{bmatrix}$$

and

$$\overline{\mathbf{F}} = \begin{bmatrix} \kappa_{21}\nu_{21} - \kappa_{11}\nu_{22} & \kappa_{22}\nu_{21} - \kappa_{12}\nu_{22} & \kappa_{23}\nu_{21} - \kappa_{13}\nu_{22} \\ -\kappa_{21}\nu_{11} + \kappa_{11}\nu_{12} & -\kappa_{22}\nu_{11} + \kappa_{12}\nu_{12} & -\kappa_{23}\nu_{11} + \kappa_{13}\nu_{13} \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

For nontrivial eigenvalue $\lambda = \frac{\omega}{k}, \mathbf{D} = [D_1, D_2, 0]^t$. Therefore,

$$(F_{11} - \lambda)D_1 + F_{12}D_2 = 0$$
$$F_{21}D_1 + (F_{22} - \lambda)D_2 = 0$$

or

$$\frac{D_1}{D_2} = \frac{-F_{12}}{(F_{11} - \lambda)} = \frac{-(F_{22} - \lambda)}{F_{21}}.$$

Hence,

$$(F_{11} - \lambda)(F_{22} - \lambda) - F_{12}F_{21} = 0$$

or

$$\lambda^2 - (F_{11} + F_{22})\lambda - F_{12}F_{21} = 0$$

Therefore, $\lambda_\pm = \frac{1}{2}(F_{11}+F_{22}) \pm \frac{1}{2}\sqrt{(F_{11} + F_{22})^2 + F_{12}F_{21}}$. Therefore, the eigenvectors can be found to within a multiplicative constant for $\lambda_\pm$. The eigenvector corresponding to $\lambda = 0$ is a vector that is orthogonal to the row space of $\overline{\mathbf{F}}$. In fact, any $\mathbf{E}_0 = \mathbf{k}e_0$ such that $\mathbf{D}_0 = \overline{\epsilon} \cdot \mathbf{k}e_0$ will be in the null space of $\overline{\mathbf{F}}$, i.e., any longitudinal wave.

§1.16

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi + i\omega\mathbf{A},$$

then,

$$\begin{aligned} \nabla \times \mathbf{B} &= \nabla \times \nabla \times \mathbf{A} \\ &= -i\omega\mu\epsilon\mathbf{E} + \mu\mathbf{J} \\ &= i\omega\mu\epsilon\nabla\phi + \omega^2\mu\epsilon\mathbf{A} + \mu\mathbf{J} \end{aligned}$$

or

$$\nabla^2\mathbf{A} + \omega^2\mu\epsilon\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - i\omega\mu\epsilon\nabla\phi - \mu\mathbf{J}$$

Using Lorentz gauge, we let $\nabla \cdot \mathbf{A} = i\omega\mu\epsilon\phi$, so that

$$[\nabla^2 + \omega^2\mu\epsilon]\mathbf{A} = -\mu\mathbf{J}$$

and

$$\mathbf{E} = i\omega\mathbf{A} - \frac{1}{i\omega\mu\epsilon}\nabla\nabla \cdot \mathbf{A} = i\omega[\mathbf{A} + \frac{1}{\omega^2\mu\epsilon}\nabla\nabla \cdot \mathbf{A}].$$

Using Green's function method, we can show that

$$\mathbf{A}(\mathbf{r}) = \mu \int_V d\mathbf{r}' g(\mathbf{r}-\mathbf{r}')\mathbf{J}(\mathbf{r}')$$

where

$$(\nabla^2 + \omega^2\mu\epsilon)g(\mathbf{r}-\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}').$$

Therefore,

$$\mathbf{E} = i\omega\mu\Big[\overline{\mathbf{I}} + \frac{1}{k^2}\nabla\nabla\Big] \cdot \int_V d\mathbf{r}' g(\mathbf{r}-\mathbf{r}')\mathbf{J}(\mathbf{r}').$$

By using the fact that $\nabla g(\mathbf{r}-\mathbf{r}') = -\nabla' g(\mathbf{r}-\mathbf{r}')$, we have

$$\mathbf{E} = i\omega\mu \int_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') \cdot \Big[\overline{\mathbf{I}} + \frac{\nabla'\nabla'}{k^2}\Big] g(\mathbf{r}-\mathbf{r}')$$

§1.17

(a)

$$\nabla \cdot p^{-1}(\mathbf{r})\nabla g(\mathbf{r},\mathbf{r}') + k^2 g(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}').$$

Multiplying the above by $s(\mathbf{r}')$ and integrating over $\mathbf{r}'$, we have

$$\nabla \cdot p^{-1}(\mathbf{r})\nabla \int_V d\mathbf{r} g(\mathbf{r},\mathbf{r}')s(\mathbf{r}') + k^2 \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}')s(\mathbf{r}') = -s(\mathbf{r}).$$

Therefore, if

$$\psi(\mathbf{r}) = -\int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}')s(\mathbf{r}'),$$

then

$$\nabla \cdot p^{-1}\nabla\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = s(\mathbf{r})$$

(b)

$$\langle \psi_1(\mathbf{r}), s_2(\mathbf{r})\rangle = \langle \psi_2(\mathbf{r}), s_1(\mathbf{r})\rangle$$

or

$$\int_V d\mathbf{r} s_2(\mathbf{r}) \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}')s_1(\mathbf{r}') = \int_V d\mathbf{r} s_1(\mathbf{r}) \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}')s_2(\mathbf{r}')$$

The two integral are similar after exchanging the order of integration. Since $s_1(\mathbf{r})$ and $s_2(\mathbf{r})$ are arbitrary, we have $g(\mathbf{r},\mathbf{r}') = g(\mathbf{r}',\mathbf{r})$.

§1.18

(a) Reciprocity holds over a bounded region as will. Therefore, $[\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')]^t = \overline{\mathbf{G}}(\mathbf{r}',\mathbf{r})$ for a dyadic Green's function defined over a bounded region.

(b)

$$\mathbf{H}(\mathbf{r}) = i\omega\epsilon \int_V d\mathbf{r}'\overline{\mathbf{G}}_m(\mathbf{r},\mathbf{r}') \cdot \mathbf{M}(\mathbf{r}').$$

Therefore,

$$\mathbf{E}_2(\mathbf{r}) = \frac{\nabla \times \mathbf{H}_2}{-i\omega\epsilon} = -\int_V d\mathbf{r}'\nabla \times \overline{\mathbf{G}}_m(\mathbf{r},\mathbf{r}') \cdot \mathbf{M}_2(\mathbf{r}'),$$

$$\mathbf{H}_1(\mathbf{r}) = \frac{\nabla \times \mathbf{E}_1}{i\omega\mu} = \int_V d\mathbf{r}'\nabla \times \overline{\mathbf{G}}_e(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}_1(\mathbf{r}').$$

Since $\langle \mathbf{M}_2, \mathbf{H}_1 \rangle = -\langle \mathbf{J}_1, \mathbf{E}_2 \rangle$, we have

$$\int d\mathbf{r}\mathbf{M}_2(\mathbf{r}) \cdot \int d\mathbf{r}'\nabla \times \overline{\mathbf{G}}_e(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}_1(\mathbf{r}')$$
$$= \int d\mathbf{r}\mathbf{J}_1(\mathbf{r}) \cdot \int d\mathbf{r}'\nabla \times \overline{\mathbf{G}}_m(\mathbf{r},\mathbf{r}') \cdot \mathbf{M}_2(\mathbf{r}')$$

or that

$$[\nabla \times \overline{\mathbf{G}}_e(\mathbf{r},\mathbf{r}')]^t = \nabla \times \overline{\mathbf{G}}_m(\mathbf{r}',\mathbf{r}).$$

§1.19

(a) A solution in a box $a \times b \times d$ with $\hat{n} \cdot \nabla\phi = 0$ on S is

$$\phi = A_0 \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right)$$

where n, m and p are integers, and $(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 + (\frac{p\pi}{d})^2 = k^2$.

(b)

$$\int_V |\nabla\phi|^2 dV = \int_V \nabla\phi^* \cdot \nabla\phi dV$$
$$= \int_V \nabla \cdot (\phi^*\nabla\phi)dV - \int_V \phi^*\nabla^2\phi dV$$
$$= \int_S \hat{n} \cdot (\phi^*\nabla\phi)dV + \int_V k^2|\phi|^2 dV$$
$$= \int_V k^2|\phi|^2 dV$$

by virtue of the boundary condition and that $\nabla^2\phi = -k^2\phi$.

(c) If

$$\phi_1 = A_1 \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right),$$
$$\phi_2 = A_2 \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right),$$

then

$$\delta\phi = \phi_1 - \phi_2 = (A_1 - A_2) \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right),$$

which satisfies the boundry condition that $\hat{n} \cdot \nabla\phi = 0$ on S.

(d) When a, b and d tend to infinity, the number of resonant frequencies satisfying $k^2 = (\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 + (\frac{p\pi}{d})^2$ becomes infinitely dense. Therefore, uniqueness is not guaranteed if we put a source inside this infinitely large box. To guarantee uniqueness, we introduce a small loss, which is equivalent to imposing the Sommerfeld radiation condition.

CHAPTER 2

EXERCISE SOLUTIONS

by G. Otto

§2.1

Using $\mu\nabla\times\mu^{-1}\nabla\times\mathbf{E}-\omega^2\mu\epsilon\mathbf{E}=0$, let $\nabla=\nabla_s+\nabla_z$, $\mathbf{E}=\mathbf{E}_s+\mathbf{E}_z$, and extract the $\hat{z}$ component

$$\begin{aligned}\left(\nabla\times\mu^{-1}\nabla\times\mathbf{E}\right)_z &= \nabla_s\times\mu^{-1}(\nabla\times\mathbf{E})_s,\\ &= \mu^{-1}\nabla_s\times\nabla_s\times\mathbf{E}_z+\mu^{-1}\nabla_s\times\nabla_z\times\mathbf{E}_s,\\ &= -\mu^{-1}\nabla_s^2\mathbf{E}_z+\mu^{-1}\nabla_z(\nabla_s\cdot\mathbf{E}_s).\end{aligned}$$

But $\nabla_s\cdot\mathbf{E}_s=-\frac{1}{\epsilon}\nabla_z\cdot\epsilon\mathbf{E}_z$ in source-free regions. Hence

$$\nabla_s^2\mathbf{E}_z+\frac{\partial}{\partial z}\epsilon^{-1}\frac{\partial}{\partial z}\epsilon\mathbf{E}_z+\omega^2\mu\epsilon\mathbf{E}_z=0,$$

and by duality

$$\nabla_s^2\mathbf{H}_z+\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\mu\mathbf{H}_z+\omega^2\mu\epsilon\mathbf{H}_z=0.$$

Since $\frac{\partial}{\partial z}\epsilon^{-1}\frac{\partial}{\partial z}\epsilon\mathbf{E}_z$ has to be finite and nonsingular, $\epsilon^{-1}\frac{\partial}{\partial z}\epsilon\mathbf{E}_z$ is continuous and then $\epsilon\mathbf{E}_z$ must also be continuous. By duality $\mu^{-1}\frac{\partial}{\partial z}\mu\mathbf{H}_z$ and $\mu\mathbf{H}_z$ are also continuous.

§2.2

Given the Fresnel reflection coefficient for TM waves (2.1.14a):

$$R^{TM}=\frac{\epsilon_2k_{1z}-\epsilon_1k_{2z}}{\epsilon_2k_{1z}+\epsilon_1k_{2z}}.$$

First, R^{TM} vanishes when

$$\begin{aligned}\epsilon_2k_{1z}&=\epsilon_1k_{2z},\\ \epsilon_2k_1\cos\theta_1&=\epsilon_1k_2\cos\theta_2,\\ k_2\cos\theta_1&=k_1\cos\theta_2.\end{aligned}$$

We also need Snell's law $k_1\sin\theta_1=k_2\sin\theta_2$. Combining these two equations yields $\sin 2\theta_1=\sin 2\theta_2$. The nontrivial solution is $\theta_1+\theta_2=\frac{\pi}{2}$. Then $\theta_1=\tan^{-1}\sqrt{\frac{k_2}{k_1}}$ is the Brewster angle.

Next, $|R^{TM}| = 1$ when $k_{2z} = i\alpha$ with $\alpha \geq 0$, then

$$\begin{aligned} k_{2z}^2 &= k_2^2 - k_{2x}^2 - k_{2y}^2 \\ &= k_2^2 - \left(k_{1x}^2 + k_{1y}^2\right); \quad \text{by phase matching} \\ &= k_2^2 - k_1^2 + k_{1z}^2 \\ k_{1z}^2 &\leq k_1^2 - k_2^2; \quad \text{since} \quad k_{2z}^2 \leq 0 \\ k_1^2 \cos^2\theta_1 &\leq k_1^2 - k_2^2. \end{aligned}$$

Then, $\theta_1 \geq \sin^{-1}(k_2/k_1)$ is known as total internal reflection.

§2.3

Given the generalized reflection coefficient for an N-layer medium (2.1.23)

$$\tilde{R}_{i,i+1} = R_{i,i+1} + \frac{T_{i,i+1}\,\tilde{R}_{i+1,i+2}\,T_{i+1,i}\,e^{2ik_{i+1,z}(d_{i+1}-d_i)}}{1 - R_{i+1,i}\,\tilde{R}_{i+1,i+2}\,e^{2ik_{i+1,z}(d_{i+1}-d_i)}}.$$

Use $T_{ij} = 1 + R_{ij}$ to show

$$\begin{aligned} \tilde{R}_{i,i+1} = &\frac{R_{i,i+1} - R_{i,i+1}R_{i+1,i}\tilde{R}_{i+1,i+2}e^{2ik_{i+1,z}(d_{i+1}-d_i)}}{1 - R_{i+1,i}\tilde{R}_{i+1,i+2}e^{2ik_{i+1,z}(d_{i+1}-d_i)}} \\ &+ \frac{(1 + R_{i,i+1})(1 + R_{i+1,i})\,\tilde{R}_{i+1,i+2}\,e^{2ik_{i+1,z}(d_{i+1}-d_i)}}{1 - R_{i+1,i}\,\tilde{R}_{i+1,i+2}\,e^{2ik_{i+1,z}(d_{i+1}-d_i)}}. \end{aligned}$$

Use $R_{ij} = -R_{ji}$ to show

$$\tilde{R}_{i,i+1} = \frac{R_{i,i+1} + \tilde{R}_{i+1,i+2}\,e^{2ik_{i+1,z}(d_{i+1}-d_i)}}{1 + R_{i,i+1}\,\tilde{R}_{i+1,i+2}\,e^{2ik_{i+1,z}(d_{i+1}-d_i)}}.$$

§2.4

(a) Given:

$$A = \left[1 - R_{21}\,\tilde{R}_{23}\,e^{2ik_{2z}(d_2-d_1)}\right]\left[1 - R_{32}\,\tilde{R}_{34}\,e^{2ik_{3z}(d_3-d_2)}\right].$$

Use (2.1.24) for $\tilde{R}_{23}$ and $R_{23} = -R_{32}$ then simplify,

$$\begin{aligned} A &= 1 - R_{32}\,\tilde{R}_{34}\,e^{2ik_{3z}(d_3-d_2)} - R_{21}\,e^{2ik_{2z}(d_2-d_1)}\left[R_{23} + \tilde{R}_{34}\,e^{2ik_{3z}(d_3-d_2)}\right], \\ &= \left[1 - R_{21}\,R_{23}\,e^{2ik_{2z}(d_2-d_1)}\right]\left[1 - \tilde{R}_{34}\,e^{2ik_{3z}(d_3-d_2)}\frac{R_{32} + R_{21}\,e^{2ik_{2z}(d_2-d_1)}}{1 - R_{21}\,R_{23}\,e^{2ik_{2z}(d_2-d_1)}}\right], \\ &= \left[1 - R_{21}\,R_{23}\,e^{2ik_{2z}(d_2-d_1)}\right]\left[1 - \tilde{R}_{32}\,\tilde{R}_{34}\,e^{2ik_{3z}(d_3-d_2)}\right]. \end{aligned}$$

(b) For multilayered media, the generalized transmission coefficient is defined by (2.1.28) and (2.1.26a).

$$\tilde{T}_{1N} = \prod_{j=1}^{N-1} e^{i\Delta_j} S_{j,j+1}\,,$$
$$= \left(\prod_{j=2}^{N-1} e^{i\Delta_j}\right)\left(\prod_{j=1}^{N-2} \frac{T_{j,j+1}}{1 - R_{j+1,j}\,\tilde{R}_{j+1,j+2}\, e^{2i\Delta_{j+1}}}\right) T_{N-1,N}\,,$$

where we define $\Delta_j = k_{jz}(d_j - d_{j-1})$ and $d_0 = d_1$. Similarly,

$$\tilde{T}_{N1} = \left(\prod_{j=2}^{N-1} e^{i\Delta_j}\right)\left(\prod_{j=2}^{N-1} \frac{T_{j+1,j}}{1 - R_{j,j+1}\,\tilde{R}_{j,j-1}\, e^{2i\Delta_j}}\right) T_{21}\,.$$

However, a change of variables results in

$$\prod_{j=2}^{N-1}\left(1 - R_{j,j+1}\,\tilde{R}_{j,j-1}\, e^{2i\Delta_j}\right) = \prod_{j=1}^{N-2}\left(1 - R_{j+1,j+2}\,\tilde{R}_{j+1,j}\, e^{2i\Delta_{j+1}}\right).$$

Now, we can use the identity derived in part (a) to move the tilde

$$\prod_{j=2}^{N-1}\left(1 - R_{j,j+1}\,\tilde{R}_{j,j-1}\, e^{2i\Delta_j}\right) = \prod_{j=1}^{N-2}\left(1 - \tilde{R}_{j+1,j+2}\,R_{j+1,j}\, e^{2i\Delta_{j+1}}\right).$$

Also, for TE waves

$$\prod_{j=1}^{N-1} T_{j+1,j} = \prod_{j=1}^{N-1} \frac{2\mu_j\, k_{j+1,z}}{\mu_j\, k_{j+1,z} + \mu_{j+1}\, k_{jz}} = \frac{\mu_1\, k_{Nz}}{\mu_N\, k_{1z}} \prod_{j=1}^{N-1} T_{j,j+1}\,.$$

Therefore, we have shown

$$\tilde{T}_{N1} = \frac{\mu_1\, k_{Nz}}{\mu_N\, k_{1z}}\tilde{T}_{1N}\,.$$

§2.5

First we need to eliminate $\tilde{R}'(z)$. Using (2.1.35) we find that

$$\tilde{R}'(z) = \frac{d}{dz}\left[\frac{-R_0(z) + \tilde{R}_0(z)}{1 - R_0(z)\tilde{R}_0(z)}\right] = \frac{\tilde{R}_0'(z)\left[1 - R_0^2(z)\right] - R_0'(z)\left[1 - \tilde{R}_0^2(z)\right]}{\left[1 - R_0(z)\tilde{R}_0(z)\right]^2}.$$

Also using (2.1.35) we can show

$$1 - \tilde{R}^2(z) = \frac{\left[1 - R_0^2(z)\right]\left[1 - \tilde{R}_0^2(z)\right]}{\left[1 - R_0(z)\tilde{R}_0(z)\right]^2}.$$

Substituting these two results into (2.1.33), yields

$$\tilde{R}_0'(z)\left[1-R_0^2(z)\right]=R_0'(z)\left[1-\tilde{R}_0^2(z)\right]$$
$$+2ik_z(z)\left[\tilde{R}_0(z)-R_0(z)\right]\left[1-R_0(z)\tilde{R}_0(z)\right]$$
$$+\frac{(k_z/p)'}{2(k_z/p)}\left[1-R_0^2(z)\right]\left[1-\tilde{R}_0^2(z)\right].$$

Now we need to differentiate (2.1.34a) to show

$$R_0'(z)=\frac{-2(k_z(z)/p(z))'(k_{0z}/p_0)}{\left[k_z(z)/p(z)+k_{0z}/p_0\right]^2}.$$

Using (2.1.34a), we can show

$$1-R_0^2(z)=\frac{4(k_z(z)/p(z))(k_{0z}/p_0)}{\left[k_z(z)/p(z)+k_{0z}/p_0\right]^2}.$$

Finally, we obtain

$$\tilde{R}_0'(z)=\frac{2ik_z(z)}{1-R_0^2(z)}\left[\tilde{R}_0(z)-R_0(z)\right]\left[1-R_0(z)\tilde{R}_0(z)\right].$$

§2.6

Equation (2.1.36) can be approximated to first order accuracy in the Runge-Kutta method (Hildebrand, 1976) yielding

$$\tilde{R}_0(z+\Delta)\cong\tilde{R}_0(z)+\Delta\frac{2ik_z(z)}{1-R_0^2(z)}\left[\tilde{R}_0(z)-R_0(z)\right]\left[1-R_0(z)\tilde{R}_0(z)\right].$$

This equation is solved at sample points $\tilde{R}_0(z_m)=\tilde{R}_0(z_0+m\Delta)$ for $m=1,2,\ldots N$ with the initial condition $\tilde{R}_0(z_0)=0$. The number of complex operations per point (ignoring additions) is 7 (including the cost of computing $R_0(z_m)$. This method approximates the continuous medium by a piecewise linear medium.

On the other hand, a finely layered medium is modeled by (2.1.24). The number of complex operations per discontinuity is dominated by the cost of the exp(·) function. This is machine-dependent and typically many complex multiplications in a Taylor series expansion polynomial approximation, or interpolation from table-lookup. In general, this method will be less efficient computationally. Notice that this method approximates the continuous medium by a piecewise constant medium.

§2.7

For a normally incident TE wave in free space ($z<0$) the field can be described by

$$E_{1y}=C_1\left(e^{ik_0z}+e^{-ik_0z}R\right),\qquad k_0=\omega^2\mu_0\epsilon_0.$$

For $z > 0$ we have the Airy equation solution

$$E_{2y} = D_1 Ai(-\eta) + D_2 Bi(-\eta), \quad \eta(z) = \left(\omega^2\mu_0 a\right)^{1/3}\left(z + \frac{\epsilon_0}{a}\right).$$

Now we can introduce a small loss. In order to have an exponentially decaying field as $z \to \infty$, it is necessary that $D_i = iD_2$. Finally, we must match the boundary conditions at z=0.

$$\text{e.g.} \qquad E_{1y} = E_{2y}, \qquad \frac{d}{dz}E_{1y} = \frac{d}{dz}E_{2y}.$$

This gives us a solution for R, the reflection coefficient,

$$R = \frac{1-\rho_0}{1+\rho_0}, \quad \text{where} \quad \rho_0 = \frac{-\left(\omega^2\mu_0 a\right)^{1/3}\left[iAi'(-\eta_0) + Bi'(-\eta_0)\right]}{ik_0\left[iAi(-\eta_0) + Bi(-\eta_0)\right]}$$

and $\eta_0 = \eta(z = 0)$.

§2.8

(a) The general solution to Equation (2.2.2) in the $\exp(j\omega t)$ time convention can be written

$$\phi(\rho, t) = e^{j\omega t}\left[C_1 H_0^{(1)}(k_\rho\rho) + C_2 H_0^{(2)}(k_\rho\rho)\right].$$

By evaluating the asymptotic approximation of the Hankel Functions as $k_\rho\rho \to \infty$, the outgoing wave can be identified as $\exp(j\omega t - jk_\rho\rho)$ so the solution satisfying the radiation condition is

$$\phi(\rho) = C_2 H_0^{(2)}(k_\rho\rho).$$

To evaluate C_2, integrate (2.2.2) over a small disk of radius δ and take the limit as $\delta \to 0$. We find

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{\Delta s} ds C_2 \nabla^2 H_0^{(2)}(k_\rho\rho) = -1, \\
&= \lim_{\delta\to 0}\oint_C dl C_2\, \hat{\rho}\cdot\nabla H_0^{(2)}(k_\rho\rho), \\
&= \lim_{\delta\to 0} 2\pi\delta C_2 \left(\frac{\partial}{\partial\rho}H_0^{(2)}(k_\rho\rho)\right)\Big|_{\rho=\delta}.
\end{aligned}$$

and using $H_0^{(2)\prime}(x) = -H_1^{(2)}(x) \sim -2j\pi^{-1}x^{-1}$ as $x \to 0$, we find $C_2 = -j/4$.

(b) Using (2.2.5) and (2.2.6) in (2.2.1) we find

$$\left[\frac{\partial^2}{\partial y^2} + k_\rho^2 - k_x^2\right]\tilde{\phi}(k_x, y) = -\delta(y),$$

so $\tilde{\phi}(k_x, y) = \frac{-j}{2k_y}e^{-jk_y|y|}$ is the outgoing wave with $k_y = \sqrt{k_\rho^2 - k_x^2}$. Then, the plane-wave expansion of the Hankel function is

$$\phi(x,y) = \frac{-j}{4\pi}\int_{-\infty}^{\infty} dk_x \frac{e^{jk_x x - jk_y|y|}}{k_y} = \frac{-j}{4}H_0^{(2)}(k_\rho \rho).$$

To introduce loss we let $\Im m[k_y] < 0$. Then the Fourier inversion contour follows the conjugate path of the contour in Figure 2.2.2.

§2.9

(a) The raising operator acts on Bessel cylinder functions $B_n(k_\rho \rho)e^{in\phi}$:

$$\begin{aligned}
&-\frac{1}{k_\rho}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] B_n(k_\rho \rho)e^{in\phi} \\
&= -B_n'(k_\rho \rho)e^{in\phi}\left[\frac{x}{\rho} + \frac{iy}{\rho}\right] - \frac{in}{k_\rho}B_n(k_\rho \rho)e^{in\phi}\left[\frac{-y}{\rho^2} + \frac{ix}{\rho^2}\right], \\
&= -B_n'(k_\rho \rho)e^{i(n+1)\phi} + \frac{n}{k_\rho \rho}B_n(k_\rho \rho)e^{i(n+1)\phi}.
\end{aligned}$$

Using the recurrence formula (1.2.34) we find

$$-\frac{1}{k_\rho}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] B_n(k_\rho \rho)e^{in\phi} = B_{n+1}(k_\rho \rho)e^{i(n+1)\phi}.$$

(b) Using the raising operator, we know

$$J_1(k_\rho \rho)e^{i\phi} = -\frac{1}{k_\rho}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] J_0(k_\rho \rho).$$

Then applying the integral representation (2.2.15)

$$\begin{aligned}
J_1(k_\rho \rho)e^{i\phi} &= -\frac{i}{2\pi}\int_0^{2\pi} d\alpha\, e^{ik_\rho \rho\cos(\alpha-\phi)} \\
&\quad \cdot \left\{\cos(\alpha-\phi)\left[\frac{x}{\rho} + \frac{iy}{\rho}\right] + \rho\sin(\alpha-\phi)\left[\frac{-y}{\rho^2} + \frac{ix}{\rho^2}\right]\right\}, \\
&= -\frac{i}{2\pi}\int_0^{2\pi} d\alpha\, e^{ik_\rho \rho\cos(\alpha-\phi)+i\alpha}, \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, e^{ik_\rho \rho\cos(\alpha-\phi)+i\alpha-i\frac{\pi}{2}}.
\end{aligned}$$

Similarly, by applying the raising operator n times we obtain

$$J_n(k_\rho\rho)e^{in\phi} = \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, e^{ik_\rho\rho\cos(\alpha-\phi)+in\alpha-in\frac{\pi}{2}}\,.$$

(c) A lowering operator acts on $B_n(k_\rho\rho)e^{in\phi}$ such that

$$\frac{1}{k_\rho}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] B_n(k_\rho\rho)e^{in\phi} = B_{n-1}(k_\rho\rho)e^{i(n-1)\phi}\,.$$

§2.10

(a) In cylindrical coordinates let

$$x = \rho\cos\phi, \qquad k_x = k_\rho\cos\alpha,$$
$$y = \rho\sin\phi, \qquad k_y = k_\rho\sin\alpha.$$

Use $k_x x + k_y y = k_\rho\rho(cos\phi\cos\alpha + \sin\phi\sin\alpha) = k_\rho\rho\cos(\alpha-\phi)$ and $dxdy = \rho d\rho d\phi$ for surface elements.

(b) Let $f(\rho,\phi) = \frac{1}{2\pi}\sum f_n(\rho)e^{in\phi+in\frac{\pi}{2}}$ and $F(k_\rho,\alpha) = \sum F_n(k_\rho)e^{in\alpha}$. Then

$$\sum_{n=-\infty}^{\infty} F_n(k_\rho)e^{in\alpha} = \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi e^{-ik_\rho\rho\cos(\alpha-\phi)}\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f_n(\rho)e^{in\phi+in\frac{\pi}{2}}\,.$$

From the plane wave expansion of the Bessel function (2.2.17) we can show

$$J_n(-k_\rho\rho)e^{in\phi} = \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, e^{-ik_\rho\rho\cos(\alpha-\phi)+in\alpha-in\frac{\pi}{2}}\,.$$

Then the Fourier transform simplifies to

$$\sum_{n=-\infty}^{\infty} F_n(k_\rho)e^{in\alpha} = \sum_{n=-\infty}^{\infty}\int_0^\infty d\rho\,\rho f_n(\rho)\,e^{-in\pi}J_n(-k_\rho\rho)e^{in\alpha}\,.$$

This equation decouples due to the orthogonality of the harmonics n. Using the identity $J_n(-k_\rho\rho) = e^{in\pi}J_n(k_\rho\rho)$, we find

$$F_n(k_\rho) = \int_0^\infty d\rho\,\rho f_n(\rho)J_n(k_\rho\rho).$$

By symmetry,

$$f_n(\rho) = \int_0^\infty dk\rho\, k_\rho F_n(k_\rho)J_n(k_\rho\rho).$$

(c) Using the Hankel transform pair we know

$$F_n(k_\rho) = \int_0^\infty d\rho\, \rho J_n(k_\rho\rho) \int_0^\infty dk'_\rho\, k'_\rho F_n(k'_\rho) J_n(k'_\rho\rho),$$
$$= \int_0^\infty dk'_\rho\, k'_\rho F_n(k'_\rho) \int_0^\infty d\rho\, \rho J_n(k_\rho\rho) J_n(k'_\rho\rho).$$

This implies that

$$\int_0^\infty d\rho\, \rho J_n(k_\rho\rho) J_n(k'_\rho\rho) = \frac{\delta(k'_\rho - k_\rho)}{k'_\rho}.$$

Likewise, by symmetry

$$\int_0^\infty dk_\rho\, k_\rho J_n(k_\rho\rho) J_n(k_\rho\rho') = \frac{\delta(\rho' - \rho)}{\rho'}.$$

§2.11

Let $k_\rho = k_0 + \delta e^{i\theta}$ and $k_0 = |k_0|e^{i\alpha}$. To first order we can approximate

$$k_z \sim (-2|k_0|\delta e^{i(\alpha+\theta)})^{1/2}$$
$$= (2|k_0|\delta)^{1/2} e^{i(\frac{\alpha+\theta+\pi}{2})}$$

The Sommerfeld branch cut is defined by $\Im m[k_z] = 0$. In Figure 2.2.7,

point A:	$\theta = -\alpha - \pi$	so that	$\Im m[k_z] = 0$ but	$\Re e[k_z] > 0$
point C:	$\theta = -\alpha + \pi$	so that	$\Im m[k_z] = 0$ but	$\Re e[k_z] < 0$
point B:	$\theta = -\alpha$	so that	$\Re e[k_z] = 0$ but	$\Im m[k_z] > 0$

Thus the points A, B, C map from the k_z plane in Figure 2.2.7b to the k_ρ plane as shown in Figure 2.2.7a.

§2.12

A branch cut can be defined by $\Re e[k_z] = 0$ with the upper Riemann sheet as $\Re e[k_z] > 0$. This condition on (2.2.32) results in the hyperbola

$$k'_\rho k''_\rho = k'_0 k''_0$$
$$k'^2_\rho - k''^2_\rho > k'^2_0 - k''^2_0$$

The mapping is shown in Figure for Exercise Solution 2.12.

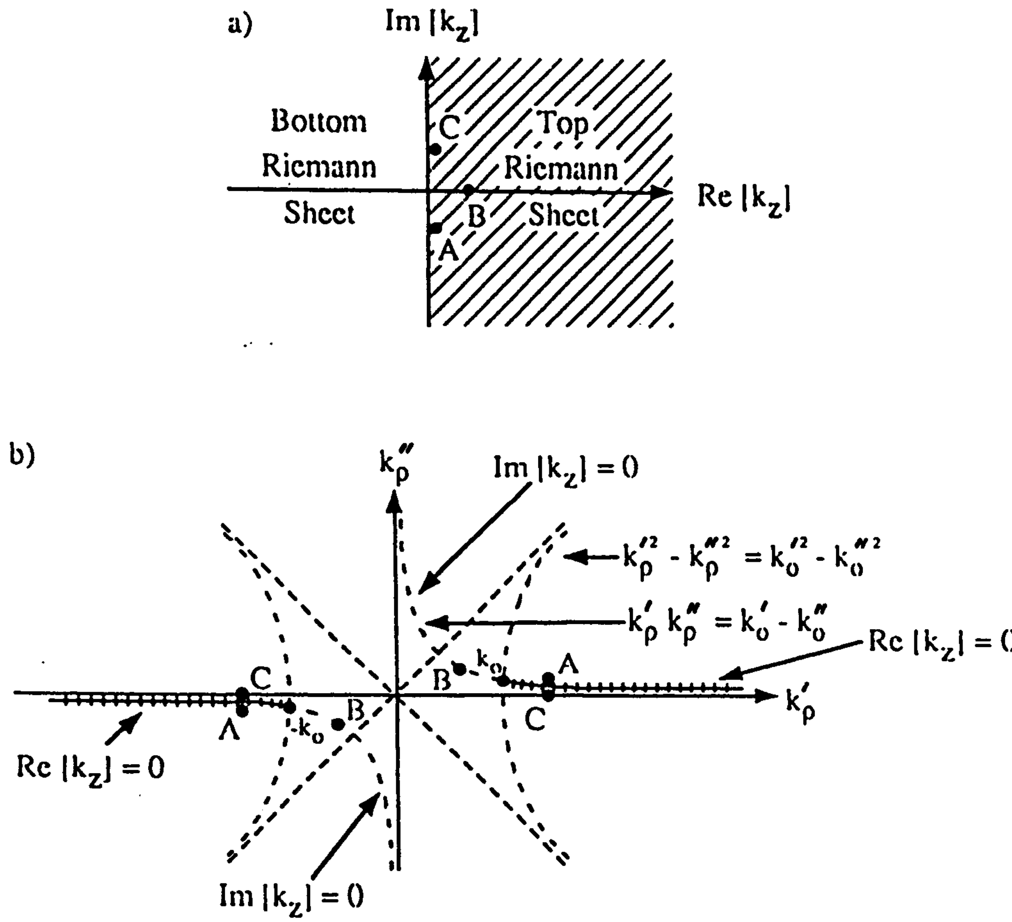

Figure for Exercise Solution 2.12

point A:	$\theta = -\alpha - 2\pi$	so that $\Re e[k_z] = 0$	but $\Im m[k_z] < 0$
point C:	$\theta = -\alpha$	so that $\Re e[k_z] = 0$	but $\Im m[k_z] > 0$
point B:	$\theta = -\alpha - \pi$	so that $\Im m[k_z] = 0$	but $\Re e[k_z] > 0$

§2.13

(a) Any field can be written as a Fourier transform so that

$$\phi(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z \tilde{\phi}(x, y, k_z) e^{ik_z z} .$$

Then substituting in (2.2.19) and using (2.2.6) for $\delta(z)$ we find

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_\rho^2\right] \tilde{\phi}(x, y, k_z) = -\delta(x)\,\delta(y),$$

where $k_\rho^2 = k_0^2 - k_z^2$. The solution satisfying the Sommerfeld radiation condition, see (2.2.4), is

$$\phi(x, y, k_z) = \frac{i}{4} H_0^{(1)}(k_\rho \rho),$$

where $\rho = \sqrt{x^2 + y^2}$. Then

$$\phi(x, y, z) = \frac{i}{8\pi} \int_{-\infty}^{\infty} dk_z H_0^{(1)}(k_\rho \rho) e^{ik_z z},$$

so the spherical wave can be expanded in cylindrical waves by

$$\frac{e^{ikr}}{r} = \frac{i}{2} \int_{-\infty}^{\infty} dk_z\, H_0^{(1)}(k_\rho \rho) e^{ik_z z}.$$

The inversion path is shown in Figure 2.2.2 (replace k_x with k_z) with $\Im m[k_\rho] > 0$ in order to satisfy the radiation condition

(b) Given the expansion (2.2.31) and the contour in Figure 2.2.5, Cauchy's theorem and Jordan's lemma allow deformation of the SIP to a clockwise integration wrapping around the branch cut (path containing points C, B, A, respectively, in Figure 2.2.7), then

$$\frac{e^{ikr}}{r} = \frac{i}{2} \int_C dk_\rho\, \frac{k_\rho}{k_z} H_0^{(1)}(k_\rho \rho) e^{ik_z |z|}.$$

By a change of variables $k_z^2 = k_0^2 - k_\rho^2$, then the differential surface element becomes $dk_z k_z = -dk_\rho k_\rho$. Now we have

$$\frac{e^{ikr}}{r} = \frac{-i}{2} \int_{\infty}^{-\infty} dk_z\, H_0^{(1)}(k_\rho \rho) e^{ik_z |z|},$$

where we require $\Im m[k_z] > 0$ to stay on the upper Riemann sheet. Finally

$$\frac{e^{ikr}}{r} = \frac{i}{2} \int_{-\infty}^{\infty} dk_z\, e^{ik_z z} H_0^{(1)}(k_\rho \rho).$$

The modulus sign has been removed by a change of variables $k_z' = -k_z$ for $z < 0$. Furthermore, the inversion path is described in part (a) above

§2.14

(a) For a current loop

$$\mathbf{J} = -\hat{\phi} I\, \delta(\rho - \rho')\, \delta(z - z').$$

In cylindrical coordinates, the $\hat{\phi}$ component of the vector wave equation (1.3.44) is

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right) - \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\phi) - \frac{1}{\rho}\frac{\partial E_\rho}{\partial \phi}\right) - k^2 E_\phi = i\omega\mu J_\phi.$$

However, since the source is axially symmetric and radiating in a homogeneous medium, $\partial/\partial\phi = 0$. Then

$$\frac{\partial}{\partial \rho}\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\phi) + \frac{\partial^2}{\partial z^2}E_\phi + k^2 E_\phi = -i\omega\mu J_\phi.$$

Thus, the current loop excites only the ϕ component of the electric field, that satisfies

$$\left[\rho\frac{\partial}{\partial \rho}\frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + k^2\right]\rho E_\phi = i\omega\mu I \rho\, \delta(\rho - \rho')\, \delta(z - z').$$

(b) The above equation can be rewritten

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + k^2\right] E_\phi = i\omega\mu I\, \delta(\rho - \rho')\, \delta(z - z').$$

This equation resembles Bessel's equation with $n = 1$. Therefore define the Hankel transform of the field by

$$E_\phi = \int_0^\infty dk_\rho\, k_\rho J_1(k_\rho \rho)\tilde{E}_\phi(k_\rho, z)$$

and use the identity for $\delta(\rho - \rho')$ derived in Exercise 2.10 (c). Then we obtain

$$\int_0^\infty dk_\rho\, k_\rho J_1(k_\rho\rho)\left[\frac{\partial^2}{\partial z^2} + k^2 - k_\rho^2\right]\tilde{E}_\phi(k_\rho, z)$$

$$= i\omega\mu I \rho'\, \delta(z - z') \int_0^\infty dk_\rho\, k_\rho J_1(k_\rho\rho) J_1(k_\rho\rho').$$

By completeness of the Hankel transform we know

$$\left[\frac{\partial^2}{\partial z^2} + k_z^2\right]\tilde{E}_\phi(k_\rho, z) = i\omega\mu I \rho'\, J_1(k_\rho\rho')\delta(z - z').$$

This is the scalar Helmholtz equation with the solution given by (2.2.9)

$$\tilde{E}_\phi(k_\rho, z) = -i\omega\mu I\rho' J_1(k_\rho\rho') \left(\frac{ie^{ik_z|z-z'|}}{2k_z}\right).$$

So the electric field is given by

$$E_\phi = \frac{\omega\mu I\rho'}{2} \int_0^\infty dk_\rho \frac{k_\rho}{k_z} J_1(k_\rho\rho') J_1(k_\rho\rho) e^{ik_z|z-z'|}.$$

(c) For a current disk

$$\mathbf{J} = \hat{\phi} J(\rho)\, \delta(z-z').$$

Using the idea of convolution, the current disk is equivalent to an integral of current loops

$$\mathbf{J} = \hat{\phi} \int_0^\infty d\rho'\, J(\rho')\delta(\rho-\rho')\, \delta(z-z').$$

The solution of part (b) above allows one to write

$$E_\phi(\rho, z) = -\frac{\omega\mu}{2} \int_0^\infty dk_\rho \frac{k_\rho}{k_z} J_1(k_\rho\rho) e^{ik_z|z-z'|} \int_0^\infty d\rho'\rho'\, J_1(k_\rho\rho') J(\rho'),$$

$$E_\phi(\rho, z) = -\frac{\omega\mu}{2} \int_0^\infty dk_\rho \frac{k_\rho}{k_z} J_1(k_\rho\rho) \tilde{J}(k_\rho) e^{ik_z|z-z'|}.$$

§2.15

(a) For a VED over a three-layer medium we can use (2.3.6) to find the field in the slab region

$$\epsilon_2 E_{2z} = \frac{-I\ell}{8\pi\omega} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{1z}} H_0^{(1)}(k_\rho\rho) A_2 \left[e^{-ik_{2z}z} + R_{23}^{TM} e^{ik_{2z}(z+2d_2)}\right].$$

The amplitude A_2 is given by (2.1.26)

$$A_2 e^{ik_{2z}d_1} = \frac{T_{12}^{TM} A_1 e^{ik_{1z}d_1}}{1 - R_{21}^{TM} R_{23}^{TM}\, e^{2ik_{2z}(d_2-d_1)}},$$

where we deduce $A_1 = 1$ from (2.3.4). Also, $R_{i,i+1}^{TM} = \frac{\epsilon_{i+1}k_{iz} - \epsilon_i k_{i+1,z}}{\epsilon_{i+1}k_{iz} + \epsilon_i k_{i+1,z}}$ and $T_{12}^{TM} = 1 + R_{12}^{TM}$.

(b) We can express the vector field as

$$\mathbf{E}(\mathbf{r}) = \int_{-\infty}^{\infty} dk_\rho \tilde{\mathbf{E}}(k_\rho, \mathbf{r}).$$

From part (a) above we know the z component

$$\tilde{E}_{2z}(k_\rho, \mathbf{r}) = -\frac{I\ell}{8\pi\omega\epsilon_2}\frac{k_\rho^3}{k_{1z}} H_0^{(1)}(k_\rho \rho) A_2 \left[e^{-ik_{2z}z} + R_{23}^{TM} e^{ik_{2z}(z+2d_2)}\right].$$

From (2.3.17a) we can relate the transverse field to the z component

$$\tilde{\mathbf{E}}_{2s}(k_\rho, \mathbf{r}) = \frac{1}{k_\rho^2}\nabla_s \frac{\partial \tilde{E}_{2z}}{\partial z}.$$

Utilizing axial symmetry we notice $\partial/\partial\phi = 0$ so $E_{2\phi} = 0$ but

$$E_{2\rho} = \frac{-iI\ell}{8\pi\omega\epsilon_2}\int_{-\infty}^{\infty} dk_\rho k_\rho^2 \frac{k_{2z}}{k_{1z}} H_1^{(1)}(k_\rho \rho) A_2 \left[e^{-ik_{2z}z} - R_{23}^{TM} e^{ik_{2z}(z+2d_2)}\right].$$

Similarly, $\tilde{\mathbf{H}}_{2s}(k_\rho, \mathbf{r}) = k_\rho^{-2} i\omega\epsilon_2 \hat{z} \times \nabla_s \tilde{E}_{2z}$. Thus $H_{2\rho} = 0$ but

$$H_{2\phi} = \frac{iI\ell}{8\pi}\int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^2}{k_{1z}} H_1^{(1)}(k_\rho \rho) A_2 \left[e^{-ik_{2z}z} - R_{23}^{TM} e^{ik_{2z}(z+2d_2)}\right].$$

§2.16

(a) For a VED embedded in a three layer medium, the field in the slab region is

$$E_{2z} = \frac{-I\ell}{8\pi\omega\epsilon_2}\int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{2z}} H_0^{(1)}(k_\rho \rho)\left[e^{ik_{2z}|z-z'|} + A_2 e^{-ik_{2z}z} + B_2 e^{ik_{2z}z}\right],$$

where z' is the source point and A_2 and B_2 are chosen to satisfy the constraint conditions at both boundaries $z = -d_1$ and $z = -d_2$ (see section 2.4)

$$A_2 e^{ik_{2z}d_1} = R_{21}^{TM}\left[e^{ik_{2z}(-d_1-z')} + B_2 e^{-ik_{2z}d_1}\right],$$

$$B_2 e^{-ik_{2z}d_2} = R_{23}^{TM}\left[e^{ik_{2z}(d_2+z')} + A_2 e^{ik_{2z}d_2}\right].$$

These two equations can be solved simultaneously yielding

$$A_2 = R_{21}^{TM} e^{-2ik_{2z}d_1} \frac{e^{-ik_{2z}z'} + R_{23}^{TM} e^{ik_{2z}(z'+2d_2)}}{1 - R_{21}^{TM} R_{23}^{TM} e^{2ik_{2z}(d_2-d_1)}},$$

$$B_2 = R_{23}^{TM} e^{2ik_{2z}d_2} \frac{e^{ik_{2z}z'} + R_{21}^{TM} e^{-ik_{2z}(z'+2d_1)}}{1 - R_{21}^{TM} R_{23}^{TM} e^{2ik_{2z}(d_2-d_1)}}.$$

(b) When regions 1 and 3 are perfect electric conductors, $|\epsilon| \to \infty$ so $R_{21}^{TM} = R_{23}^{TM} = 1$. Then

$$A_2 = e^{-2ik_{2z}d_1} \frac{e^{-ik_{2z}z'} + e^{ik_{2z}(z'+2d_2)}}{1 - e^{2ik_{2z}(d_2-d_1)}},$$

$$B_2 = e^{2ik_{2z}d_2} \frac{e^{ik_{2z}z'} + e^{-ik_{2z}(z'+2d_1)}}{1 - e^{2ik_{2z}(d_2-d_1)}}.$$

After simplifying we find

$$E_{2z} = \frac{-I\ell}{8\pi\omega\epsilon_2} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{2z}} H_0^{(1)}(k_\rho\rho) \frac{2e^{ik_{2z}(d_2-d_1)}}{1 - e^{2ik_{2z}(d_2-d_1)}} \cdot [\cos(|z-z'| + d_1 - d_2) + \cos(z + z' + d_1 + d_2)].$$

§2.17

(a) Given a HED in free space

$$\mathbf{J} = \hat{x} I\ell\delta(x - x')\delta(y - y')\delta(z).$$

We know the TM component is characterized by (2.3.7a)

$$E_z = \frac{iI\ell}{4\pi\omega\epsilon} \frac{\partial^2}{\partial z \partial x} \frac{e^{ikr}}{r}.$$

Using the Weyl identity (2.2.27) the integral representation is

$$E_z = \frac{\pm I\ell}{8\pi^2\omega\epsilon} \iint_{-\infty}^{\infty} dk_x dk_y k_x e^{ik_x(x-x')+ik_y(y-y')+ik_z|z|}, \quad z \gtrless 0$$

(b) Given a current sheet in free space

$$\mathbf{J} = \hat{x}\delta(z) J_s(x, y).$$

Using the idea of convolution, the current sheet is equivalent to an integral of HEDs

$$\mathbf{J} = \hat{x} \iint_{-\infty}^{\infty} dx' dy' \delta(x - x')\delta(y - y')\delta(z) J_s(x', y').$$

Then the TM component is

$$E_z = \iint_{-\infty}^{\infty} dx' dy' J_s(x', y') \cdot \left\{ \pm \frac{1}{8\pi^2\omega\epsilon} \iint_{-\infty}^{\infty} dk_x dk_y k_x e^{ik_x(x-x')+ik_y(y-y')+ik_z|z|} \right\}, \qquad z \gtrless 0.$$

We can exchange the order of integration and recognize the two-dimensional Fourier-transform $\tilde{J}_s(k_x,k_y)$ of $J_s(x,y)$. Then,

$$E_z = \frac{\pm 1}{8\pi^2\omega\epsilon}\iint\limits_{-\infty}^{\infty} dk_x dk_y k_x \tilde{J}_s(k_x,k_y) e^{ik_x x + ik_y y + ik_z|z|}, \quad z \gtrless 0.$$

§2.18

(a) Substitute (2.4.5) into (2.4.2) for source region m in Figure 2.4.1

$$F(z,z') = e^{ik_{mz}|z-z'|} + e^{-ik_{mz}(z+d_{m-1})}\tilde{R}_{m,m-1}\tilde{M}_m\Big[e^{-ik_{mz}(z'+d_{m-1})} + e^{ik_{mz}(d_m-d_{m-1})}\tilde{R}_{m,m+1} + e^{ik_{mz}(z'+d_m)}\Big] + e^{ik_{mz}(z+d_m)}\tilde{R}_{m,m+1}\tilde{M}_m \cdot \Big[e^{+ik_{mz}(z'+d_m)} + e^{ik_{mz}(d_m-d_{m-1})}\tilde{R}_{m,m-1}e^{-ik_{mz}(z'+d_{m-1})}\Big].$$

Next, bring all the terms under a common denominator

$$F(z,z') = \tilde{M}_m\Big\{e^{ik_{mz}|z-z'|} - e^{ik_{mz}|z-z'|}\tilde{R}_{m,m+1}\tilde{R}_{m,m-1}e^{2ik_{mz}(d_m-d_{m-1})} + \tilde{R}_{m,m-1}e^{-ik_{mz}(z+z'+2d_{m-1})} + \tilde{R}_{m,m+1}e^{ik_{mz}(z+z'+2d_m)} + \tilde{R}_{m,m-1}\tilde{R}_{m,m+1}\Big[e^{-ik_{mz}(z-z'-2d_m+2d_{m-1})} + e^{ik_{mz}(z-z'+2d_m-2d_{m-1})}\Big]\Big\}.$$

Now, we can identify 2 solutions in the source region m

$$F_+(z,z') = \tilde{M}_m\Big[e^{ik_{mz}z} + e^{-ik_{mz}(z+2d_{m-1})}\tilde{R}_{m,m-1}\Big]\Big[e^{-ik_{mz}z'} + e^{ik_{mz}(z'+2d_m)}\tilde{R}_{m,m+1}\Big], \qquad z > z'$$

$$F_-(z,z') = \tilde{M}_m\Big[e^{-ik_{mz}z} + e^{ik_{mz}(z+2d_m)}\tilde{R}_{m,m+1}\Big]\Big[e^{ik_{mz}z'} + e^{-ik_{mz}(z'+2d_{m-1})}\tilde{R}_{m,m-1}\Big], \qquad z < z'$$

(b) For regions $n < m$ we use the generalized transmission coefficient and constraint conditions as in (2.4.13) to arrive at

$$A_n^+ = \Big[1 - \tilde{R}_{n,n+1}\tilde{R}_{n,n-1}e^{2ik_{nz}(d_n-d_{n-1})}\Big]^{-1} e^{ik_{nz}d_n}\tilde{T}_{mn}e^{-ik_{mz}d_{m-1}}A_m^+.$$

By using (2.4.9), which should state (incorrect in the first edition)

$$A_m^+ = \Big[e^{-ik_{mz}z'} + e^{ik_{mz}(z'+2d_m)}\tilde{R}_{m,m+1}\Big]\tilde{M}_m.$$

Then the field above the source has the form

$$F_+(z,z') = \tilde{M}_m\tilde{M}_n\left[e^{ik_{nz}z} + \tilde{R}_{n,n-1}e^{-ik_{nz}(z+2d_{n-1})}\right]e^{ik_{nz}d_n} \cdot \tilde{T}_{mn}e^{-ik_{mz}d_{m-1}}\left[e^{-ik_{mz}z'} + \tilde{R}_{m,m+1}e^{ik_{mz}(z'+2d_m)}\right], \qquad z > z'.$$

In a similar manner, for $n > m$ we define A_n^- as in (2.4.10). From (2.4.6b) we notice

$$A_m^- = \left[e^{ik_{mz}z'} + e^{-ik_{mz}(z'+2d_{m-1})}\tilde{R}_{m,m-1}\right]\tilde{M}_m.$$

Once again we use the constraint conditions and the generalized reflection coefficient to solve for A_n^-

$$A_n^- = \left[1 - \tilde{R}_{n,n-1}\tilde{R}_{n,n+1}e^{2ik_{nz}(d_n-d_{n-1})}\right]^{-1} e^{-ik_{nz}d_{n-1}}\tilde{T}_{mn}e^{ik_{mz}d_m}A_m^-.$$

Finally, the field below the source has the form

$$F_-(z,z') = \tilde{M}_m\tilde{M}_n\left[e^{-ik_{nz}z} + \tilde{R}_{n,n+1}e^{ik_{nz}(z+2d_n)}\right]e^{-ik_{nz}d_{n-1}} \cdot \tilde{T}_{mn}e^{ik_{mz}d_m}\left[e^{ik_{mz}z'} + e^{-ik_{mz}(z'+2d_{m-1})}\tilde{R}_{m,m-1}\right], \qquad z < z'.$$

(c) The reciprocal nature of the solution can be investigated by evaluating

$$\frac{F_+(z,z')}{F_-(z',z)}.$$

Using the generalized transmission coefficient solutions (2.4.15) and (2.4.16) we find

$$\frac{F_+(z,z')}{F_-(z',z)} = \frac{\tilde{T}_{mn}}{\tilde{T}_{nm}}.$$

We have previously shown in Exercise 2.4b that the generalized transmission coefficients for TE waves are reciprocal such that

$$\frac{\mu_1}{k_{1z}}\tilde{T}_{1N} = \frac{\mu_N}{k_{Nz}}\tilde{T}_{N1}.$$

Then a TE field is reciprocal such that

$$\frac{\mu_m}{k_{mz}}F_+(z,z') = \frac{\mu_n}{k_{nz}}F_-(z',z), \quad z' \in R_m,\ z \in R_n$$

where R_i stands for region i.

§2.19

Given an HED above an interface with $d_1 = 0$ in (2.3.9), then the field above the dipole ($z > 0$) is

$$E_{1z} = \frac{iI\ell}{8\pi\omega\epsilon_1}\cos\phi\int_{-\infty}^{\infty} dk_\rho k_\rho^2 H_1^{(1)}(k_\rho\rho)[e^{ik_{1z}z} - \tilde{R}_{12}^{TM}e^{ik_{1z}z}],$$

$$H_{1z} = \frac{iI\ell}{8\pi}\sin\phi\int_{-\infty}^{\infty} dk_\rho\frac{k_\rho^2}{k_{1z}}H_1^{(1)}(k_\rho\rho)[e^{ik_{1z}z} + \tilde{R}_{12}^{TE}e^{ik_{1z}z}].$$

Write

$$H_1^{(1)}(k_\rho\rho) = \left[H_1^{(1)}(k_\rho\rho)/H_0^{(1)}(k_\rho\rho)\right] H_0^{(1)}(k_\rho\rho),$$

where $H_1^{(1)}(k_\rho\rho)/H_0^{(1)}(k_\rho\rho) \sim e^{-i\pi/2}$ as $\rho \to \infty$ is slowly varying in space and $H_0^{(1)}(k_\rho\rho)e^{ik_{1z}z} \sim (0.5\pi k_\rho\rho)^{-\frac{1}{2}} e^{i(-\pi/4+k_\rho\rho+k_{1z}z)}$ as $\rho \to \infty$ is rapidly varying in space. Now the stationary phase point of the rapidly varying part is given by

$$\frac{d}{dk_\rho}\left[k_\rho\rho + k_{1z}z\right]\bigg|_{k_\rho=k_{\rho s}} = 0.$$

Using $k_{1z} = \sqrt{k_1^2 - k_\rho^2}$ and defining $\theta = \tan^{-1}(\rho/z)$ we arrive at

$$k_{\rho s} = k_1 \sin\theta.$$

We can evaluate the slowly vary part of the integrand at the stationary part, resulting in

$$E_{1z} \sim \frac{iI\ell}{8\pi\omega\epsilon_1}\cos\phi\frac{H_1^{(1)}(k_{\rho s}\rho)}{H_0^{(1)}(k_{\rho s}\rho)}k_{\rho s}k_{1zs}\left[1 - \tilde{R}_{12}^{TM}(k_{1zs})\right] \cdot \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho}{k_{1z}} H_0^{(1)}(k_\rho\rho)e^{ik_{1z}z}, \qquad \rho \to \infty$$

$$H_{1z} \sim \frac{iI\ell}{8\pi}\sin\phi\frac{H_1^{(1)}(k_{\rho s}\rho)}{H_0^{(1)}(k_{\rho s}\rho)}k_{\rho s}\left[1 + \tilde{R}_{12}^{TE}(k_{1zs})\right] \cdot \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho}{k_{1z}} H_0^{(1)}(k_\rho\rho)e^{ik_{1z}z}, \qquad \rho \to \infty$$

Finally, we use the Sommerfeld identity (2.2.31) to arrive at the leading order stationary phase expansion

$$E_{1z} \sim \frac{-iI\ell k_1^2}{4\pi\omega\epsilon_1}\cos\phi\sin\theta\cos\theta\left[1 - \tilde{R}_{12}^{TM}(k_1\cos\theta)\right]\frac{e^{ik_1 r}}{r}, \qquad \rho \to \infty$$

$$H_{1z} \sim \frac{-iI\ell k_1}{4\pi}\sin\phi\sin\theta\left[1 + \tilde{R}_{12}^{TE}(k_1\cos\theta)\right]\frac{e^{ik_1 r}}{r}, \qquad \rho \to \infty$$

§2.20

(a) We can show the odd integrals (2.5.20a) vanish by a change of variables $t = -s$

$$\int_{-\infty}^{\infty} s^{2m+1} e^{-\lambda s^2} ds = \int_{-\infty}^{0} s^{2m+1} e^{-\lambda s^2} ds + \int_{0}^{\infty} s^{2m+1} e^{-\lambda s^2} ds,$$

$$= (-1)^{2m+2} \int_{\infty}^{0} t^{2m+1} e^{-\lambda t^2} dt + \int_{0}^{\infty} s^{2m+1} e^{-\lambda s^2} ds,$$

$$= 0, \qquad \text{for } m = 0, 1, 2, \ldots$$

The even integrals (2.5.20b) are found by integrating by parts m times. Fortunately, the surface terms vanish as $\exp(-\lambda s^2)$ at $s = \pm\infty$. We also need the indefinite integral

$$\int x e^{-\alpha x^2} dx = \frac{-1}{2\alpha} e^{-\alpha x^2}.$$

Then, we can show

$$\int_{-\infty}^{\infty} s^{2m} e^{-\lambda s^2} ds = -(2m-1) \int_{-\infty}^{\infty} s^{2m-2} \left(-\frac{1}{2\lambda}\right) e^{-\lambda s^2} ds,$$

$$= \frac{(2m-1)(2m-3)}{(2\lambda)^2} \int_{-\infty}^{\infty} s^{2m-4} e^{-\lambda s^2} ds,$$

$$= \cdots$$

Recursively, we can reduce the above integral to the case $m = 0$ and we can evaluate the Gaussian integral in closed form

$$\int_{-\infty}^{\infty} e^{-\lambda s^2} ds = \sqrt{\frac{\pi}{\lambda}} \quad .$$

Then,

$$\int_{-\infty}^{\infty} s^{2m} e^{-\lambda s^2} ds = \frac{(2m)!}{m! 2^{2m} \lambda^m} \sqrt{\frac{\pi}{\lambda}} \quad , \qquad \text{for} \quad m = 0, 1, 2, \cdots$$

(b) Given the Taylor series expansion (2.5.19)

$$I_s \sim e^{\lambda h(t_0)} \int_{-\infty}^{\infty} e^{-\lambda s^2} \sum_{n=0}^{\infty} s^n F^{(n)}(0)/n! \, ds, \qquad \lambda \to \infty.$$

We use (2.5.20) to find the asymptotic expansion

$$I_s \sim e^{\lambda h(t_o)} \sqrt{\frac{\pi}{\lambda}} \sum_{n=0}^{\infty} \frac{F^{(2n)}(0)}{n! 2^{2n} \lambda^n}, \qquad \lambda \to \infty.$$

where the leading-order term has a coefficient given by

$$F(0) = \lim_{s \to 0} f(t) \frac{dt}{ds}.$$

However, as $s \to 0, t \to t_o$ and the first term in the Taylor series expansion to $h(t)$ in (2.5.12) dominates so we can approximate

$$\lim_{s \to 0} s^2 \approx -(t - t_0)^2 \frac{h''(t_o)}{2}.$$

Then

$$\lim_{s \to 0} \frac{dt}{ds} = \sqrt{\frac{-2}{h''(t_o)}} \quad .$$

Therefore, the leading-order approximation is

$$I_s \sim e^{\lambda h(t_o)} f(t_0) \sqrt{\frac{-2\pi}{\lambda h''(t_0)}} \quad , \qquad \lambda \to \infty.$$

§2.21

In order to derive (2.5.30), we need to change variables $t = \lambda s$

$$\int_0^\infty e^{-\lambda s} s^{n+\alpha} ds = \int_0^\infty e^{-t} \left(\frac{t}{\lambda}\right)^{n+\alpha} \frac{dt}{\lambda}$$
$$= \left(\frac{1}{\lambda}\right)^{n+1+\alpha} \int_0^\infty e^{-t} t^{n+\alpha} dt.$$

Now we use the definition of the Gamma function $\Gamma(x)$ from Abramowitz and Stegun, 1965.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Then, we obtain the formula

$$\int_0^\infty e^{-\lambda s} s^{n+\alpha} ds = \frac{\Gamma(n+1+\alpha)}{\lambda^{n+1+\alpha}}, \quad n \neq -\alpha - 1, -\alpha - 2, \cdots$$

Applying the above formula in the Taylor series expansion (2.5.29) results in the asymptotic expansion

$$I_a \sim e^{\lambda h(a)} \sum_{n=0}^{\infty} \frac{F^{(n)}(0)\,\Gamma(n+1+\alpha)}{n!\;\lambda^{n+1+\alpha}}, \qquad \lambda \to \infty$$

where the leading-order term has a coefficient given by

$$F(0) = \lim_{s\to 0} s^{-\alpha} f(t) \frac{dt}{ds}.$$

However, as $s \to 0$, $t \to a$ and the dominant term in the Taylor series approximation of $h(t)$ yields

$$\lim_{s\to 0} s \approx -(t-a)h'(a),$$

and the function $f(t)$ has a factorizable algebraic branch point at $t = a$

$$f(t) \sim (t-a)^{\alpha} B.$$

Then, the leading order term coefficient is

$$\begin{aligned} F(0) &= [-h'(a)]^{-\alpha} \cdot B \cdot [-h'(a)], \\ &= \left[\frac{-1}{h'(a)}\right]^{\alpha+1} B. \end{aligned}$$

§2.22

The function $s^n \exp(-\lambda s^2)$ in (2.5.19) peaks at

$$\frac{d}{ds}\left(s^n e^{-\lambda s^2}\right) = e^{-\lambda s^2}\left[n s^{n-1} - 2\lambda s^{n+1}\right] = 0,$$

or

$$s = \sqrt{n/2\lambda} \quad .$$

Similarly, the function $s^{n+\alpha} \exp(-\lambda s)$ in (2.5.29) peaks at

$$\frac{d}{ds}\left(s^{n+\alpha} e^{-\lambda s}\right) = e^{-\lambda s}\left[(n+\alpha)s^{n-1+\alpha} - \lambda s^{n+\alpha}\right] = 0,$$

or

$$s = (n+\alpha)/\lambda.$$

Thus, the radius of convergence of the asymptotic expansion (2.5.21) is limited by the distance of the nearest singularity to the origin. Hence, the truncation term n must satisfy

$$\sqrt{\frac{n}{2\lambda}} \ll |s_p|,$$

where $|s_p|$ is the distance of the nearest singularity from the origin on the complex s plane. Similarly, (2.5.31) should have a truncation term n that satisfies

$$\frac{n+\alpha}{\lambda} \ll |s_p|.$$

§2.23

(a) Given an integral of the form (2.5.37):

$$I = e^{\lambda h(t_0)} \int_{C'} (s - s_b)^k F(s) e^{-\lambda s^2} ds.$$

The saddle-point contribution is found by expanding $(s - s_b)^k F(s)$ in a Taylor series about the saddle point $s = 0$. That is

$$(s - s_b)^k F(s) \approx (-s_b)^k F(0) + s \frac{\partial}{\partial s} \left[(s - s_b)^k F(s) \right] \bigg|_{s=0} + \cdots$$

The leading-order term dominates as $\lambda \to \infty$, resulting in

$$I_s \sim e^{\lambda h(t_0)} (-s_b)^k F(0) \int_{-\infty}^{\infty} e^{-\lambda s^2} ds, \qquad \lambda \to \infty.$$

Using the integral formula (2.5.20b) with $m = 0$ we find

$$I_s \sim \sqrt{\frac{\pi}{\lambda}}\, e^{\lambda h(t_0)} F(0) (-s_b)^k, \qquad \lambda \to \infty.$$

(b) The singularity contribution to the integral of the form (2.5.37) is found by first transforming $t = s^2 - s_b^2$, then

$$I = e^{\lambda h(t_0) - \lambda s_b^2} \int_C e^{-\lambda t}\, t^k\, G(t) dt,$$

where C is an infinite contour, the image of C' on the complex t plane. This contribution is of the form (2.5.27) where $G(t)$ is defined by

$$\begin{aligned} G(t) &= \left(\frac{s - s_b}{t} \right)^k F(s) \frac{ds}{dt}, \\ &= \left(\frac{1}{s + s_b} \right)^k F(s) \frac{1}{2s}. \end{aligned}$$

Clearly, the singularity contribution comes from $t = 0$. The dominant term is the first term in a Taylor series expansion of $G(t)$, hence

$$I_a \sim e^{\lambda h(t_0) - \lambda s_b^2} G(0) \int_{SD} e^{-\lambda t}\, t^k dt, \qquad \lambda \to \infty.$$

where SD is the infinite steepest-descent contour passing through the singularity and

$$G(0) = \lim_{s \to s_b} \left(\frac{1}{s+s_b} \right)^k F(s) \frac{1}{2s},$$
$$= F(s_b) \frac{1}{(2s_b)^{k+1}}.$$

Finally, using the fact that $h(t_b) = h(t_0) - s_b^2$ we arrive at

$$I_a \sim e^{\lambda h(t_b)} \frac{F(s_b)}{(2s_b)^{k+1}} \int_{SD} e^{-\lambda t} t^k dt, \qquad \lambda \to \infty.$$

As a word of caution, the integrand in the above equation is not analytic for negative integers k. Special care must be taken in this case. Also, the path SD is an infinite contour that follows the semi-infinite steepest descent path, passes through the singularity, and then follows the semi-infinite steepest descent path again. This path is also called Hankel's Contour [Abramowitz and Stegun, 1965 p. 255].

First consider the case $k = 1/2$ (an algebraic branch point). Here, $G(0)$ assumes different values after the contour SD passes through the branch point, that is

$$I_B \sim e^{\lambda h(t_b)} \left[G(0^+) - G(0^-) \right] \int_0^\infty e^{-\lambda t} t^k dt, \quad \lambda \to \infty,$$

where

$$G(0^+) - G(0^-) = \left(\frac{1}{2s_b} \right)^k F(s_b) \left[\frac{1}{2s} - \frac{-1}{2s} \right],$$
$$= \frac{F(s_b)}{2^k s_b^{k+1}}.$$

We can use the integral formula (2.5.30) to simplify

$$I_B \sim e^{\lambda h(t_b)} \frac{\Gamma\left(\frac{3}{2}\right)}{2^{\frac{1}{2}} \lambda^{\frac{3}{2}} s_b^{\frac{3}{2}}} F(s_b), \quad \lambda \to \infty.$$

We notice that this is identically equal to the second term in the uniform asymptotic expansion (2.5.54) since $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\pi^{\frac{1}{2}}$ and $\Gamma\left(-\frac{1}{2}\right) = -2\pi^{\frac{1}{2}}$.

Next consider the case $k = -1$ (a single pole). In this case, we compute the residue [Abramowitz and Stegun, 1965 p 255]

$$\int_{SD} t^{-1} e^{-\lambda t} dt = 2\pi i.$$

Then

$$I_P \sim e^{\lambda h(t_b)} 2\pi i F(s_b), \qquad \lambda \to \infty.$$

We notice that this is identically equal to the second term in (2.5.54) when $k = -1$. Thus, we conclude that the uniform asymptotic expansion (2.5.54) is a consequence of the leading-order saddle-point contribution, plus the leading-order contribution of the singularity.

§2.24

(a) Given a VMD over a half-space when $d_1 = 0$

$$H_{1z} = -\frac{iIA}{8\pi} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{1z}} H_0^{(1)}(k_\rho \rho) e^{ik_{1z}z} \frac{2k_{1z}}{k_{1z}+k_{2z}}, \qquad z > 0.$$

For large ρ we can use the large argument approximation of the Hankel function to show that the integrand

$$\tilde{H}_{1z} \sim e^{ik_\rho \rho + ik_{1z}z}, \qquad \rho \to \infty$$

The stationary point is given by

$$\frac{\partial}{\partial k_\rho}\left(ik_\rho \rho + ik_{1z}z\right)\Big|_{k_{\rho s}} = 0,$$

so

$$k_{\rho s} = k_1 \frac{\rho}{\sqrt{\rho^2+z^2}}.$$

The steepest-descent path is then given by (2.5.13)

$$\begin{aligned} -\lambda s^2 &= ik_\rho \rho + ik_{1z}z - \left(ik_1 \rho \frac{\rho}{\sqrt{\rho^2+z^2}} + ik_1 z\sqrt{1-\frac{\rho^2}{\rho^2+z^2}}\right), \\ &= ik_\rho \rho + ik_{1z}z - ik_1 r, \end{aligned}$$

where $r = \sqrt{\rho^2+z^2}$.

(b) On the SDP, s is purely-real. This path on the complex k_ρ plane intercepts the real k_ρ axis at two points. For $k_\rho < k_1$, k_{1z} is real so the SDP requires

$$ik_\rho \rho + iz\sqrt{k_1^2 - k_\rho^2} - ik_1 r = 0,$$

$$k_\rho = k_1 \frac{\rho}{r} = k_1 \sin\theta,$$

where we defined $\sin\theta = \rho/r$. Notice that this is also the stationary point in this problem. For $k_\rho > k_1$, k_{1z} is imaginary so the SDP requires

$$ik_\rho \rho - ik_1 r = 0,$$

$$k_\rho = k_1 \frac{r}{\rho} = k_1/\sin\theta.$$

The SDP intercepts the imaginary k_ρ axis when k_{1z} is real and greater than k_1, so the SDP requires

$$iz\sqrt{k_1^2 - k_\rho^2} - ik_1 r = 0,$$

$$k_\rho = ik_1 \frac{\rho}{z} = ik_1 \tan\theta.$$

(c) To find asymptotes of the SDP we must follow the integration contour. Starting at $s = -\infty$ on the top k_{1z} Riemann sheet ($\Im m[k_{1z}] > 0$ to satisfy the radiation condition), we must have $k_{1z} = -ik_\rho$ for $\Re e[k_\rho] < 0$. The SDP asymptote is then

$$-\lambda s^2 \sim ik_\rho \rho + k_\rho z, \quad s \to -\infty, z > 0$$
$$k_\rho \sim \lambda s^2 \frac{i\rho - z}{r^2}, \quad s \to -\infty, z > 0$$

Eventually, the SDP will intercept the imaginary k_ρ axis at $ik_1 \tan\theta$, after which it will cross the k_{1z} branch cut. Before the first real k_ρ axis crossing at $k_1 \sin\theta$ the SDP must once again cross the k_{1z} branch cut since $k_1 \sin\theta < k_1$. Here the SDP is back on the top k_{1z} Riemann sheet. To maintain analytic continuity, the SDP must go below the real axis before re-crossing the real axis at $k_1/\sin\theta$. Clearly this point is beyond k_1 so the remaining contour must stay on the top k_{1z} Riemann sheet.

The final asymptote as $s \to \infty$ is thus on the same k_{1z} Riemann sheet, where $k_{1z} = ik_\rho$ for $\Re e[k_\rho] > 0$

$$-\lambda s^2 \sim ik_\rho \rho - k_\rho z, \quad s \to +\infty, z > 0$$
$$k_\rho \sim \lambda s^2 \frac{i\rho + z}{r^2}, \quad s \to +\infty, z > 0$$

The contour is drawn in Figure 2.6.1 (replace θ_I with $\theta = \sin^{-1}(\rho/r)$).

§2.25

(a) Given a VMD over a half space, the reflected wave field has a branch cut contribution (2.6.9)

$$H_{1z}^{RB} \sim \frac{iIA}{2\pi}\sqrt{\frac{2}{i\pi\rho}} \int_{S_2} dk_\rho k_\rho^{\frac{5}{2}} e^{ik_\rho \rho + ik_{1z}(z+2d_1)} \frac{k_{2z}}{k_1^2 - k_2^2}, \quad \rho \to \infty.$$

Using a linear transformation as in (2.5.26) for semi-infinite integrals around the factorizable algebraic branch point $k_\rho = k_2$ we obtain the steepest descent path at k_2

$$-\lambda s = ik_\rho \rho + ik_{1z}(z + 2d_1) - ik_2 \rho - i\sqrt{k_1^2 - k_2^2}(z + 2d_1).$$

Then we have

$$H_{1z}^{RB} \sim \frac{IAe^{\frac{i\pi}{4}}}{\pi\sqrt{2\pi\rho}} \frac{e^{ik_2\rho-\sqrt{k_2^2-k_1^2}(z+2d_1)}}{k_1^2-k_2^2} \int_0^\infty ds\, e^{-\lambda s} s^{\frac{1}{2}} G(s), \quad \lambda \to \infty$$

where we have defined

$$s^{\frac{1}{2}} G(s) = k_\rho^{\frac{5}{2}} k_{2z} \frac{dk_\rho}{ds}.$$

We have factored the $s^{\frac{1}{2}}$ term since we know that a branch-point singularity exists at $k_\rho = k_2$. Near the branch point we can approximate

$$\begin{aligned} s^{\frac{1}{2}} G(s) &= k_\rho^{\frac{5}{2}} \sqrt{(k_2-k_\rho)(k_2+k_\rho)} \frac{dk_\rho}{ds}, \\ &\sim ik_2^3\sqrt{2}\,(k_\rho-k_2)^{\frac{1}{2}} \left.\frac{dk_\rho}{ds}\right|_{s=0}, \quad k_\rho \to k_2 \end{aligned}$$

Then we identify $B = ik_2^3\sqrt{2}$ and $\alpha = \frac{1}{2}$ in the leading order term (2.5.32) and we can evaluate

$$\left.\frac{dk_\rho}{ds}\right|_{s=0} = \frac{-1}{i\rho - i(z+2d_1)\frac{k_2}{\sqrt{k_1^2-k_2^2}}}.$$

Now we can use the first term of the Taylor series expansion (2.5.31) for semi-infinite integrals to find

$$H_{1z}^{RB} \sim \frac{IAe^{\frac{i\pi}{4}}}{2\pi\sqrt{\rho}} e^{ik_2\rho-\sqrt{k_2^2-k_1^2}(z+2d_1)} \cdot \left[\frac{i}{\rho + \frac{ik_2}{\sqrt{k_2^2-k_1^2}}(z+2d_1)}\right]^{\frac{3}{2}} \frac{ik_2^3}{k_1^2-k_2^2}, \qquad \rho, z \to \infty$$

where we have used $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\pi^{\frac{1}{2}}$ and $\lambda = 1$.

(b) This branch-point contribution is a conical surface wave propagating in region 2 along the interface, while being evanescent in region 1. The decay of ρ^{-2} as $z, d_1 \to 0$ is due to interference near the interface.

§2.26

Given the transmitted field for a VMD over a half-space with $d_1 = 0$

$$H_{2z}^T = -\frac{iIA}{8\pi} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{1z}} H_0^{(1)}(k_\rho\rho) T_{12}^{TE} e^{-ik_{2z}z}, \qquad z < 0$$

(a) The large argument approximation of the Hankel function yields

$$H_{2z}^T \sim -\frac{iIA}{8\pi}\sqrt{\frac{2}{i\pi\rho}}\int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^{\frac{5}{2}}}{k_{1z}} T_{12}^{TE} e^{ik_\rho\rho - ik_{2z}z}, \qquad \rho \to \infty, \quad z < 0$$

The stationary phase point is given by

$$\frac{\partial}{\partial k_\rho}\left[ik_\rho\rho - i\sqrt{k_2^2 - k_\rho^2}\, z\right]\Bigg|_{k_{\rho s}} = 0,$$

so

$$k_{\rho s} = k_2 \frac{\rho}{\sqrt{\rho^2 + z^2}} = k_2 \sin\theta.$$

(b) The constant-phase path for this infinite integral is governed by

$$\begin{aligned} -\lambda s^2 &= ik_\rho\rho - ik_{2z}z - i\left(k_2\frac{\rho}{r}\right)\rho + i\left(k_2^2 - k_2^2\frac{\rho^2}{r^2}\right)^{\frac{1}{2}} z, \quad z < 0 \\ &= ik_\rho\rho - ik_{2z}z - ik_2 r, \qquad z < 0 \end{aligned}$$

where $r = \sqrt{\rho^2 + z^2}$.

The asymptotes of the SDP must be on the top k_{1z} Riemann sheet ($\Im m[k_{1z}] > 0$) so $k_{1z} = -ik_\rho$ for $\Re e[k_\rho] < 0$. Then

$$\begin{aligned} -\lambda s^2 &\sim ik_\rho\rho - k_\rho z, \qquad s \to -\infty, z < 0 \\ k_\rho &\sim \lambda s^2 \frac{i\rho + z}{r^2}, \qquad s \to -\infty, z < 0 \end{aligned}$$

Similarly $k_{1z} = ik_\rho$ for $\Re e[k_\rho] > 0$, then the asymptote is

$$\begin{aligned} -\lambda s^2 &\sim ik_\rho\rho + k_\rho z, \qquad s \to +\infty, z < 0 \\ k_\rho &\sim \lambda s^2 \frac{i\rho - z}{r^2}, \qquad s \to +\infty, z < 0 \end{aligned}$$

The SDP is shown in Figure 2.6.11 for $k_2 \sin\theta < k_1$. However, if $k_2 \sin\theta > k_1$ a branch point contribution from k_1 must be included, (Similar to the path drawn in Figure 2.6.7 around the branch point k_3).

(c) After regrouping the terms in (2.6.2) we can identify a slowly varying part and a rapidly oscillating part

$$H_{2z}^T = -\frac{iIA}{8\pi}\int_{-\infty}^{\infty} dk_\rho \underbrace{\left(\frac{k_\rho^2}{k_{1z}} T_{12}^{TE} k_{2z}\right)}_{\text{slowly varying}} \underbrace{\frac{k_\rho}{k_{2z}} H_0^{(1)}(k_\rho\rho) e^{-ik_{2z}z}}_{\text{rapidly varying}}, \qquad z < 0.$$

Using the stationary phase method this becomes

$$H_{2z}^T \sim -\frac{iIA}{8\pi}\left(\frac{k_2^3\sin^2\theta\cos\theta\, T_{12}^{TE}(k_2\sin\theta)}{\sqrt{k_1^2-k_2^2\sin^2\theta}}\right)\int_{-\infty}^{\infty} dk_\rho \frac{k_\rho}{k_{2z}} H_0^{(1)}(k_\rho\rho)e^{ik_{2z}|z|}, \quad r\to\infty$$

$$\sim -\frac{IA}{4\pi}\frac{k_2^3\sin^2\theta\cos\theta\, T_{12}^{TE}(k_2\sin\theta)}{\sqrt{k_1^2-k_2^2\sin^2\theta}}\frac{e^{ik_2 r}}{r}, \quad r\to\infty$$

Notice the spherical wave dependence $\exp(ik_2r)/r$.

(d) The SDP path is given by

$$-\lambda s^2 = ik_\rho\rho - i\sqrt{k_2^2-k_\rho^2}\,z - ik_2r, \quad z<0.$$

Since s is real on the SDP, we have two crossings: one when $k_\rho < k_2$ and another when $k_\rho > k_2$. These two points can be shown to be $k_\rho = k_2\sin\theta$ and $k_\rho = k_2/\sin\theta$ [see exercise 2.24(b)].

In order to deform the SIP to the SDP we must define a closed contour. However, if $k_2\sin\theta > k_1$ we end up on the wrong Riemann sheet of k_{1z}. Here, we must include a branch point contribution about k_1. This is given by

$$H_{2z}^{TB} = -\frac{iIA}{8\pi}\int_{S_1} dk_\rho \frac{k_\rho^3}{k_{1z}} H_0^{(1)}(k_\rho\rho)T_{12}^{TE}e^{ik_{2z}|z|}, \qquad z<0$$

where S_1 is a contour that wraps around k_1 and has an asymptote given by $k_\rho = \lambda s^2(i\rho+z)/r^2$, $z<0$. (This contour is similar to S_2 drawn in Figure 2.6.7 around branch point k_3).

Now, the integration path S_1 on different Riemann sheets of k_{1z} can be combined to a single integral by noting that k_{1z} assumes opposite signs on the different Riemann sheets. This yields

$$H_{2z}^{TB} = -\frac{iIA}{8\pi}\int_{k_1}^{\infty} dk_\rho k_\rho^3 H_0^{(1)}(k_\rho\rho)e^{ik_{2z}|z|}\left[\frac{T_{12}^{TE+}}{k_{1z}^+} - \frac{T_{12}^{TE-}}{k_{1z}^-}\right], \quad z<0,$$

$$= -\frac{iIA}{8\pi}\int_{k_1}^{\infty} dk_\rho k_\rho^3 H_0^{(1)}(k_\rho\rho)e^{ik_{2z}|z|}\left[\frac{-4k_{1z}}{k_2^2-k_1^2}\right], \quad z<0,$$

$$\sim \frac{iIA}{2\pi}\sqrt{\frac{2}{i\pi\rho}}\frac{1}{k_2^2-k_1^2}\int_{k_1}^{\infty} dk_\rho k_\rho^{\frac{5}{2}} k_{1z} e^{ik_\rho\rho - ik_{2z}z} \qquad \rho\to\infty.$$

For this semi-infinite integral we perform the linear transformation

$$-\lambda s = ik_\rho\rho - ik_{2z}z - ik_1\rho + i(k_2^2-k_1^2)^{\frac{1}{2}}z, \qquad z<0.$$

We can factor the branch-point singularity in the integrand so that

$$s^{\frac{1}{2}}F(s) = k_\rho^{\frac{5}{2}}\sqrt{(k_1-k_\rho)(k_1+k_\rho)}\,\frac{dk_\rho}{ds},$$
$$\sim ik_1^3\sqrt{2}\,(k_\rho-k_1)^{\frac{1}{2}}\left.\frac{dk_\rho}{ds}\right|_{s=0}, \qquad k_\rho \to k_1.$$

Thus, we identify $B = ik_1^3\sqrt{2}$ and $\alpha = \frac{1}{2}$ in the leading order term (2.5.32) and

$$\left.\frac{dk_\rho}{ds}\right|_{s=0} = \frac{-1}{i\rho + iz\frac{k_1}{\sqrt{k_2^2-k_1^2}}}$$

Now we can use the first term of the Taylor series expansion (2.5.31) for semi-infinite integrals to obtain

$$H_{2z}^{TB} \sim \frac{-iIA}{2\pi}\,\frac{k_1^3}{k_2^2-k_1^2}\,\frac{1}{\sqrt{\rho}}\,\frac{1}{\left[\rho + zk_1(k_2^2-k_1^2)^{-\frac{1}{2}}\right]^{\frac{3}{2}}}\,e^{ik_1\rho - i\sqrt{k_2^2-k_1^2}\,z},$$
$$\rho \to \infty, \quad z < 0$$

where we have used $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\pi^{\frac{1}{2}}$ and $\lambda = 1$.

Physically, this represents a conical lateral wave. It is induced by a spherical wave in region 1. Hence, it is only observed for angles $\theta > \sin^{-1}(k_1/k_2)$ because it is a critically refracted wave from region 1.

§2.27

(a) First, consider a HED $\mathbf{J} = \hat{x}\delta(x)\delta(y)\delta(z)$ in free-space. The TE component is characterized by (2.3.7b)

$$H_z^{HED} = -\frac{\partial}{\partial y}\frac{e^{ikr}}{4\pi r}.$$

We can use the Weyl identity (2.2.27) to find the integral representation

$$H_z^{HED} = \frac{1}{8\pi^2}\iint_{-\infty}^{\infty} dk_x dk_y\,\frac{k_y}{k_z}\,e^{ik_x x + ik_y y + ik_z|z|}.$$

Now consider a current sheet and use the idea of convolution to show

$$\mathbf{J} = \hat{x}\delta(z)J_x(x,y),$$
$$= \hat{x}\iint_{-\infty}^{\infty} dx'dy'\delta(x-x')\delta(y-y')\delta(z)J_x(x',y').$$

We can convolve the fields and exchange the order of integration to result in

$$H_z^{\text{sheet}} = \frac{1}{8\pi^2} \iint_{-\infty}^{\infty} dk_x dk_y \, \frac{k_y}{k_z} \, \tilde{J}_x(k_x, k_y) e^{ik_x x + ik_y y + ik_z |z|} ,$$

where the two-dimensional Fourier transform $\tilde{J}_x(k_x, k_y)$ is

$$\tilde{J}_x(k_x, k_y) = \iint_{-\infty}^{\infty} dx' dy' e^{-ik_x x' - ik_y y'} \, J_x(x', y').$$

(b) For a current sheet on a dielectric slab, we have $d_1 = 0$ in Figure 2.3.1, so we can separate the TM and TE fields in region 1

$$E_{1z} = \frac{1}{8\pi^2 \omega \epsilon_1} \iint_{-\infty}^{\infty} d\mathbf{k}_s k_x \tilde{J}_x(k_x, k_y) e^{i\mathbf{k}_1 \cdot \mathbf{r}} \left[1 - R^{TM}(k_x, k_y)\right] ,$$

$$H_{1z} = \frac{1}{8\pi^2} \iint_{-\infty}^{\infty} d\mathbf{k}_s \, \frac{k_y}{k_{1z}} \, \tilde{J}_x(k_x, k_y) e^{i\mathbf{k}_1 \cdot \mathbf{r}} \left[1 + R^{TE}(k_x, k_y)\right] ,$$

where we have used the results of Exercise 2.17(b) and we have defined $\mathbf{k}_1 \cdot \mathbf{r} = k_x x + k_y y + k_{1z} z$ and $d\mathbf{k}_s = dk_x dk_y$.

(c) For a dielectric slab backed by a perfect electric conductor as in the Figure for Exercise 2.27, R^{TM} and R^{TE} are the generalized reflection coefficients at the interface (see 2.1.21) and we can use $R_{23}^{TM} = +1$, $R_{23}^{TE} = -1$ to simplify

$$R^{TM} = R_{12}^{TM} + \frac{T_{12}^{TM} T_{21}^{TM} e^{2ik_{2z}t}}{1 - R_{21}^{TM} e^{2ik_{2z}t}} ,$$

$$R^{TE} = R_{12}^{TE} - \frac{T_{12}^{TE} T_{21}^{TE} e^{2ik_{2z}t}}{1 + R_{21}^{TE} e^{2ik_{2z}t}} .$$

Here, t is the thickness of the slab and $T_{ij} = 1 + R_{ij}$ as given by (2.1.13) and (2.1.14).

(d) The stationary phase point in two-dimensions is given by

$$\frac{\partial}{\partial k_x} \mathbf{k}_1 \cdot \mathbf{r} = 0 \quad \text{and} \quad \frac{\partial}{\partial k_y} \mathbf{k}_1 \cdot \mathbf{r} = 0,$$

$$x - \frac{k_x}{k_{1z}} z = 0 \quad \text{and} \quad y - \frac{k_y}{k_{1z}} z = 0.$$

These equations are satisfied simultaneously when $x k_y = y k_x$ so

$$k_{xs} = k_1 \frac{x}{r} = k_1 \sin\theta \cos\phi,$$

$$k_{ys} = k_1 \frac{y}{r} = k_1 \sin\theta \sin\phi.$$

We can evaluate the slowly varying part of the integrand at the stationary phase point to show

$$E_{1z} \sim \frac{1}{8\pi^2\omega\epsilon_1}\tilde{J}_x(k_{xs},k_{ys})\left[1-R^{TM}(k_{xs},k_{ys})\right]k_{xs}k_{1zs}\iint d\mathbf{k}_s\frac{e^{i\mathbf{k}_1\cdot\mathbf{r}}}{k_{1z}}, \quad x,y\to\infty$$

$$\sim \frac{-i\eta_1 k_1}{4\pi}\tilde{J}_x(k_{xs},k_{ys})\left[1-R^{TM}(k_{xs},k_{ys})\right]\cos\theta\sin\theta\cos\phi\frac{e^{ik_1r}}{r}, \quad x,y\to\infty$$

$$H_{1z} \sim \frac{1}{8\pi^2}\tilde{J}_x(k_{xs},k_{ys})\left[1+R^{TE}(k_{xs},k_{ys})\right]k_{ys}\iint d\mathbf{k}_s\frac{e^{i\mathbf{k}_1\cdot\mathbf{r}}}{k_{1z}}, \quad x,y\to\infty$$

$$\sim \frac{-ik_1}{4\pi}\tilde{J}_x(k_{xs},k_{ys})\left[1+R^{TE}(k_{xs},k_{ys})\right]\sin\theta\sin\phi\frac{e^{ik_1r}}{r}, \quad x,y\to\infty.$$

In the above we have assumed that t is small so that we can detour the contour from the Weyl integral path defined by $\Im m[k_{1z}]>0$ and $\Re e[k_{1z}]>0$ to the steepest-descent path, without containing any singularities. In other words, if t is small, then there can be no resonant modes in the slab, (except for the TM_0 mode described in part (f)) and the dominant far-fields are due to the direct fields.

(e) In the far-field the rapid space variations are smoothed out and propagate like plane waves. Expand $\hat{z}=\hat{r}\cos\theta-\hat{\theta}\sin\theta$ and ignore the radial field component as it cannot describe a plane wave propagating in the $\hat{r}$ direction. In other words, $E_{1z}=-E_{1\theta}\sin\theta$, $H_{1z}=-H_{1\theta}\sin\theta$. Then

$$E_{1\theta}=-E_{1z}/\sin\theta,$$
$$H_{1\theta}=-H_{1z}/\sin\theta.$$

Also, the fields of plane waves are related by the right-hand-rule such that

$$\frac{E_\theta}{H_\phi}=\eta, \qquad \frac{E_\phi}{H_\theta}=-\eta.$$

$$H_{1\phi}=-\frac{E_{1z}}{\eta_1\sin\theta}, \qquad E_{1\phi}=\frac{\eta_1 H_{1z}}{\sin\theta}.$$

Here the Pointing vector is

$$\begin{aligned}\mathbf{S}_1&=\mathbf{E}_1\times\mathbf{H}_1^*=\hat{r}\left[E_{1\theta}H_{1\phi}^*-E_{1\phi}H_{1\theta}^*\right]\\&=\hat{r}\left[\eta_1^{-1}\left|E_{1z}\right|^2+\eta_1\left|H_{1z}\right|^2\right]/\sin^2\theta\\&\sim\hat{r}\frac{\eta_1k_1^2}{(4\pi r)^2}\left|\tilde{J}_x(k_{xs},k_{ys})\right|^2\left\{\left|1-R^{TM}(k_{xs},k_{ys})\right|^2\cos^2\theta\cos^2\phi\right.\\&\qquad\left.+\left|1+R^{TE}(k_{xs},k_{ys})\right|^2\sin^2\phi\right\}, \qquad x,y\to\infty\end{aligned}$$

(f) Resonance occurs for TM modes in a dielectric slab when

$$1 - R_{21}^{TM} R_{23}^{TM} e^{2ik_{2z}t} = 0.$$

One wall of this slab waveguide is backed by a perfect electric conductor so $R_{23}^{TM} = +1$. The other waveguide wall is open. In order to satisfy this guidance condition R_{21}^{TM} must have the form $R_{21}^{TM} = e^{-2i\phi_{21}}$. Then

$$2k_{2z}t - 2\phi_{21} = 2m\pi, \qquad m = 0, 1, 2, \cdots$$

Using the results of Exercise 2.2 we find $|R_{21}^{TM}| = 1$ when $k_{1z} = i\alpha_{1z}$ with $\alpha_{1z} > 0$. In other words, the field must be evanescent in region 1 and propagating in region 2. Thus

$$R_{21}^{TM} = \frac{\epsilon_1 k_{2z} - i\epsilon_2 \alpha_{1z}}{\epsilon_1 k_{2z} + i\epsilon_2 \alpha_{1z}}, \qquad k_1^2 \leq \left(k_x^2 + k_y^2\right) \leq k_2^2.$$

Hence $\phi_{21} = \tan^{-1}(\epsilon_2\alpha_{1z}/\epsilon_1 k_{2z})$. We can rewrite the guidance condition as

$$\alpha_{1z}t = \frac{\epsilon_1}{\epsilon_2} k_{2z}t \tan(k_{2z}t).$$

Cutoff occurs when $\alpha_{1z} = 0$ and the wave is no longer evanescent in region 1, that is $k_{2z}t = \sqrt{k_2^2 - k_1^2}t = m\pi$. The cutoff wavenumbers are thus

$$k_{cm} = \frac{m\pi}{t\sqrt{1 - \mu_1\epsilon_1/\mu_2\epsilon_2}}, \qquad m = 0, 1, 2, \cdots$$

Thus the TM_0 mode has no cut-off frequency for this waveguide, and R^{TM} always has a pole. This pole can be enclosed between the Weyl integral path defined by $\Re e[k_{1z}] > 0$, $\Im m[k_{1z}] > 0$ and the steepest-descent path since $\epsilon_2 > \epsilon_1$. In this case the pole will contribute a residue. The pole location is given by

$$\epsilon_1 k_{2z} + i\epsilon_2\alpha_{1z} = 0,$$

$$1 + i\tan(k_{2z}t) = 0.$$

The roots of the above equation must be solved numerically for complex values of $k_P^2 = \left(k_x^2 + k_y^2\right)$ given t and complex k_2.

Assuming the pole contributes, we evaluate the residue in the first integral for E_{1z}, resulting in

$$E_{1z}^P = \frac{i}{4\pi\omega\epsilon_1} e^{-z(k_P^2 - k_1^2)^{1/2}} \int_{-\infty}^{\infty} dk_x k_x \cdot \tilde{J}_z\left(k_x, \sqrt{k_P^2 - k_x^2}\right) A(k_x) e^{ik_x x + iy(k_P^2 - k_x^2)^{1/2}},$$

where the residue is

$$A(k_x) = \lim_{k_y \to \sqrt{k_P^2 - k_x^2}} \left(k_y - \sqrt{k_P^2 - k_x^2}\right) \left[1 - R^{TM}(k_x, k_y)\right] .$$

§2.28

A VED in free space produces a field characterized by (2.3.4)

$$E_z = \frac{-I\ell}{8\pi\omega\epsilon} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_z} H_0^{(1)}(k_\rho \rho) e^{ik_z|z|} .$$

When confined between two parallel plates, the field can be written

$$E_{2z} = \frac{-I\ell}{8\pi\omega\epsilon_2} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{2z}} H_0^{(1)}(k_\rho \rho) \left[e^{ik_{2z}|z|} + A_2 e^{-ik_{2z}z} + B_2 e^{ik_{2z}z}\right] .$$

Applying the constraint condition at $z = d_1$ and $z = -d_2$ we have

$$\begin{aligned} A_2 e^{-ik_{2z}d_1} &= R_{21}^{TM} \left[e^{ik_{2z}|d_1|} + B_2 e^{ik_{2z}d_1}\right] , \\ B_2 e^{-ik_{2z}d_2} &= R_{23}^{TM} \left[e^{ik_{2z}|-d_2|} + A_2 e^{ik_{2z}d_2}\right] . \end{aligned}$$

For metallic plates $R_{21}^{TM} = R_{23}^{TM} = 1$. These equations are satisfied by

$$\begin{aligned} A_2 &= \frac{1 + e^{2ik_{2z}d_2}}{1 - e^{2ik_{2z}(d_1+d_2)}} e^{2ik_{2z}d_1} , \\ B_2 &= \frac{1 + e^{2ik_{2z}d_1}}{1 - e^{2ik_{2z}(d_1+d_2)}} e^{2ik_{2z}d_2} . \end{aligned}$$

(a) We can collect the k_{2z} dependence in the integrand and define the function

$$\begin{aligned} f(k_{2z}) =& \frac{e^{ik_{2z}|z|} + A_2 e^{-ik_{2z}z} + B_2 e^{ik_{2z}z}}{k_{2z}} \\ =& \frac{e^{ik_{2z}|z|}}{k_{2z}} + \frac{e^{-ik_{2z}z}\left(e^{2ik_{2z}d_1} + e^{2ik_{2z}(d_1+d_2)}\right) + e^{ik_{2z}z}\left(e^{2ik_{2z}d_2} + e^{2ik_{2z}(d_1+d_2)}\right)}{k_{2z}\left(1 - e^{2ik_{2z}(d_1+d_2)}\right)} . \end{aligned}$$

Consider the case $z > 0$ then we can show

$$\begin{aligned} f(-k_{2z}) &= \frac{e^{-ik_{2z}z}\left(1 + e^{-2ik_{2z}d_2}\right) + e^{ik_{2z}z}\left(e^{-2ik_{2z}d_1} + e^{-2ik_{2z}(d_1+d_2)}\right)}{-k_{2z}\left(1 - e^{-2ik_{2z}(d_1+d_2)}\right)} , \qquad z > 0 \\ &= \frac{e^{-ik_{2z}z}\left(e^{2ik_{2z}d_1} + e^{2ik_{2z}(d_1+d_2)}\right) + e^{ik_{2z}z}\left(1 + e^{2ik_{2z}d_2}\right)}{k_{2z}\left(1 - e^{2ik_{2z}(d_1+d_2)}\right)} , \\ &= f(k_{2z}) \end{aligned}$$

By symmetry we also find $f(-k_{2z}) = f(k_{2z})$ for $z < 0$. Since the integrand f is a single-valued function of k_{2z}, there are no branch point at $k_\rho = k_2$ and the only singularities must be poles.

(b) The pole locations are given by

$$1 - e^{2ik_{2z}(d_1+d_2)} = 0,$$

$$k_{2z}(d_1 + d_2) = n\pi, \quad n = 0, 1, 2, \cdots$$

$$k_{\rho n} = \sqrt{k_2^2 - \left(\frac{n\pi}{d_1 + d_2}\right)^2}, \quad n = 0, 1, 2, \cdots$$

When $(d_1 + d_2) \to \infty$ the poles migrate from $k_{\rho n} = \pm i\infty$ along the imaginary k_ρ-axis towards the origin. When $n\pi/(d_1 + d_2) < \Re e[k_2]$ the poles cross to the real k_ρ-axis and approach $\pm k_2$.

(c) By Cauchy's theorem, the integral path can be deformed from the SIP to C plus the enclosed pole residues. (The contour C is drawn in Figure 2.2.3). Jordan's lemma forces the contribution from C to zero. Hence the fields can be written

$$E_{2z} = \frac{-I\ell}{8\pi\omega\epsilon_2} \sum_{n=1}^{\infty} k_{\rho n}^3 H_0^{(1)}(k_{\rho n}\rho) 2\pi i A_n ,$$

where the residue is

$$A_n = \lim_{k_\rho \to k_{\rho n}} (k_\rho - k_{\rho n}) f(k_{2z}).$$

§2.29

The reflected field for a VMD over a slab is given by

$$H_{1z}^R = -\frac{iIA}{8\pi} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{1z}} H_0^{(1)}(k_\rho \rho) e^{ik_{1z}(z+2d_1)} \tilde{R}_{12}^{TE} .$$

The above integrand could have a leaky pole k_l near the saddle point $k_1 \sin\theta_I$ if $k_l < k_1$ where k_l is a solution of (2.6.25). In this case we must use the uniform asymptotic expansion to include the effect of the leaky pole. For large ρ we find

$$H_{1z}^R \sim -\frac{iIA}{8\pi}\sqrt{\frac{2}{i\pi\rho}} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^{\frac{5}{2}}}{k_{1z}} \tilde{R}_{12}^{TE}(k_\rho)\, e^{ik_\rho\rho + ik_{1z}(z+2d_1)}, \qquad \rho \to \infty.$$

The stationary phase point is given by

$$\frac{d}{dk_\rho}\Big(k_\rho\rho + k_{1z}(z + 2d_1)\Big)\Big|_{k_\rho = k_{\rho s}} = 0 ,$$

$$k_{\rho s} = k_1 \frac{\rho}{\sqrt{\rho^2 + (z^2 + 2d_1)^2}} = k_1 \sin\theta_I .$$

For infinite integrals the steepest-descent path requires a change of variables such that

$$\begin{aligned} -s^2 &= ik_\rho\rho + ik_{1z}(z + 2d_1) - ik_1 \sin\theta_I \rho - ik_1 \cos\theta_I (z + 2d_1) , \\ &= ik_\rho\rho + ik_{1z}(z + 2d_1) - ik_1 r_I , \end{aligned}$$

where $r_I = (\rho^2 + (z + 2d_1)^2)^{1/2}$. Then we obtain the approximation

$$H_{1z}^R \sim -\frac{IAe^{\frac{i\pi}{4}}}{2^{\frac{5}{2}}\pi^{\frac{3}{2}}\rho^{\frac{1}{2}}} e^{ik_1 r_I} \int_{C'} (s - s_b)^{-1} F(s)\, e^{-s^2} ds , \qquad \rho \to \infty$$

where C' is above the origin if $k_1 \sin\theta_I > k_l$ or below the origin if $k_1 \sin\theta_I < k_l$. This integral has the form of (2.5.37). Furthermore, s_b is the image of k_l on the complex s plane and

$$F(s) = (s - s_b) \frac{k_\rho^{\frac{5}{2}}}{k_{1z}} \tilde{R}_{12}^{TE}(k_\rho) \frac{dk_\rho}{ds} .$$

The two critical points are the saddle point at $s = 0$ and the singularity at $s = s_b$. According to (2.5.47) the leading-order uniform asymptotic expansion is

$$H_{1z}^R \sim \frac{-IAe^{i\frac{\pi}{4}}}{8\pi^{\frac{3}{2}}\rho^{\frac{1}{2}}} e^{ik_1 r_I - s_b^2} \left[\sqrt{2}\,\gamma_0 W_{-1}\left(\sqrt{2}\, s_b\right) + \gamma_1 W_0\left(\sqrt{2}\, s_b\right)\right] , \; \rho, z \to \infty$$

where the W function is defined by (2.5.45) and the singularity is

$$-s_b^2 = ik_l\rho + i\sqrt{k_1^2 - k_l^2}\,(z + 2d_1) - ik_1 r_I .$$

We can simplify γ_0 evaluated at the leaky pole (2.5.40)

$$\begin{aligned} \gamma_0 &= F(s_b) , \\ &= \lim_{\substack{s \to s_b \\ k_\rho \to k_l}} (s - s_b) \frac{k_\rho^{\frac{5}{2}}}{k_{1z}} \tilde{R}_{12}^{TE}(k_\rho) \frac{dk_\rho}{ds} , \\ &= \frac{k_l^{\frac{5}{2}}}{\sqrt{k_1^2 - k_l^2}} \lim_{\substack{s \to s_b \\ k_\rho \to k_l}} (s - s_b) \tilde{R}_{12}^{TE}(k_\rho) \frac{dk_\rho}{ds} , \\ &= \frac{k_l^{\frac{5}{2}}}{\sqrt{k_1^2 - k_l^2}} \lim_{k_\rho \to k_l} (k_\rho - k_l) \tilde{R}_{12}^{TE}(k_\rho) . \end{aligned}$$

Also, γ_1 is given by (2.5.40)

$$\begin{aligned}\gamma_1 &= \frac{F(s_b) - F(0)}{s_b}, \\ &= s_b^{-1}\gamma_0 + \lim_{\substack{s\to 0\\ k_\rho \to k_1 \sin\theta_I}} \frac{k_\rho^{\frac{5}{2}}}{k_{1z}} \tilde{R}_{12}^{TE}(k_\rho)\frac{dk_\rho}{ds}, \\ &= s_b^{-1}\gamma_0 + k_1^{\frac{3}{2}} \frac{\sin^{\frac{5}{2}}\theta_I}{\cos\theta_I} \tilde{R}_{12}^{TE}(k_1 \sin\theta_I) \lim_{s\to 0}\left(\frac{dk_\rho}{ds}\right).\end{aligned}$$

Moreover, the steepest-descent transformation is quadratic around $k_{\rho s}$ so

$$-s^2 \approx \frac{h''(k_{\rho s})}{2}(k_\rho - k_{\rho s})^2 .$$

Then, the Jacobian is approximately

$$\lim_{s\to 0}\left(\frac{dk_\rho}{ds}\right) \approx \lim_{s\to 0}\left(\frac{(k_\rho - k_{\rho s})^2}{s^2}\right)^{\frac{1}{2}} = \sqrt{\frac{-2}{h''(k_{\rho s})}}$$

We can evaluate the quadratic coefficient in closed form by

$$\begin{aligned}h''(k_\rho) &= \frac{d^2}{dk_\rho^2}\Big(ik_\rho\rho + ik_{1z}(z+2d_1)\Big), \\ &= -i(z+2d_1)\frac{k_1^2}{k_{1z}^3}.\end{aligned}$$

Combining these results we find

$$\gamma_1 = s_b^{-1}\gamma_0 + \frac{\sqrt{2}\, e^{-\frac{i\pi}{4}} k_1^2 \sin^{\frac{5}{2}}\theta_I \cos^{\frac{1}{2}}\theta_I}{(z+2d_1)^{\frac{1}{2}}} \tilde{R}_{12}^{TE}(k_1\sin\theta_I).$$

Now we use $\sin\theta_I = \rho/r_I$ and $\cos\theta_I = (z+2d_1)/r_I$

$$\gamma_1 = s_b^{-1}\gamma_0 + \sqrt{2}\, e^{-\frac{i\pi}{4}} k_1^2 \frac{\rho^{\frac{5}{2}}}{r_I^3} \tilde{R}_{12}^{TE}(k_1\sin\theta_I) .$$

Finally, the leading-order uniform asymptotic expansion can be written

$$\begin{aligned}H_{1z}^R \sim \frac{-IA}{8\pi^{\frac{3}{2}}\rho} e^{ik_l\rho + i(k_1^2 - k_l^2)^{\frac{1}{2}}(z+2d_1)} \Big[&2^{\frac{1}{2}} k_l^{\frac{3}{2}} (k_1^2/k_l^2 - 1)^{-\frac{1}{2}} \tilde{A}_{12}^{TE}(k_l) \\ &\cdot W_{-1}(\sqrt{2}\, s_b) + k_l^{\frac{3}{2}}(k_1^2/k_l^2 - 1)^{-\frac{1}{2}} \tilde{A}_{12}^{TE}(k_l) s_b^{-1} W_0(\sqrt{2}\, s_b) \\ &+ 2^{\frac{1}{2}} k_1^2 \sin^3\theta_I \tilde{R}_{12}^{TE}(k_1\sin\theta_I) W_0(\sqrt{2}\, s_b)\Big], \qquad \rho, z \to \infty\end{aligned}$$

where the residue is

$$\tilde{A}_{12}^{TE}(k_l) = e^{i\frac{\pi}{4}} \rho^{\frac{1}{2}} \lim_{k_\rho \to k_l} (k_\rho - k_l)\tilde{R}_{12}^{TE}(k_\rho).$$

§2.30

(a) The m-th image term in the geometric optic series (2.6.26) is

$$H_{1z}^{Rm} = -\frac{iIA}{8\pi} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{1z}} H_0^{(1)}(k_\rho \rho) e^{2imk_{2z}d_2} T_{12} R_{23}^m R_{21}^{m-1} T_{21} ,$$

where $z = d_1 = 0$. For large ρ we use the large argument expansion of the Hankel function. The stationary-phase point here is given by

$$\frac{d}{dk_\rho}\Big(ik_\rho \rho + 2imk_{2z}d_2\Big)\Big|_{k_\rho = k_{\rho sm}} = 0 ,$$

$$k_{\rho sm} = k_2 \frac{\rho}{\sqrt{\rho^2 + 4m^2 d_2^2}} = k_2 \frac{\rho}{r_m} = k_2 \sin\theta_m , \qquad \rho \to \infty.$$

For an N-layer medium, one can show that there are branch-point contributions from only k_1 and k_N (see Subsection 2.7.1 for further discussions). Therefore in the case $k_2 > k_1$ the saddle point can be close to the branch point at $k_\rho = k_1$, that is there may exist an m such that

$$k_2 \sin\theta_m \approx k_1$$

(b) In the case of a branch point near the saddle point we must use the uniform asymptotic expansion for the leading-order term. For large ρ we find

$$H_{1z}^{Rm} \sim -\frac{iIA}{8\pi}\sqrt{\frac{2}{i\pi\rho}} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^{\frac{5}{2}}}{k_{1z}} e^{ik_\rho \rho + 2ik_{2z}md_2} T_{12} R_{23}^m R_{21}^{m-1} T_{21} , \qquad \rho \to \infty.$$

For infinite integrals the steepest descent path requires a change of variables such that

$$\begin{aligned} -s^2 &= ik_\rho \rho + 2ik_{2z}md_2 - ik_2 \sin\theta_m \rho - 2ik_2 \cos\theta_m md_2 , \\ &= ik_\rho \rho + 2ik_{2z}md_2 - ik_2 r_m . \end{aligned}$$

Under this change of variables we rewrite

$$H_{1z}^{Rm} \sim \frac{-IAe^{\frac{i\pi}{4}}}{2^{\frac{5}{2}}\pi^{\frac{3}{2}}\rho^{\frac{1}{2}}} e^{ik_2 r_m} \int_{C'} (s - s_b)^{-\frac{1}{2}} F(s) e^{-s^2} ds, \qquad \rho \to \infty$$

where C' is above the origin if $k_2 \sin\theta_m > k_1$ or below the origin if $k_2 \sin\theta_m < k_1$. Furthermore, s_b is the image of the branch point k_1 on the complex s plane and we define

$$F(s) = (s - s_b)^{\frac{1}{2}} \frac{k_\rho^{\frac{5}{2}}}{k_{1z}} T_{12} R_{23}^m R_{21}^{m-1} T_{21} \frac{dk_\rho}{ds} .$$

The two critical points are the saddle point at $s = 0$ and the singularity at $s = s_b$. According to (2.5.47) the leading-order uniform asymptotic expansion is

$$H_{1z}^{Rm} \sim \frac{-IAe^{\frac{i\pi}{4}}}{8\pi^{\frac{3}{2}}\rho^{\frac{1}{2}}} e^{ik_2 r_m - s_b^2} \left[2^{\frac{1}{4}}\gamma_0 W_{-\frac{1}{2}}\left(\sqrt{2}\, s_b\right) + 2^{-\frac{1}{4}}\gamma_1 W_{\frac{1}{2}}\left(\sqrt{2}\, s_b\right)\right], \qquad \rho \to \infty$$

where the W function is defined by (2.5.45) and the singularity is

$$-s_b^2 = ik_1\rho + 2i\sqrt{k_2^2 - k_1^2}\, md_2 - ik_2 r_m .$$

We can simplify γ_0 evaluated at the branch point:

$$\begin{aligned}
\gamma_0 &= F(s_b) \\
&= \lim_{\substack{s \to s_b \\ k_\rho \to k_1}} (s - s_b)^{\frac{1}{2}} \frac{k_\rho^{\frac{5}{2}}}{k_{1z}} T_{12} R_{23}^m R_{21}^{m-1} T_{21} \frac{dk_\rho}{ds} \\
&= \left[k_\rho^{\frac{5}{2}} T_{12} R_{23}^m R_{21}^{m-1} T_{21}\right]\Big|_{k_\rho = k_1} \cdot \lim_{\substack{s \to s_b \\ k_\rho \to k_1}} \frac{(s - s_b)^{\frac{1}{2}}}{(k_1^2 - k_\rho^2)^{\frac{1}{2}}} \frac{dk_\rho}{ds} .
\end{aligned}$$

Moreover, we can evaluate the Jacobian in closed form

$$\frac{dk_\rho}{ds} \approx \frac{2is}{\rho}, \qquad \rho \gg md_2.$$

This allows us to show

$$\gamma_0 = \frac{-i}{\sqrt{2}} k_1^2 \left[T_{12} R_{23}^m R_{21}^{m-1} T_{21}\right]\Big|_{k_\rho = k_1} \cdot \sqrt{\frac{2is_b}{\rho}}, \qquad \rho \gg md_2.$$

Next, we can simplify γ_1 given by (2.5.40)

$$\begin{aligned}
\gamma_1 &= \frac{F(s_b) - F(0)}{s_b}, \\
&= s_b^{-1}\gamma_0 - is_b^{-\frac{1}{2}} \lim_{\substack{s \to 0 \\ k_\rho \to k_2 \sin\theta_m}} \frac{k_\rho^{\frac{5}{2}}}{k_{1z}} T_{12} R_{23}^m R_{21}^{m-1} T_{21} \frac{dk_\rho}{ds}, \\
&= s_b^{-1}\gamma_0 - is_b^{-\frac{1}{2}} \frac{k_2^{\frac{5}{2}} \sin^{\frac{5}{2}}\theta_m}{\sqrt{k_1^2 - k_2^2 \sin^2\theta_m}} \\
&\qquad \cdot \left[T_{12} R_{23}^m R_{21}^{m-1} T_{21}\right]\Big|_{k_\rho = k_2 \sin\theta_m} \cdot \lim_{s \to 0} \left(\frac{dk_\rho}{ds}\right) .
\end{aligned}$$

Moreover, the steepest-descent transformation is quadratic around $k_{\rho sm}$ so

$$-s^2 \approx \frac{h''(k_{\rho sm})}{2}(k_\rho - k_{\rho sm})^2 .$$

Then, the Jacobian is approximately

$$\lim_{s\to 0}\frac{dk_\rho}{ds}\approx\lim_{s\to 0}\left(\frac{(k_\rho-k_{\rho sm})^2}{s^2}\right)^{\frac{1}{2}}=\sqrt{\frac{-2}{h''(k_{\rho sm})}}\quad .$$

The quadratic term $h''(k_\rho)$ is given by

$$\begin{aligned}h''(k_\rho)&=\frac{d^2}{dk_\rho^2}\left(ik_\rho\rho+2ik_{2z}md_2\right),\\&=-2imd_2\frac{k_2^2}{k_{2z}^3}.\end{aligned}$$

Combining these results we find

$$\gamma_1=s_b^{-1}\gamma_0-s_b^{-\frac{1}{2}}\frac{e^{\frac{i\pi}{4}}k_2^3\sin^{\frac{5}{2}}\theta_m\cos^{\frac{3}{2}}\theta_m}{(md_2)^{\frac{1}{2}}\sqrt{k_1^2-k_2^2\sin^2\theta_m}}\left[T_{12}R_{23}^mR_{21}^{m-1}T_{21}\right]\big|_{k_\rho=k_2\sin\theta_m}\quad .$$

Now we use $\sin\theta_m=\rho/r_m$ and $\cos\theta_m=2md_2/r_m$

$$\gamma_1=s_b^{-1}\gamma_0-s_b^{-\frac{1}{2}}2^{\frac{1}{2}}e^{\frac{i\pi}{4}}k_2^2\frac{\rho^{\frac{5}{2}}}{r_m^3}\frac{k_2\cos\theta_m}{\sqrt{k_1^2-k_2^2\sin^2\theta_m}}\left[T_{12}R_{23}^mR_{21}^{m-1}T_{21}\right]\big|_{k_\rho=k_2\sin\theta_m}.$$

Finally, the leading-order asymptotic expansion can be written

$$\begin{aligned}H_{1z}^{Rm}\sim\frac{-IA}{8\pi^{\frac{3}{2}}\rho}e^{ik_1\rho+2i(k_2^2-k_1^2)^{\frac{1}{2}}md_2}&\Big[2^{\frac{1}{4}}s_b^{\frac{1}{2}}k_1^2T_m(k_1)W_{-\frac{1}{2}}\left(\sqrt{2}\,s_b\right)\\&+2^{-\frac{1}{4}}s_b^{-\frac{1}{2}}k_1^2T_m(k_1)W_{\frac{1}{2}}\left(\sqrt{2}\,s_b\right)-i2^{\frac{1}{4}}s_b^{-\frac{1}{2}}k_2^2\sin^3\theta_m\\&\cdot\frac{k_2\cos\theta_m}{\sqrt{k_1^2-k_2^2\sin^2\theta_m}}T_m\left(k_2\sin\theta_m\right)W_{\frac{1}{2}}(\sqrt{2}\,s_b)\Big]\,,\quad\rho\to\infty\end{aligned}$$

where

$$T_m(k_\rho)=T_{12}(k_\rho)R_{23}^m(k_\rho)R_{21}^{m-1}(k_\rho)T_{21}(k_\rho).$$

§2.31

Given a VMD over a half-space (2.6.35)

$$H_{1z}^R\sim\frac{-iIA}{8\pi}\sqrt{\frac{2}{i\pi\rho}}\int_\Gamma d\alpha\,(k_1\sin\alpha)^{\frac{5}{2}}R_{12}^{TE}e^{ik_1r_I\cos(\theta_I-\alpha)}\,.$$

where Γ is shown in Figure 2.6.13. The stationary point is given by

$$\frac{d}{d\alpha}\Big(\cos(\theta_I-\alpha)\Big)\Big|_{\alpha=\alpha_s}=0$$

so

$$\alpha_s = \theta_I.$$

The stationary phase path is given by

$$-\lambda s^2 = ik_1 r_I \cos(\theta_I - \alpha) - ik_1 r_I .$$

Using a change of variables we find

$$H_{1z}^R \sim \frac{-iIA}{8\pi}\sqrt{\frac{2}{i\pi\rho}}e^{ik_1 r_I}\int_{C'} e^{-\lambda s^2}F(s)ds.$$

Here we have defined

$$F(s) = (k_1 \sin\alpha)^{\frac{5}{2}} R_{12}^{TE}\frac{d\alpha}{ds}.$$

and we have assumed that the branch point is not contributing, that is $k_2 \sin\theta_I < k_1$. The leading-order term is given by (2.5.21)

$$H_{1z}^R \sim \frac{-iIA}{8\pi}\sqrt{\frac{2}{i\pi\rho}}e^{ik_1 r_I}\sqrt{\frac{\pi}{ik_1 r_I}}F(0), \qquad k_1 r \to \infty$$

where the stationary phase contribution is

$$\begin{aligned} F(0) &= (k_1 \sin\alpha_s)^{\frac{5}{2}} R_{12}^{TE}(k_1 \sin\alpha_s)\lim_{s\to 0}\left(\frac{d\alpha}{ds}\right), \\ &= (k_1 \sin\theta_I)^{\frac{5}{2}} R_{12}^{TE}(k_1 \sin\theta_I)\sqrt{\frac{-2}{h''(t_0)}}, \\ &= (k_1 \sin\theta_I)^{\frac{5}{2}} R_{12}^{TE}(k_1 \sin\theta_I)\sqrt{2}. \end{aligned}$$

In the above, we have used $h(t) = \cos(\theta_I - \alpha)$. Finally, we arrive at

$$H_{1z}^R \sim \frac{-IA}{4\pi}(k_1 \sin\theta_I)^2 R_{12}^{TE}(k_1 \sin\theta_I)\frac{e^{ik_1 r_I}}{r_I}, \quad k_1 r_I \to \infty$$

and we have used $\rho = r_I \sin\theta_I$. Notice that we have arrived at (2.6.6) using the angular spectrum representation.

§2.32

A guided TE mode of an inhomogeneous slab satisfies the equation

$$\left[\mu\frac{d}{dz}\mu^{-1}\frac{d}{dz} + k^2 - k_\rho^2\right]\phi(z) = 0.$$

where μ, ϵ, and k are functions of z and k_ρ is the wavenumber for a TE wave Sommerfeld integrand $\phi(z)$ as in (2.3.10a).

(a) We multiply the above TE wave equation by $\phi^*(z)\mu^{-1}(z)$ integrate from $-\infty$ to $+\infty$ and find

$$\int_{-\infty}^{\infty} dz\phi^*(z)\frac{d}{dz}\mu^{-1}(z)\frac{d}{dz}\phi(z) + \int_{-\infty}^{\infty} dz\mu^{-1}(z)k^2(z)|\phi(z)|^2 - k_\rho^2\int_{-\infty}^{\infty} dz\mu^{-1}(z)|\phi(z)|^2 = 0.$$

Now we can simplify the first integral term above using integration by parts and ignoring the surface terms as we look for guided modes whose field vanishes as $|z| \to \infty$. Then, we obtain

$$-\int_{-\infty}^{\infty} dz\mu^{-1}(z)\left|\frac{d\phi(z)}{dz}\right|^2 + \int_{-\infty}^{\infty} dz\mu^{-1}(z)k^2(z)\,|\phi(z)|^2 - k_\rho^2\int_{-\infty}^{\infty} dz\mu^{-1}(z)\,|\phi(z)|^2 = 0.$$

For lossless $\mu(z) > 0$ we deduce that the real part of the above equation must satisfy the inequality

$$\int_{-\infty}^{\infty} dz\mu^{-1}(z)\Re e\left(k^2(z)\right)|\phi(z)|^2 > \Re e\left(k_\rho^2\right)\int_{-\infty}^{\infty} dz\mu^{-1}(z)\,|\phi(z)|^2 .$$

These two terms have bounds given by

$$\int_{-\infty}^{\infty} dz\mu^{-1}(z)\Re e\left(k^2(z)\right)|\phi(z)|^2 \le \max\,\mu^{-1}\Re e(k^2)\int_{-\infty}^{\infty} dz\,|\phi(z)|^2 ,$$

$$\Re e(k_\rho^2)\int_{-\infty}^{\infty} dz\mu^{-1}(z)\,|\phi(z)|^2 \ge \Re e(k_\rho^2)\,\min\,\mu^{-1}\int_{-\infty}^{\infty} dz\,|\phi(z)|^2 , \quad \text{for } \Re e(k_\rho^2) > 0,$$

where min or max are the minimum or maximum of a function over $-\infty < z < \infty$. These inequalities imply that

$$\Re e\left(k_\rho^2\right)\min\,\mu^{-1} < \max\,\mu^{-1}\Re e(k^2), \quad \text{for} \quad \Re e(k_\rho^2) > 0.$$

The above inequality defines an upper bound on $\Re e(k_\rho^2)$ for positive $\Re e(k_\rho^2)$. Similarly, we can deduce that the imaginary part of the TE wave equation for lossless μ must satisfy the equation

$$\int_{-\infty}^{\infty} dz\mu^{-1}(z)\Im m\left(k^2(z)\right)|\phi(z)|^2 = \Im m(k_\rho^2)\int_{-\infty}^{\infty} dz\mu^{-1}(z)\,|\phi(z)|^2 .$$

These two terms have bounds given by

$$\int_{-\infty}^{\infty} dz\mu^{-1}(z)\Im m\left(k^2(z)\right)|\phi(z)|^2 \geq \min\, \mu^{-1}\Im m(k^2)\int_{\infty}^{\infty} dz|\phi(z)|^2\,,$$

$$\Im m(k_\rho^2)\int_{-\infty}^{\infty} dz\mu^{-1}(z)\,|\phi(z)|^2 \leq \Im m(k_\rho^2)\,\max \mu^{-1}\int_{-\infty}^{\infty} dz\,|\phi(z)|^2\,,$$

$$\text{for}\quad \Im m\left(k_\rho^2\right) > 0.$$

These equations can be combined to yield

$$\Im m(k_\rho^2)\,\max\mu^{-1} \geq \min\mu^{-1}\Im m(k^2), \quad \text{for } \Im m(k_\rho^2) > 0\,.$$

This equation defines the lower bound on $\Im m(k_\rho^2)$ for all $\Re e(k_\rho^2)$ since min $\mu^{-1}\Im m(k^2)/\max\mu^{-1}$ is a positive number for passive media.

Furthermore, the imaginary part of the TE wave equation has additional bounds given by

$$\int_{-\infty}^{\infty} dz\mu^{-1}(z)\Im m\left(k^2(z)\right)|\phi(z)|^2 \leq \max\mu^{-1}\Im m(k^2)\int_{-\infty}^{\infty} dz|\phi(z)|^2\,,$$

$$\Im m(k_\rho^2)\int_{-\infty}^{\infty} dz\mu^{-1}(z)\,|\phi(z)|^2 \geq \Im m(k_\rho^2)\,\min\,\mu^{-1}\int_{-\infty}^{\infty} dz\,|\phi(z)|^2\,,$$

$$\text{for}\quad \Im m(k_\rho^2) > 0.$$

These inequalities are used to show

$$\Im m(k_\rho^2)\min\mu^{-1} \leq \max\mu^{-1}\Im m(k^2), \quad \text{for } \Im m(k_\rho^2) > 0\,.$$

Finally, this equation defines the upper bound on $\Im m(k_\rho^2)$ for all $\Re e(k_\rho^2)$. In summary, the poles of the Sommerfeld integrand for a TE wave can be located in the shaded region of the Figure for Exercise Solution 2.32.

(b) We can map the complex k_ρ plane to the complex k_ρ^2 plane by

$$k_\rho^2 = (\Re e[k_\rho] + i\Im m[k_\rho])^2 = \Re e[k_\rho]^2 - \Im m[k_\rho]^2 + 2i\Re e[k_\rho]\Im m[k_\rho].$$

Thus the line $\Re e[k_\rho^2]$ = constant on the complex k_ρ^2 plane maps to a hyperbola on the complex k_ρ plane given by $\Re e[k_\rho]^2 - \Im m[k_\rho]^2 =$ constant. This hyperbola is centered at the origin and has foci on the $\Re e[k_\rho]$ axis.

Similarly, the line $\Im m[k_\rho^2]$ = constant on the complex k_ρ^2 plane maps to a hyperbola on the complex k_ρ plane given by $2i\Re e[k_\rho]\Im m[k_\rho]$ = constant. This hyperbola is centered at the origin with foci along the line $\Re e[k_\rho] =$

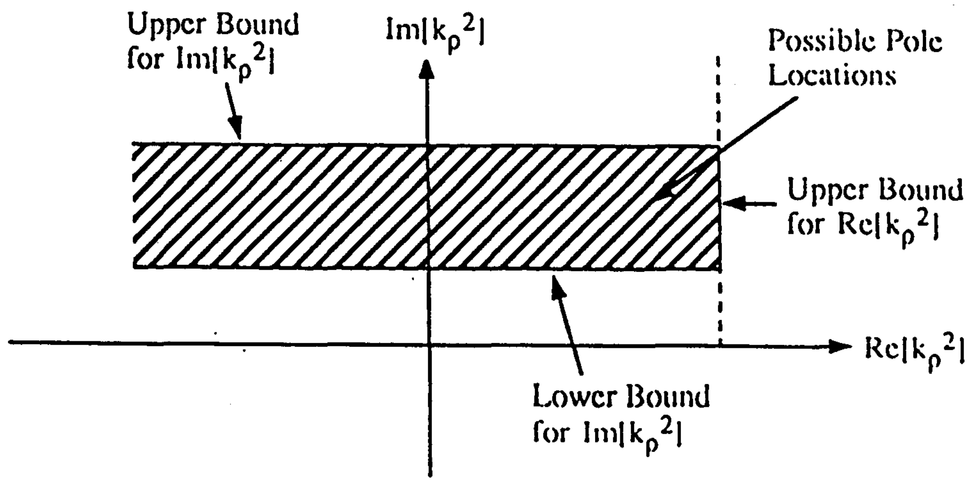

Figure for Exercise Solution 2.32

$\Im m[k_\rho]$. The asymptotes for this hyperbola are the real and imaginary axes.

Thus, the possible locations of the poles on the complex k_ρ plane are shown in Figure 2.7.4.

§2.33

(a) Given an integrand $G(k_\rho)$ with a pole at $k_\rho = k_P$ such that $G(k_\rho) \sim \frac{A}{k_\rho - k_P}$ as $k_\rho \to k_P$, first evaluate

$$\begin{aligned}\lim_{k_\rho \to k_P} (k_\rho - k_P)F(k_\rho) &= \lim_{k_\rho \to k_P} (k_\rho - k_P)G(k_\rho) \\ &\quad - \lim_{k_\rho \to k_P} (k_\rho - k_P)\frac{2Ak_P}{k_\rho^2 - k_P^2}\frac{J_0(k_\rho a)}{J_0(k_P a)}, \\ &= \lim_{k_\rho \to k_P} A\left(1 - \frac{2k_P}{k_\rho + k_P}\frac{J_0(k_\rho a)}{J_0(k_P a)}\right) = 0.\end{aligned}$$

Thus, $F(k_\rho)$ has no pole at $k_\rho = k_P$. Then the integral can be rewritten

$$\begin{aligned}\phi &\triangleq \int_0^\infty dk_\rho k_\rho J_0(k_\rho \rho)G(k_\rho), \\ &= \int_0^\infty dk_\rho k_\rho J_0(k_\rho \rho)F(k_\rho) + \frac{2Ak_P}{J_0(k_P a)}\int_0^\infty dk_\rho \frac{k_\rho J_0(k_\rho \rho)J_0(k_\rho a)}{k_\rho^2 - k_P^2}.\end{aligned}$$

(b) The second term above can be evaluated in closed form

$$I_P = \frac{2Ak_P}{J_0(k_P a)} \int_0^\infty dk_\rho \frac{k_\rho J_0(k_\rho \rho) J_0(k_\rho a)}{k_\rho^2 - k_P^2}.$$

Recall the Bessel identity $J_0(x) = \frac{1}{2}\left[H_0^{(1)}(x) + H_0^{(2)}(x)\right]$ then

$$I_P = \frac{Ak_P}{J_0(k_P a)} \int_0^\infty dk_\rho \frac{k_\rho H_0^{(1)}(k_\rho \rho) J_0(k_\rho a)}{k_\rho^2 - k_P^2} + \frac{Ak_P}{J_0(k_P a)} \int_0^\infty dk_\rho \frac{k_\rho H_0^{(2)}(k_\rho \rho) J_0(k_\rho a)}{k_\rho^2 - k_P^2}.$$

Now, use a change of variables in the second integral above such that $k_\rho = -k'_\rho$ and recall the Hankel identity $H_0^{(1)}(x) = -H_0^{(2)}(-x)$ to unfold the integral to

$$I_P = \frac{Ak_P}{J_0(k_P a)} \int_{-\infty}^\infty dk_\rho \frac{k_\rho H_0^{(1)}(k_\rho \rho)(J_0(k_\rho a) + J_0(-k_\rho a))}{k_\rho^2 - k_P^2}.$$

Then, we recall the Bessel identity $J_0(x) = J_0(-x)$ and we find

$$I_P = \frac{2Ak_P}{J_0(k_P a)} \int_{-\infty}^\infty dk_\rho \frac{k_\rho J_0(k_\rho a) H_0^{(1)}(k_\rho \rho)}{k_\rho^2 - k_P^2},$$

where the contour of integration is the SIP. Using Cauchy's theorem and Jordan's lemma this integral is exactly equal to the residue at k_P so

$$I_P = \frac{2Ak_P}{J_0(k_P a)} 2\pi i \lim_{k_\rho \to k_P} (k_\rho - k_P) \frac{k_\rho J_0(k_\rho a) H_0^{(1)}(k_\rho \rho)}{k_\rho^2 - k_P^2},$$
$$I_P = 2\pi i A k_P H_0^{(1)}(k_P \rho).$$

§2.34

(a) Assume $\phi(\mathbf{r}) = A \exp(i\omega\tau(\mathbf{r}))$ then

$$\nabla^2\phi = \left(i\omega\nabla^2\tau - \omega^2\nabla\tau \cdot \nabla\tau\right)\phi,$$

so the 3-D scalar wave equation can be rewritten

$$i\omega\nabla^2\tau(\mathbf{r}) - \omega^2\nabla\tau(\mathbf{r}) \cdot \nabla\tau(\mathbf{r}) + k^2(\mathbf{r}) = 0.$$

Now we use a perturbation series such that

$$\tau(\mathbf{r}) = \tau_0(\mathbf{r}) + \frac{1}{\omega}\tau_1(\mathbf{r}) + \cdots, \qquad \omega \to \infty.$$

The leading-order terms compose the eikonal equation:

$$\omega^2 \nabla\tau_0(\mathbf{r}) \cdot \nabla\tau_0(\mathbf{r}) = k^2(\mathbf{r}).$$

Similarly, the first-order terms describe the transport equation:

$$i\nabla^2\tau_0(\mathbf{r}) - 2\nabla\tau_0(\mathbf{r}) \cdot \nabla\tau_1(\mathbf{r}) = 0.$$

(b) These equations are solved numerically by using a ray-tracing algorithm. The ray path is given by the equation

$$\nabla c = \frac{d}{ds} c \frac{d\mathbf{r}}{ds}$$

where c is the velocity and ds is along a ray path. The eikonal equation can be rewritten as $\nabla\tau_0(\mathbf{r}) = S(\mathbf{r})\hat{s}$ where $\hat{s}$ is a unit vector in the direction of the ray at the point $\mathbf{r}$ and $S(\mathbf{r})$ is the slowness. In the ray coordinate, this is $\frac{d}{ds}\tau_0(\mathbf{r}) = S(\mathbf{r})$. When integrated along the ray we find

$$\tau_0(\mathbf{r}) = \pm \int^{\mathbf{r}} S(\mathbf{r}')ds' + C_0 .$$

Similarly, the transport equation can be rewritten $-2S\frac{\partial}{\partial s}\tau_1(\mathbf{r}) + i\frac{\partial}{\partial s}S = 0$ and solved by

$$\tau_1(\mathbf{r}) = \frac{i}{2}\ln S(\mathbf{r}) + C_{1\pm} .$$

§2.35

(a) Assume a TE incident wave $ae^{-ik_{0z}z}$ from $z = +\infty$. Here, reflections are generated by two features. First, the step discontinuity at $z = d$ provides a reflection coefficient given by

$$R_{01} = \frac{k_{0z} - k_{1z}}{k_{0z} + k_{1z}} .$$

Second, the smooth dielectric profile also produces reflections. For the case $\kappa d \gg 1$ the smooth dielectric profile resembles Figure 2.8.1 with $\epsilon_1 > \epsilon_0 > \epsilon_2$. Far away from the origin we can use the WKB method to approximate the field using (2.8.33)

$$\phi_I(z) \sim \sqrt{\frac{\omega}{k_{1z}}} \left[A_+ e^{ik_{1z}z + i\alpha} + A_- e^{-ik_{1z}z - i\alpha}\right] , \qquad z \gg 0$$

where (2.8.32) gives the phase term

$$a = \int_{\zeta}^{d} [k(z') - k_{1z}]\, dz' - k_{1z}\zeta.$$

Now using asymptotic matching we find $A_+/A_- = -i$ from (2.8.38). Thus, the smooth dielectric profile is characterized by a reflection coefficient referenced at $z = 0$ given by

$$R_{12} = \frac{A_+}{A_-} e^{2ia} = -ie^{2ia}.$$

Finally, the generalized reflection coefficient due to multiple reflections between the two features is

$$\tilde{R}_{01} = R_{01} + \frac{T_{01} R_{12} T_{10} e^{2ik_{1z}d}}{1 + R_{01} R_{12} e^{2ik_{1z}d}}.$$

(b) The transverse resonance guidance condition is $1 + R_{01} R_{12} e^{2ik_{1z}d} = 0$ or

$$1 - \frac{k_{0z} - k_{1z}}{k_{0z} + k_{1z}} ie^{2ik_{1z}d + 2ia} = 0.$$

§2.36

For a half-space the state vector in region 1 is given by (2.9.14)

$$\mathbf{V}_1(z) = A_{1-} e^{-ik_{1z}z} \mathbf{a}_{1-} + R A_{1-} e^{ik_{1z}z} \mathbf{a}_{1+} = \bar{\mathbf{a}}_1 \cdot e^{i\bar{\mathbf{K}}_1 z} \cdot \begin{bmatrix} R \\ 1 \end{bmatrix} A_{1-}.$$

Similarly the state vector in Region 2 has to be of the form

$$\mathbf{V}_2(z) = T A_{1-} e^{-ik_{2z}z} \mathbf{a}_{2-} = \bar{\mathbf{a}}_2 \cdot e^{i\bar{\mathbf{K}}_2 z} \cdot \begin{bmatrix} 0 \\ T \end{bmatrix} A_{1-}.$$

Because of the definitions of ϕ and ψ in (2.9.1) and (2.9.2), they are continuous quantities across a discontinuity. Then, we know

$$\bar{\mathbf{a}}_1 \cdot \begin{bmatrix} R \\ 1 \end{bmatrix} = \bar{\mathbf{a}}_2 \cdot \begin{bmatrix} 0 \\ T \end{bmatrix},$$

If we normalize the eigenvectors such that $\mathbf{a}^\dagger \cdot \mathbf{a} = 1$ we can test the above equation with $\bar{\mathbf{a}}_1^\dagger$ resulting in

$$\begin{bmatrix} R \\ 1 \end{bmatrix} = \bar{\mathbf{a}}_1^\dagger \cdot \bar{\mathbf{a}}_2 \cdot \begin{bmatrix} 0 \\ T \end{bmatrix},$$

$$\begin{bmatrix} R \\ 1 \end{bmatrix} = T \begin{bmatrix} \mathbf{a}_{1+}^\dagger \cdot \mathbf{a}_{2-} \\ \mathbf{a}_{1-}^\dagger \cdot \mathbf{a}_{2-} \end{bmatrix},$$

$$R = \mathbf{a}_{1+}^\dagger \cdot \mathbf{a}_{2-} \left(\mathbf{a}_{1-}^\dagger \cdot \mathbf{a}_{2-}\right)^{-1},$$

$$T = (\mathbf{a}_{1-}^\dagger \cdot \mathbf{a}_{2-})^{-1}.$$

§2.37

By writing (2.10.8) in the form (2.10.9) we know that the state equation has matrix elements described the following operators

$$\overline{\mathbf{H}}_{11} = (-i\hat{z} \times \overline{\boldsymbol{\mu}}_{sz} \cdot \nu_{zz}\overline{\mathbf{k}}_s \times) - (i\hat{z} \times \overline{\mathbf{k}}_s \times \kappa_{zz}\overline{\boldsymbol{\epsilon}}_{zs} \cdot),$$

$$\overline{\mathbf{H}}_{12} = (-i\omega\hat{z} \times \overline{\boldsymbol{\mu}}_s \cdot) + (i\omega\hat{z} \times \overline{\boldsymbol{\mu}}_{sz} \cdot \overline{\boldsymbol{\mu}}_{zs} \cdot \nu_{zz}) - \left(\frac{i\hat{z}}{\omega} \times \overline{\mathbf{k}}_s \times \kappa_{zz}\overline{\mathbf{k}}_s \times\right),$$

with $\mathbf{H}_{21}$ and $\mathbf{H}_{22}$ given by duality. In Cartesian coordinates we know $\overline{\mathbf{k}}_s = \hat{x}k_x + \hat{y}k_y$ and we can simplify this expression using vector identities. In particular we recall

$$\begin{aligned}\hat{z} \times \overline{\mathbf{k}}_s \times \hat{z} &= \overline{\mathbf{k}}_s(\hat{z} \cdot \hat{z}) - \hat{z}(\hat{z} \cdot \overline{\mathbf{k}}_s) \\ &= \overline{\mathbf{k}}_s\end{aligned}$$

where we have used $\hat{z} \cdot \overline{\mathbf{k}}_s = 0$. Now we can rewrite the state equation operators as

$$\overline{\mathbf{H}}_{11} = (-i\hat{z} \times \overline{\boldsymbol{\mu}}_{sz} \cdot \nu_{zz}\overline{\mathbf{k}}_s \times) - (i\overline{\mathbf{k}}_s\kappa_{zz}\hat{z} \cdot \overline{\boldsymbol{\epsilon}}_{zs} \cdot),$$

$$\overline{\mathbf{H}}_{12} = (-i\omega\hat{z} \times \overline{\boldsymbol{\mu}}_s \cdot) + (i\omega\hat{z} \times \overline{\boldsymbol{\mu}}_{sz} \cdot \overline{\boldsymbol{\mu}}_{zs} \cdot \nu_{zz}) - \left(\frac{i}{\omega}\overline{\mathbf{k}}_s\kappa_{zz}\hat{z} \cdot \overline{\mathbf{k}}_s \times\right).$$

Finally, in Cartesian coordinates we define the state vector as $\mathbf{V}^t = (E_x, E_y, H_x, H_y)$. With this definition the state equation operators can be written in matrix form as

$$\overline{\mathbf{H}}_{11} = i\begin{bmatrix} -\mu_{yz}\nu_{zz}k_y - k_x\kappa_{zz}\epsilon_{zx}, & \mu_{yz}\nu_{zz}k_x - k_x\kappa_{zz}\epsilon_{zy} \\ \mu_{xz}\nu_{zz}k_y - k_y\kappa_{zz}\epsilon_{zx}, & -\mu_{xz}\nu_{zz}k_x - k_y\kappa_{zz}\epsilon_{zy} \end{bmatrix},$$

$$\overline{\mathbf{H}}_{12} =$$
$$i\omega\begin{bmatrix} \mu_{yx} - \mu_{yz}\mu_{zx}\nu_{zz} + \frac{1}{\omega^2}k_x\kappa_{zz}k_y, & \mu_{yy} - \mu_{yz}\mu_{zy}\nu_{zz} - \frac{1}{\omega^2}k_x\kappa_{zz}k_x \\ -\mu_{xx} + \mu_{xz}\mu_{zx}\nu_{zz} + \frac{1}{\omega^2}k_y\kappa_{zz}k_y, & -\mu_{xy} + \mu_{xz}\mu_{zy}\nu_{zz} - \frac{1}{\omega^2}k_y\kappa_{zz}k_x \end{bmatrix}.$$

Similarly, we can find $\overline{\mathbf{H}}_{21}$ and $\overline{\mathbf{H}}_{22}$ by duality.

§2.38

(a) Given two half-spaces, the reflection and transmission matrices are related by (2.10.21)

$$\overline{\mathbf{a}}_1 \cdot \begin{bmatrix} \overline{\mathbf{R}}_{12} \\ \overline{\mathbf{I}} \end{bmatrix} = \overline{\mathbf{a}}_2 \cdot \begin{bmatrix} 0 \\ \overline{\mathbf{T}}_{12} \end{bmatrix}.$$

Now we can multiply by $\overline{\mathbf{a}}_1^\dagger$ and solve for the two unknown matrices

$$\begin{bmatrix} \overline{\mathbf{R}}_{12} \\ \overline{\mathbf{I}} \end{bmatrix} = \left[\overline{\mathbf{a}}_1^\dagger \cdot \overline{\mathbf{a}}_1\right]^{-1} \cdot \overline{\mathbf{a}}_1^\dagger \cdot \overline{\mathbf{a}}_2 \cdot \begin{bmatrix} 0 \\ \overline{\mathbf{T}}_{12} \end{bmatrix},$$

$$\begin{bmatrix} \overline{\mathbf{R}}_{12} \\ \overline{\mathbf{I}} \end{bmatrix} \triangleq \begin{bmatrix} \overline{\mathbf{D}}_{11} & \overline{\mathbf{D}}_{12} \\ \overline{\mathbf{D}}_{21} & \overline{\mathbf{D}}_{22} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \overline{\mathbf{T}}_{12} \end{bmatrix},$$

$$\overline{\mathbf{R}}_{12} = \overline{\mathbf{D}}_{12} \cdot \overline{\mathbf{D}}_{22}^{-1},$$

$$\overline{\mathbf{T}}_{12} = \overline{\mathbf{D}}_{22}^{-1}.$$

(b) For homogeneous and isotropic media, one can show that the state equation (2.10.9) simplifies. In this case $\overline{\mu} = \mu_1\overline{\mathbf{I}}$ and $\epsilon = \epsilon_1\overline{\mathbf{I}}$ where $\overline{\mathbf{I}}$ is the identity tensor. Then, $\overline{\mathbf{H}}_{11} = \overline{\mathbf{H}}_{22} = 0$ in the state equation. Furthermore, for a $\mathbf{k}$ vector in the xz plane one can show

$$\overline{\mathbf{H}}_{12} = \begin{bmatrix} 0 & h_1 \\ h_2 & 0 \end{bmatrix},$$
$$\overline{\mathbf{H}}_{21} = \begin{bmatrix} 0 & h_3 \\ h_4 & 0 \end{bmatrix}.$$

Here, we have defined the state vector as $\mathbf{V}^t = (E_x, E_y, H_x, H_y)$. In this case, the eigensolutions of (2.10.12) separate into TE and TM waves such that the general solution is of the form

$$\mathbf{V}(z) = \mathbf{a}_1\left[A_1 e^{i\beta_1 z} + A_3 e^{-i\beta_1 z}\right] + \mathbf{a}_2\left[A_2 e^{i\beta_2 z} + A_4 e^{-i\beta_2 z}\right],$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal. It is appropriate to normalize the eigenvectors such that $\mathbf{a}_1^\dagger\mathbf{a}_1 = \mathbf{a}_2^\dagger\mathbf{a}_2 = 1$.

In the two region problem, the TE and TM waves are still orthogonal. Thus, the matrices $\overline{\mathbf{D}}_{12}$ and $\overline{\mathbf{D}}_{22}$ defined above are diagonal, and hence so are $\overline{\mathbf{R}}_{12}$ and $\overline{\mathbf{T}}_{12}$.

(c) The reflection and transmission matrices in this case are given by

$$\overline{\mathbf{R}}_{12} = \begin{bmatrix} R_{12}^{TE} & 0 \\ 0 & R_{12}^{TM} \end{bmatrix}, \qquad \overline{\mathbf{T}}_{12} = \begin{bmatrix} T_{12}^{TE} & 0 \\ 0 & T_{12}^{TM} \end{bmatrix}.$$

§2.39

As a series we can write (2.10.30)

$$\begin{aligned}\tilde{\overline{\mathbf{R}}}_{12} &= \overline{\mathbf{R}}_{12} + \overline{\mathbf{T}}_{21}\cdot e^{i\overline{\beta}_{2+}d_2}\cdot\overline{\mathbf{R}}_{23}\cdot e^{i\overline{\beta}_{2-}d_2}\cdot\overline{\mathbf{T}}_{12} \\ &+ \overline{\mathbf{T}}_{21}\cdot e^{i\overline{\beta}_{2+}d_2}\cdot\overline{\mathbf{R}}_{23}\cdot e^{i\overline{\beta}_{2-}d_2}\cdot\overline{\mathbf{R}}_{21}\cdot e^{i\overline{\beta}_{2+}d_2}\cdot\overline{\mathbf{R}}_{23}\cdot e^{i\overline{\beta}_{2-}d_2}\cdot\overline{\mathbf{T}}_{12} + \cdots\end{aligned}$$

where we have expanded $(1-x)^{-1} = 1 + x + x^2 + \cdots$. The physical interpretation of the first term in the above is just the result of a single reflection off the first interface. The n-th term above is a consequence of the n-th reflection from the three-layer medium. Thus, the series above can be thought of as a **ray** or **geometric optics** series, as a consequence of multiple reflections and transmissions in region 2. Notice the difference between the above series and (2.1.22) is that the above generalized reflection matrix incorporates both type I and type II waves.

CHAPTER 3

EXERCISE SOLUTIONS

By J. H. Lin and C. C. Lu

§3.1

The wave equation for electric field is

$$\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = 0.$$

Using the vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla\nabla \cdot \mathbf{E} - \nabla^2\mathbf{E},$$

and the fact that in the homogeneous and source free region, $\nabla \cdot \mathbf{E} = 0$, we obtain

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0. \tag{1}$$

Therefore, for the z component,

$$\nabla^2 E_z + k^2 E_z = 0.$$

Now, forming the dot product of Equation (1) with $\hat{\rho}$, we have

$$\cos\phi\nabla^2 E_x + \sin\phi\nabla^2 E_y + k^2 E_\rho = 0.$$

Since $E_x = \cos\phi E_\rho - \sin\phi E_\phi$ and $E_y = \sin\phi E_\rho + \cos\phi E_\phi$, the above equation becomes

$$\begin{aligned}
&\cos\phi\Big[\frac{\cos\phi}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial E_\rho}{\partial\rho}\Big) - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial E_\phi}{\partial\rho}\Big) - \frac{\cos\phi}{\rho^2}E_\rho \\
&+ \frac{\cos\phi}{\rho^2}\frac{\partial^2 E_\rho}{\partial\phi^2} + \frac{\sin\phi}{\rho^2}E_\phi - \frac{\sin\phi}{\rho^2}\frac{\partial^2 E_\phi}{\partial\phi^2} + \cos\phi\frac{\partial^2 E_\rho}{dz^2} \\
&- \sin\phi\frac{\partial^2 E_\phi}{\partial z^2}\Big] + \sin\phi\Big[\frac{\sin\phi}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial E_\rho}{\partial\rho}\Big) + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial E_\phi}{\partial\rho}\Big) - \frac{\sin\phi}{\rho^2}E_\rho \\
&+ \frac{\sin\phi}{\rho^2}\frac{\partial^2 E_\rho}{\partial\phi^2} - \frac{\cos\phi}{\rho^2}E_\phi + \frac{\cos\phi}{\rho^2}\frac{\partial^2 E_\phi}{\partial\phi^2} + \sin\phi\frac{\partial^2 E_\rho}{\partial z^2} \\
&+ \cos\phi\frac{\partial^2 E_\phi}{\partial z^2}\Big] + k^2 E_\rho = 0.
\end{aligned}$$

After some arrangements of the above equation, we obtain

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial E_\rho}{\partial\rho}\Big) - \frac{E_\rho}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_\rho}{\partial\phi^2} + \frac{\partial^2 E_\rho}{\partial z^2} + k^2 E_\rho = 0,$$

which is actually of the form

$$\nabla^2 E_\rho + k^2 E_\rho - \frac{E_\rho}{\rho^2} = 0.$$

Same derivation can be applied for the ϕ component, and it leads to

$$\nabla^2 E_\phi + k^2 E_\phi - \frac{E_\phi}{\rho^2} = 0.$$

So, the ϕ and ρ components do not satisfy the same equation as z component does. Same arguments can be applied in the case of **H** field as well.

§3.2

Since (3.1.2) are linear and second order PDE which can be decomposed into three second order ODE's by the method of separation of variables, we then have two linearly independent solutions for every n.

So, owing to the linearity of Bessel's equations, any linear combination of two linearly-independent solutions is still a solution to the equation for each n and since Bessel's functions form a complete set, their linear superposition is still a complete set.

§3.3

In a cylindrically layered medium, since it is translationally invariant in the z direction, if one region has $e^{ik_z z}$ dependence, the k_z must be the same in every region.

In order to satisfy the phase matching condition at each interface of these cylindrically layers, $e^{ik_z z}$ dependence must be assumed in each region.

§3.4

(3.1.4) are

$$\mathbf{E}_s = \frac{1}{k_\rho^2}\left[ik_z \nabla_s E_z - i\omega\mu\hat{z}\times\nabla_s H_z\right],$$
$$\mathbf{H}_s = \frac{1}{k_\rho^2}\left[ik_z \nabla_s H_z + i\omega\epsilon\hat{z}\times\nabla_s E_z\right].$$

Consider the ϕ component,

$$E_\phi = \frac{1}{k_\rho^2}\left[ik_z \frac{1}{\rho}\frac{\partial E_z}{\partial\phi} - i\omega\mu\frac{\partial H_z}{\partial\rho}\right],$$
$$H_\phi = \frac{1}{k_\rho^2}\left[ik_z \frac{1}{\rho}\frac{\partial H_z}{\partial\phi} + i\omega\epsilon\frac{\partial E_z}{\partial\rho}\right].$$

Since E_z, H_z have $e^{in\phi}$ dependence, $\frac{\partial}{\partial\phi}$ operation leads to multiplications by in. Thus,

$$E_\phi = \frac{1}{k_\rho^2}\left[-\frac{nk_z}{\rho}E_z - i\omega\mu\frac{\partial H_z}{\partial\rho}\right],$$
$$H_\phi = \frac{1}{k_\rho^2}\left[-\frac{nk_z}{\rho}H_z + i\omega\epsilon\frac{\partial E_z}{\partial\rho}\right].$$

Written in matrix form, it appears as

$$\begin{bmatrix} H_\phi \\ E_\phi \end{bmatrix} = \frac{1}{k_\rho^2}\left\{-\frac{nk_z}{\rho}\begin{bmatrix} H_z \\ E_z \end{bmatrix} + i\omega\begin{bmatrix} 0 & \epsilon\frac{\partial}{\partial\rho} \\ -\mu\frac{\partial}{\partial\rho} & 0 \end{bmatrix}\begin{bmatrix} H_z \\ E_z \end{bmatrix}\right\}.$$

For region 1

$$\begin{bmatrix} H_{1z} \\ E_{1z} \end{bmatrix} = H_n^{(1)}(k_{1\rho}\rho)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot \mathbf{a}_1 + J_n(k_{1\rho}\rho)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\overline{\mathbf{R}}_{12}\cdot\mathbf{a}_1.$$

Then

$$\begin{aligned}\begin{bmatrix} H_{1\phi} \\ E_{1\phi} \end{bmatrix} &= \frac{1}{k_{1\rho}^2}\left\{-\frac{nk_z}{\rho}H_n^{(1)}(k_{1\rho}\rho)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\mathbf{a}_1 - \frac{nk_z}{\rho}J_n(k_{1\rho}\rho)\right.\\
&\quad\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\overline{\mathbf{R}}_{12}\cdot\mathbf{a}_1 + i\omega k_{1\rho}\begin{bmatrix} 0 & \epsilon_1 \\ -\mu_1 & 0 \end{bmatrix}H_n^{'(1)}(k_{1\rho}\rho)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\mathbf{a}_1\\
&\quad\left. + i\omega k_{1\rho}\begin{bmatrix} 0 & \epsilon_1 \\ -\mu_1 & 0 \end{bmatrix}J_n'(k_{1\rho}\rho)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\overline{\mathbf{R}}_{12}\cdot\mathbf{a}_1\right\}\\
&= \frac{1}{k_{1\rho}^2}\left\{\begin{bmatrix} i\omega\epsilon_1 k_{1\rho}H_n^{'(1)}(k_{1\rho}\rho) & -\frac{nk_z}{\rho}H_n^{(1)}(k_{1\rho}\rho) \\ -\frac{nk_z}{\rho}H_n^{(1)}(k_{1\rho}\rho) & -i\omega\mu_1 k_{1\rho}H_n^{'(1)}(k_{1\rho}\rho) \end{bmatrix}\cdot\mathbf{a}_1\right.\\
&\quad\left. + \begin{bmatrix} i\omega\epsilon_1 k_{1\rho}J_n'(k_{1\rho}\rho) & -\frac{nk_z}{\rho}J_n(k_{1\rho}\rho) \\ -\frac{nk_z}{\rho}J_n(k_{1\rho}\rho) & -i\omega\mu_1 k_{1\rho}J_n'(k_{1\rho}\rho) \end{bmatrix}\cdot\overline{\mathbf{R}}_{12}\cdot\mathbf{a}_1\right\}. \end{aligned} \tag{1}$$

For region 2,

$$\begin{bmatrix} H_{2z} \\ E_{2z} \end{bmatrix} = H_n^{(1)}(k_{2\rho}\rho)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\overline{\mathbf{T}}_{12}\cdot\mathbf{a}_1,$$

then

$$\begin{aligned}\begin{bmatrix} H_{2\phi} \\ E_{2\phi} \end{bmatrix} &= \frac{1}{k_{2\rho}^2}\left\{-\frac{nk_z}{\rho}H_n^{(1)}(k_{2\rho}\rho)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\overline{\mathbf{T}}_{12}\cdot\mathbf{a}_1 + \right.\\
&\quad\left. i\omega k_{2\rho}\begin{bmatrix} 0 & \epsilon_2 \\ -\mu_2 & 0 \end{bmatrix}H_n^{'(1)}(k_{2\rho}\rho)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\overline{\mathbf{T}}_{12}\cdot\mathbf{a}_1\right\}\\
&= \frac{1}{k_{2\rho}^2}\left\{\begin{bmatrix} i\omega\epsilon_2 k_{2\rho}H_n^{'(1)}(k_{2\rho}\rho) & -\frac{nk_z}{\rho}H_n^{'(1)}(k_{2\rho}\rho) \\ -\frac{nk_z}{\rho}H_n^{(1)}(k_{2\rho}\rho) & -i\omega\mu_2 H_n^{'(1)}(k_{2\rho}\rho) \end{bmatrix}\cdot\overline{\mathbf{T}}_{12}\cdot\mathbf{a}_1\right\}. \end{aligned} \tag{2}$$

(1) and (2) can be expressed as

$$\begin{bmatrix} H_{1\phi} \\ E_{1\phi} \end{bmatrix} = \overline{\mathbf{H}}_n^{(1)}(k_{1\rho}\rho)\cdot\mathbf{a}_1 + \overline{\mathbf{J}}_n(k_{1\rho}\rho)\cdot\overline{\mathbf{R}}_{12}\cdot\mathbf{a}_1,$$
$$\begin{bmatrix} H_{2\phi} \\ E_{2\phi} \end{bmatrix} = \overline{\mathbf{H}}_n^{(1)}(k_{2\rho}\rho)\cdot\overline{\mathbf{T}}_{12}\cdot\mathbf{a}_1,$$

where

$$\overline{\mathbf{B}}_n(k_{i\rho}\rho) = \frac{1}{k_{i\rho}^2\rho}\begin{bmatrix} i\omega\epsilon_i k_{i\rho}\rho B_n'(k_{i\rho}\rho) & -nk_z B_n(k_{i\rho}\rho) \\ -nk_z B_n(k_{i\rho}\rho) & -i\omega\mu_i k_{i\rho}\rho B_n'(k_{i\rho}\rho) \end{bmatrix}.$$

Here, B_n is either $H_n^{(1)}$ or J_n.

§3.5

(a) Bessel's equation is of the form,

$$P'' + \frac{1}{x}P' + \left(1 - \frac{n^2}{x^2}\right)P = 0.$$

If P_1 and P_2 are two linearly independent solutions to the above equation, then

$$P_1'' + \frac{1}{x}P_1' + \left(1 - \frac{n^2}{x^2}\right)P_1 = 0, \tag{1}$$
$$P_2'' + \frac{1}{x}P_2' + \left(1 - \frac{n^2}{x^2}\right)P_2 = 0. \tag{2}$$

Multiplying (1) by P_2, and (2) by P_1, then subtracting the second equation from first equation yields,

$$P_1''P_2 - P_1P_2'' + \frac{1}{x}(P_1'P_2 - P_1P_2') = 0.$$

Recognizing that $P_1''P_2 - P_1P_2'' = (P_1'P_2 - P_1P_2')'$ and let $P_1'P_2 - P_1P_2' = f$, we have

$$f' = \frac{-f}{x}, \quad \longrightarrow \quad \frac{df}{f} = -\frac{dx}{x}.$$

Thus,

$$\ln f = -\ln x + c \quad \longrightarrow \quad f = \frac{\text{const.}}{x}.$$

So,

$$H_n^{(1)}(x)J_n'(x) - J_n(x)H_n^{(1)\prime}(x) = \frac{\text{const.}}{x}.$$

(b) Since $H_n^{(1)}(x)J_n'(x) - J_n(x)H_n'^{(1)}(x) = \frac{\text{const.}}{x}$ holds for every x, we let $x \to 0$. Then we have

$$
\begin{aligned}
& -\frac{i(n-1)!}{\pi}\left(\frac{2}{x}\right)^n\left[\frac{(x/2)^n}{n!}\right]' - \frac{(x/2)^n}{n!}\left[-\frac{i(n-1)!}{\pi}\left(\frac{2}{x}\right)^n\right]' \\
&= -\frac{i(n-1)!}{\pi}\left(\frac{2}{x}\right)^n \frac{x^{n-1}}{2^n(n-1)!} + \frac{x^n}{2^n n!}\frac{i(n-1)!}{\pi}\frac{2^n}{x^{n+1}}(-n) \\
&= \frac{-i}{\pi x} - \frac{i}{\pi x} = \frac{-2i}{\pi x}.
\end{aligned}
$$

So this constant is $\frac{-2i}{\pi}$.

§3.6

(a) Since a guided mode exists in the cylinder without the external excitation, $\mathbf{a}_1$ in (3.1.6) can be set to zero. But there is still a wave existing in the cylinder, which requires that determinant of $\overline{\mathbf{R}}_{12}$ be infinite.

For the case of (3.1.14), $\overline{\mathbf{T}}_{21}$ should possess an infinite determinantal values in order for $[E_{1z}\ \ H_{1z}]^T$ exists without $\mathbf{a}_2$. And judging from (3.1.16a), $\overline{\mathbf{T}}_{21}$ is related to $\overline{\mathbf{R}}_{21}$ by some multiplicative and additive finite constants. So $\overline{\mathbf{R}}_{21}$ should have an infinite determinant.

(b) $\overline{\mathbf{D}}$ is the same both in (3.1.11a) and (3.1.17a). So in order for $\overline{\mathbf{R}}_{12}$ and $\overline{\mathbf{R}}_{21}$ having infinite determinant, $\overline{\mathbf{D}}^{-1}$ should have infinite determinant, i.e. $\overline{\mathbf{D}}$ has zero determinant.

(c) Since both cases deal with the same structure of circular cylinder except for different locations of external excitations, and resonant conditions are independent of those sources, we obtain the same guidance condition with (3.1.11a) and (3.1.17a).

(d) When $\mathbf{a}_3$ is zero, in order for $\mathbf{a}_2$ to exist, from (3.2.17), the determinant of $(\overline{\mathbf{I}} - \overline{\mathbf{R}}_{23}\cdot\overline{\mathbf{R}}_{21})^{-1}$ should be infinite. In other words, $\overline{\mathbf{I}} - \overline{\mathbf{R}}_{23}\cdot\overline{\mathbf{R}}_{21}$ should have a zero of determinant value.

Also, $\mathbf{a}_1 = \overline{\mathbf{T}}_{21}\cdot\mathbf{a}_2 = \overline{\mathbf{T}}_{21}\cdot(\overline{\mathbf{I}} - \overline{\mathbf{R}}_{23}\cdot\overline{\mathbf{R}}_{21})^{-1}\cdot\overline{\mathbf{T}}_{32}\cdot\mathbf{a}_3$. So it is clear that $\det(\overline{\mathbf{I}} - \overline{\mathbf{R}}_{23}\cdot\overline{\mathbf{R}}_{21}) = 0$ is the guidance condition. Maybe one will argue that $\overline{\mathbf{T}}_{32}$ being infinte can yield the resonance as well, but an infinite $\overline{\mathbf{R}}_{23}$ ensues when we make the above assumption, which does not guarantee that $(\overline{\mathbf{I}} - \overline{\mathbf{R}}_{23}\cdot\overline{\mathbf{R}}_{21})^{-1}\cdot\overline{\mathbf{T}}_{32}$ is infinite.

§3.7

(a) Let

$$e^{ikx} = e^{ik\rho\cos\phi} = \sum_{n=-\infty}^{\infty} a_n e^{in\phi} J_n(k\rho).$$

Multiplying both sides of the above by $e^{im\phi}$ and integrating with respect to ϕ from 0 to 2π, we have

$$\int_0^{2\pi} e^{ik\rho\cos\phi+im\phi}\,d\phi = \sum_{n=-\infty}^{\infty} a_n J_n(k\rho)\int_0^{2\pi} e^{i(n+m)\phi}\,d\phi.$$

The only term that survives in the right-hand side is $n=-m$.
Hence,

$$\int_0^{2\pi} e^{ik\rho\cos\phi+im\phi}\,d\phi = a_{-m}J_{-m}(k\rho)\cdot 2\pi.$$

Invoking the fact that $J_m(k\rho)e^{i\frac{m\pi}{2}}\cdot 2\pi = \int_0^{2\pi} e^{ik\rho\cos\phi+im\phi}\,d\phi$, the above equation becomes

$$2\pi e^{i\frac{m\pi}{2}}J_m(k\rho) = a_{-m}J_{-m}(k\rho)\cdot 2\pi = a_{-m}(-1)^m J_m(k\rho)\cdot 2\pi.$$

Therefore, $a_{-m}=(-1)^m e^{i\frac{m\pi}{2}}$, $\quad a_m = e^{im\pi}e^{-i\frac{m\pi}{2}} = e^{i\frac{m\pi}{2}}$.

(b) Since the field transversed to z can be expressed as

$$\mathbf{E}_s = \frac{1}{k_x^2+k_y^2}\left[ik_z\nabla_s E_z - i\omega\mu\hat{z}\times\nabla_s H_z\right],$$
$$\mathbf{H}_s = \frac{1}{k_x^2+k_y^2}\left[ik_z\nabla_s H_z + i\omega\epsilon\hat{z}\times\nabla_s E_z\right]$$

in terms of H_z and E_z, any arbitrarily polarized plane wave can be decomposed into TM to z and TE to z plane wave.

(c) For an arbitrarily polarized incident plane wave, from (b), we can characterize it by E_z and H_z as

$$\begin{bmatrix} E_z \\ H_z \end{bmatrix} = \begin{bmatrix} E_0 \\ H_0 \end{bmatrix} e^{ik_x x+ik_y y+ik_z z} = \begin{bmatrix} E_0 \\ H_0 \end{bmatrix} e^{ik_\rho\rho\cos(\phi-\phi_{in})+ik_z z}.$$

¿From (a), we can express this incident plane wave in terms of cylindrical wave as

$$\begin{bmatrix} E_z \\ H_z \end{bmatrix} = \begin{bmatrix} E_0 \\ H_0 \end{bmatrix}\left(\sum_{n=-\infty}^{\infty} e^{i\frac{n\pi}{2}}e^{in(\phi-\phi_{in})}J_n(k\rho)\right)e^{ik_z z}.$$

Then, for the n-th harmonic with $e^{in(\phi-\phi_{in})}$ dependence, applying the recursive formula (3.2.19) to obtain the generalized reflection matrix $\overline{\mathbf{R}}_{N(N-1)}$, we have the reflected wave

$$\begin{bmatrix} E_z^R \\ H_z^R \end{bmatrix}_n = H_n^{(1)}(k_{N\rho}\rho)\tilde{\overline{\mathbf{R}}}_{N(N-1)}\cdot\begin{bmatrix} E_0 \\ H_0 \end{bmatrix} e^{i\frac{n\pi}{2}}.$$

So, the total reflected wave is

$$\begin{bmatrix} E_z^R \\ H_z^R \end{bmatrix} = \sum_{n=-\infty}^{\infty} e^{i\frac{n\pi}{2}} e^{in(\phi-\phi_{in})} H_n^{(1)}(k_{N_\rho}\rho) \tilde{\overline{\mathbf{R}}}_{N(N-1)} \cdot \begin{bmatrix} E_0 \\ H_0 \end{bmatrix} e^{ik_z z}.$$

Note that $\tilde{\overline{\mathbf{R}}}_{N(N-1)}$ is a matrix depending on n.

§3.8

(a) When $n \to -n$, (3.1.9) becomes

$$\begin{aligned}
&\overline{\mathbf{B}}_{-n}(k_{i\rho}\rho) \\
&\quad = \frac{1}{k_{i\rho}^2\rho} \begin{bmatrix} i\omega\epsilon_i k_{i\rho}\rho(-1)^n B_n'(k_{i\rho}\rho) & nk_z(-1)^n B_n(k_{i\rho}\rho) \\ nk_z(-1)^n B_n(k_{i\rho}\rho) & -i\omega\mu_i k_{i\rho}\rho(-1)^n B_n'(k_{i\rho}\rho) \end{bmatrix} \\
&\quad = \frac{(-1)^n}{k_{i\rho}^2\rho} \begin{bmatrix} i\omega\epsilon_i k_{i\rho}\rho B_n'(k_{i\rho}\rho) & nk_z B_n(k_{i\rho}\rho) \\ nk_z B_n(k_{i\rho}\rho) & -i\omega\mu_i k_{i\rho}\rho B_n'(k_{i\rho}\rho) \end{bmatrix} \\
&\quad \triangleq (-1)^n \overline{\mathbf{B}}_n^{od}(k_{i\rho}\rho) ,
\end{aligned}$$

where $\overline{\mathbf{B}}_n^{od}$ is defined as the matrix whose off-diagonal elements are those of $\overline{\mathbf{B}}_n$ by changing sign.

Therefore,

$$\begin{aligned}
\overline{\mathbf{D}} &= \left[\overline{\mathbf{J}}_{-n}(k_{1\rho}a)H_{-n}^{(1)}(k_{2\rho}a) - \overline{\mathbf{H}}_{-n}^{(1)}(k_{2\rho}a)J_{-n}(k_{1\rho}a)\right] \\
&= \left[(-1)^n\overline{\mathbf{J}}_n^{od}(k_{1\rho}a)(-1)^n H_n^{(1)}(k_{2\rho}a) - (-1)^n\overline{\mathbf{H}}_n^{od}(k_{2\rho}a)(-1)^n J_n(k_{1\rho}a)\right] \\
&= \left[\overline{\mathbf{J}}_n^{od}(k_{1\rho}a)H_n^{(1)}(k_{2\rho}a) - \overline{\mathbf{H}}_n^{od}(k_{2\rho}a)J_n(k_{1\rho}a)\right] , \\
\overline{\mathbf{D}}^{-1} &= \left[\overline{\mathbf{J}}_n^{od}(k_{1\rho}a)H_n^{(1)}(k_{2\rho}a) - \overline{\mathbf{H}}_n^{od}(k_{2\rho}a)J_n(k_{1\rho}a)\right]^{-1}
\end{aligned}$$

whose off-diagonal elements are just those elements of the opposite sign when $\overline{\mathbf{D}}^{-1}$ is the function of n.

Looking at these forms of $\overline{\mathbf{R}}_{12}$, $\overline{\mathbf{T}}_{12}$, $\overline{\mathbf{R}}_{21}$ and $\overline{\mathbf{T}}_{21}$ in (3.1.11) and (3.1.17), which actually are similar to that of $\overline{\mathbf{D}}$, it is easy to verify that only the off-diagonal elements change sign when $n \to -n$.

When $k_z \to -k_z$, it is readily seen that

$$\overline{\mathbf{B}}_n(k_{i\rho}\rho) = \overline{\mathbf{B}}_n^{od}(k_{i\rho}\rho).$$

So does off-diagonal elements of $\overline{\mathbf{R}}_{12}$, $\overline{\mathbf{T}}_{12}$, $\overline{\mathbf{R}}_{21}$ and $\overline{\mathbf{T}}_{21}$ change signs when $k_z \to -k_z$.

As for $\tilde{\overline{\mathbf{R}}}_{12}, \tilde{\overline{\mathbf{R}}}_{21}, \tilde{\overline{\mathbf{T}}}_{12}$, and $\tilde{\overline{\mathbf{T}}}_{21}$, since these matrices are constructed from $\overline{\mathbf{R}}_{i,i\pm1}$ and $\overline{\mathbf{T}}_{i,i\pm1}$ whose off-diagonal elements change signs when $n \to -n$

or $k_z \to -k_z$, these generalized reflection and transmission matrices have the same behavior.

(b) When $\rho = a$, from (3.1.6), $\mathbf{a}_1$ can be expressed in terms of $[E_{1z} \quad H_{1z}]^T$ as

$$\mathbf{a}_1 = \left[H_n^{(1)}(k_{1\rho}a)\bar{\mathbf{I}} + J_n(k_{1\rho}a)\overline{\mathbf{R}}_{12}\right]^{-1} \cdot \begin{bmatrix} E_{1z} \\ H_{1z} \end{bmatrix}_{\rho=a}.$$

Then,

$$\begin{aligned}
\begin{bmatrix} E_{2z} \\ H_{2z} \end{bmatrix}_{\rho=a} &= H_n^{(1)}(k_{2\rho}\rho)\overline{\mathbf{T}}_{12} \cdot \mathbf{a} \\
&= H_n^{(1)}(k_{2\rho}\rho)\overline{\mathbf{T}}_{12} \cdot \left[H_n^{(1)}(k_{1\rho}a)\bar{\mathbf{I}} + J_n(k_{1\rho}a)\overline{\mathbf{R}}_{12}\right]^{-1} \cdot \begin{bmatrix} E_{1z} \\ H_{1z} \end{bmatrix}_{\rho=a} \\
&\triangleq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} E_{1z} \\ H_{1z} \end{bmatrix}_{\rho=a}.
\end{aligned}$$

But when $n \to -n$, or $k_z \to -k_z$, according to the proof in (a), $[E_{2z} \; H_{2z}]^T_{\rho=a}$ and $[E_{1z} \; H_{1z}]^T_{\rho=a}$ are only related by $\begin{bmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}$. Therefore, $\begin{bmatrix} E_{2z} \\ H_{2z} \end{bmatrix}_{\rho=a} = \begin{bmatrix} a_{11}E_{1z} - a_{12}H_{1z} \\ -a_{21}E_{1z} + a_{22}H_{1z} \end{bmatrix}$.

E_{2z} contains $a_{11}E_{1z} - a_{12}H_{1z}$, which means that odd symmetric TE wave couples only to even symmetric TM wave; H_{2z} contains $-a_{21}E_{1z} + a_{22}H_{1z}$, which means that odd symmetric TM wave couples only to even symmetric TE wave.

Also note that $k_z \to -k_z$ is equivalent to $z \to -z$, and $n \to -n$ is equivalent to $\phi \to -\phi$.

§3.9

(a) From Section 2.2.1, we have already known that the solution for

$$\left[\nabla_s^2 + k_\rho^2\right] \Phi(\rho, \phi) = -\delta(\boldsymbol{\rho} - \boldsymbol{\rho}') = -\frac{\delta(\rho - \rho')}{\rho}\delta(\phi - \phi') \tag{1}$$

is

$$\Phi = \frac{i}{4}H_0^{(1)}(k_\rho|\boldsymbol{\rho} - \boldsymbol{\rho}'|). \tag{2}$$

When $\rho > \rho'$, we can express Φ as

$$\Phi = \sum_n a_n f(\rho', \phi')H_n^{(1)}(k_\rho\rho)e^{in\phi}, \tag{3}$$

and for $\rho < \rho'$,

$$\Phi = \sum_n b_n g(\rho', \phi')J_n(k_\rho\rho)e^{in\phi}. \tag{4}$$

But from the rotational invariance of the problem, Φ must be proportional to $(\phi - \phi')$. Hence, Φ is of the form,

$$\Phi = \begin{cases} \sum\limits_n a_n J_n(k_\rho \rho') H_n^{(1)}(k_\rho \rho) e^{in(\phi-\phi')} , & \rho > \rho', \\ \sum\limits_n a_n H_n^{(1)}(k_\rho \rho') J_n(k_\rho \rho) e^{in(\phi-\phi')} , & \rho < \rho', \end{cases}$$

where the coefficients are the same for both expressions ($a_n = b_n$) owing to the continuity of Φ at ρ'.

If we examine Equation (1) closely, it is a second order differential equation whose singularity comes from the discontinuity of the first derivative of the wave function at the source point. Substituting the above two expressions in (1), integrating from $(\rho' - \epsilon)$ to $(\rho' + \epsilon)$, where $\epsilon \to 0$ and recognizing that $\delta(\phi - \phi') = \frac{1}{2\pi} \sum\limits_{n=-\infty}^{\infty} e^{in(\phi-\phi')}$, we obtain that

$$a_n k_\rho [J_n(k_\rho \rho') H_n'^{(1)}(k_\rho \rho) - H_n^{(1)}(k_\rho \rho') J_n'(k_\rho \rho)] = -\frac{1}{2\pi \rho'}.$$

Invoking the Wronskian property of Bessel functions, the term in the square bracket can be simplified to $\frac{2i}{\pi k_\rho \rho'}$. Hence,

$$a_n = \frac{i}{4}.$$

So,

$$H_0^{(1)}(k_\rho |\overline{\rho} - \overline{\rho}'|) = \begin{cases} \sum\limits_n J_n(k_\rho \rho') H_n^{(1)}(k_\rho \rho) e^{in(\phi-\phi')}, \rho > \rho', \\ \sum\limits_n H_n^{(1)}(k_\rho \rho') J_n(k_\rho \rho) e^{in(\phi-\phi')}, \rho < \rho'. \end{cases}$$

(b) By using the raising operator given by (2.2.16)

$$-\frac{1}{k_\rho} \left[\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right] B_n(k_\rho \rho) e^{in\phi} = B_{n+1}(k_\rho \rho) e^{i(n+1)\phi}$$

m times on the addition theorem derived in (a), we have

$$\left\{ -\frac{1}{k_\rho} \left[\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right] \right\}^m H_0^{(1)}(k_\rho |\rho - \rho'|)$$
$$= \begin{cases} \sum\limits_n J_n(k_\rho \rho') e^{-in\phi'} \left(-\frac{1}{k_\rho} \left[\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right] \right)^m H_n^{(1)}(k_\rho \rho) e^{in\phi} , & \rho > \rho', \\ \sum\limits_n H_n^{(1)}(k_\rho \rho') e^{-in\phi'} \left(-\frac{1}{k_\rho} \left[\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right] \right)^m J_n(k_\rho \rho) e^{in\phi} , & \rho < \rho', \end{cases}$$

$$\rightarrow H_m^{(1)}(k_\rho|\boldsymbol{\rho}-\boldsymbol{\rho}'|)e^{im\phi}$$
$$=\begin{cases}\sum_n J_n(k_\rho\rho')e^{-in\phi'}H_{n+m}^{(1)}(k_\rho\rho)e^{i(n+m)\phi}\,, & \rho>\rho',\\ \sum_n H_n^{(1)}(k_\rho\rho')e^{-in\phi'}J_{n+m}(k_\rho\rho)e^{i(n+m)\rho'}\,, & \rho<\rho',\end{cases}$$
$$=\begin{cases}\sum_{n=-\infty}^{\infty} J_{n-m}(k_\rho\rho')H_n^{(1)}(k_\rho\rho)e^{in\phi-i(n-m)\phi'}\,, & \rho>\rho',\\ \sum_{n=-\infty}^{\infty} H_{n-m}^{(1)}(k_\rho\rho')J_n(k_\rho\rho)e^{in\phi-i(n-m)\phi'}\,, & \rho<\rho'.\end{cases}$$

§3.10

(a) When $\hat{\alpha}=\hat{z}$, the operator D' can be expressed as

$$D'=\begin{bmatrix}k^2+\frac{\partial^2}{\partial z'^2}\\ 0\end{bmatrix}.$$

Then (3.3.6) becomes

$$\begin{bmatrix}E_z\\ H_z\end{bmatrix}=\frac{iI\ell}{4\pi\omega\epsilon}\sum_{n=-\infty}^{\infty}e^{in(\phi-\phi')}\int_{-\infty}^{\infty}dk_z\begin{bmatrix}k^2+\frac{\partial^2}{\partial z'^2}\\ 0\end{bmatrix}$$
$$e^{ik_z(z-z')}J_n(k_\rho\rho_<)H_n^{(1)}(k_\rho\rho_>),$$

$$\rightarrow\quad\begin{bmatrix}E_z\\ H_z\end{bmatrix}=$$
$$\begin{bmatrix}\frac{iI\ell}{4\pi\omega\epsilon}\sum_{n=-\infty}^{\infty}e^{in(\phi-\phi')}\int_{-\infty}^{\infty}dk_z(k^2-k_z^2)e^{ik_z(z-z')}J_n(k_\rho\rho_<)H_n^{(1)}(k_\rho\rho_>)\\ 0\end{bmatrix}.$$

(b) When $\hat{\alpha}=\hat{x}$, and $\phi'=0$, D' becomes

$$D'=\begin{bmatrix}(\hat{z}k^2+\frac{\partial}{\partial z'}\nabla')\cdot\hat{\alpha}\\ i\omega\epsilon\hat{\alpha}\cdot\hat{z}\times\nabla'\end{bmatrix}$$
$$=\begin{bmatrix}(\hat{z}k^2+\frac{\partial}{\partial z'}\nabla')\cdot(\cos\phi\hat{\rho}-\sin\phi\hat{\phi})\\ i\omega\epsilon(\cos\phi\hat{\rho}-\sin\phi\hat{\phi})\cdot\left(\hat{\phi}\frac{\partial}{\partial\rho'}-\hat{\rho}\frac{1}{\rho'}\frac{\partial}{\partial\phi'}\right)\end{bmatrix}$$
$$=\begin{bmatrix}\frac{\partial}{\partial z'}\left(\cos\phi\frac{\partial}{\partial\rho'}-\sin\phi\frac{1}{\rho'}\frac{\partial}{\partial\phi'}\right)\\ -i\omega\epsilon\left(\frac{\cos\phi}{\rho'}\frac{\partial}{\partial\phi'}+\sin\phi\frac{\partial}{\partial\rho'}\right)\end{bmatrix}.$$

Thus,

$$\begin{bmatrix}E_z\\ H_z\end{bmatrix}=\frac{iI\ell}{4\pi\omega\epsilon}\begin{bmatrix}\cos\phi\frac{\partial^2}{\partial z'\partial\rho'}-\frac{\sin\phi}{\rho'}\frac{\partial^2}{\partial z'\partial\phi'}\\ -\frac{i\omega\epsilon\cos\phi}{\rho'}\frac{\partial}{\partial\phi'}-i\omega\epsilon\sin\phi\frac{\partial}{\partial\rho'}\end{bmatrix}\sum_{n=-\infty}^{\infty}e^{in\phi}\,.$$

$$\int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} J_n(k_\rho \rho_<) H_n^{(1)}(k_\rho \rho_>).$$

If $\rho' < \rho$, then

$$\begin{bmatrix} E_z \\ H_z \end{bmatrix}$$
$$= \frac{iI\ell}{4\pi\omega\epsilon} \begin{bmatrix} \sum_{n=-\infty}^{\infty} e^{in\phi} \int_{-\infty}^{\infty} dk_z(-ik_z) e^{ik_z(z-z')} \cos\phi k_\rho J_n'(k_\rho\rho') H_n^{(1)}(k_\rho\rho) \\ -i\omega\epsilon \sum_{n=-\infty}^{\infty} \sin\phi e^{in\phi} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} k_\rho J_n'(k_\rho\rho') H_n^{(1)}(k_\rho\rho) \end{bmatrix}$$
$$= \frac{I\ell}{4\pi\omega\epsilon} \begin{bmatrix} \cos\phi \sum_{n=-\infty}^{\infty} e^{in\phi} \int_{-\infty}^{\infty} dk_z k_z k_\rho e^{ik_z(z-z')} J_n'(k_\rho\rho') H_n^{(1)}(k_\rho\rho) \\ \omega\epsilon \sin\phi \sum_{n=-\infty}^{\infty} e^{in\phi} \int_{-\infty}^{\infty} dk_z k_\rho e^{ik_z(z-z')} J_n'(k_\rho\rho') H_n^{(1)}(k_\rho\rho) \end{bmatrix}.$$

If $\rho' > \rho$, then

$$\begin{bmatrix} E_z \\ H_z \end{bmatrix} = \frac{I\ell}{4\pi\omega\epsilon}$$
$$\begin{bmatrix} \cos\phi \sum_{n=-\infty}^{\infty} e^{in\phi} \int_{-\infty}^{\infty} dk_z k_z k_\rho e^{ik_z(z-z')} J_n(k_\rho\rho) H_n'^{(1)}(k_\rho\rho') \\ \omega\epsilon \sin\phi \sum_{n=-\infty}^{\infty} e^{in\phi} \int_{-\infty}^{\infty} dk_z k_\rho e^{ik_z(z-z')} J_n(k_\rho\rho) H_n'^{(1)}(k_\rho\rho') \end{bmatrix}.$$

§3.11

Substituting (3.3.11) and (3.3.12) into the expression

$$\left[J_n(k_{j\rho}\rho_<) H_n^{(1)}(k_{j\rho}\rho_>)\overline{\mathbf{I}} + H_n^{(1)}(k_{j\rho}\rho)\overline{\mathbf{a}}_{jn} + J_n(k_{j\rho}\rho)\overline{\mathbf{b}}_{jn}\right] \cdot \overleftarrow{\mathbf{D}}_j'$$

leads to, for $\rho' < \rho$,

$$\Big\{ J_n(k_{j\rho}\rho') H_n^{(1)}(k_{j\rho}\rho)\overline{\mathbf{I}} + H_n^{(1)}(k_{j\rho}\rho) \left[\overline{\mathbf{I}} - \tilde{\overline{\mathbf{R}}}_{j,j-1} \cdot \tilde{\overline{\mathbf{R}}}_{j,j+1}\right]^{-1}$$
$$\cdot \tilde{\overline{\mathbf{R}}}_{j,j-1} \cdot \left[H_n^{(1)}(k_{j\rho}\rho')\overline{\mathbf{I}} + J_n(k_{j\rho}\rho')\tilde{\overline{\mathbf{R}}}_{j,j+1}\right] + J_n(k_{j\rho}\rho) \left[\overline{\mathbf{I}} - \tilde{\overline{\mathbf{R}}}_{j,j+1} \cdot \tilde{\overline{\mathbf{R}}}_{j,j-1}\right]^{-1}$$
$$\cdot \tilde{\overline{\mathbf{R}}}_{j,j+1} \cdot \left[J_n(k_{j\rho}\rho')\overline{\mathbf{I}} + H_n^{(1)}(k_{j\rho}\rho')\tilde{\overline{\mathbf{R}}}_{j,j-1}\right] \Big\} \cdot \overleftarrow{\mathbf{D}}_j'.$$

Since $\tilde{\overline{\mathbf{R}}}_{i,j}$'s are symmetric, we define $\tilde{\overline{\mathbf{M}}}_{j\pm}$ as

$$\tilde{\overline{\mathbf{M}}}_{j\pm} \triangleq \left(\overline{\mathbf{I}} - \tilde{\overline{\mathbf{R}}}_{j,j\mp1} \cdot \overline{\mathbf{R}}_{j,j\pm1}\right)^{-1}.$$

Thus, (3.3.13) becomes

$$\begin{aligned}
&\Big\{J_n(k_{j\rho}\rho')H_n^{(1)}(k_{j\rho}\rho)\left[\bar{\mathbf{I}}+\widetilde{\overline{\mathbf{M}}}_{j+}\cdot\tilde{\overline{\mathbf{R}}}_{j,j-1}\cdot\tilde{\overline{\mathbf{R}}}_{j,j+1}\right]\\
&\quad+H_n^{(1)}(k_{j\rho}\rho)H_n^{(1)}(k_{j\rho}\rho')\widetilde{\overline{\mathbf{M}}}_{j+}\cdot\tilde{\overline{\mathbf{R}}}_{j,j-1}+J_n(k_{j\rho}\rho)J_n(k_{j\rho}\rho')\widetilde{\overline{\mathbf{M}}}_{j+}\cdot\tilde{\overline{\mathbf{R}}}_{j,j+1}\\
&\qquad+J_n(k_{j\rho}\rho)H_n^{(1)}(k_{j\rho}\rho')\widetilde{\overline{\mathbf{M}}}_{j+}\cdot\tilde{\overline{\mathbf{R}}}_{j,j+1}\cdot\tilde{\overline{\mathbf{R}}}_{j,j-1}\Big\}\cdot\overleftarrow{\mathbf{D}}'_j\\
&=\Big\{J_n(k_{j\rho}\rho')H_n^{(1)}(k_{j\rho}\rho)\widetilde{\overline{\mathbf{M}}}_{j+}+H_n^{(1)}(k_{j\rho}\rho)H_n^{(1)}(k_{j\rho}\rho')\widetilde{\overline{\mathbf{M}}}_{j+}\cdot\tilde{\overline{\mathbf{R}}}_{j,j-1}\\
&\quad+J_n(k_{j\rho}\rho)J_n(k_{j\rho}\rho')\widetilde{\overline{\mathbf{M}}}_{j+}\cdot\tilde{\overline{\mathbf{R}}}_{j,j+1}\\
&\qquad+J_n(k_{j\rho}\rho)H_n^{(1)}(k_{j\rho}\rho')\widetilde{\overline{\mathbf{M}}}_{j+}\cdot\tilde{\overline{\mathbf{R}}}_{j,j+1}\cdot\tilde{\overline{\mathbf{R}}}_{j,j-1}\Big\}\cdot\overleftarrow{\mathbf{D}}'_j\\
&=\Big\{H_n^{(1)}(k_{j\rho}\rho)\widetilde{\overline{\mathbf{M}}}_{j+}\cdot\left(J_n(k_{j\rho}\rho')\bar{\mathbf{I}}+H_n^{(1)}(k_{j\rho}\rho')\tilde{\overline{\mathbf{R}}}_{j,j-1}\right)\\
&\quad+J_n(k_{j\rho}\rho)\widetilde{\overline{\mathbf{M}}}_{j+}\cdot\tilde{\overline{\mathbf{R}}}_{j,j+1}\cdot\left(J_n(k_{j\rho}\rho')\bar{\mathbf{I}}+H_n^{(1)}(k_{j\rho}\rho')\tilde{\overline{\mathbf{R}}}_{j,j-1}\right)\Big\}\cdot\overleftarrow{\mathbf{D}}'_j\\
&=\Big\{\left[H_n^{(1)}(k_{j\rho}\rho)\bar{\mathbf{I}}+J_n(k_{j\rho}\rho)\tilde{\overline{\mathbf{R}}}_{j,j+1}\right]\cdot\widetilde{\overline{\mathbf{M}}}_{j+}\\
&\qquad\cdot\left[J_n(k_{j\rho}\rho')\bar{\mathbf{I}}+H_n^{(1)}(k_{j\rho}\rho')\tilde{\overline{\mathbf{R}}}_{j,j-1}\right]\Big\}\cdot\overleftarrow{\mathbf{D}}'_j
\end{aligned}$$

Same derivation applied to the case for $\rho' > \rho$ yields

$$\begin{aligned}
&\Big\{\left[J_n(k_{j\rho}\rho)\bar{\mathbf{I}}+H_n^{(1)}(k_{j\rho}\rho)\tilde{\overline{\mathbf{R}}}_{j,j-1}\right]\cdot\widetilde{\overline{\mathbf{M}}}_{j-}\\
&\qquad\qquad\cdot\left[H_n^{(1)}(k_{j\rho}\rho')\bar{\mathbf{I}}+J_n(k_{j\rho}\rho')\tilde{\overline{\mathbf{R}}}_{j,j+1}\right]\Big\}\cdot\overleftarrow{\mathbf{D}}'_j
\end{aligned}$$

§3.12

(a) Since a_{in} and a_{jn} are the amplitudes of outgoing wave in regions i and j respectively, according to (3.2.11) these two amplitudes are related by

$$\mathbf{a}_{in}=\tilde{\overline{\mathbf{T}}}_{ji}\cdot\mathbf{a}_{jn}$$

if the ith region is the outermost region.

But generally it is not necessarily so. We have to multiply an additional factor to account for the multiple-reflection. So, we modify the above equation as

$$\mathbf{a}_{in}=\widetilde{\overline{\mathbf{M}}}_{i+}\cdot\tilde{\overline{\mathbf{T}}}_{ji}\cdot\mathbf{a}_{jn}$$

(b) Using Equations (3.3.16) and (3.3.17) in Equation (3.3.15), we have

$$\begin{aligned}\begin{bmatrix} E_{iz} \\ H_{iz} \end{bmatrix} =& \frac{iI\ell}{4\pi\omega\epsilon_i} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \\ & \left[H_n^{(1)}(k_{i\rho}\rho)\overline{\mathbf{I}} + J_n(k_{i\rho}\rho)\tilde{\overline{\mathbf{R}}}_{i,i+1}\right] \cdot \tilde{\overline{\mathbf{M}}}_{i+} \cdot \tilde{\overline{\mathbf{T}}}_{ji} \cdot \mathbf{a}_{jn} \\ =& \frac{iI\ell}{4\pi\omega\epsilon_i} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \Big[H_n^{(1)}(k_{i\rho}\rho)\overline{\mathbf{I}} \\ & + J_n(k_{i\rho}\rho)\tilde{\overline{\mathbf{R}}}_{i,i+1}\Big] \\ & \cdot \tilde{\overline{\mathbf{M}}}_{i+} \cdot \tilde{\overline{\mathbf{T}}}_{ji} \cdot \tilde{\overline{\mathbf{M}}}_{j+} \cdot \left[J_n(k_{j\rho}\rho')\overline{\mathbf{I}} + H_n^{(1)}(k_{j\rho}\rho')\tilde{\overline{\mathbf{R}}}_{j,j-1}\right] \cdot \overleftarrow{\mathbf{D}}'_j. \end{aligned} \quad (3.3.18)$$

(c) Similar to (3.3.17), we identify

$$a_{jn} = \tilde{\overline{\mathbf{M}}}_{j-} \cdot \left[H_n^{(1)}(k_{j\rho}\rho')\overline{\mathbf{I}} + J_n(k_{j\rho}\rho')\tilde{\overline{\mathbf{R}}}_{j,j+1}\right] \cdot \overleftarrow{\mathbf{D}}_j,$$

Thus, (3.3.19) becomes

$$\begin{aligned}\begin{bmatrix} E_{iz} \\ H_{iz} \end{bmatrix} =& \frac{iI\ell}{4\pi\omega\epsilon_i} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \\ & \left[J_n(k_{i\rho}\rho)\overline{\mathbf{I}} + H_n^{(1)}(k_{i\rho}\rho)\tilde{\overline{\mathbf{R}}}_{i,i-1}\right] \cdot \tilde{\overline{\mathbf{M}}}_{i-} \cdot \tilde{\overline{\mathbf{T}}}_{ji} \cdot \tilde{\overline{\mathbf{M}}}_{j-} \\ & \cdot \left[H_n^{(1)}(k_{j\rho}\rho')\overline{\mathbf{I}} + J_n(k_{j\rho}\rho')\tilde{\overline{\mathbf{R}}}_{j,j+1}\right] \cdot \overleftarrow{\mathbf{D}}_j. \end{aligned} \quad (3.3.20)$$

§3.13

Note that (3.3.21) with (3.3.22), (3.3.23) and (3.3.24) can be simply expressed as the following form

$$\begin{bmatrix} E_z \\ H_z \end{bmatrix} = \frac{iI\ell}{4\pi\epsilon_j} \sum_{n=-\infty}^{\infty} e^{in\phi} \int_{-\infty}^{\infty} dk_z e^{ik_z z} \mathbf{P}_n(\rho),$$

where

$$\mathbf{P}_n(\rho) = \left[\mathbf{A}_n H_n^{(1)}(k_{j\rho}\rho) + \mathbf{B}_n J_n(k_{j\rho}\rho)\right],$$

and $\mathbf{A}_n$, $\mathbf{B}_n$ are the functions of Bessel Functions and Hankel Functions and their derivatives whose arguments are radial distances of all layered media or that of the source location.

For those layers other than the outmost layer, $\mathbf{P}_n$ is a linear combination of $H_n^{(1)}$ and J_n and remains unchanged no matter which branch of $k_{j\rho}$'s is chosen since according to the uniqueness theorem, $\mathbf{A}_n$ and $\mathbf{B}_n$ adjust automatically so that there is one and only one solution in this case.

But for the outermost layer N, $\mathbf{P}_n$ is just $\mathbf{A}_n H_n^{(1)}(k_{j\rho}\rho)$, $k_{j\rho}$'s of different branches will yield different results some of which even exhibit the exponential growth when $\rho \to \infty$.

So the solution of a source in a cylindrically layered medium has branch points only at $k_\rho = \pm k_N$ on the complex k_z plane, where k_N is the wave member of the outermost region.

§3.14

(a) For a vertical electric dipole pointing in the z direction, we need only consider the E_z component. Since there is only one perfectly conducting cylinder, from Equation (3.1.17),

$$R_{21} = \frac{J_\nu(k_{2\rho}a)}{H_\nu^{(1)}(k_{2\rho}a)} \quad \text{and} \quad R_{23} = 0.$$

Then, from (3.3.11) and (3.3.12),

$$a_\nu = \frac{J_\nu(k_{2\rho}a)}{H_\nu^{(1)}(k_{2\rho}a)} H_\nu^{(1)}(k_{2\rho}\rho') \quad \text{and} \quad b_\nu = 0.$$

(b) Since from (a), $H_\nu^{(1)}(k_{2\rho}a)$ appears in the denominator in a_ν, the poles on the complex ν plane are the zeros of $H_\nu^{(1)}(k_{2\rho}a)$. Since $H_{-\nu}^{(1)}(z) = e^{i\pi\nu}H_\nu^{(1)}(z)$, if ν_p is a pole, so is $-\nu_p$.

(c) From (9.3.31) and (9.3.32) in the book by Abramowitz and Stegun, $H_\nu^{(1)}(\nu)$ can be approximated as

$$\frac{a}{\nu^{\frac{1}{3}}} - \frac{b}{\nu^{\frac{5}{3}}}\beta_0 + i(-\frac{\sqrt{3}a}{\nu^{\frac{1}{3}}} - \frac{\sqrt{3}b}{\nu^{\frac{5}{3}}}\beta_0),$$

where $a = 0.44731$, $b = 0.41085$ and $\beta_0 = 0.014286$. Consequently, if ν is large, the above equation goes to zero. So the zeros of $H_\nu^{(1)}(k_\rho a)$ are approximately at $\nu \simeq k_\rho a$.

(d) We can use contour integration technique to evaluate the integral over ν. For $\phi - \phi' + 2n\pi > 0$, the integrand is exponentially small as $\Im m[\nu] \to \infty$. So we can deform the path to the upper half complex ν plane and by Jordan's lemma, the integration for ν over the contour in the upper half ν plane vanishes. Then, by Cauchy's theorem, the integration over the contour is the same as integration over the pole locations enclosed in the contour. Similarly for $\phi - \phi' + 2n\pi < 0$, but the path has to be deformed to the lower half complex ν plane. Assuming the pole locations are denoted by ν_p and the residues of the poles by β_p, from (3.3.30), we obtain

$$\begin{bmatrix} E_{jz}^R \\ H_{jz}^R \end{bmatrix} = \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} e^{i\nu_p(\phi-\phi'+2n\pi)}\beta_p.$$

If we assume that ν_p has a positive imaginary part, for $\phi - \phi' > 0$, the dominant terms are those for $n = 0$ and $n = -1$ since $e^{2in\nu_p\pi}$ is often small for $|n| > 1$. Also from the distribution of the zeros of $H_n^{(1)}(z)$ in page 373 of the book by Abramowitz and Stegun, it shows that the zeros have larger imaginary part. Therefore, the above equation can be approximated as

$$\begin{bmatrix} E_{jz}^R \\ H_{jz}^R \end{bmatrix} \approx \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')}[e^{i\nu_1(\phi-\phi')} + e^{i\nu_1(2\pi-\phi+\phi')}]\beta_1.$$

(e) For part (a), the equation corresponding to (3.3.30) can be written as

$$E_z^R = \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu e^{i\nu(\phi-\phi')+i2n\nu\pi} \cdot \left[\frac{-J_\nu(k_{2\rho}a)}{H_\nu^{(1)}(k_{2\rho}a)} H_\nu^{(1)}(k_{2\rho}\rho') H_\nu^{(1)}(k_{2\rho}\rho) k_{2\rho}^2\right].$$

The above equation can be approximated as

$$E_z^R \approx \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')}[e^{ik_{2\rho}a(\phi-\phi')} + e^{ik_{2\rho}a(2\pi-\phi+\phi')}]\beta_1(k_{2\rho}), \qquad (1)$$

where

$$\beta_1 = -2\pi i k_{2\rho}^2 \lim_{\nu\to k_{2\rho}a} \left[(\nu - k_{2\rho}a)\frac{J_\nu(k_{2\rho}a)}{H_\nu^{(1)}(k_{2\rho}a)} H_\nu^{(1)}(k_{2\rho}\rho') H_\nu^{(1)}(k_{2\rho}\rho)\right]$$

and note that $k_{2\rho} = \sqrt{k_2^2 - k_z^2}$.

For a fixed k_z, Equation (1) describes a wave propagating azimuthally. The first term in square brackets corresponds to a wave emanating from the source and the second term corresponds to that from the image source at $2\pi + \phi'$, which can be viewed as a creeping wave travelling around the circumference of the cylinder by a distance less that 2π and then radiating.

§3.15

Assume

$$\overline{\mathbf{H}}(\rho) = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \qquad (1)$$

$\overline{\mathbf{H}}$ is defined by (3.4.9) as

$$\frac{d}{d\rho} \begin{bmatrix} E_z \\ E_\phi \\ H_z \\ H_\phi \end{bmatrix} = \overline{\mathbf{H}}(\rho) \begin{bmatrix} E_z \\ E_\phi \\ H_z \\ H_\phi \end{bmatrix} \qquad (2)$$

We can easily find from (3.4.7a):

$$h_{11}=0, \qquad h_{12}=0, \qquad h_{13}=-\frac{ik_z n}{\omega\epsilon\rho}, \qquad h_{14}=-i\omega\mu+\frac{ik_z^2}{\omega\epsilon}. \tag{3}$$

From (3.4.8a),

$$h_{31}=\frac{ik_z n}{\omega\mu\rho}, \qquad h_{32}=i\omega\epsilon-\frac{ik_z^2}{\omega\mu}, \qquad h_{33}=0, \qquad h_{34}=0. \tag{4}$$

Since $\frac{1}{\rho}\frac{d}{d\rho}\rho E_\phi=\frac{1}{\rho}E_\phi+\frac{dE_\phi}{d\rho}$, so that from (3.4.7b),

$$h_{21}=0, \qquad h_{22}=-\frac{1}{\rho}, \qquad h_{23}=i\omega\mu-\frac{in^2}{\omega\epsilon\rho^2}, \qquad h_{24}=\frac{ink_z}{\omega\epsilon\rho}. \tag{5}$$

Similarly,

$$h_{41}=-i\omega\epsilon+\frac{in^2}{\omega\mu\rho^2}, \qquad h_{42}=-\frac{ink_z}{\omega\mu\rho}, \qquad h_{43}=0, \qquad h_{44}=\frac{-1}{\rho} \tag{6}$$

To sum up,

$$\overline{\mathbf{H}}(\rho)=$$
$$\begin{bmatrix} 0 & 0 & -\frac{ink_z}{\omega\epsilon\rho} & -i\omega\mu+\frac{ik_z^2}{\omega\epsilon} \\ 0 & -\frac{1}{\rho} & i\omega\epsilon-\frac{in^2}{\omega\mu\rho^2} & \frac{ink_z}{\omega\epsilon\rho} \\ \frac{ink_z}{\omega\mu\rho} & i\omega\epsilon-\frac{ik_z^2}{\omega\mu} & 0 & 0 \\ -i\omega\epsilon+\frac{in^2}{\omega\mu\rho^2} & -\frac{ink_z}{\omega\mu\rho} & 0 & -\frac{1}{\rho} \end{bmatrix}. \tag{7}$$

§3.16

(a) In order to be clear, we denote the elements in $\mathbf{a}_i$ $(i = 1,2,3,4)$ as, $a_{i1}, a_{i2}, a_{i3}, a_{i4}$, such that

$$\mathbf{a}_i=(a_{i1},a_{i2},a_{i3},a_{i4})^t, \qquad i=1,2,3,4. \tag{1}$$

We can see that

$$a_{13}=a_{21}=a_{33}=a_{41}=0.$$

Vector $\mathbf{b}_1$ is orthogonal to $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$, this is to say

$$\mathbf{b}_1^t\cdot\mathbf{a}_2=0, \quad \mathbf{b}_1^t\cdot\mathbf{a}_3=0, \quad \mathbf{b}_1^t\cdot\mathbf{a}_4=0. \tag{2}$$

Assume that $\mathbf{b}_1^t = (b_{11}\ b_{12}\ b_{13}\ b_{14})$, and substitute into (2), we obtain three linear equations:

$$b_{11}\cdot 0+b_{12}\cdot a_{22}+b_{13}\cdot a_{23}+b_{14}\cdot a_{24}=0, \tag{3}$$
$$b_{11}\cdot a_{31}+b_{12}\cdot a_{32}+b_{13}\cdot 0+b_{14}\cdot a_{34}=0, \tag{4}$$
$$b_{11}\cdot 0+b_{12}\cdot a_{42}+b_{13}\cdot a_{43}+b_{14}\cdot a_{44}=0. \tag{5}$$

This is an undetermined system, because its number of unknowns are more than that of equations.

This means that one of the unknowns in $(b_{11}, b_{12}, b_{13}, b_{14})$ can be arbitrary. For simplicity, we may at first choose $b_{12} = 0$ (or $b_{14} = 0$, but we can show this is not a solution). As a result, (3)–(5) become

$$b_{13}a_{23} + b_{14} \cdot a_{24} = 0 \tag{6}$$

$$b_{11}a_{31} + b_{14} \cdot a_{34} = 0 \tag{7}$$

$$b_{13}a_{43} + b_{14} \cdot a_{44} = 0 \tag{8}$$

The coefficient matrix for the above system is

$$\overline{\mathbf{A}} = \begin{bmatrix} 0 & a_{23} & a_{24} \\ a_{31} & 0 & a_{34} \\ 0 & a_{43} & a_{44} \end{bmatrix}, \tag{9}$$

$$|\overline{\mathbf{A}}| = -a_{31}(a_{23}a_{44} - a_{43}a_{24}) = -a_{31} \cdot \left\{ J_n \cdot \frac{-nk_z}{k_\rho^2 \rho} H_n^{(1)} + \frac{nk_z}{k_\rho^2 \rho} H_n^{(1)} J_n \right\} \equiv 0, \tag{10}$$

Equation (10) guarantees that (6)–(8) have non-zero solutions, since these equations are homogeneous, we can choose any one of the unknown (say b_{14}) to be a constant C_1, therefore

$$b_{14} = C_1, \tag{11}$$

$$b_{11} = -C_1 a_{34}/a_{31}. \tag{12}$$

$$b_{13} = -C_1 a_{44}/a_{43}. \tag{13}$$

So that, we have solutions for $\mathbf{b}_1$:

$$\mathbf{b}_1^t = \left(-C_1 \frac{a_{34}}{a_{31}}, \quad 0, \quad -C_1 \frac{a_{44}}{a_{43}}, \quad C_1 \right). \tag{14}$$

The constant in $\mathbf{b}_1$ can be determined by the fact that

$$\mathbf{b}_1^t \cdot \mathbf{a}_1 = 1. \tag{15}$$

Hence,

$$C_1 \left[-\frac{a_{11}a_{34}}{a_{31}} + a_{14} \right] = 1,$$

$$C_1 = \frac{a_{31}}{a_{14}a_{31} - a_{11}a_{34}}, \tag{16}$$

$$\mathbf{b}_1^t = \frac{1}{a_{14}a_{31} - a_{11}a_{34}} (-a_{34}, 0, -a_{44}, a_{31}). \tag{17}$$

In deriving the above, we have used the relation that $a_{31} = a_{43}$. By using the Wronskian for Bessel functions [see page 165], we have

$$\begin{aligned} a_{14}a_{31} - a_{11}a_{34} &= \frac{i\omega\epsilon}{k_\rho} J_n'(k_\rho\rho)H_n^{(1)}(k_\rho\rho) - \frac{i\omega\epsilon}{k_\rho} J_n(k_\rho\rho)H_n^{(1)\prime}(k_\rho\rho) \\ &= \frac{i\omega\epsilon}{k_\rho}\left[J_n'(k_\rho\rho)H_n^{(1)}(k_\rho\rho) - J_n(k_\rho\rho)H_n^{(1)\prime}(k_\rho\rho)\right] \\ &= \frac{i\omega\epsilon}{k_\rho}\cdot\left[-\frac{2i}{\pi k_\rho\rho}\right] = \frac{2\omega\epsilon}{\pi k_\rho^2\rho}. \end{aligned} \tag{18}$$

(b) In the same way, we let

$$\mathbf{b}_2^t = (b_{21}, b_{22}, b_{23}, b_{24}),$$

with orthogonal relations:

$$\mathbf{b}_2^t \cdot \mathbf{a}_1 = 0,\ \mathbf{b}_2^t \cdot \mathbf{a}_3 = 0,\ \mathbf{b}_2^t \cdot \mathbf{a}_4 = 0,$$

and choose

$$b_{24} = 0,\ b_{22} = C_2,$$

to find

$$b_{21} = -C_2 a_{32}/a_{31},\ b_{23} = -C_2 a_{42}/a_{43}.$$

We normalize:

$$\mathbf{b}_2^t \cdot \mathbf{a}_2 = 1.$$

Then

$$C_2 = \frac{a_{43}}{a_{22}a_{43} - a_{42}a_{23}} = \frac{a_{43}}{\left(\frac{2\omega\mu}{\pi k_\rho^2\rho}\right)},$$

$$\mathbf{b}_2^t = \left(\frac{\pi k_\rho^2\rho}{2\omega\mu}\right)\left[-\frac{nk_z}{k_\rho^2\rho}H_n^{(1)}(k_\rho\rho),\ H_n^{(1)}(k_\rho\rho),\ +\frac{i\omega\mu}{k_\rho}H_n^{(1)\prime}(k_\rho\rho), 0\right]. \tag{19}$$

Similarly,

$$\begin{aligned} \mathbf{b}_3^t &= \left(\frac{-2\omega\epsilon}{\pi k_\rho^2\rho}\right)^{-1}\left[-\frac{i\omega\epsilon}{k_\rho}J_n'(k_\rho\rho),\ 0,\ -\frac{nk_z}{k_\rho\rho}H_n^{(1)}(k_\rho\rho),\ \frac{i\omega\epsilon}{k_\rho}H_n^{(1)\prime}(k_\rho\rho)\right], \\ \mathbf{b}_4^t &= \left(\frac{2\omega\mu}{\pi k_\rho^2\rho}\right)^{-1}\left[+\frac{nk_z}{k_\rho^2\rho}H_n^{(1)}(k_\rho\rho),\ J_n(k_\rho\rho),\ +\frac{i\omega\mu}{k_\rho}J_n'(k_\rho\rho),\ 0\right] \end{aligned} \tag{20}$$

(c) $\mathbf{a}^{-1}$, the left inverse of $\mathbf{a}$ is $\mathbf{a}^{-1} = (\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3,\ \mathbf{b}_4)^t$. In detail, it is

$$\mathbf{a}^{-1} = \begin{bmatrix} a_{34}/\Delta_1 & 0 & a_{44}/\Delta_1 & -a_{31}/\Delta_1 \\ a_{32}/\Delta_2 & -a_{43}/\Delta_2 & a_{42}/\Delta_2 & 0 \\ -a_{14}/\Delta_1 & 0 & -a_{24}/\Delta_1 & a_{11}/\Delta_1 \\ a_{12}/\Delta_2 & -a_{23}/\Delta_2 & a_{22}/\Delta_2 & 0 \end{bmatrix},$$

$$\Delta_1 = a_{11}a_{34} - a_{31}a_{14} = \frac{-2\omega\epsilon}{\pi k_\rho^2 \rho},$$
$$\Delta_2 = a_{22}a_{43} - a_{23}a_{42} = \frac{-2\omega\mu}{\pi k_\rho^2 \rho}.$$

$$\begin{aligned}
\overline{\mathbf{P}}(\rho,\rho') &= \mathbf{a}(\rho)\mathbf{a}^{-1}(\rho') \\
&= \begin{bmatrix} a_{11}(\rho) & 0 & a_{31}(\rho) & 0 \\ a_{12}(\rho) & a_{22}(\rho) & a_{32}(\rho) & a_{42}(\rho) \\ 0 & a_{23}(\rho) & 0 & a_{43}(\rho) \\ a_{14}(\rho) & a_{24}(\rho) & a_{34}(\rho) & a_{44}(\rho) \end{bmatrix} \\
&\cdot \begin{bmatrix} \frac{a_{34}^{(\rho')}}{\Delta_1(\rho')} & 0 & \frac{a_{44}(\rho')}{\Delta_1(\rho')} & -\frac{a_{31}(\rho')}{\Delta_1(\rho')} \\ \frac{a_{32}(\rho')}{\Delta_2(\rho')} & \frac{a_{43}(\rho')}{\Delta_2(\rho')} & -\frac{a_{42}(\rho')}{\Delta_2(\rho')} & 0 \\ -\frac{a_{14}(\rho')}{\Delta_1(\rho')} & 0 & \frac{a_{24}(\rho')}{\Delta_1(\rho')} & \frac{a_{11}(\rho')}{\Delta_1(\rho')} \\ \frac{a_{12}(\rho')}{\Delta_2(\rho')} & -\frac{a_{23}(\rho')}{\Delta_2(\rho')} & \frac{a_{22}(\rho')}{\Delta_2(\rho')} & 0 \end{bmatrix} \\
&= \begin{bmatrix} P_{11}(\rho,\rho') & P_{12}(\rho,\rho') & P_{13}(\rho,\rho') & P_{14}(\rho,\rho') \\ P_{21}(\rho,\rho') & P_{22}(\rho,\rho') & P_{23}(\rho,\rho') & P_{24}(\rho,\rho') \\ P_{31}(\rho,\rho') & P_{32}(\rho,\rho') & P_{33}(\rho,\rho') & P_{34}(\rho,\rho') \\ P_{41}(\rho,\rho') & P_{42}(\rho,\rho') & P_{43}(\rho,\rho') & P_{44}(\rho,\rho') \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
P_{11}(\rho,\rho') &= a_{11}(\rho)\cdot\frac{a_{34}(\rho')}{\Delta_1(\rho')} - a_{31}(\rho)\cdot\frac{a_{14}(\rho')}{\Delta_1(\rho')} \\
&= [a_{11}(\rho)a_{34}(\rho') - a_{31}(\rho)a_{14}(\rho')]\,/\Delta_1(\rho'), \\
P_{12}(\rho,\rho') &= 0, \\
P_{13}(\rho,\rho') &= [a_{11}(\rho)a_{44}(\rho') - a_{31}(\rho)a_{24}(\rho')]\,/\Delta_1(\rho'), \\
P_{14}(\rho,\rho') &= [-a_{11}(\rho)a_{31}(\rho') - a_{31}(\rho)a_{11}(\rho')]\,/\Delta_1(\rho'), \\
P_{21}(\rho,\rho') &= \frac{1}{\Delta_1(\rho')}a_{12}(\rho)a_{34}(\rho') + \frac{1}{\Delta_2}a_{22}(\rho) \\
&\qquad - \frac{1}{\Delta_1}a_{32}(\rho)a_{14}(\rho') + \frac{1}{\Delta_2}a_{42}(\rho)a_{12}(\rho'), \\
P_{22}(\rho,\rho') &= [a_{22}(\rho)a_{43}(\rho') - a_{42}(\rho)a_{23}(\rho')]\,/\Delta_2(\rho'), \\
P_{23}(\rho,\rho') &= \left[\frac{1}{\Delta_1}a_{12}(\rho)a_{44}(\rho') - \frac{1}{\Delta_2}a_{22}(\rho)a_{42}(\rho')\right. \\
&\qquad \left. -\frac{1}{\Delta_1}a_{32}(\rho)a_{24}(\rho') + \frac{1}{\Delta_2}a_{42}(\rho)a_{22}(\rho')\right], \\
P_{24}(\rho,\rho') &= [-a_{12}(\rho)a_{31}(\rho') + a_{32}(\rho)a_{11}(\rho')]\,/\Delta_1(\rho'), \\
P_{31}(\rho,\rho') &= [a_{23}(\rho)a_{32}(\rho') + a_{43}(\rho)a_{12}(\rho')]\,/\Delta_2(\rho'), \\
P_{32}(\rho,\rho') &= [a_{23}(\rho)a_{43}(\rho') + a_{43}(\rho)a_{43}(\rho')]\,/\Delta_2(\rho'), \\
P_{33}(\rho,\rho') &= [a_{23}(\rho)a_{42}(\rho') + a_{43}(\rho)a_{22}(\rho')]\,/\Delta_2(\rho'), \\
P_{34}(\rho,\rho') &= 0,
\end{aligned}$$

$$P_{41}(\rho,\rho') = [a_{14}(\rho)a_{34}(\rho') - a_{34}(\rho)a_{14}(\rho')] \\ /\Delta_1(\rho') + [a_{24}(\rho)a_{32}(\rho') + a_{42}(\rho)a_{12}(\rho')]/\Delta_2(\rho'),$$
$$P_{42}(\rho,\rho') = [a_{24}(\rho')a_{43}(\rho') - a_{44}(\rho)a_{23}(\rho')]/\Delta_2(\rho'),$$
$$P_{43}(\rho,\rho') = [a_{14}(\rho)a_{44}(\rho') - a_{34}(\rho)a_{24}(\rho')] \\ /\Delta_1(\rho') + [a_{24}(\rho)a_{42}(\rho') + a_{44}(\rho)a_{22}(\rho')]/\Delta_2(\rho'),$$
$$P_{44}(\rho,\rho') = [-a_{14}(\rho')a_{31}(\rho') + a_{34}(\rho)a_{11}(\rho')]/\Delta_1(\rho').$$

§3.17

(a) Outgoing wave case:

From Equation. (3.4.10), we know that $\mathbf{a}_1$ and $\mathbf{a}_2$ are standing waves, $\mathbf{a}_3$ and $\mathbf{a}_4$ are outgoing waves. If there is an exciting source in region 1, it generates an outgoing wave which on the one hand reflects in region 1, and on the other hand, transmits to region 2 and 3. This transmitted wave will excite both standing wave and outgoing wave in region 2, and only an outgoing wave in region 3.

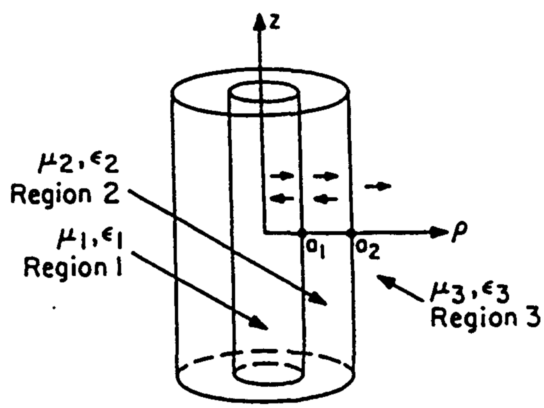

Figure for Exercise Solution 3.17

Assume the reflection coefficients for the outgoing wave in region 1 are R_1 (TM) and R_2 (TE), and also the transmission coefficients of outgoing wave from region 1 to region 3 are T_3 (TM) and T_4 (TE), respectively. Hence, the state vector $\mathbf{V}$ in region 1 can be written as

$$\mathbf{V}_1(\rho) = (\mathbf{a}_1 R_1 + \mathbf{a}_2 R_2 + \mathbf{a}_3 + \mathbf{a}_4)A_1 \\ = \bar{\mathbf{a}}_1 \cdot \begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} A_1. \quad (1)$$

Since state vectors are composed of tangential field components, they are continuous across the boundary. Hence, using the propagator $\overline{\mathbf{P}}(\rho,\rho')$, we have:

$$\mathbf{V}_1(a_1) = \mathbf{V}_2(a_1) = \overline{\mathbf{P}}(a_1,a_2)\mathbf{V}_2(a_2) = \overline{\mathbf{P}}(a_1,a_2)\mathbf{V}_3(a_2). \tag{2}$$

In region 3, there is only an outgoing wave which is the result of the transmission of the outgoing wave in region 1. Therefore,

$$\mathbf{V}_3(\rho) = (T_3\mathbf{a}_3 + T_4\mathbf{a}_4)A_1 = \overline{\mathbf{a}}_3 \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & T_3 & \\ & & & T_4 \end{bmatrix} A_1. \tag{3}$$

Substituting (1) and (3) into (2), and eliminating the constant $A_1(\neq 0)$, we have

$$\overline{\mathbf{a}}_1(a_1)\begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} = \overline{\mathbf{P}}(a_1,a_2)\cdot\overline{\mathbf{a}}_3(a_2)\cdot\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & T_3 & \\ & & & T_4 \end{bmatrix}. \tag{4}$$

(4) has four equations which can be used to solve for 4 unknowns R_1, R_2, T_3, T_4. Knowing R_1, R_2 and T_1, T_4 we can analyze reflection and transmission of an outgoing wave through layered medium. Furthermore, the above process can be easily extended to multilayer problems.

(b) Standing wave case:

In this case, the excitation is the standing wave in region 3. It will cause reflection of outgoing wave in region 3 and transmission wave to region 1 and 2. We can write the state vector in region 3 as

$$\mathbf{V}_3 = (\mathbf{a}_1 + \mathbf{a}_2 + R_3\mathbf{a}_3 + R_4\mathbf{a}_4)A_3 = \overline{\mathbf{a}}_3\cdot\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & R_3 & \\ & & & R_4 \end{bmatrix} A_3. \tag{5}$$

In region 1, there will be only standing waves since the outgoing wave is singular. The state vector for region 1 is

$$\mathbf{V}_1 = (T_1\mathbf{a}_1 + T_2\mathbf{a}_2 + 0 + 0)A_3 = \overline{\mathbf{a}}_1\cdot\begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} A_3. \tag{6}$$

$\mathbf{V}_1$ and $\mathbf{V}_3$ are related by the propagator $\mathbf{P}$ such that

$$V_3(a_2) = V_2(a_2) = \overline{\mathbf{P}}(a_2,a_1)V_2(a_1) = \overline{\mathbf{P}}(a_2,a_1)V_1(a_1). \tag{7}$$

Hence, we have

$$\overline{\mathbf{P}}(a_2,a_1)\cdot\overline{\mathbf{a}}_1\begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & 0 & \\ & & & 0\end{bmatrix} = \overline{\mathbf{a}}_3\cdot\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & R_3 & \\ & & & R_4\end{bmatrix}. \quad (8)$$

Solve (8), we can find T_1, T_2, R_3 and R_4, which can be used to analyze the reflection and transmission of standing waves in a layered medium.

§3.18

$$\frac{d\mathbf{V}}{d\rho} = \frac{1}{\Delta}[\mathbf{V}(\rho_{i+1}) - \mathbf{V}(\rho_i)], \qquad \Delta = \rho_{i+1} - \rho_i$$

Substituting into (3.4.9), we have

$$\frac{1}{\Delta}[\mathbf{V}(\rho_{i+1}) - \mathbf{V}(\rho_i)] = \mathbf{H}(\rho_i)\cdot\mathbf{V}(\rho_i).$$

This is equivalent to

$$\mathbf{V}(\rho_{i+1}) = \left(\Delta\cdot\overline{\mathbf{H}}(\rho_i) + \overline{\mathbf{I}}\right)\mathbf{V}(\rho_i) = \overline{\mathbf{T}}(\rho_i)\cdot\mathbf{V}(\rho_i).$$

By using the above equation recursively, we arrive at

$$\begin{aligned}\mathbf{V}(\rho_N) &= \left(\Delta\overline{\mathbf{H}}(\rho_{N-1}) + \overline{\mathbf{I}}\right)\mathbf{V}(\rho_{N-1}) \\ &= \left(\Delta\overline{\mathbf{H}}(\rho_{N-1}) + \overline{\mathbf{I}}\right)\cdot\left(\Delta\overline{\mathbf{H}}(\rho_{N-2}) + \overline{\mathbf{I}}\right)\mathbf{V}(\rho_{N-2}) \\ &= \overline{\mathbf{T}}(\rho_{N-1})\cdot\overline{\mathbf{T}}(\rho_{N-2})\cdot(\rho_{N-2}) \\ &= \cdots \\ &= \overline{\mathbf{T}}(\rho_{N-1})\cdot\overline{\mathbf{T}}(\rho_{N-2})\cdot\overline{\mathbf{T}}(\rho_{N-3})\cdots\overline{\mathbf{T}}(\rho_0)\cdot\mathbf{V}(\rho_0) \\ &\equiv \overline{\mathbf{P}}(\rho_N,\rho_0)\cdot\mathbf{V}(\rho_0).\end{aligned}$$

Hence,

$$\overline{\mathbf{P}}(\rho_N,\rho_0) = \prod_{j=N}^{1}\left(\overline{\mathbf{I}} + \Delta\overline{\mathbf{H}}(\rho_{N-j})\right).$$

This is a numerical scheme to find $\overline{\mathbf{P}}(\rho_N,\rho_0)$.

§3.19

Considering (3.4.16), substituting $\Delta_s = i\mathbf{k}_s = \frac{in}{\rho}\hat{\phi} + ik_z\hat{z}$ into (3.4.16), we have

$$\begin{aligned}\hat{\rho}\times\frac{d\mathbf{E}_s}{d\rho} &= i\omega\overline{\boldsymbol{\mu}}_s\cdot\mathbf{H}_s + \overline{\boldsymbol{\mu}}_{s\rho}\cdot\nu_{\rho\rho}\left(\frac{in}{\rho}\hat{\phi} + ik_z\hat{z}\right)\times\mathbf{E}_s \\ &- i\omega\overline{\boldsymbol{\mu}}_{s\rho}\cdot\overline{\boldsymbol{\mu}}_{\rho s}\cdot\nu_{\rho\rho}\mathbf{H}_s + \frac{1}{i\omega}\left(\frac{in}{\rho}\hat{\phi} + ik_z\hat{z}\right)\times\kappa_{\rho\rho}\left(\frac{in}{\rho}\hat{\phi} + ik_z\hat{z}\right)\times\mathbf{H}_s \\ &+ \left(\frac{in}{\rho}\hat{\phi} + \hat{z}ik_z\right)\times\kappa_{\rho\rho}\cdot\overline{\boldsymbol{\epsilon}}_{\rho s}\cdot\mathbf{E}_s. \quad (1)\end{aligned}$$

In the above,

$$\left(\frac{in}{\rho}\hat{\phi}+ik_z\hat{z}\right)\times\left(\frac{in}{\rho}\hat{\phi}+ik_z\hat{z}\right)\times\mathbf{H}_s$$
$$=\left(\frac{in}{\rho}\hat{\phi}+ik_z\hat{z}\right)\left[\frac{in}{\rho}\hat{\phi}+ik_z\hat{z}\right]\cdot\mathbf{H}_s-\left[\left(\frac{in}{\rho}\right)^2+k_z^2\right]\mathbf{H}_s.$$

Cross multiplying (1) by $\hat{\rho}$, and using the following identities,

$$\mathbf{a}\times\mathbf{b}\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$
$$\mathbf{a}\times[\overline{\mathbf{d}}\cdot\mathbf{c}]=(\mathbf{a}\times\overline{\mathbf{d}})\cdot\mathbf{c},$$

we can get

$$-\frac{d\mathbf{E}_s}{d\rho}=(i\omega\hat{\rho}\times\overline{\boldsymbol{\mu}}_s)\cdot\mathbf{H}_s+\hat{\rho}\times\left[\overline{\boldsymbol{\mu}}_{s\rho}\times\nu_{\rho\rho}\left(\frac{in}{\rho}\hat{\phi}+ik_z\hat{z}\right)\right]\cdot\mathbf{E}_s$$
$$-\left[i\omega\hat{\rho}\times\left(\overline{\boldsymbol{\mu}}_{s\rho}\cdot\overline{\boldsymbol{\mu}}_{\rho s}\right)\nu_{\rho\rho}\right]\cdot\mathbf{H}_s+\left[\frac{\kappa_{\rho\rho}}{i\omega}\left(\frac{in}{\rho}\hat{z}-ik_z\hat{\phi}\right)\cdot\right.$$
$$\left.\left(\frac{in}{\rho}\hat{\phi}+ik_z\hat{z}\right)\right]\cdot\mathbf{H}_s-(\hat{\rho}\times\overline{\mathbf{I}}\,k_s^2)\cdot\mathbf{H}_s \quad (2)$$

Combining the terms involving $\mathbf{H}_s(\mathbf{E}_s)$, we can rewrite (2) as

$$\frac{d\mathbf{E}_s}{d\rho}=\overline{\mathbf{h}}_{11}\cdot\mathbf{E}_s+\overline{\mathbf{h}}_{12}\cdot\mathbf{H}_s \quad (3)$$

where

$$\overline{\mathbf{h}}_{11}=-\hat{\rho}\times\left[\overline{\boldsymbol{\mu}}_{s\rho}\times\nu_{\rho\rho}\left(\frac{in}{\rho}\hat{\phi}+ik_z\hat{z}\right)\right] \quad (4)$$
$$\overline{\mathbf{h}}_{12}=-i\omega\hat{\rho}\times\overline{\boldsymbol{\mu}}_s+i\omega\nu_{\rho\rho}\hat{\rho}\times\left(\overline{\boldsymbol{\mu}}_{s\rho}\cdot\overline{\boldsymbol{\mu}}_{\rho s}\right)$$
$$-\frac{\kappa_{\rho\rho}}{i\omega}\left(\frac{in}{\rho}\hat{z}-ik_z\hat{\phi}\right)\left(\frac{in}{\rho}\hat{\phi}+ik_z\hat{z}\right)+\hat{\rho}\times\overline{\mathbf{I}}\,k_s^2. \quad (5)$$

In the same way, we can find

$$\frac{d}{d\rho}\mathbf{H}_s=\overline{\mathbf{h}}_{21}\cdot\mathbf{E}_s+\overline{\mathbf{h}}_{22}\cdot\mathbf{H}_s, \quad (6)$$

where

$$\overline{\mathbf{h}}_{21}=i\omega\hat{\rho}\times\overline{\boldsymbol{\epsilon}}_s-i\omega\kappa_{\rho\rho}\hat{\rho}\times\left(\overline{\boldsymbol{\epsilon}}_{s\rho}\cdot\overline{\boldsymbol{\epsilon}}_{\rho s}\right)$$
$$\frac{\nu_{\rho\rho}}{i\omega}\left(\frac{in}{\rho}\hat{z}-ik_z\hat{\phi}\right)\left(\frac{in}{\rho}\hat{\phi}+ik_z\hat{z}\right)+\hat{\rho}\times\overline{\mathbf{I}}\,k_s^2 \quad (7)$$
$$\overline{\mathbf{h}}_{22}=-\hat{\rho}\times\left[\overline{\boldsymbol{\epsilon}}_{s\rho}\times\kappa_{\rho\rho}\left(\frac{in}{\rho}\hat{\phi}+ik_z\hat{z}\right)\right]. \quad (8)$$

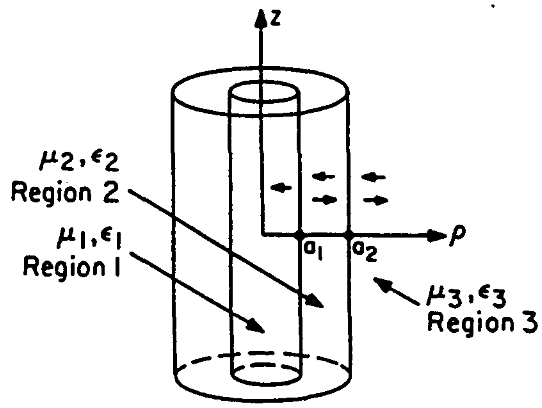

Figure for Exercise Solution 3.20

Hence

$$\overline{\mathbf{H}}(\rho) = \begin{bmatrix} \overline{\mathbf{h}}_{11} & \overline{\mathbf{h}}_{12} \\ \overline{\mathbf{h}}_{21} & \overline{\mathbf{h}}_{22} \end{bmatrix}.$$

§3.20

(a) Assume that $\mathbf{V}_N = \mathbf{V}(\rho_N)$, $\mathbf{V}_0 = \mathbf{V}(\rho_0)$. First divide the region between ρ_0 and ρ_N into N subregions, each has the same width Δ (this condition is not necessary). When $k\Delta \ll 1$, or $\Delta \ll \lambda$ (wavelength), we can approximate,

$$\left(\frac{d\mathbf{V}}{d\rho}\right)_{\rho_i} \approx \frac{1}{\Delta}\left[\mathbf{V}_{i+1} - \mathbf{V}_i\right], i = 0, 1, 2, \cdots, N-1 \tag{1}$$

Substituting this equation into (3.4.26), we have

$$\frac{1}{\Delta}\left[\mathbf{V}_{i+1} - \mathbf{V}_i\right] = \overline{\mathbf{H}}(\rho_i) \cdot \mathbf{V}_i, \text{or} \mathbf{V}_{i+1} = \left(\Delta\overline{\mathbf{H}}_i + \overline{\mathbf{I}}\right)\mathbf{V}_i. \tag{2}$$

Using (2) recursively, we can find

$$\begin{aligned}\mathbf{V}_n &= \left(\Delta\overline{\mathbf{H}}_{N-1} + \overline{\mathbf{I}}\right) \cdot \mathbf{V}_{N-1} = \left(\Delta\overline{\mathbf{H}}_{N-1} + \overline{\mathbf{I}}\right) \cdot \left(\Delta\overline{\mathbf{H}}_{N-2} + \overline{\mathbf{I}}\right)\mathbf{V}_{N-2} = \cdots \\ &= \left(\Delta\overline{\mathbf{H}}_{N-1} + \overline{\mathbf{I}}\right) \cdot \left(\Delta\overline{\mathbf{H}}_{N-2} + \overline{\mathbf{I}}\right) \cdot \left(\Delta\overline{\mathbf{H}}_{N-3} + \overline{\mathbf{I}}\right) \cdots \left(\Delta\overline{\mathbf{H}}_0 + \overline{\mathbf{I}}\right) \cdot \mathbf{V}_0 \\ &= \overline{\mathbf{p}} \cdot (\rho_N, \rho_0) \cdot \mathbf{V}_0. \end{aligned} \tag{3}$$

Hence, we can compute $\overline{\mathbf{P}}(\rho_n, \rho_0)$ numerically using the following formula

$$\overline{\mathbf{P}}(\rho_N, \rho_0) = \prod_{j=N}^{1} \left(\Delta\overline{\mathbf{H}}_{N-j} + \overline{\mathbf{I}}\right). \tag{4}$$

(b) Knowing the propagator matrix in (4), we can study the transmission and reflection of waves through the an isotropic layer. As shown the Figure.

In this case, the state vector (which is infinite dimensional) will still be composed of standing wave harmonics ϕ_s and outgoing wave ϕ_0. Hence if there is an outgoing wave in region 1, it will cause reflected standing wave in region 1 and a transmitted outgoing wave in region 3. V_1 can be written as

$$\mathbf{V}_1 = \left(\overline{\mathbf{I}} + \overline{\mathbf{R}}_1\right) \cdot \mathbf{V}_{10}$$

where, $\mathbf{V}_{10}$ is the outgoing wave in region 1, which is determined by the source in region 1.

$\overline{\mathbf{R}}_1$ is the reflection matrix at boundary $\rho = a_1$. Assuming the state vector in region 2 and 3 are $\mathbf{V}_2$ and $\mathbf{V}_3$ respectively, and that $\overline{\mathbf{T}}_3$ is the transmission matrix from region 1 to region 3, then, we have

$$\mathbf{V}_3 = \overline{\mathbf{T}}_3 \cdot \mathbf{V}_{10}.$$

Using the propagator matrix, we can relate $\mathbf{V}_1, \mathbf{V}_2$ and $\mathbf{V}_3$ as follows,

$$\begin{aligned} \mathbf{V}_2 &= \overline{\mathbf{P}}(\rho_2, \rho_1) \cdot \mathbf{V}_1 \triangleq \overline{\mathbf{P}}_{21} \cdot \mathbf{V}_1, \\ \mathbf{V}_3 &= \overline{\mathbf{P}}_{31} \cdot \mathbf{V}_1 = \overline{\mathbf{P}}_{32} \cdot \mathbf{V}_2. \end{aligned}$$

Hence, we have

$$\begin{aligned} \overline{\mathbf{P}}_{21} \cdot \left(\overline{\mathbf{I}} + \overline{\mathbf{R}}_1\right) V_{10} &= \mathbf{V}_2, \\ \overline{\mathbf{T}}_3 \mathbf{V}_{10} &= \overline{\mathbf{P}}_{31} \cdot \left(\overline{\mathbf{I}} + \overline{\mathbf{R}}_1\right) \cdot \mathbf{V}_{10}, \\ \overline{\mathbf{T}}_3 \cdot \mathbf{V}_{10} &= \overline{\mathbf{P}}_{32} \cdot \mathbf{V}_2. \end{aligned}$$

Solving the above equation, we can find $\mathbf{V}_2$, $\overline{\mathbf{R}}_1$, $\overline{\mathbf{T}}_3$ in terms of $\mathbf{V}_{10}$, $\overline{\mathbf{P}}_{21}$, $\overline{\mathbf{P}}_{32}$ and $\overline{\mathbf{P}}_{31}$.

§3.21

For $\mathbf{E}^{TE}$ case,

$$\mathbf{E}^{TE} = \nabla \times \mathbf{r}\pi_m = \nabla\pi_m \times \mathbf{r}, \quad (\text{because} \quad \nabla \times \mathbf{r} = 0), \tag{1}$$

and

$$\begin{aligned} \nabla \times \mathbf{E}^{TE} &= \nabla \times [\nabla\pi_m \times \mathbf{r}] \\ &= (\mathbf{r} \cdot \nabla)\nabla\pi_m - (\nabla\pi_m \cdot \nabla)\mathbf{r} + \nabla\pi_m(\nabla \cdot \mathbf{r}) - \mathbf{r}(\nabla \cdot \nabla\pi_m), \end{aligned}$$

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E}^{TE} &= \nabla \times [(\mathbf{r} \cdot \nabla)\nabla\pi_m] - \nabla \times [(\nabla\pi_m \cdot \nabla)\mathbf{r}] \\ &\quad + \nabla \times [\nabla\pi_m(\nabla \cdot \mathbf{r})] - \nabla \times [-k^2\pi_m\mathbf{r}]. \end{aligned} \tag{2}$$

In the above, we have used the fact that

$$\nabla^2\pi_m + k^2\pi_m = 0.$$

Furthermore, we can show that

$$\nabla \times [(\mathbf{r}\cdot\nabla)\nabla\pi_m] = 0, \quad \nabla \times [(\nabla\pi_m \cdot \nabla)\mathbf{r}] = 0, \quad \nabla \times [\nabla\pi_m(\nabla\cdot\mathbf{r})] = 0.$$

Hence, (2) becomes

$$\nabla \times \nabla \times \mathbf{E}^{TE} = k^2\nabla\pi_m \times \mathbf{r} + k^2\pi_m\nabla \times \mathbf{r} = k^2\nabla\pi_m \times \mathbf{r} = k^2.$$

Since $\nabla \times \nabla \times \mathbf{E}^{TE} = \nabla\nabla\cdot\mathbf{E}^{TE} - \nabla^2\mathbf{E}^{TE}$ and $\nabla\cdot\mathbf{E}^{TE} = \nabla\cdot(\nabla\mathbf{r}\pi_m) = 0$, we have $-\nabla^2\mathbf{E}^{TE} = k^2\mathbf{E}^{TE}$, or

$$(\nabla^2 + k^2)\mathbf{E}^{TE} = 0. \tag{3}$$

In the same way, we can show that if π_e satisfies

$$(\nabla^2 + k^2)\pi_e = 0,$$

then

$$\mathbf{H}^{TM} = \nabla \times (\mathbf{r}\pi_e)$$

will satisfy

$$(\nabla^2 + k^2)\mathbf{H}^{TM} = 0.$$

§3.22

(a) From (3.5.4a),

$$\mathbf{H} = \nabla \times (\mathbf{r}\pi_e) + \frac{1}{i\omega\mu}\nabla \times \nabla \times (\mathbf{r}\pi_m), \tag{1}$$

$$\begin{aligned}\nabla \times (\mathbf{r}\pi_e) &= \nabla \times (\hat{r}r\pi_e)\\ &= \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\pi_e)\hat{\theta} - \frac{1}{r}\frac{\partial}{\partial\theta}(r\pi_e)\hat{\phi},\end{aligned} \tag{2}$$

$$\begin{aligned}&\nabla \times \nabla \times (\mathbf{r}\pi_m)\\ =&\nabla \times \left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\pi_m)\hat{\theta} - \frac{1}{r}\frac{\partial}{\partial\theta}(r\pi_m)\hat{\phi}\right\}\\ =&\frac{-1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}(r\pi_m)\right] + \frac{\partial}{\partial\phi}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\pi_m)\right]\right\}\hat{r}\\ &+ \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial\theta}(r\pi_m)\right]\hat{\theta} + \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\pi_m)\right]\hat{\phi}.\end{aligned} \tag{3}$$

Extracting $\hat{r}$-component from (1), we have

$$\begin{aligned}(i\omega\mu)H_r &= \frac{-1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}(r\pi_m)\right] + \frac{\partial}{\partial\phi}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\pi_m)\right]\right\}\\ &= \frac{-1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}(r\pi_m)\right] - \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}(r\pi_m)\\ &= r\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\pi_m}{\partial r}\right] - \nabla^2\pi_m\right\}\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial\pi_m}{\partial r}\right) + rk^2\pi_m = \frac{\partial^2}{\partial r^2}(r\pi_m) + k^2r\pi_m.\end{aligned}$$

Hence,

$$H_r = \frac{1}{i\omega\mu}\left[\frac{\partial^2}{\partial r^2}(r\pi_m) + k^2(r\pi_m)\right]. \tag{4}$$

Similarly,

$$E_r = -\frac{1}{i\omega\epsilon}\left[\frac{\partial^2}{\partial r^2}(r\pi_e) + k^2(r\pi_e)\right]. \tag{5}$$

Since π_e and π_m are the solutions of the scalar wave equation, their general solutions are of the form

$$\left\{\begin{matrix} j_n(kr) \\ h_n^{(1)}(kr)\end{matrix}\right\} P_n^m(\cos\theta)\left\{\begin{matrix}\cos m\phi \\ \sin m\phi\end{matrix}\right\}.$$

In this case,

$$\frac{\partial^2}{\partial r^2}\begin{bmatrix} r\pi_e \\ r\pi_m \end{bmatrix} = -\left[k^2 - \frac{n(n+1)}{r^2}\right]\begin{bmatrix} r\pi_e \\ r\pi_m \end{bmatrix}.$$

Therefore, for the n-th harmonics,

$$H_r = \frac{1}{i\omega\mu}\frac{n(n+1)}{r}\pi_m,$$

$$E_r = -\frac{1}{i\omega\epsilon}\frac{n(n+1)}{r}\pi_e.$$

(b) Substituting (3) and (2) into (1), equating the $\hat{s}$-directed components on both sides, we have

$$\begin{aligned}\mathbf{H}_s &= -\mathbf{r}\times\nabla_s\pi_e + \frac{1}{i\omega\mu}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial\theta}(r\pi_m)\hat\theta + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\pi_m)\hat\phi\right]\right\} \\ &= -\mathbf{r}\times\nabla_s\pi_e + \frac{1}{i\omega\mu}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{r\partial\theta}\hat\theta + r^2\frac{\hat\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\hat\phi\right]\pi_m\right\} \\ &= -\mathbf{r}\times\nabla_s\pi_e + \frac{1}{i\omega\mu}\frac{\partial}{r\partial r}(r^2\nabla_s\pi_m).\end{aligned}$$

Likewise,

$$\mathbf{E}_s = -\mathbf{r}\times\nabla_s\pi_m - \frac{1}{i\omega\epsilon}\frac{\partial}{r\partial r}(r^2\nabla_s\pi_e).$$

§3.23

The matrix $\overline{\mathbf{P}}_n^m(\cos\theta)$ is defined as

$$\overline{\mathbf{P}}_n^m(\cos\theta) = \begin{bmatrix} \frac{d}{d\theta}P_n^m(\cos\theta) & -\frac{im}{\sin\theta}P_n^m(\cos\theta) \\ \frac{im}{\sin\theta}P_n^m(\cos\theta) & \frac{d}{d\theta}P_n^m(\cos\theta)\end{bmatrix}. \tag{1}$$

Therefore,

$$\overline{\mathbf{M}} \equiv \int_0^\pi d\theta\sin\theta\,\overline{\mathbf{P}}_n^m(\cos\theta)\overline{\mathbf{P}}_{n'}^m(\cos\theta) \tag{2}$$

is a 2×2 matrix. From (1), we have

$$M_{11} = M_{22} = \int_0^\pi d\theta \sin\theta \left[\frac{dP_n^m}{d\theta}\frac{dP_{n'}^m}{d\theta} + \frac{m^2}{\sin^2\theta}P_n^m P_{n'}^m\right]. \tag{3}$$

Changing the variable $x = \cos\theta$, we have

$$\begin{aligned} M_{11} &= \int_{-1}^{1} dx \left[(1-x^2)\frac{dP_n^m}{dx}\frac{dP_{n'}^m}{dx} + \frac{m^2}{1-x^2}P_n^m P_{n'}^m\right] \\ &= \int_{-1}^{1} dx \left[-P_n^m \frac{d}{dx}(1-x^2)\frac{dP_{n'}^m}{dx} + \frac{m^2}{1-x^2}P_n^m P_{n'}^m\right]. \end{aligned} \tag{4}$$

In the above, integration by parts has been applied. Here, $P_{n'}^m$ satisfies the equation of

$$\frac{d}{dx}(1-x^2)\frac{dP_{n'}^m}{dx} + \left[n'(n'+1) - \frac{m^2}{1-x^2}\right] P_{n'}^m = 0. \tag{5}$$

Using (5) in (4), we obtain

$$\begin{aligned} M_{11} &= \int_{-1}^{1} dx\, n'(n'+1)P_n^m P_{n'}^m \\ &= \frac{2n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nn'} = M_{22}. \end{aligned} \tag{6}$$

For the off-diagonal terms, we have

$$\begin{aligned} M_{12} = -M_{21} &= -\int_0^\pi d\theta \sin\theta \left[\frac{dP_{n'}^m}{d\theta}\frac{im}{\sin\theta}P_n^m + \frac{im}{\sin\theta}P_{n'}^m \frac{dP_n^m}{d\theta}\right] \\ &= im \int_{-1}^{1} dx \left[\frac{dP_{n'}^m}{dx}P_n^m + P_{n'}^m \frac{dP_{n'}^m}{dx}\right] \\ &= im \int_{-1}^{1} dx \frac{d}{dx}\left[P_{n'}^m P_n^m\right] \\ &= im P_{n'}^m(x) P_n^m(x) \Big|_{-1}^{1} = 0. \end{aligned} \tag{7}$$

Since for $m = 0$, the above is zero, for $m \neq 0$,

$$P_n^m(x) = \frac{(1-x^2)^{\frac{m}{2}}}{2^n n'}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^l$$

Thus

$$P_n^m(\pm 1) = 0.$$

Combining (7) and (6), we have,

$$\int_0^\pi d\theta \sin\theta \overline{\mathbf{P}}_n^m(\cos\theta)\overline{\mathbf{P}}_n^m(\cos\theta) = \delta_{nn'} \frac{2n(n+1)}{(2n+1)} \frac{(n+m)!}{(n-m)!}\overline{\mathbf{I}}$$

In Equation (3.1.15), $|m|$ should be replaced by m.

§3.24

It is easy to show that:

(1)

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix}, \quad \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & d^{-1} \\ c^{-1} & 0 \end{bmatrix}.$$

The inverse of a diagonal matrix is still a diagonal matrix. The inverse of an off diagonal matrix is still an off diagonal matrix.

(2)

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} ax & 0 \\ 0 & by \end{bmatrix},$$
$$\begin{bmatrix} 0 & d \\ c & 0 \end{bmatrix}\begin{bmatrix} 0 & r \\ u & 0 \end{bmatrix} = \begin{bmatrix} ud & 0 \\ 0 & cv \end{bmatrix},$$
$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & d \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & ad \\ bc & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & d \\ c & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & ac \\ db & 0 \end{bmatrix}.$$

The product of two off-diagonal matrices is a diagonal matrix.

We first show that $\overline{\mathbf{R}}_{12}$ and $\overline{\mathbf{T}}_{12}$ are diagonal matrices. First,

$$\overline{\mathbf{h}}_{ne}^{(1)} \text{ is diagonal} \Longrightarrow \left.\begin{matrix} \overline{\mathbf{h}}_{ne}^{(1)-1}(k_1a): & \text{diagonal} \\ \overline{\mathbf{j}}_{ne}(k_2a): & \text{diagonal} \end{matrix}\right\}$$
$$\Longrightarrow \overline{\mathbf{h}}_{ne}^{(1)-1}\cdot\overline{\mathbf{j}}_{ne} \quad \text{diagonal},$$
$$\overline{\mathbf{h}}_{nh}^{(1)} \text{is off diagnal} \Longrightarrow \left.\begin{matrix} \overline{\mathbf{h}}_{nh}^{(1)-1}: & \text{off diagonal} \\ \overline{\mathbf{j}}_{nh}: & \text{off diagonal} \end{matrix}\right\}$$
$$\Longrightarrow \overline{\mathbf{h}}_{nh}^{-1}\overline{\mathbf{j}}_{nh}: \quad \text{diagonal}.$$

Hence,

$$\overline{\mathbf{A}} = \left[\overline{\mathbf{h}}_{ne}^{(1)-1}(k_2a)\overline{\mathbf{j}}_{ne}(k_1a) - \overline{\mathbf{h}}_{nh}^{(1)-1}(k_2a)\overline{\mathbf{j}}_{nh}(k_1a)\right]^{-1} \quad \text{is diagonal},$$
$$\overline{\mathbf{B}} = \left[\overline{\mathbf{h}}_{ne}^{(1)-1}(k_2a)\overline{\mathbf{h}}_{ne}^{(1)}(k_1a) - \overline{\mathbf{h}}_{nh}^{(1)-1}(k_2a)\overline{\mathbf{h}}_{nh}^{(1)}(k_1a)\right] \quad \text{is diagonal}.$$

Therefore, $\overline{\mathbf{R}}_{12} = \overline{\mathbf{A}} \cdot \overline{\mathbf{B}}$ is a diagonal matrix.

In the same way, we can show $\overline{\mathbf{T}}_{12}$ is a diagonal matrix. To find the elements for $\overline{\mathbf{R}}_{12}$ and $\overline{\mathbf{T}}_{12}$, we denote that

$$\overline{\mathbf{A}} = \overline{\mathbf{h}}_{ne}^{(1)^{-1}}(k_2a) \cdot \overline{\mathbf{j}}_{ne}(k_1a), \qquad \overline{\mathbf{B}} = \overline{\mathbf{h}}_{nh}^{(1)^{-1}}(k_2a) \cdot \overline{\mathbf{j}}_{nh}(k_1a),$$
$$\overline{\mathbf{C}} = \overline{\mathbf{h}}_{ne}^{(1)^{-1}}(k_2a) \cdot \overline{\mathbf{h}}_{ne}^{(1)}(k_1a), \qquad \overline{\mathbf{D}} = \overline{\mathbf{h}}_{nh}^{(1)^{-1}}(k_2a) \cdot \overline{\mathbf{h}}_{nh}^{(1)}(k_1a),$$

$$\hat{J}_n(x) = xj_n(x), \quad \hat{H}_n^{(1)}(x) = xh_n^{(1)}(x).$$

Therefore,

$$\overline{\mathbf{h}}_{ne}^{(1)}(k_2a) = \begin{bmatrix} \frac{i}{\omega\epsilon_2}\frac{1}{r}\frac{\partial}{\partial r}rh_n^{(1)}(k_2r) & 0 \\ 0 & -h_n^{(1)}(k_2r) \end{bmatrix}_{r=a}$$
$$= \begin{bmatrix} i\sqrt{\frac{\mu_2}{\epsilon_2}}\frac{1}{k_2a}\hat{H}_n^{(1)'}(k_2a) & 0 \\ 0 & -h_n^{(1)}(k_2a) \end{bmatrix}$$
$$= \frac{1}{k_2a}\begin{bmatrix} i\sqrt{\frac{\mu_2}{\epsilon_2}}\hat{H}_n^{(1)'}(k_2a) & 0 \\ 0 & -H_n^{(1)}(k_2a) \end{bmatrix},$$
$$\overline{\mathbf{h}}_{nh}^{(1)}(k_2a) = \frac{1}{k_2a}\begin{bmatrix} 0 & -i\sqrt{\frac{\epsilon_2}{\mu_2}}\hat{H}_n^{(1)'}(k_2a) \\ -\hat{H}_n^{(1)}(k_2a) & 0 \end{bmatrix},$$
$$\overline{\mathbf{j}}_{ne}(k_1a) = \frac{1}{k_1a}\begin{bmatrix} i\sqrt{\frac{\mu_1}{\epsilon_1}}\hat{J}_n'(k_1a) & 0 \\ 0 & -\hat{J}_n(k_1a) \end{bmatrix},$$
$$\overline{\mathbf{j}}_{nh}(k_1a) = \frac{1}{k_1a}\begin{bmatrix} 0 & -i\sqrt{\frac{\epsilon_1}{\mu_1}}\hat{J}_n'(k_1a) \\ -\hat{J}_n(k_1a) & 0 \end{bmatrix},$$

$$
\begin{aligned}
\overline{\mathbf{A}} &= \frac{k_2 a}{k_1 a}\begin{bmatrix} \sqrt{\frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}}\hat{J}_n'(k_2a)/\hat{H}_n^{(1)'}(k_2a) & 0 \\ 0 & \hat{J}_n(k_1a)/\hat{H}_n^{(1)}(k_2a)\end{bmatrix} \\
&= \frac{k_2}{k_1}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \\
\overline{\mathbf{B}} &= \frac{k_2 a}{k_1 a}\begin{bmatrix} \hat{J}_n(k_1a)/\hat{H}_n^{(1)}(k_2a) & 0 \\ 0 & \sqrt{\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}}\hat{J}_n'(k_1a)/\hat{H}_n^{(1)'}(k_2a)\end{bmatrix} \\
&= \frac{k_2}{k_1}\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \\
\overline{\mathbf{C}} &= \frac{k_2 a}{k_1 a}\begin{bmatrix} \sqrt{\frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}}\hat{H}_n^{(1)'}(k_1a)/\hat{H}_n^{(1)'}(k_2a) & 0 \\ 0 & \hat{H}_n^{(1)}(k_1a)/\hat{H}_n^{(1)}(k_2a)\end{bmatrix} \\
&= \frac{k_2}{k_1}\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}, \\
\overline{\mathbf{D}} &= \frac{k_2 a}{k_1 a}\begin{bmatrix} \hat{H}_n^{(1)}(k_1a)/\hat{H}_n^{(1)}(k_2a) & 0 \\ 0 & \sqrt{\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}}\hat{H}_n^{(1)'}(k_1a)/\hat{H}^{(1)}(k_2a)\end{bmatrix} \\
&= \frac{k_2}{k_1}\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\overline{\mathbf{R}}_{12} &= -[\overline{\mathbf{A}}-\overline{\mathbf{B}}]^{-1}[\overline{\mathbf{C}}-\overline{\mathbf{D}}] = -\begin{bmatrix} \frac{1}{a_1-b_1} & 0 \\ 0 & \frac{1}{a_2-b_2}\end{bmatrix}\begin{bmatrix} c_1-d_1 & 0 \\ 0 & c_2-d_2\end{bmatrix} \\
&= -\begin{bmatrix} \frac{c_1-d_1}{a_1-b_1} & 0 \\ 0 & \frac{c_2-d_2}{a_2-b_2}\end{bmatrix} = \begin{bmatrix} R_{12}^{TM} & 0 \\ 0 & R_{12}^{TE}\end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
R_{12}^{TM} &= -\frac{c_1-d_1}{a_1-b_1} = -\frac{\sqrt{\frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}}\hat{H}_n^{(1)'}(k_2a)/\hat{H}_n^{(1)'}(k_2a) - \hat{H}_n^{(1)}(k_1a)/\hat{H}_n^{(1)'}(k_2a)}{\sqrt{\frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}}\hat{J}_n'(k_1a)/\hat{H}_n^{(1)'}(k_2a) - \hat{J}_n(k_1a)/\hat{H}_n^{(1)}(k_2a)} \\
&= \frac{\sqrt{\mu_1\epsilon_2}\hat{H}_n^{(1)'}(k_1a)\hat{H}_n^{(1)}(k_2a) - \sqrt{\mu_2\epsilon_1}\hat{H}_n^{(1)'}(k_2a)\hat{H}_n^{(1)}(k_1a)}{\sqrt{\epsilon_1\mu_2}\hat{J}_n(k_1a)\hat{H}_n^{(1)'}(k_2a) - \sqrt{\epsilon_2\mu_1}\hat{H}_n^{(1)}(k_2a)\hat{J}_n'(k_1a)},
\end{aligned}
$$

$$
\begin{aligned}
R_{12}^{TE} &= -\frac{c_2-d_2}{a_2-b_2} = -\frac{\hat{H}_n^{(1)}(k_1a)/\hat{H}_n^{(1)}(k_2a) - \sqrt{\frac{\mu_2\epsilon_1}{\mu_1\epsilon_2}}\hat{H}_n^{(1)'}(k_1a)/\hat{H}_n^{(1)}(k_2a)}{\hat{J}_n(k_1a)/\hat{H}_n^{(1)}(k_2a) - \sqrt{\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}}\hat{J}_n'(k_1a)/\hat{H}_n^{(1)'}(k_2a)} \\
&= \frac{\sqrt{\epsilon_1\mu_2}\hat{H}_n^{(1)'}(k_1a)\hat{H}_n^{(1)}(k_2a) - \sqrt{\mu_1\epsilon_2}\hat{H}_n^{(1)}(k_1a)\hat{H}_n^{(1)'}(k_2a)}{\sqrt{\epsilon_2\mu_1}\hat{J}_n(k_1a)\hat{H}_n^{(1)'}(k_2a) - \sqrt{\epsilon_1\mu_2}\hat{J}_n'(k_1a)\hat{H}_n^{(1)}(k_2a)}.
\end{aligned}
$$

In order to find $\overline{\mathbf{T}}_{12}$, we define

$$\overline{\mathbf{E}} = \bar{\mathbf{j}}_{ne}^{-1}(k_1a)\hat{h}_{ne}^{(1)}(k_2a)$$
$$= \frac{k_1}{k_2}\begin{bmatrix} \frac{1}{i\sqrt{\frac{\mu_1}{\epsilon_1}}\hat{J}_n'(k_1a)} & 0 \\ 0 & -\frac{1}{\hat{J}_n(k_1a)} \end{bmatrix}\begin{bmatrix} i\sqrt{\frac{\mu_2}{\epsilon_2}}\hat{H}_n^{(1)'}(k_2a) & 0 \\ 0 & -\hat{H}_n^{(1)}(k_2a) \end{bmatrix}$$
$$= \frac{k_1}{k_2}\begin{bmatrix} \sqrt{\frac{\mu_2\epsilon_1}{\mu_1\epsilon_2}}\hat{H}_n^{(1)'}(k_2a)/\hat{J}_n'(k_1a) & 0 \\ 0 & \hat{H}_n^{(1)}(k_2a)/\hat{J}_n(k_1a) \end{bmatrix} = \frac{k_1}{k_2}\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix},$$

$$\overline{\mathbf{F}} = \bar{\mathbf{j}}_{nh}^{-1}(k_1a)\cdot\overline{\mathbf{h}}_{nh}^{(1)}(k_2a)$$
$$= \frac{k_1}{k_2}\begin{bmatrix} 0 & -\frac{1}{\hat{J}_n(k_1a)} \\ \frac{1}{i\sqrt{\frac{\epsilon_1}{\mu_1}}\hat{J}_n'(k_1a)} & 0 \end{bmatrix}\begin{bmatrix} 0 & -i\sqrt{\frac{\epsilon_2}{\mu_2}}\hat{H}_n^{(1)'}(k_2a) \\ -\hat{H}_n^{(1)}(k_2a) & 0 \end{bmatrix}$$
$$= \frac{k_1}{k_2}\begin{bmatrix} \hat{H}_n^{(1)}(k_2a)/\hat{J}_n(k_1a) & 0 \\ 0 & \sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}}\hat{H}_n^{(1)'}(k_2a)/\hat{J}_n'(k_1a) \end{bmatrix} = \frac{k_1}{k_2}\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix},$$

$$\overline{\mathbf{G}} = \bar{\mathbf{j}}_{ne}^{-1}(k_1a)\hat{h}_{ne}^{(1)}(k_1a)$$
$$= \begin{bmatrix} \hat{H}_n^{(1)'}(k_1a)/\hat{J}_n'(k_1a) & 0 \\ 0 & \hat{H}_n^{(1)}(k_1a)/\hat{J}_n(k_1a) \end{bmatrix} = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix},$$
$$\hat{H} = \bar{\mathbf{j}}_{nh}^{-1}(k_1a)\overline{\mathbf{h}}_{nh}^{(1)}(k_1a)$$
$$= \begin{bmatrix} \overline{\mathbf{H}}_n^{(1)}(k_1a)/\hat{J}_n(k_1a) & 0 \\ 0 & \overline{\mathbf{H}}_n^{(1)'}(k_1a)/\hat{J}_n'(k_1a) \end{bmatrix} = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix},$$

$$\overline{\mathbf{T}}_{12} = \left[\overline{\mathbf{E}} - \overline{\mathbf{F}}\right]^{-1}\left[\overline{\mathbf{G}} - \overline{\mathbf{H}}\right] = \begin{bmatrix} \frac{1}{e_1-f_1} & 0 \\ 0 & \frac{1}{e_2-f_2} \end{bmatrix}\cdot\frac{k_2}{k_1}\cdot\begin{bmatrix} g_1-h_1 & 0 \\ 0 & g_2-h_2 \end{bmatrix}$$
$$= \frac{k_2}{k_1}\begin{bmatrix} \frac{g_1-f_1}{e_1-f_1} & 0 \\ 0 & \frac{g_2-h_2}{e_2-f_2} \end{bmatrix} = \begin{bmatrix} T_{12}^{TM} & 0 \\ 0 & T_{12}^{TE} \end{bmatrix},$$
$$T_{12}^{TM} = \frac{k_2}{k_1}\frac{\hat{H}_n^{(1)'}(k_1a)/\hat{J}_n'(k_1a) - \hat{H}_n^{(1)}(k_1a)/\hat{J}_n(k_1a)}{\sqrt{\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}}\hat{H}_n^{(1)'}(k_2a)/\hat{J}_n'(k_1a) - \hat{H}_n^{(1)}(k_2a)/\hat{J}_n(k_1a)}$$
$$= \frac{i\epsilon_2\sqrt{\frac{\mu_2}{\epsilon_1}}}{\sqrt{\epsilon_1\mu_2}\hat{J}_n(k_1a)\hat{H}_n^{(1)'}(k_2a) - \sqrt{\epsilon_2\mu_1}\hat{J}_n'(k_1a)\hat{H}_n^{(1)}(k_2a)},$$
$$T_{12}^{TE} = \frac{i\mu_2\sqrt{\frac{\epsilon_2}{\mu_1}}}{\sqrt{\epsilon_2\mu_1}\hat{J}_n(k_1a)\hat{H}_n^{(1)'}(k_2a) - \sqrt{\epsilon_1\mu_2}\hat{J}_n'(k_1a)\hat{H}_n^{(1)}(k_2a)}.$$

§3.25

Define that

$$\hat{J}_n(x) = xj_n(x), \quad \hat{H}_n^{(1)}(x) = xh_n^{(1)}(x). \tag{1}$$

Then,

$$\hat{J}_n'(x) = \frac{d}{dx}\left[xj_n(x)\right], \quad \hat{H}_n^{(1)'}(x) = \frac{d}{dx}\left[xh_n^{(1)}(x)\right], \tag{2}$$

$$\frac{\partial}{\partial a}[aj_n(k_1a)] = \frac{\partial}{\partial u}[uj_n(u)] = \hat{J}_n'(k_1a), \tag{3}$$

$$\frac{\partial}{\partial a}[ah_n^{(1)}(k_1a)] = \hat{H}_n^{(1)'}(k_1a), \tag{4}$$

$$j_n(k_2a) = \frac{1}{k_2a}\hat{J}_n(k_2a), \quad h_n^{(1)}(k_2a) = \frac{1}{k_2a}\hat{H}_n^{(1)}(k_2a). \tag{5}$$

Substituting the above relations to (3.5.26a), we have

$$\begin{aligned} R_{21}^{TM} &= \frac{\frac{\epsilon_2}{k_2a^2}\hat{J}_n(k_2a)\hat{J}_n'(k_1a) - \frac{1}{k_1a^2}\epsilon_1\hat{J}_n(k_1a)\hat{J}_n'(k_2a)}{\frac{\epsilon_1}{k_1a^2}\hat{J}_n(k_1a)\hat{H}_n^{(1)}(k_2a) - \frac{\epsilon_2}{d_2a^2}\hat{H}_n^{(1)}(k_2a)\hat{J}_n'(k_1a)} \\ &= \frac{\sqrt{\epsilon_2\mu_1}\hat{J}_n(k_2a)\hat{J}_n'(k_1a) - \sqrt{\epsilon_1\mu_2}\hat{J}_n(k_1a)\hat{J}_n'(k_2a)}{\sqrt{\epsilon_1\mu_2}\hat{J}_n(k_1a)\hat{H}_n^{(1)'}(k_2a) - \sqrt{\epsilon_2\mu_1}\hat{H}_n^{(1)'}(k_2a)\hat{J}_n'(k_1a)}. \end{aligned} \tag{6}$$

Substituting (1)–(5) into (3.5.26b), we have

$$\begin{aligned} T_{21}^{TM} &= \frac{\frac{\epsilon_1}{k_2a^2}\hat{J}_n(k_2a)\hat{H}_n^{(1)'}(k_2a) - \frac{\epsilon_1}{k_2a^2}\hat{H}_n^{(1)}(k_2a)\hat{J}_n'(k_2a)}{\frac{\epsilon_1}{k_1a^2}\hat{J}_n(k_1a)\hat{H}_n^{(1)'}(k_2a) - \frac{\epsilon_1}{k_2a^2}\hat{H}_n^{(1)}(k_2a)\hat{J}_n'(k_1a)} \\ &= \frac{\epsilon_1\sqrt{\frac{\mu_1}{\epsilon_2}}\left[\hat{J}_n(k_2a)\hat{H}_n^{(1)'}(k_2a) - \hat{J}_n'(k_2a)\hat{H}_n^{(1)}(k_2a)\right]}{\sqrt{\epsilon_1\mu_2}\hat{J}_n(k_1a)\hat{H}_n^{(1)'}(k_2a) - \sqrt{\mu_1\epsilon_2}\hat{H}_n^{(1)}(k_2a)\hat{J}_n'(k_1a)}. \end{aligned} \tag{7}$$

Using Wronskian for spherical Bessel function, we have

$$\hat{J}_n(x)\hat{H}_n^{(1)'}(x) - \hat{J}_n'(x)\hat{H}_n^{(1)}(x) = i. \tag{8}$$

We find that

$$T_{12}^{TM} = \frac{i\epsilon_1\sqrt{\frac{\mu_1}{\epsilon_2}}}{\sqrt{\epsilon_1\mu_2}\hat{J}_n(k_1a)\hat{H}_n^{(1)'}(k_2a) - \sqrt{\mu_1\epsilon_2}\hat{H}_n^{(1)}(k_2a)\hat{J}_n'(k_1a)}. \tag{9}$$

In the same way, we can verify (3.5.28).

§3.26

In region i, the Debye potential is of the form:

$$\pi_i = a_i[h_n^{(1)}(k_ir) + \tilde{R}_{i,i+1}j_n(k_ir)] \tag{1}$$

where $\tilde{R}_{i,i+1}$ is a generalized reflection coefficient.

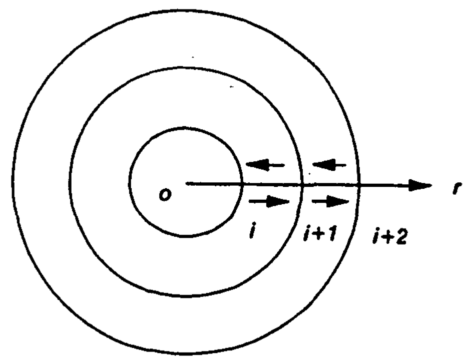

Figure 1 for Exercise Solution 3.26

In regions $i+1$ and $i+2$

$$\pi_{i+1} = a_{i+1}\left[h_n^{(1)}(k_{i+1}r) + \tilde{R}_{i+1,i+2}j_n(k_{i+1}r)\right], \tag{2}$$

$$\pi_{i+1} = a_{i+1}\left[h_n^{(1)}(k_{i+2}r) + \tilde{R}_{i+2,i+3}j_n(k_{i+2}r)\right]. \tag{3}$$

Since an outgoing wave in region $i+1$ is a consequence of transmission of an outgoing wave in region i plus the reflection of a standing wave in region $i+1$, we have

$$a_{i+1} = T_{i,i+1}a_i + R_{i+1,i}\tilde{R}_{i+1,i+2}\cdot a_{i+1}. \tag{4}$$

Furthermore, the standing wave in region i is a result of the reflection of an outgoing wave in region i plus the transmission of a standing wave in region $i+1$, we have

$$\tilde{R}_{i,i+1}a_i = R_{i,i+1}a_i + T_{i+1,i}\tilde{R}_{i+1,i+2}a_{i+1}. \tag{5}$$

Solving (4) yields

$$a_{i+1} = (1 - R_{i+1,i}\tilde{R}_{i+1,i+2})^{-1}T_{i,i+1}a_i. \tag{6}$$

Substituting (6) into (5), and eliminate constant $a_i(\neq 0)$, we have

$$\tilde{R}_{i,i+1} = R_{i,i+1} + \frac{T_{i,i+1}T_{i+1,i}\tilde{R}_{i+1,i+2}}{1 - R_{i+1,i}\tilde{R}_{i+1,i+2}}. \tag{7}$$

If we define

$$S_{i,i+1} = \frac{T_{i,i+1}}{1 - R_{i+1,i}\tilde{R}_{i+1,i+2}}, \tag{8}$$

then

$$a_{i+1} = S_{i,i+1}a_i. \tag{9}$$

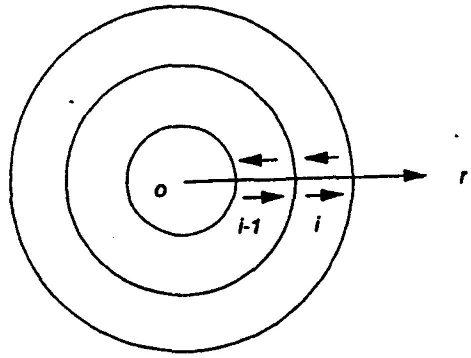

Figure 2 for Exercise Solution 3.26

For the standing wave case, the solution in the respective regions are

Region i:

$$\pi_i = [j_n(k_i r) + \tilde{R}_{i,i-1} h_n^{(1)}(k_i r)]a_i. \tag{10}$$

Region $i-1$:

$$\pi_{i-1} = [j_n(k_{i-1} r) + \tilde{R}_{i-1,i-2} h_n^{(1)}(k_{i-1} r)]a_{i-1}. \tag{11}$$

By the constraint condition, a standing wave in region $i-1$ is a consequence of the transmission of the standing wave in region i plus the reflection of the outgoing wave in region 2. Hence,

$$a_{i-1} = T_{i,i-1}a_i + R_{i-1,i}\tilde{R}_{i-1,i-2}a_{i-1}. \tag{12}$$

On the other hand, the outgoing wave in region i is a consequence of reflection of the standing wave in region i plus the transmission of the outgoing wave in region $i-1$. Therefore,

$$\tilde{R}_{i,i-1}a_i = R_{i,i-1}a_i + T_{i-1,i}\tilde{R}_{i-1,i-2}a_{i-1}. \tag{13}$$

From (12), we get

$$a_{i-1} = (1 - R_{i-1}\tilde{R}_{i-1,i-2})^{-1}T_{i,i-1}a_i = S_{i,i-1}a_i. \tag{14}$$

Applying (14) in (13), we have

$$\tilde{R}_{i,i-1} = R_{i,i-1} + \frac{T_{i,i-1}T_{i-1,i}\tilde{R}_{i-1,i-2}}{1 - R_{i-1,i}\tilde{R}_{i-1,i-2}}. \tag{15}$$

A generalized transmission coefficient is

$$\tilde{T}_{N1} = S_{N,N-1} \cdot S_{N-1,N-2} \cdots S_{32}S_{21}.$$

Since $S_{21} = T_{21}$,

$$\tilde{T}_{N1} = S_{N,n_1}S_{N-1,N-2} \cdots S_{32}T_{21}. \tag{16}$$

§3.27

To solve the following equation in spherical coordinates,

$$(\nabla^2 + k^2)\psi(r,\theta,\phi) = -\frac{1}{r^2\sin\theta}\delta(\mathbf{r}-\mathbf{r}')\delta(\theta-\theta')\delta(\phi-\phi'), \qquad (1)$$

we assume $\psi(r,\theta,\phi)$ can be expressed as linear combination of spherical harmonics. That is

$$\psi(r,\theta,\phi) = \sum_{mn} f(r)P_n^m(\cos\theta)e^{im\phi}. \qquad (2)$$

Substituting (2) into (1), we have

$$\sum_{mn}\left\{\frac{1}{r^2}\frac{d}{dr}[r^2f'(r)]P_n^m e^{im\phi} + \frac{f(r)e^{im\phi}}{r^2\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dP_n^m}{d\theta}\right]\right.$$
$$\left. -\frac{m^2e^{im\phi}}{r^2\sin^2\theta}f(r)P_n^m + k^2f(r)P_n^m e^{im\phi}\right\}$$
$$= -\frac{1}{r^2\sin\theta}\delta(r-\mathbf{r}')\delta(\theta-\theta')\delta(\phi-\phi'). \qquad (3)$$

Since

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dP_n^m}{d\theta}\right] = \left[\frac{m^2}{\sin^2\theta} - n(n+1)\right]P_n^m, \qquad (4)$$

(3) becomes

$$\sum_{mn}\left\{\frac{d}{dr}\left[r^2\frac{df(r)}{dr}\right] + [k^2r^2 - n(n+1)]f(r)\right\}P_n^m(\cos\theta)e^{im\phi}$$
$$= -\frac{\delta(r-r')}{r^2\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi'). \qquad (5)$$

Multiplying both sides of (5) by $\sin\theta P_{n'}^{m'}(\cos\theta)e^{-im'\phi}$ and integrating over θ and ϕ from 0 to π and 0 to 2π respectively, and making use of the orthogonal properties of P_n^m, we have

$$\int_0^\pi \sin\theta P_n^m(\cos\theta)P_{n'}^m(\cos\theta)d\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nn'} = 2N_{nm}^{-1}\delta_{nn'}, \qquad (6)$$

we find that

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{df(r)}{dr}\right] + [kr^2 - n\frac{(n+1)}{r^2}]f(r) = -N_{nm}P_{nm}^m(\cos\theta')e^{-im\phi'}\delta(r-r')$$
$$= -\frac{C_{mn}}{r^2}\delta(r-r'). \qquad (7)$$

When $r \neq r'$, (7) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{df(r)}{dr}\right] + \left[kr^2 - n\frac{(n+1)}{r^2}\right]f(r) = 0. \tag{8}$$

For

$$r > r': \qquad f(r) = a_n h_n^{(1)}(kr), \tag{9a}$$
$$r < r': \qquad f(r) = b_n j_n(kr). \tag{9b}$$

Matiching the solution at $r = r'$, we have

$$\begin{aligned} a_n h_n^{(1)}(kr') - b_n j_n(kr') &= 0, \\ a_n h_n^{(1)\prime}(kr') - b_n j_n'(kr) &= -C_{mn}/k. \end{aligned} \tag{10}$$

Solving the above for a_n and b_n yields

$$\begin{aligned} a_n &= \frac{1}{k\Delta}C_{mn}j_n(kr') \\ b_n &= \frac{1}{k\Delta}C_{mn}h_n^{(1)}(kr'), \end{aligned} \tag{11}$$

where

$$\Delta = j_n'(kr')h_n^{(1)}(kr') - j_n(kr')h_n^{(1)\prime}(kr') = -\frac{i}{(kr')^2}. \tag{12}$$

Substituting (12) and (11) into (9), we get

$$f(r) = ikC_{mn}\begin{cases} h_n^{(1)}(kr)j_n(kr') & r > r', \\ h_n^{(1)}(kr')j_n(kr) & r < r'. \end{cases} \tag{13}$$

Finally, we have

$$\begin{aligned} &\psi(r,\theta,\phi) \\ &= \sum_{mn} N_{mn}P_n^m(\cos\theta)e^{im(\phi-\phi')}P_n^m(\cos\theta')\begin{cases} h_n^{(1)}(kr)j_n(kr'), & r > r', \\ h_n^{(1)}(kr')j_n(kr), & r < r'. \end{cases} \end{aligned} \tag{14}$$

Denoting

$$\begin{aligned} r_< &= \min(r, r'), \\ r_> &= \max(r, r'), \end{aligned}$$

we have

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \sum_{mn} N_{mn}j_n(kr_<)h_n^{(1)}(kr_>)P_n^m(\cos\theta)P_n^m(\cos\theta')e^{im(\phi-\phi')}. \tag{15}$$

§3.28

(a)

$$\begin{aligned}
\mathbf{E} &= i\omega\mu\left(\bar{\mathbf{I}} + \frac{1}{k^2}\nabla\nabla\right)\cdot\hat{\alpha}Ilg(\mathbf{r}-\mathbf{r}') \\
&= i\omega\mu\cdot k^{-2}\left[k^2\hat{\alpha}Ilg(\mathbf{r}-\mathbf{r}') + \nabla\times\nabla\times\hat{\alpha}Ilg(\mathbf{r}-\mathbf{r}') + \nabla^2\hat{\alpha}Ilg\right] \\
&= \frac{i\omega\mu}{\omega^2\epsilon\mu}\left\{\hat{\alpha}Il[(\nabla^2+k^2)g(\mathbf{r}-\mathbf{r}')] + Il\nabla\times\nabla\times\hat{\alpha}g(\mathbf{r}-\mathbf{r}')\right\} \\
&= \frac{iI\ell}{\omega\epsilon}\nabla\times\nabla\times\hat{\alpha}g(\mathbf{r}-\mathbf{r}'), \quad (\mathbf{r}\neq\mathbf{r}'). \qquad (1)
\end{aligned}$$

In deriving the above, we have used identities:

$$\nabla\times\nabla\times\mathbf{A} = \nabla\nabla\cdot\mathbf{A} - \nabla^2\mathbf{A},$$

and

$$(\nabla^2+k^2)g(\mathbf{r}-\mathbf{r}') = 0, \qquad (\mathbf{r}\neq\mathbf{r}').$$

(b) Reciprocity theorem says that $\langle\mathbf{E}_1,\mathbf{J}_2\rangle = \langle\mathbf{E}_2,\mathbf{J}_1\rangle$, where $\mathbf{E}_1$ is due to $\mathbf{J}_1$, $\mathbf{E}_2$ is due to $\mathbf{J}_2$.

Assume

$$\mathbf{J}_1 = \hat{\alpha}\delta(r'), \qquad \mathbf{J}_2 = \hat{r}\delta(r),$$

then

$$\mathbf{E}_1 = C\nabla\times\nabla\times\hat{\alpha}g(\mathbf{r}-\mathbf{r}'), \qquad \mathbf{E}_2 = C\nabla'\times\nabla'\times\hat{r}g(\mathbf{r}'-\mathbf{r}),$$

where C is a constant.

Hence

$$\hat{r}\cdot\nabla\times\nabla\times\hat{\alpha}g(\mathbf{r}-\mathbf{r}') = \hat{\alpha}\cdot\nabla'\times\nabla'\times\hat{r}g(\mathbf{r}'-\mathbf{r}),$$

i.e. (3.7.8) is equal to (3.7.7).

(c) Assume $\mathbf{r}-\mathbf{r}'=\mathbf{R}$, $\nabla' f(\mathbf{R}) = \frac{\partial f(R)}{\partial R}[\nabla' R] = \frac{\partial f}{\partial R}\hat{R}$, then

$$\begin{aligned}
(\mathbf{r}-\mathbf{r}')\cdot g(\mathbf{r}-\mathbf{r}') &= \frac{1}{4\pi}\mathbf{R}\frac{e^{ikR}}{R} = \frac{1}{4\pi}e^{ikR}\hat{R} \\
&= \frac{1}{4\pi}\frac{\partial}{\partial R}\left[\frac{e^{ikR}}{ik}\right]\hat{R} \\
&= \nabla'\left[\frac{1}{4\pi ik}e^{ikR}\right].
\end{aligned}$$

Therefore,

$$\nabla'\times(\mathbf{r}-\mathbf{r}')g(\mathbf{r}-\mathbf{r}') = \nabla'\times\nabla'\left[\frac{e^{ikR}}{4\pi ik}\right] \equiv 0.$$

§3.29

(a) For an electrical dipole with $\hat{\alpha} = \hat{r}$,

$$\mathbf{E} = i\omega\mu \left(\bar{\mathbf{I}} + \frac{1}{k^2}\nabla\nabla\right) \cdot \hat{r} I l g(\mathbf{r} - \mathbf{r}')$$
$$= \frac{iI\ell}{\omega\epsilon} \nabla \times \nabla \times \hat{r} g(\mathbf{r} - \mathbf{r}'). \tag{1}$$

Then,

$$\mathbf{r} \cdot \mathbf{E} = rE_r = \frac{iI\ell}{\omega\epsilon} \mathbf{r} \cdot \nabla \times \nabla \times \hat{r} g(\mathbf{r} - \mathbf{r}')$$
$$= \frac{iI\ell}{\omega\epsilon} \hat{r} \cdot \nabla' \times \nabla' \times \mathbf{r} g(\mathbf{r} - \mathbf{r}')$$
$$= \frac{iI\ell}{\omega\epsilon} \hat{r} \cdot \nabla' \times \nabla' \times \mathbf{r}' g(\mathbf{r} - \mathbf{r}'),$$

where

$$g(\mathbf{r} - \mathbf{r}') = ik \sum_n j_n(kr_<) A_n(\theta, \phi, \theta', \phi').$$

Consequently,

$$rE_r = \frac{-kIl}{\omega\epsilon} \hat{r} \cdot \nabla' \times \nabla' \times \mathbf{r}' \sum_n j_n(kr_<) h_n^{(1)}(kr_>) A_n(\theta, \phi, \theta', \phi'),$$

and

$$\pi_e = \frac{i\omega\epsilon}{(n+1)n} rE_r = ikIl\hat{r} \cdot \nabla' \times \nabla' \times \mathbf{r}' \sum_n j_n(kr_<) h_n^{(1)}(kr_>) A_n.$$

Since

$$\mathbf{H} = \nabla \times \hat{r} I l g(\mathbf{r} - \mathbf{r}'),$$
$$H_r = 0 \implies \pi_m = 0. \tag{3.7.5b}$$

Hence, we only need to know π_e to find all the field components.

(b) If a source is put into the j-th region, the Debye potention in the j-th region is

$$\pi_j = D_j' \sum_n \left[j_n(kr_<) h_n^{(1)}(kr_>) + a_{jn} h_n^{(1)}(k_j r) + b_{jn}(k_j r) \right] \frac{A_n(\theta, \phi, \theta', \phi')}{n(n+1)}. \tag{2}$$

Here, we have taken into account the additional reflected waves. We use generalized reflection coefficients to determine a_{jn} and b_{jn}

The outgoing wave is related to standing wave by

$$a_{jn} = \tilde{R}_{jj-1} \left[h_n^{(1)}(k_j r') + b_{jn} \right]. \tag{3a}$$

The standing wave is related to outgoing wave by

$$b_{jn} = \tilde{R}_{jj-1}\left[j_n(k_j r') + a_{jn}\right]. \tag{3b}$$

Solving (3a) and (3b) for a_{jn} and b_{jn}, we get

$$a_{jn} = \tilde{R}_{jj-1}\left[h_n^{(1)}(k_j r') + \tilde{R}_{jj+1} j_n(k_j r')\right]\tilde{M}_j, \tag{4}$$

$$b_{jn} = \tilde{R}_{jj+1}\left[j_n(k_j r') + \tilde{R}_{jj-1} h_n^{(1)}(k_j r')\right]\tilde{M}_j, \tag{5}$$

where

$$\tilde{M}_g = \left(1 - \tilde{R}_{jj-1}\tilde{R}_{jj+1}\right)^{-1}. \tag{6}$$

Substituting (4) and (5) back into (2) yields

$$\pi_j = D'_j \sum_n \left\{ j_n(k_j r_>) h_n^{(1)}(k_j r_>) + \tilde{R}_{jj-1}\left[h_n^{(1)}(k_j r') + \tilde{R}_{jj+1} j_n(k_j r')\right]\tilde{M}_j \right.$$
$$\left. + \tilde{R}_{jj+1}\left[j_n(k_j r') + \tilde{R}_{jj-1} h_n^{(1)}(k_j r')\right]\tilde{M}_j \right\} \frac{A_n(\theta,\phi;\theta',\phi')}{n(n+1)}, \tag{7}$$

or

$$\pi_j = D'_j \sum_n \left\{ j_n(k_j r_<) h_n^{(1)}(k_j r_>) \left[1 - \tilde{R}_{jj-1}\tilde{R}_{jj+1}\right] \right.$$
$$+ \tilde{R}_{jj-1}\left[h_n^{(1)}(k_j r') + \tilde{R}_{jj+1} j_n(k_j r')\right]$$
$$\left. + \tilde{R}_{jj+1}\left[j_n(k_j r') + \tilde{R}_{jj-1} h_n^{(1)}(k_j r')\right]\right\} \tilde{M}_j \frac{A_n(\theta,\phi;\theta',\phi')}{n(n+1)}. \tag{8}$$

Since

$$\tilde{M}_j = 1 - \tilde{R}_{jj-1}\tilde{R}_{jj+1},$$

$$\pi_j = D'_j \sum_n \left\{ j_n(k_j r') h_n^{(1)}(k_j r_>) \left[1 - \tilde{R}_{jj-1}\tilde{R}_{jj+1}\right] \right.$$
$$+ \tilde{R}_{jj-1}\left[h_n^{(1)}(k_j r') + \tilde{R}_{jj+1} j_n(k_j r')\right]$$
$$\left. + \tilde{R}_{jj+1}\left[j_n(k_j r') + \tilde{R}_{jj-1} h_n^{(1)}(k_j r')\right]\right\} \tilde{M}_j \frac{A_n(\theta,\phi,\theta',\phi')}{n(n+1)}.$$

(c) In $i(> j)$-th region, the Debye potential can be written as

$$\pi_i = D'_j \sum_n a_{in}(r') \left[h_n^{(1)}(k_i r) + \tilde{R}_{i,i+1} j_n(k_i r)\right] \frac{A_n(\theta,\phi,\theta',\phi')}{n(n+1)}.$$

The wave amplitude in region i, $D'_j a_{in}$, is related to wave amplitude in region j via generalized transmission operator $\tilde{T}_{ji}$. But $\tilde{T}_{ji}$ does not include the multiple reflections in region i. Hence, we must add a factor $\tilde{M}_i$ to take into account the multiple reflection. Multiple reflections can be represented by series of the form:

$$1 + \tilde{R}_{ii-1}\tilde{R}_{ii+1} + (\tilde{R}_{ii-1}\tilde{R}_{ii+1})^2 + (R_{ii-1}R_{ii+1})^3 + \cdots = \frac{1}{1 - \tilde{R}_{ii-1}\tilde{R}_{ii+1}}.$$

Therefore

$$D_j' a_{in} = \tilde{T}_{ji}\frac{1}{1 - \tilde{R}_{ii-1}\tilde{R}_{ii+1}} D_j' a_{jn} = \tilde{T}_{ji}\tilde{M}_i D_j' a_{jn}.$$

§3.30

The vector wave equation is

$$\nabla \times \epsilon^{-1}\nabla \times \mathbf{H} - \omega^2\mu\mathbf{H} = 0 \qquad (1)$$

Extracting the r component, we have

$$(\nabla \times \epsilon^{-1}\nabla \times \mathbf{H})_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\sin\theta\epsilon^{-1}(\nabla \times \mathbf{H})_\phi - \frac{\partial}{\partial\phi}\epsilon^{-1}(\nabla \times \mathbf{H})_\theta\right].$$

But

$$(\nabla \times \mathbf{H})_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}rH_\theta - \frac{\partial}{\partial\theta}H_r\right], \qquad (3a)$$

$$(\nabla \times \mathbf{H})_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}H_r - \frac{\partial}{\partial r}rH_\phi\right]. \qquad (3b)$$

Therefore, (2) becomes

$$\begin{aligned}
&(\nabla \times \epsilon^{-1}\nabla \times \mathbf{H})_r \\
&= \frac{\epsilon^{-1}}{r\sin\theta}\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial\theta}\sin\theta H_\theta - \frac{1}{r}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}H_r\right. \\
&\qquad \left. -\frac{1}{r\sin\theta}\frac{\partial^2}{\partial\phi^2}H_r + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial\phi}H_\phi\right] \\
&= \frac{\epsilon^{-1}}{r\sin\theta}\left[-\frac{1}{r}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}H_r - \frac{1}{r\sin\theta}\frac{\partial^2}{\partial\phi^2}H_r\right. \\
&\qquad \left. +\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{\partial}{\partial\theta}\sin\theta H_\theta + \frac{\partial}{\partial\phi}H_\phi\right)\right].
\end{aligned} \qquad (4)$$

The above is the same as

$$(\nabla \times \epsilon^{-1}\nabla \times \mathbf{H})_r = \epsilon^{-1}\left[-\nabla_s^2 H_r + \frac{1}{r^2}\frac{\partial}{\partial r}r^2\nabla_s \cdot \mathbf{H}_s\right]. \qquad (5)$$

Since

$$\nabla_s \cdot \mathbf{H}_s = -\mu^{-1}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\mu_s H_r, \tag{6}$$

we have

$$(\nabla \times \epsilon^{-1}\nabla \times \mathbf{H})_r = \epsilon^{-1}\left[-\nabla_s^2 H_r - \frac{1}{r^2}\frac{\partial}{\partial r}\mu^{-1}\frac{\partial}{\partial r}r^2\mu H_r\right], \tag{7}$$

and (1) becomes

$$\nabla_s^2 H_r + \frac{1}{r^2}\frac{\partial}{\partial r}\mu^{-1}\frac{\partial}{\partial r}r^2\mu H_r + k^2 H_r = 0,$$

or

$$\nabla_s^2 r\mu H_r + \mu\frac{1}{r}\frac{\partial}{\partial r}\mu^{-1}\frac{\partial}{\partial r}r^2\mu H_r + k^2 r\mu H_r = 0.$$

Define a π_m according to Equation (3.5.8), we have

$$\nabla_s^2\pi_m + \mu\frac{1}{r}\frac{\partial}{\partial r}\mu^{-1}\frac{\partial}{\partial r}r\pi_m + k^2\pi_m = 0.$$

By duality,

$$\nabla_s^2\pi_e + \epsilon\frac{1}{r}\frac{\partial}{\partial r}\epsilon^{-1}\frac{\partial}{\partial r}r\pi_e + k^2\pi_e = 0.$$

§3.31

To find $\overline{\mathbf{a}}^{-1}$, where

$$\overline{\mathbf{a}}(\mathbf{r}) = [\mathbf{a}_1\ \mathbf{a}_2], \quad \mathbf{a}_1 = \begin{bmatrix} \frac{1}{p}\frac{d}{dr}rj_n(kr) \\ rj_n(kr) \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} \frac{1}{p}\frac{d}{dr}rh_n^{(1)}(kr) \\ rh_n^{(1)}(kr) \end{bmatrix},$$

we first construct vector $\mathbf{b}_1$ such that

$$\mathbf{b}_1^t \cdot \mathbf{a}_2 = 0, \tag{1}$$
$$\mathbf{b}_1 \cdot \mathbf{a}_1 = 1. \tag{2}$$

Assume that $\mathbf{b}^t = (b_{11}, b_{12})$, then

$$b_{11}a_{21} + b_{12}a_{22} = 0, \tag{3}$$
$$b_{11}a_{11} + b_{12}a_{12} = 1. \tag{4}$$

Solving (3), (4) we get $b_{11} = \frac{\pm a_{22}}{\Delta}$, $b_2 = \frac{-a_{21}}{\Delta}$ where

$$\begin{aligned}\Delta &= a_{11}a_{22} - a_{12}a_{21} \\ &= \left[\frac{1}{p}\frac{d}{dr}rj_n(kr)\right]\left[rh_n^{(1)}(kr)\right] - \left[\frac{1}{p}\frac{d}{dr}rh_n^{(1)}(kr)\right]\left[rj_n(kr)\right] \\ &= \frac{1}{pk}\left[\hat{J}_n'(kr)\hat{H}_n^{(1)}(kr) - J_n(kr)H_n^{(1)\prime}(kr)\right] = \frac{i}{kp}.\end{aligned}$$

Therefore

$$\mathbf{b}^t = \left(\frac{i}{kp}\right)^{-1} \left[r h_n^{(1)}(kr), \quad -\frac{1}{p}\frac{d}{dr} r h_n^{(1)}(kr) \right].$$

In the same way, we find $\mathbf{b}_2 = \left(\frac{kp}{i}\right)\left[-r j_n(kr), \quad \frac{1}{p}\frac{d}{dr} r j_n(kr)\right]$, such that

$$\mathbf{b}_2 \cdot \mathbf{a}_1 = 0,$$
$$\mathbf{b}_2 \cdot \mathbf{a}_2 = 1.$$

Hence,

$$\overline{\mathbf{a}}^{-1} = \frac{kp}{i} \begin{bmatrix} r h_n^{(1)}(kr) & -r j_n(kr) \\ \frac{-1}{p}\frac{d}{dr} r h_n^{(1)}(kr) & \frac{1}{p}\frac{d}{dr} r j_n(kr) \end{bmatrix} = -i \begin{bmatrix} \hat{H}_n^{(1)}(kr) & -\hat{J}_n(kr) \\ -\hat{H}_n^{(1)\prime}(kr) & \hat{J}_n'(kr) \end{bmatrix},$$

where, $\hat{J}_n(x) = x j_n(x), \quad \hat{H}_n^{(1)}(x) = x h_n^{(1)}(x).$

CHAPTER 4

EXERCISE SOLUTIONS

by R. Wagner

§4.1

The proof is the same as the derivation of the Kramers-Kronig relations found in §§4.1.1

§4.2

(a)

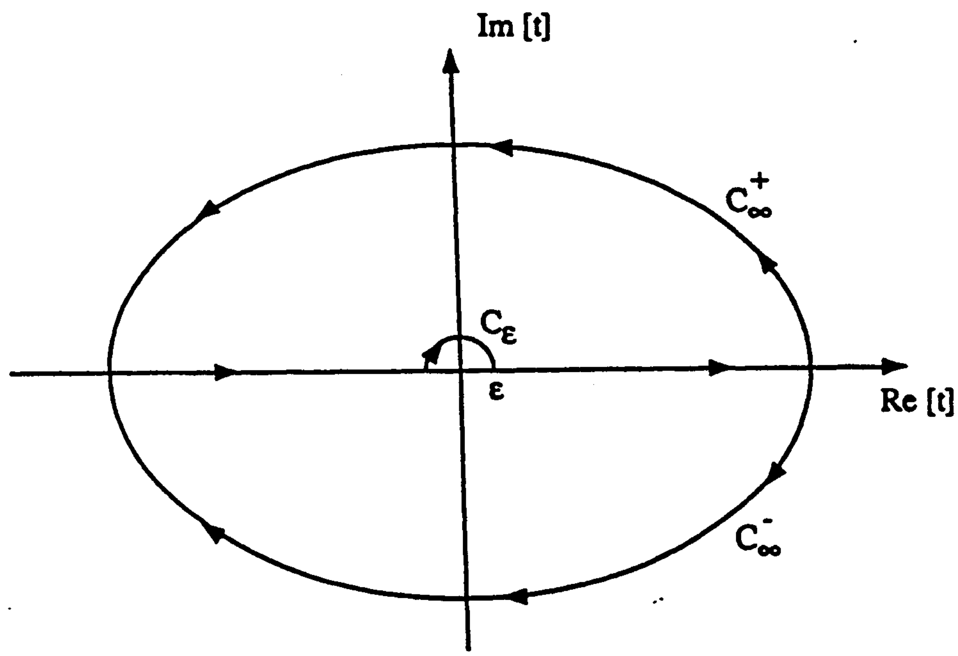

Figure for Exercise Solution 4.2

By definition $P.V. \int\limits_{-\infty}^{\infty} \frac{e^{i\omega t}}{t}\, dt = \lim\limits_{\epsilon \to 0}\left[\int\limits_{-\infty}^{\epsilon} \frac{e^{i\omega t}}{t}\, dt + \int\limits_{\epsilon}^{\infty} \frac{e^{i\omega t}}{t}\, dt\right].$

The P.V. integral can be computed by complex contour integration. Note that the residue of $\frac{e^{i\omega t}}{t}$ at $t = 0$ is 1.

Thus,

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} \frac{e^{i\omega t}}{t}\,dt = -\pi i \cdot 1,$$

where C_ϵ is shown in the figure. For $\omega > 0, e^{i\omega t} \to 0$ as $t \to +i\infty$, and $\int_{C_\infty^+} \frac{e^{i\omega t}}{t}\,dt = 0$ by Jordan's lemma. Applying the residue theorem yields

$$P.V.\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{t}\,dt + \lim_{\epsilon\to 0}\int_{C_\epsilon} \frac{e^{i\omega t}}{t}\,dt + \int_{C_\infty^+} \frac{e^{i\omega t}}{t}\,dt = 0,$$

or

$$P.V.\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{t}\,dt = -(-\pi i) + 0 = \pi i, \quad \omega > 0.$$

For $\omega < 0, e^{i\omega t} \to 0$ as $t \to -i\infty$, and the integral over C_∞^- vanishes. Applying the residue theorem in this case yields

$$P.V.\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{t}\,dt + \lim_{\epsilon\to 0}\int_{C_\epsilon} \frac{e^{i\omega t}}{t}\,dt + \int_{C_\infty^-} \frac{e^{i\omega t}}{t}\,dt = -2\pi i \cdot 1,$$

or,

$$P.V.\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{t}\,dt = -(-\pi i) - 0 - 2\pi i = -\pi i, \qquad \omega < 0.$$

When $\omega = 0$, the positive and negative portions of the P.V. integral cancel, so the total integral is zero. Combining these results,

$$P.V.\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{t}\,dt = \begin{cases} \pi i, & \omega > 0 \\ 0, & \omega = 0 \\ -\pi i, & \omega < 0 \end{cases}$$

or

$$\frac{1}{\pi} P.V.\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{t}\,dt = i\,\mathrm{sgn}(\omega).$$

(b) Since $g(t) = f(t) * (-\frac{1}{\pi} P.V. \frac{1}{t})$, application of the Fourier time convolution theorem immediately yields

$$g(\omega) = -i\,\mathrm{sgn}(\omega) f(\omega),$$

where $f(\omega)$ is the Fourier transform of $F(t)$ and $g(\omega)$ is the Fourier transform of $g(t)$, the Hilbert transform of $f(t)$.

§4.3

(a) See the solution to problem 4.2 (a). The identity here is obtained from the identity in 4.2 (a) by interchanging ω and t, and taking the complex conjugate.

(b) Equation (4.1.8) is

$$\epsilon'(\omega) - \epsilon(\infty) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\epsilon''(\omega')}{\omega' - \omega}\, d\omega', \tag{4.1.8}$$

where $\epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$ is the complex permittivity. Taking the inverse Fourier transform gives

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon'(\omega)\, e^{-i\omega t}\, d\omega - \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon(\infty)\, e^{-i\omega t}\, d\omega \\ =& \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\epsilon''(\omega')}{\omega' - \omega}\, d\omega'. \end{aligned}$$

Evaluating the first two integrals and changing the order of integration on the third,

$$\epsilon'(t) - \epsilon(\infty)\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \epsilon''(\omega') \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega' - \omega}.$$

Let $u = \omega - \omega'$. Then $du = d\omega$ and $\omega = u + \omega'$, so the right hand side of the above becomes

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \epsilon''(\omega') \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} du\, \frac{e^{-iut}}{-u} \cdot e^{-i\omega' t} \\ =& \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \epsilon''(\omega') \cdot i \operatorname{sgn}(t) e^{-i\omega' t} = i \operatorname{sgn}(t) \epsilon''(t). \end{aligned}$$

Thus the inverse Fourier transform of (4.1.8) reads

$$\epsilon'(t) - \epsilon(\infty)\, \delta(t) = i \operatorname{sgn}(t) \epsilon''(t).$$

Similarly, (4.1.9) is

$$\epsilon''(\omega) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\epsilon'(\omega') - \epsilon(\infty)}{\omega' - \omega}\, d\omega'. \tag{4.1.9}$$

Taking the inverse Fourier transform,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \epsilon''(\omega)\, e^{-i\omega t}\, d\omega = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \cdot \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\epsilon'(\omega') - \epsilon(\infty)}{\omega' - \omega}\, d\omega'.$$

Following the same steps as for (4.1.8), this becomes

$$\epsilon''(t) = -i\, \text{sgn}(t)[\epsilon'(t) - \epsilon(\infty)\delta(t)].$$

(c) Using the results just established,

$$\epsilon(t) - \epsilon(\infty)\,\delta(t) = \epsilon'(t) - \epsilon(\infty)\,\delta(t) + i\epsilon''(t) = i\, \text{sgn}(t)\epsilon''(t) + i\epsilon''(t)\,.$$

Thus, $\epsilon(t) - \epsilon(\infty)\,\delta(t) = i[\,\text{sgn}(t)\epsilon''(t) + \epsilon''(t)] = 0, \quad t < 0.$
So, $\epsilon(t) - \epsilon(\infty)\delta(t)$ is causal if (4.1.8) and (4.1.9) hold.

§4.4

Suppose the singularities in (4.1.10) extend above the real axis. Consider only pole singularities for simplicity.

From (4.1.10),

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \tilde{\phi}(\omega). \tag{4.1.10}$$

Consider a closed integration path consisting of the original Fourier inversion contour C and the arc C_∞, defined similarly to C_∞^+ in the figure for Exercise 4.2. Assuming $\tilde{\phi}(\omega) \to 0$ when $|\omega| \to \infty$, then for $t < 0$ the integral over C_∞ vanishes by Jordan's lemma. Thus, applying the residue theorem, for $t < 0$,

$$\frac{1}{2\pi}\int_{C+C_\infty} d\omega e^{-i\omega t}\, \tilde{\phi}(\omega) = \frac{1}{2\pi}\int_{C} d\omega e^{-i\omega t}\, \tilde{\phi}(\omega) = \phi(t) = \sum_{n=1}^{N} A_n e^{-i\omega_{pn} t},$$

where $\Re e[\omega_{pn}] > 0, \quad n = 1, \ldots, N.$
Thus, if C lies below any singularities, $\phi(t) \neq 0$ for $t < 0$.

However, if C is moved above all the singularities, then for $t < 0$,

$$\int_{C+C_\infty} d\omega e^{-i\omega t}\, \tilde{\phi}(\omega) = \int_{C} d\omega e^{-i\omega t}\, \tilde{\phi}(\omega) = \phi(t) = 0,$$

and causality is not violated.

Now consider the solution for $\phi(t)$ expressed by (4.1.13),

$$\phi(t) = \frac{1}{2\pi}\sum_{i=1}^{N} A_i e^{-i\omega_{pi} t} + \frac{1}{2\pi}\sum_{i=1}^{N} f_i(t) e^{-i\omega_{bi} t}. \tag{4.1.13}$$

If, for an active medium, any of the poles or branch points ω_{pi} or ω_{bi} have an imaginary part greater than zero, say $\omega = \omega' + i\omega''$, $\omega'' > 0$, then the corresponding term in $\phi(t)$ will be $e^{-i\omega t} = e^{-i(\omega'+i\omega'')t} = e^{-i\omega' t} \cdot e^{\omega'' t}$. The magnitude of this term approaches infinity as $t \to \infty$, if $\omega'' > 0$.

§4.5

The transformation between the complex t and s_x planes is given by

$$t = s_x x + (s_0^2 - s_x^2)^{\frac{1}{2}} |y|, \tag{4.2.7}$$

$$s_x = \frac{t}{\rho} \cos\phi \pm \sin\phi (s_0^2 - \frac{t^2}{\rho^2})^{\frac{1}{2}}, \tag{4.2.9}$$

where $\rho = \sqrt{x^2 + y^2}, \cos\phi = x/\rho, \sin\phi = y/\rho$. If t is real and $t > s_0 \rho$, then

$$s_x = \frac{t}{\rho} \cos\phi \pm i \sin\phi (\frac{t^2}{\rho^2} - s_0^2)^{\frac{1}{2}}.$$

So, $\{\Re e[s_x]\}^2 = \frac{t^2}{\rho^2} \cos^2\phi$ and $\{\Im m[s_x]\}^2 = \left(\frac{t^2}{\rho^2} - s_0^2\right) \sin^2\phi$.
Thus,

$$\frac{\{\Re e[s_x]\}^2}{\cos^2\phi} - \frac{\{\Im m[s_x]\}^2}{\sin^2\phi} = \frac{t^2}{\rho^2} - \left(\frac{t^2}{\rho^2} - s_0^2\right) = s_0^2.$$

Therefore,

$$\frac{\{\Re e[s_x]\}^2}{(s_0 \cos\phi)^2} - \frac{\{\Im m[s_x]\}^2}{(s_0 \sin\phi)^2} = 1,$$

which is the equation of a hyperbola which crosses the real axis at $\Re e[s_x] = s_0 \cos\phi$ and has asymptotes $\Im m[s_x] = \pm \tan\phi \Re e[s_x]$. The hyperbola is shown in Figure 4.2.1.

§4.6

The source on the wire can be written as a linear superposition of impulsive point sources:

$$\mathbf{J}(\mathbf{r}', t') = \hat{z} I \int_{t''=0}^{T} \int_{z''=-\ell/2}^{\ell/2} \delta(\rho') \delta(z' - z'') \delta(t' - t'') \, dz'' dt''.$$

The response at $\rho = a$ is therefore the integration of the responses due to the individual point sources, i.e. an integration of the three-dimensional Green's function for the Helmholtz equation,

$$g(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta(t - t' - s_0 |\mathbf{r} - \mathbf{r}'|)}{4\pi |\mathbf{r} - \mathbf{r}'|},$$

where $(\mathbf{r}, t)$ =observation point, and $(\mathbf{r}', t')$ = source point. The effect of a source at $(\mathbf{r}', t')$ is thus seen at $t = t' + s_0 |\mathbf{r} - \mathbf{r}'|$. The first arrival at

$\rho = a$ will be due to the point source at $z' = 0, t' = 0$, and will appear at $t = 0 + s_0 a$. The final arrival will be due to the point sources at $z' = \pm\ell/2, t' = T$, and will arrive at $t = T + s_o\sqrt{a^2 + (\ell/2)^2}$.

So, the field is nonzero for $s_0 a \leq t \leq T + s_0\sqrt{a^2 + (\ell/2)^2}$.

§4.7

Equation (4.2.20) states

$$\tilde{g}(\omega, r) = \frac{i\omega}{8\pi^2} \int_{-\infty}^{\infty} ds_y \int_{-\infty}^{\infty} ds_x \frac{e^{i\omega(s_x\rho + s_z|z|)}}{s_z}. \tag{4.2.20}$$

Interchanging the order of integration,

$$\tilde{g}(\omega, r) = \frac{i\omega}{8\pi^2} \int_{-\infty}^{\infty} ds_x\, e^{i\omega s_x \rho} \int_{-\infty}^{\infty} ds_y \frac{e^{i\omega\sqrt{s_0^2 - s_x^2 - s_y^2}\,|z|}}{\sqrt{s_0^2 - s_x^2 - s_y^2}}.$$

Assume $\omega > 0$, real. Then for the solution to decay as $|z| \to \infty$, we must have $\Im m(s_0^2 - s_x^2 - s_y^2)^{\frac{1}{2}} > 0$. Thus, there is a branch cut along the contour $\Im m(s_0^2 - s_x^2 - s_y^2)^{\frac{1}{2}} = 0$, or $s_0^2 - s_x^2 = s_y^2$. The branch points are at $s_y = \pm\sqrt{s_0^2 - s_x^2}$, which are located on either the real or imaginary axis. A typical case is shown in Figure 1.

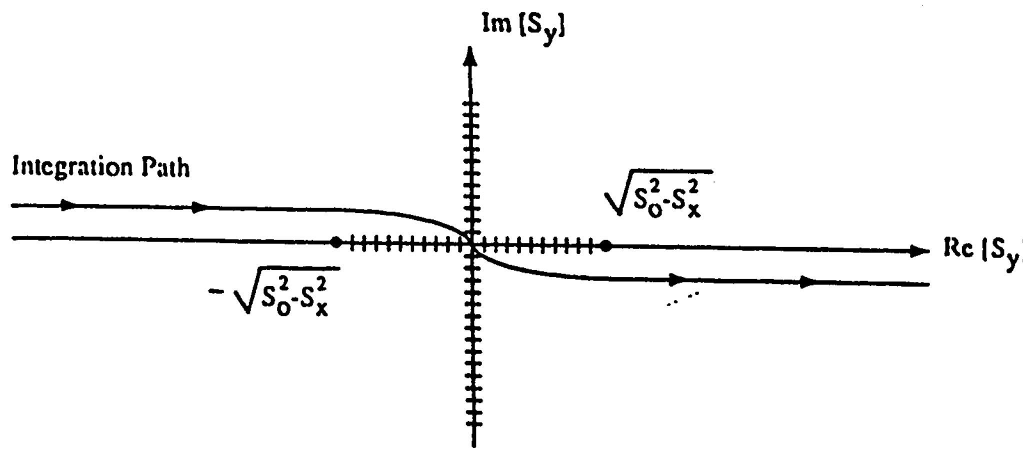

Figure 1 for Exercise Solution 4.7

Since $e^{i\omega\sqrt{s_0^2 - s_x^2 - s_y^2}}$ approaches zero exponentially as $s_y \to \pm\infty$, the path of integration can be deformed to the contour shown in Figure 2.

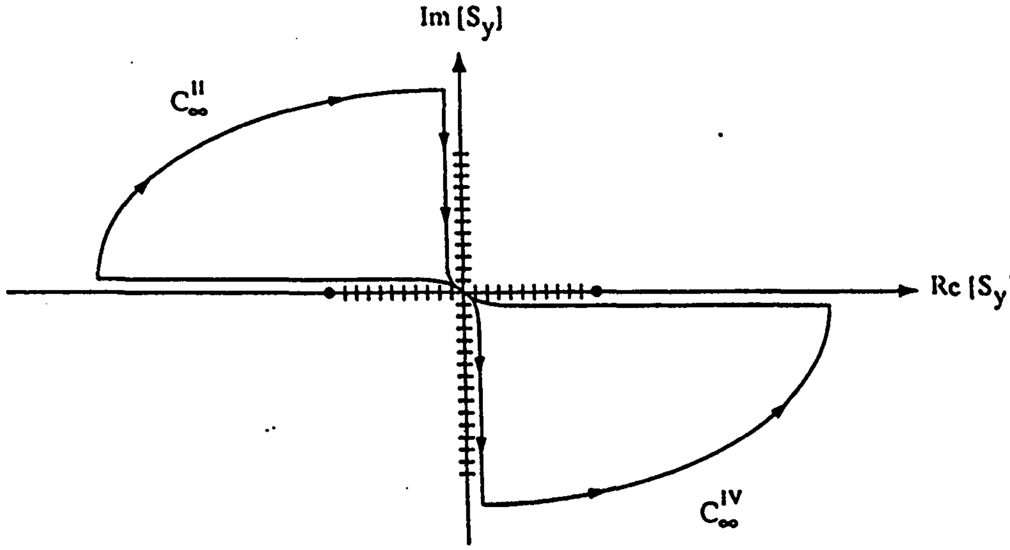

Figure 2 for Exercise Solution 4.7

The deformation is permitted by Cauchy's theorem, and the integral over C_∞^{II} and C_∞^{IV} vanishes by Jordan's lemma. This leaves only the integral along the imaginary axis from $s_y = +i\infty$ to $s_y = -i\infty$.

Changing variables: $s_y = -iq, \quad q = is_y, \quad dq = ids_y,$

$$\tilde{g}(\omega, r) = \frac{\omega}{8\pi^2} \int_{-\infty}^{\infty} ds_x \, e^{i\omega s_x \rho} \int_{-\infty}^{\infty} dq \, \frac{e^{i\omega\sqrt{s_0^2+q^2-s_x^2}\,|z|}}{\sqrt{s_0^2+q^2-s_x^2}}.$$

Changing the order of integration once again,

$$\tilde{g}(\omega, r) = \frac{\omega}{8\pi^2} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} ds_x \, \frac{e^{i\omega(s_x\rho + s_z|z|)}}{s_z},$$

$$s_z = \sqrt{s_0^2 + q^2 - s_x^2}. \tag{4.2.21}$$

Recall that we assumed $\omega > 0$, real, to prove the above equation. A parallel result holds for $\omega < 0$, real, which proves (4.2.21).

§4.8

The Green's function for a line source satisfies

$$\left(\nabla^2 - s_0^2 \frac{\partial^2}{\partial t^2}\right) g(t, \rho) = -\delta(x)\delta(y)\delta(t)$$

$$= \int_{-\infty}^{\infty} -\delta(x)\delta(y)\delta(z - z')\delta(t)dz'.$$

From (4.2.26), the solution to

$$\left(\nabla^2 - s_0^2\frac{\partial^2}{\partial t^2}\right)h(t,r) = -\delta(x)\delta(y)\delta(z)\delta(t) \quad \text{is}$$

$$h(t,r) = \frac{\delta(t-s_0r)}{4\pi r}, \quad \text{where } r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}.$$

Applying linear superposition, the solution for $g(t,\rho)$ is thus

$$\begin{aligned} g(t,\rho) &= \int_{-\infty}^{\infty} \frac{\delta\left(t - s_0\sqrt{\rho^2+z'^2}\right)}{4\pi\sqrt{\rho^2+z'^2}}\,dz' \\ &= 2\int_0^{\infty} \frac{\delta\left(t - s_0\sqrt{\rho^2+z'^2}\right)dz'}{4\pi\sqrt{\rho^2+z'^2}}. \end{aligned}$$

The geometry is shown in the figure.

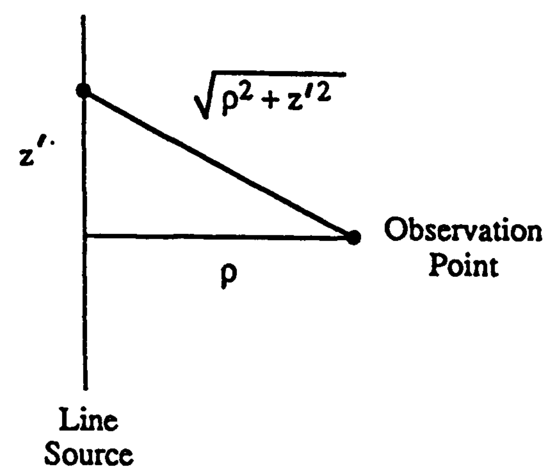

Figure for Exercise Solution 4.8

Make a change of variables: $u = s_0\sqrt{\rho^2+z'^2}$, $z' = \sqrt{\left(\frac{u}{s_0}\right)^2 - \rho^2}$.
Then

$$du = \frac{s_0 z'}{\sqrt{\rho^2+z'^2}}dz' = \frac{\sqrt{u^2 - s_0^2\rho^2}\,dz'}{\sqrt{\rho^2+z'^2}}.$$

So,

$$\frac{dz'}{\sqrt{\rho^2+z'^2}} = \frac{du}{\sqrt{u^2-s_0^2\rho^2}}.$$

Thus,

$$g(t,\rho) = \frac{1}{2\pi}\int_{s_0\rho}^{\infty} \frac{\delta(t-u)du}{\sqrt{u^2 - s_0^2\rho^2}} = \frac{H(t - s_0\rho)}{2\pi\sqrt{t^2 - s_0^2\rho^2}},$$

which is the line source Green's function (4.2.15).

§4.9

This problem is very similar to problem (4.7), the main difference being the presence of the reflection coefficient R_{12}^{TM} in the integrand. The solution thus proceeds along the same lines, but we now must consider the locations in the complex s_y plane of the singularities of

$$R_{12}^{TM} = \frac{\epsilon_2 s_{1z} - \epsilon_1 s_{2z}}{\epsilon_2 s_{1z} + \epsilon_1 s_{2z}},$$

where

$$s_{iz} = \sqrt{s_i^2 - s_x^2 - s_y^2}, \quad s_i = \sqrt{\mu_i \epsilon_i}, \ i = 1,2.$$

Taking the $\Im m[s_{iz}] = 0$ branch cut for $i = 1,2$ gives two branch cuts similar to that in problem (4.7). The cuts extend along the real and imaginary axes of the s_y plane.

All that remains is to locate the poles of R_{12}^{TM} in the complex s_y plane. The condition for a pole is

$$\epsilon_2 s_{1z} + \epsilon_1 s_{2z} = 0.$$

In the Cagniard-de Hoop method, we are concerned with real, positive permittivities, so ϵ_1 and ϵ_2 are both real and positive. Thus, to have a pole, the real and imaginary parts of s_{1z} and s_{2z} must be opposite in sign. The integral in (4.2.45 a) is carried out on the top Riemann sheet of s_{1z} and s_{2z}, where $\Im m[s_{iz}] > 0$, $i = 1,2$. So, there are no poles in the second or fourth quadrants of the complex s_y plane.

Thus, the contour deformation to the complex s_y axis can be carried out as shown in problem (4.7).

For more discussion of the poles and zeros of R_{12}^{TM}, see §§ 2.6.2.

§4.10

(a) The problem geometry is similar to that shown in Figure 2.3.1, but with an $\hat{x}$ oriented dipole located at $z = 0$ and a single interface at $z = -d$.

For a horizontal electric dipole, the primary field in the $\hat{z}$ direction is given by

$$E_z = \frac{iIl}{4\pi\omega\epsilon}\frac{\partial^2}{\partial z \partial x}\frac{e^{ikr}}{r}, \tag{2.3.7a}$$

$$H_z = -\frac{Il}{4\pi}\frac{\partial}{\partial y}\frac{e^{ikr}}{r}. \tag{2.3.7b}$$

Using the Weyl identity,

$$\frac{e^{ikr}}{r} = \frac{i}{2\pi}\iint_{-\infty}^{\infty} dk_x dk_y\, \frac{e^{ik_x x+ik_y y+ik_z|z|}}{k_z}, \tag{2.2.27}$$

$$E_z = \frac{\pm Il}{8\pi^2\omega\epsilon}\iint_{-\infty}^{\infty} dk_x dk_y k_x e^{ik_x x+ik_y y+ik_z|z|},$$

$$H_z = \frac{Il}{8\pi^2}\iint_{-\infty}^{\infty} dk_x dk_y\, \frac{k_y}{k_z}\, e^{ik_x x+ik_y y+ik_z|z|},$$

where the $\pm$ sign in E_z is for $z \gtrless 0$.

In the presence of a half-space,

$$E_{1z} = \frac{Il}{8\pi^2\omega\epsilon_1}\iint_{-\infty}^{\infty} dk_x dk_y k_x \left[\pm e^{ik_x x+ik_y y+ik_{1z}|z|} - R_{12}^{TM} e^{ik_x x+ik_y y+ik_{1z}(z+2d)}\right]$$

$$H_{1z} = \frac{Il}{8\pi^2}\iint_{-\infty}^{\infty} dk_x dk_y \frac{k_y}{k_{1z}}\left[e^{ik_x x+ik_y y+ik_{1z}|z|} + R_{12}^{TE} e^{ik_x x+ik_y y+ik_{1z}(z+2d)}\right].$$

From (2.3.17a), $\tilde{E}_{1x} = \frac{1}{k_x^2+k_y^2}\left[\frac{\partial^2}{\partial x\partial z}\tilde{E}_{1z} + i\omega\mu\frac{\partial}{\partial y}\tilde{H}_{1z}\right]$, where $\tilde{E}$ denotes the integrand in the spectral representation above.
So,

$$E_{1x} = \frac{Il}{8\pi^2}\iint_{-\infty}^{\infty}\frac{dk_x dk_y}{k_x^2+k_y^2}\left\{\frac{k_x^2 k_{1z}}{\omega\epsilon_1}\left[-e^{ik_x x+ik_y y+ik_{1z}|z|} + R_{12}^{TM} e^{ik_x x+ik_y y+ik_{1z}(z+2d)}\right] - \frac{\omega\mu_1 k_y^2}{k_{1z}}\left[e^{ik_x x+ik_y y+ik_{1z}|z|} + R_{12}^{TE} e^{ik_x x+ik_y y+ik_{1z}(z+2d)}\right]\right\}.$$

Combining some terms above, this simplifies to

$$\begin{aligned} E_{1x} =& \frac{-Il\omega\mu_1}{8\pi^2}\iint_{-\infty}^{\infty} dk_x dk_y\, \frac{e^{ik_x x+ik_y y}}{k_x^2+k_y^2}\left\{\left[\frac{k_x^2 k_{1z}}{k_1^2} + \frac{k_y^2}{k_{1z}}\right] e^{ik_{1z}|z|}\right. \\ &\left. + \left(\frac{-k_x^2 k_{1z}}{k_1^2}\, R_{12}^{TM} + \frac{k_y^2}{k_{1z}}\, R_{12}^{TE}\right) e^{ik_{1z}(z+2d)}\right\} \\ =& \frac{-Il}{8\pi^2\omega\epsilon_1}\iint_{-\infty}^{\infty} dk_x dk_y \frac{k_1^2-k_x^2}{k_{1z}} e^{ik_x x+ik_y y}\left[e^{ik_{1z}|z|} + \tilde{R} e^{ik_{1z}(z+2d)}\right], \end{aligned}$$

where

$$\tilde{R} = \frac{1}{k_x^2 + k_y^2} \cdot \frac{k_1^2 k_y^2 R_{12}^{TE} - k_x^2 k_{1z}^2 R_{12}^{TM}}{k_1^2 - k_x^2}.$$

The reflection coefficients R_{12}^{TM} and R_{12}^{TE} are given by

$$R_{12}^{TM} = \frac{\epsilon_2 k_{1z} - \epsilon_1 k_{2z}}{\epsilon_2 k_{1z} + \epsilon_1 k_{2z}}, \qquad (2.1.14a)$$

$$R_{12}^{TE} = \frac{\mu_2 k_{1z} - \mu_1 k_{2z}}{\mu_2 k_{1z} + \mu_1 k_{2z}}, \qquad (2.1.13a)$$

$$k_{iz} = \sqrt{k_i^2 - k_x^2 - k_y^2}.$$

When $k_x = ik_y$, $k_{iz} = k_i = \sqrt{\mu_i \epsilon_i}$. In this case,

$$R_{12}^{TM} = \frac{\epsilon_2\sqrt{\mu_1\epsilon_1} - \epsilon_1\sqrt{\mu_2\epsilon_2}}{\epsilon_2\sqrt{\mu_1\epsilon_1} + \epsilon_1\sqrt{\mu_2\epsilon_2}} \cdot \frac{\sqrt{\frac{\mu_1\mu_2}{\epsilon_1\epsilon_2}}}{\sqrt{\frac{\mu_1\mu_2}{\epsilon_1\epsilon_2}}}$$

$$= \frac{\mu_1\sqrt{\mu_2\epsilon_2} - \mu_2\sqrt{\mu_1\epsilon_1}}{\mu_1\sqrt{\mu_2\epsilon_2} + \mu_2\sqrt{\mu_1\epsilon_1}} = -R_{12}^{TE}.$$

Thus,

$$\lim_{k_x \to ik_y} \tilde{R} = \lim_{k_x \to ik_y} \frac{1}{k_x^2 + k_y^2} \cdot \frac{k_1^2 k_y^2 + k_{1z}^2 k_x^2}{k_1^2 - k_x^2} \cdot R_{12}^{TE}$$

$$= \frac{k_1^2}{k_1^2 + k_y^2} R_{12}^{TE}, \quad \text{which is a finite number.}$$

Therefor, there is no pole at $k_x = ik_y$.

(b) We can write

$$E_{1x} = \frac{-Il}{8\pi^2\omega\epsilon_1}\left(k_1^2 + \frac{\partial^2}{\partial x^2}\right) \iint_{-\infty}^{\infty} dk_x dk_y \frac{e^{ik_x x + ik_y y}}{k_{1z}} \left[e^{ik_{1z}|z|} + \tilde{R}e^{ik_{1z}(z+2d)}\right].$$

Let $k_x = \beta_x \cos\phi - \beta_y \sin\phi$, $k_y = \beta_x \sin\phi + \beta_y \cos\phi$, a coordinate rotation, where $x = \rho\cos\phi$, $y = \rho\sin\phi$. Then

$$E_{1x} = \frac{-Il}{8\pi^2\epsilon_1}\left(k_1^2 + \frac{\partial^2}{\partial x^2}\right) \iint_{-\infty}^{\infty} d\beta_x d\beta_y \frac{e^{i\beta_x\rho}}{\omega\beta_{1z}} \left[e^{i\beta_{1z}|z|} + \tilde{R}e^{i\beta_{1z}(z+2d)}\right],$$

where

$$\tilde{R} = \frac{1}{\beta_x^2 + \beta_y^2} \cdot \frac{k_1^2(\beta_x \sin\phi + \beta_y \cos\phi)^2 R_{12}^{TE} - \beta_{1z}^2(\beta_x\cos\phi - \beta_y\sin\phi)^2 R_{12}^{TM}}{k_1^2 - (\beta_x\cos\phi - \beta_y\sin\phi)^2},$$

and

$$\beta_{1z} = \sqrt{k_1^2 - \beta_x^2 - \beta_y^2}\,.$$

(c) Let $\beta_x = \omega s_x, \quad \beta_y = \omega s_y, \quad s_{iz} = \sqrt{s_i^2 - s_x^2 - s_y^2}, \quad s_i = \frac{1}{c_i}.$

Then

$$E_{1z} = \frac{-Il}{8\pi^2\epsilon_1}\left(k_1^2 + \frac{\partial^2}{\partial x^2}\right)\iint_{-\infty}^{\infty} ds_x ds_y \frac{e^{i\omega s_x \rho}}{s_{1z}}\left[e^{i\omega s_{1z}|z|} + \tilde{R}\, e^{i\omega s_{1z}(z+2d)}\right].$$

Then, deform the s_y integral to the imaginary axis, and let $s_y = -iq$, so that

$$E_{1z} = \frac{-Il}{8\pi^2\epsilon_1}\left(k_1^2 + \frac{\partial^2}{\partial x^2}\right)(-i)\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} ds_x \frac{e^{i\omega s_x \rho}}{s_{1z}}\left[e^{i\omega s_{1z}|z|} + \tilde{R} e^{i\omega s_{1z}(z+2d)}\right],$$

where

$$s_{iz} = \sqrt{s_i^2 + q^2 - s_x^2}\,,$$

$$\tilde{R} = \frac{1}{(s_x^2 - q^2)} \cdot \frac{s_1^2(s_x \sin\phi - iq\cos\phi)^2 R_{12}^{TE} - s_{1z}^2(s_x\cos\phi + iq\sin\phi)^2\, R_{12}^{TM}}{s_1^2 - (s_x\cos\phi + iq\sin\phi)^2},$$

$$R_{12}^{TE} = \frac{\mu_2 s_{1z} - \mu_1 s_{2z}}{\mu_2 s_{1z} + \mu_1 s_{2z}}, \qquad R_{12}^{TM} = \frac{\epsilon_2 s_{1z} - \epsilon_1 s_{2z}}{\epsilon_2 s_{1z} + \epsilon_1 s_{2z}}.$$

For the direct field term,

$$P_D = -i\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} ds_x \frac{e^{i\omega(s_x\rho + s_{1z}|z|)}}{s_{1z}}.$$

Let

$$t = s_x\rho + s_{1z}|z|, \quad s_x = \frac{t}{r}\sin\theta \pm \cos\theta\left[\alpha_1^2 - \frac{t^2}{r^2}\right]^{\frac{1}{2}},$$

where

$$\theta = \sin^{-1}\left(\frac{\rho}{r}\right), \quad r = \sqrt{z^2 + \rho^2}, \quad \alpha_1 = \sqrt{s_1^2 + q^2}.$$

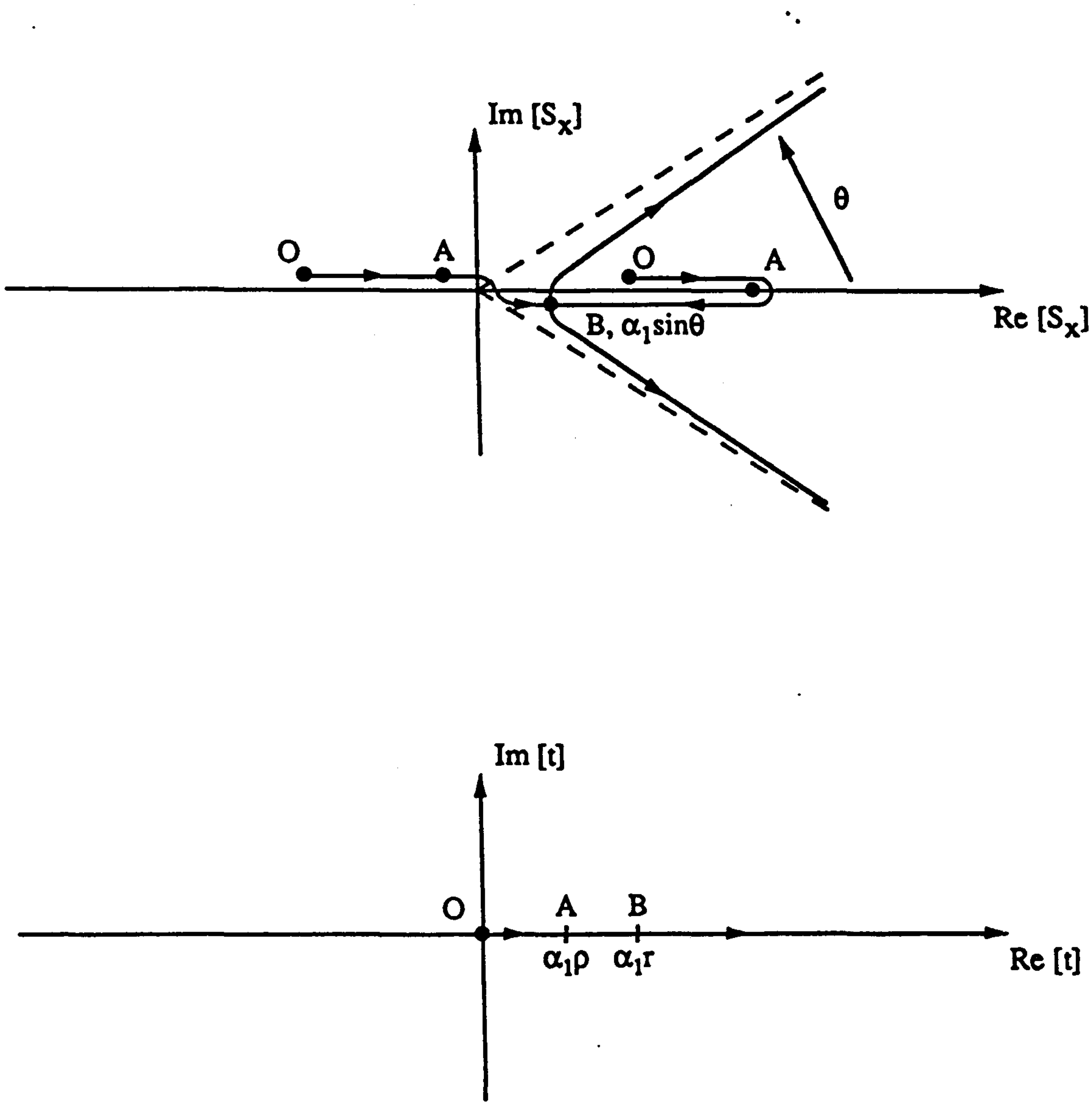

Figure for Exercise Solution 4.10

The mapping between s_x and the real t axis is given in the figure.

For the reflected field,

$$P_R = -i \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} ds_x \frac{\tilde{R}}{s_{1z}} e^{i\omega(s_x\rho + s_{1z}(z+2d))} .$$

Let

$$t = s_x\rho + s_{1z}(z+2d),$$
$$s_x = \frac{t}{r_I}\sin\theta_I \pm \cos\theta_I \left[\alpha_1^2 - \frac{t^2}{r_I^2}\right]^{\frac{1}{2}},$$

where

$$r_I = \sqrt{(z+2d)^2+\rho^2}, \quad \theta_I = \sin^{-1}\left(\frac{\rho}{r_I}\right), \quad \alpha_1 = \sqrt{s_1^2+q^2}.$$

The geometry is shown in Figure 2.6.2.

The mapping is the same as in the figure, with $\theta \to \theta_I, \quad r \to r_I$.

(d) The direct field term has a branch cut only for s_{1z}, so the integration path is deformed from the SIP to P_U and P_L as shown in Figure 4.2.5, but without the excursion around $s_x = \alpha_2$.

$$P_D = -i \int_{-\infty}^{\infty} dq \int_{\sqrt{s_1^2+q^2}r}^{\infty} dt \left[\left(\frac{ds_x}{dt}\frac{1}{s_{1z}}\right)_L - \left(\frac{ds_x}{dt}\frac{1}{s_{1z}}\right)_U\right] e^{i\omega t},$$

where

$$\frac{ds_x}{dt}\cdot\frac{1}{s_{1z}} = \frac{\pm i}{[t^2-(s_1^2+q^2)r^2]^{\frac{1}{2}}}.$$

So,

$$P_D(\omega,\mathbf{r}) = 2\int_{-\infty}^{\infty} dq \int_{\sqrt{s_1^2+q^2}r}^{\infty} dt \frac{e^{i\omega t}}{[t^2-(s_1^2+q^2)r^2]^{\frac{1}{2}}}$$

$$= \int_{s_1 r}^{\infty} dt\, e^{i\omega t} \int_{-\sqrt{\frac{t^2}{r^2}-s_1^2}}^{+\sqrt{\frac{t^2}{r^2}-s_1^2}} dq \frac{2}{[t^2-(s_1^2+q^2)r^2]^{\frac{1}{2}}}$$

$$= \frac{2\pi}{r}\int_{s_1 r}^{\infty} dt e^{i\omega t} \equiv \int_{-\infty}^{\infty} dt e^{i\omega t} P_D(t,\mathbf{r}).$$

Therefore,

$$P_D(t,\mathbf{r}) = \frac{2\pi}{r} H(t - s_1 r).$$

For the reflected field term, there are two branch cuts, one for s_{1z} and one for s_{2z}, with branch points at $\pm\sqrt{s_1^2+q^2}$ and $\pm\sqrt{s_2^2+q^2}$, respectively. If $s_2 < s_1$ (i.e. the lower half-space is optically less dense than the upper), then the integration path must be deformed as shown in Figure 4.2.5.

Then,

$$P_R = -i \int_{-\infty}^{\infty} dq \int_{\tau}^{\infty} dt \left[\left(\frac{ds_x}{dt} \frac{\tilde{R}}{s_{1z}} \right)_L - \left(\frac{ds_x}{dt} \frac{\tilde{R}}{s_{1z}} \right)_U \right] e^{i\omega t},$$

$$\tau = \sqrt{s_2^2+q^2}\,\rho + \sqrt{s_1^2 - s_2^2}\,(z+2d).$$

As before,

$$\frac{ds_x}{dt}\frac{1}{s_{1z}} = \begin{cases} \dfrac{\pm 1}{r_I\left[(s_1^2+q^2) - \frac{t^2}{r_I^2}\right]^{\frac{1}{2}}}, & \frac{t}{r_I} < \sqrt{s_1^2+q^2} \\ \dfrac{\pm i}{r_I\left[\frac{t^2}{r_I^2} - (s_1^2+q^2)\right]^{\frac{1}{2}}}, & \frac{t}{r_I} > \sqrt{s_1^2+q^2} \end{cases}.$$

Also,

$$P_R(\omega,\mathbf{r}) = \int_{\tau_0}^{\infty} dt e^{i\omega t} \int_{-q_0}^{+q_0} dq(-i) \left[\left(\frac{ds_x}{dt} \frac{\tilde{R}}{s_{1z}} \right)_L - \left(\frac{ds_x}{dt} \frac{\tilde{R}}{s_{1z}} \right)_U \right],$$

$$\tau_0 = s_2\rho + \sqrt{s_1^2 - s_2^2}(z+2d),$$

$$q_0 = \frac{1}{\rho}\left\{ \left[t - \sqrt{s_1^2 - s_2^2}\,(z+2d) \right]^2 - s_2^2\rho^2 \right\}^{\frac{1}{2}}.$$

Therefore

$$P_R(t,\mathbf{r}) = H(t-\tau_0) \int_{-q_0}^{q_0} dq(-i) \left[\left(\frac{ds_x}{dt} \frac{\tilde{R}}{s_{1z}} \right)_L - \left(\frac{ds_x}{dt} \frac{\tilde{R}}{s_{1z}} \right)_U \right].$$

Then

$$E_{1x}(\omega,\mathbf{r}) = \frac{-I(\omega)l}{8\pi^2\epsilon_1} \left(\frac{\omega^2}{c_1^2} + \frac{\partial^2}{\partial x^2} \right) \left[P_D(\omega,\mathbf{r}) + P_R(\omega,\mathbf{r}) \right],$$

so,

$$E_{1x}(t,\mathbf{r}) = \frac{-I(t)l}{8\pi^2\epsilon_1} * \left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right] \left[P_D(t,\mathbf{r}) + P_R(t,\mathbf{r}) \right].$$

$$\text{Since} \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E_{1x}(t,\mathbf{r})=0$$

outside the source region, this can also be written as

$$E_{1x}(t,\mathbf{r})=\frac{I(t)l}{8\pi^2\epsilon_1}*\left[\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right][P_D(t,\mathbf{r})+P_R(t,\mathbf{r})].$$

§4.11

With $d_1 = y = 0$, (4.3.3) is

$$E_{1z}^{Rm}=\frac{-\omega\mu_1 I}{4\pi}\int_{-\infty}^{\infty} ds_x \frac{1}{s_{1y}} T_{12}R_{23}^m R_{21}^{m-1}T_{21}e^{i\omega[s_x x+2ms_{2y}d_2]}.$$

Notice that there are branch points at $s_x = \pm s_1, \pm s_2$, and $\pm s_3$.

To make this look like a Fourier inverse transform, let

$$t = s_x x + 2m\, s_{2y}\, d_2.$$

The inverse of this transformation is

$$s_x=\frac{t}{\rho_m}\cos\phi_m \pm \sin\phi_m\left(s_2^2-\frac{t^2}{\rho_m^2}\right)^{\frac{1}{2}},$$

where

$$\rho_m=\sqrt{x^2+(2md_2)^2}, \quad \text{and} \quad \phi_m=\cos^{-1}\left(\frac{x}{\rho_m}\right).$$

The contour P on the complex s_x plane that is the mapping of the real-t axis will look like Figure 4.2.3 with ϕ_I in the figure replaced by ϕ_m. Also, the hyperbolic portion of the contour P will meet the real s_x axis at $s_x = s_2\cos\phi_m$. If s_1 and s_3 are less than s_2, as would be the case for a slab wave guide, then it may be necessary to detour the integration contour P around the branch points at $s_x = s_1$ and s_3.

Assuming for definiteness that $s_1 \le s_3$, one obtains

$$E_{1z}^{Rm}=\frac{-\omega\mu_1 I}{4\pi}\int_{\tau}^{\infty} dt\left[\left(\frac{ds_x}{dt}\frac{T_{12}R_{23}^m R_{21}^{m-1}T_{21}}{s_{1y}}\right)_L - \left(\frac{ds_x}{dt}\frac{T_{12}R_{23}^m R_{23}^{m-1}T_{21}}{s_{1y}}\right)_U\right]e^{i\omega t},$$

where

$$\tau=\min\left[s_2\cos\phi_m,\ s_1 x+2md_2\sqrt{s_2^2-s_1^2}\right].$$

Now,

$$\frac{1}{s_{1y}}\frac{ds_x}{dt} = \frac{\pm i s_{2y}}{s_{1y}(t^2 - s_2^2\rho_m^2)^{\frac{1}{2}}}.$$

Also, $T_{12}R_{23}^m R_{21}^{m-1}T_{21}$ takes on conjugate values on the different Riemann sheets, so

$$\left(\frac{ds_x}{dt}\frac{T_{12}R_{23}^m R_{21}^{m-1}T_{21}}{s_{1y}}\right)_L - \left(\frac{ds_x}{dt}\frac{T_{12}R_{23}^m R_{21}^{m-1}T_{21}}{s_{1y}}\right)_U$$
$$=2i\Im m\left[\frac{ds_x}{dt}\frac{T_{12}R_{23}^m R_{21}^{m-1}T_{21}}{s_{1y}}\right]_L.$$

Thus, the inverse Fourier transform of $\frac{E_{1z}^{Rm}}{i\omega I(\omega)}$ is

$$f(t) = \frac{\mu_1}{2\pi}\Im m\left[\frac{ds_x}{dt}\frac{T_{12}R_{23}^m R_{21}^{m-1}T_{21}}{s_{1y}}\right]_L H(t-\tau),$$

where τ is given above.

Then,

$$E_{1z}^{Rm} = -\frac{\partial}{\partial t}I(t) * f(t).$$

§4.12

(a) Using the transformation (4.4.7), it is easily shown that $\frac{d\tau}{ds_{1z}}\frac{ds_\rho}{d\tau}\frac{s_\rho}{s_{1z}} = -1$, so that (4.4.11) maybe written as

$$F(t) = \frac{-2}{\pi}\Im m\int\limits_C \frac{ds_{1z}}{[(t - s_{1z}z)^2 - s_\rho^2\rho^2]^{\frac{1}{2}}},$$

where C is the image of the segment of the real τ axis $s_1 r < \tau < t$ on the complex s_{1z} plane.

First, note that $F(t) = 0$ for $t < s_1 r$. This is because $F(t)$ is an integral to be carried out over the P contour defined by (4.4.7), with τ real and $\tau \geq s_1 r$. This is also required by causality.

For $t > s_1 r$, $F(t)$ can be written in closed form. Using $s_\rho^2 = s_1^2 - s_{1z}^2$,

$$F(t) = -\frac{2}{\pi}\Im m\int\limits_C \frac{ds_{1z}}{[(s_{1z}r - t\cos\theta)^2 + (t^2 - s_1^2 r^2)\sin^2\theta]^{\frac{1}{2}}},$$

where

$$\cos\theta = \frac{z}{r},\quad \sin\theta = \frac{\rho}{r},\quad r = \sqrt{\rho^2 + z^2}.$$

Integrating,

$$F(t) = \frac{-2}{\pi}\cdot\frac{1}{r}\Im m\left\{\sinh^{-1}\left[\frac{s_{1z}r - t\cos\theta}{(t^2 - s_1^2 r^2)^{\frac{1}{2}}\sin\theta}\right]\Big|_{\tau=s_1 r}^{\tau=t}\right\}.$$

From (4.4.8b),

$$s_{1z} = \frac{\tau}{r}\cos\theta \mp i\sin\theta\left(\frac{\tau^2}{r^2} - s_1^2\right)^{\frac{1}{2}}. \tag{4.4.8b}$$

So,

$$\begin{aligned} F(t) &= -\frac{2}{\pi r}\Im m\left\{\sinh^{-1}\left[\frac{-i\sin\theta(t^2 - s_1^2r^2)^{\frac{1}{2}}}{\sin\theta(t^2 - s_1^2r^2)^{\frac{1}{2}}}\right]\right. \\ &\qquad\qquad \left. - \sinh^{-1}\left[\frac{(s_1r - t)\cos\theta}{(t^2 - s_1^2r^2)^{\frac{1}{2}}\sin\theta}\right]\right\} \\ &= \frac{-2}{\pi r}\Im m\left\{\sinh^{-1}[-i] - \sinh^{-1}\left[\frac{(s_1r - t)\cos\theta}{(t^2 - s_1^2r^2)^{\frac{1}{2}}\sin\theta}\right]\right\}. \end{aligned}$$

For $t > s_1 r$ the second term is real, so

$$F(t) = \frac{-2}{\pi r}\cdot\Im m\left\{\sinh^{-1}(-i)\right\} = \frac{1}{r},$$

since $\sinh^{-1}(-i) = -\frac{i\pi}{2}$. So, we have

$$F(t) = \frac{H(t - s_1 r)}{r}.$$

To relate this to (4.2.25), note that

$$F(\omega) = \frac{1}{2}\int_P ds_\rho \frac{s_\rho}{s_{1z}} H_0^{(1)}(\omega s_\rho \rho) e^{i\omega s_{1z} z}. \tag{4.4.6}$$

From (2.2.31), this is equal to $\frac{e^{ik_1 r}}{r \cdot i\omega}$.

According to (4.2.25), the inverse Fourier transform of the above is $\frac{H(t-s_1 r)}{r}$, which is the result obtained above for $F(t)$.

(b) The time-harmonic response of a VED on top of a half-space is given by

$$E_{1z} = \frac{-Il}{8\pi\omega\epsilon_1}\int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{1z}} H_0^{(1)}(k_\rho\rho)\left[e^{ik_{1z}|z|} + R_{12}^{TM} e^{ik_{1z}z + 2ik_{1z}d}\right], \tag{2.3.5}$$

where $z = -d$ is the location of the interface, and the source is at $z = 0$. Working with the reflected field only, and letting $k_\rho = \omega s_\rho$, $k_{1z} = \omega s_{1z}$, we have

$$E_{1z}^R = \frac{-I\omega^2 l}{8\pi\epsilon_1}\int_{-\infty}^{\infty} ds_\rho \frac{s_\rho^3}{s_{1z}} H_0^{(1)}(\omega s_\rho \rho) e^{i\omega s_{1z}(z+2d)} \cdot R_{12}^{TM}.$$

Deforming to the P contour, as shown in Figure (4.2.5), on page 226 of the text,

$$\frac{E^R_{1z}}{-\omega^2 I(\omega)\cdot l} = \frac{1}{8\pi\epsilon_1}\int_P ds_\rho \frac{s_\rho^3}{s_{1z}} H_0^{(1)}(\omega s_\rho \rho) e^{i\omega s_{1z}(z+2d)}\cdot R_{12}^{TM}.$$

Here, the P contour is defined by

$$s_\rho \rho + s_{1z}(z+2d) = \tau,$$

where τ is real. Consequently,

$$s_\rho = \frac{\tau}{r_I}\sin\theta_I \pm i\cos\theta_I\left(\frac{\tau^2}{r_I^2} - s_1^2\right)^{\frac{1}{2}},$$
$$s_{1z} = \frac{\tau}{r_I}\cos\theta_I \mp i\sin\theta_I\left(\frac{\tau^2}{r_I^2} - s_1^2\right)^{\frac{1}{2}},$$

where $r_I = \sqrt{\rho^2 + (z+2d)^2}$, $\theta_I = \sin^{-1}\left(\frac{\rho}{r_I}\right)$.

The inverse Fourier transform of $F(\omega) = \frac{E^R_{1z}(\omega)}{-\omega^2 I(\omega)\cdot l}$ is then

$$F(t) = \frac{1}{16\pi^2\epsilon_1}\int_P ds_\rho \frac{s_\rho^3}{s_{1z}} R_{12}^{TM}(s_\rho)\int_{-\infty}^{\infty} d\omega e^{-i\omega(t - s_{1z}(z+2d))} H_0^{(1)}(\omega s_\rho \rho)$$
$$= \frac{1}{4i\pi^2\epsilon_1}\int_P ds_\rho \frac{s_\rho^3}{s_{1z}} R_{12}^{TM}\cdot \frac{H(t - s_{1z}(z+2d) - s_\rho\rho)}{\left[(t - s_{1z}(z+2d))^2 - s_\rho^2\rho^2\right]^{\frac{1}{2}}}.$$

Folding the P contour and combining the parts associated with P_U and P_L,

$$F(t) = \frac{1}{2\pi^2\epsilon_1}\Im m\int_{\tau_0}^{t} d\tau \frac{ds_\rho}{d\tau}\frac{s_\rho^3}{s_{1z}}\frac{R_{12}^{TM}}{\left[(t - s_{1z}z)^2 - s_\rho^2\rho^2\right]^{\frac{1}{2}}}.$$

Assuming $s_2 < s_1$, as in Figure (4.2.5), τ_0 is the value of τ at the branch point for s_{2z}, i.e. at $s_\rho = s_2$, or

$$\tau_0 = s_2\rho + \sqrt{s_1^2 - s_2^2}\,(z+2d).$$

§4.13

(a) Computing $\nabla_s^2 \hat{g}(x,y,t)$ yields

$$\begin{aligned}
\nabla_s^2 \hat{g}(x,y,t) =& \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\left(\frac{2AH(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}\right) \\
=& \frac{2A}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(\frac{\delta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}+\frac{H(ct-\rho)\rho}{(c^2t^2-\rho^2)^{\frac{3}{2}}}\right)\right] \\
=& \frac{2A}{\rho}\left[\frac{-\delta(ct-\rho)c^2t^2}{(c^2t^2-\rho^2)^{\frac{3}{2}}}+\frac{\delta'(ct-\rho)\rho}{\sqrt{c^2t^2-\rho^2}}\right. \\
&\left. -\frac{\delta(ct-\rho)\rho^2}{(c^2t^2-\rho^2)^{\frac{3}{2}}}+H(ct-\rho)\cdot\frac{2\rho c^2t^2+\rho^3}{(c^2t^2-\rho^2)^{\frac{5}{2}}}\right] \\
=& 2A\left[-\delta(ct-\rho)\frac{(c^2t^2+\rho^2)}{\rho(c^2t^2-\rho^2)^{\frac{3}{2}}}+\frac{\delta'(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}\right. \\
&\left. +H(ct-\rho)\frac{(2c^2t^2+\rho^2)}{(c^2t^2-\rho^2)^{\frac{5}{2}}}\right].
\end{aligned}$$

Computing the time derivative,

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{g}(x,y,t) =& \frac{2A}{c^2}\frac{\partial}{\partial t}\left[\frac{\delta(ct-\rho)c}{\sqrt{c^2t^2-\rho^2}}-H(ct-\rho)\frac{c^2t}{(c^2t^2-\rho^2)^{\frac{3}{2}}}\right] \\
=& \frac{2A}{c^2}\left[\frac{-\delta(ct-\rho)c^3t}{(c^2t^2-\rho^2)^{\frac{3}{2}}}+\frac{\delta'(ct-\rho)\cdot c^2}{\sqrt{c^2t^2-\rho^2}}\right. \\
&\left. \frac{-\delta(ct-\rho)c^3t}{(c^2t^2-\rho^2)^{\frac{3}{2}}}+H(ct-\rho)c^2\frac{2c^2t^2+\rho^2)}{(c^2t^2-\rho^2)^{\frac{5}{2}}}\right] \\
=& 2A\left[\frac{-\delta(ct-\rho)2ct}{(c^2t^2-\rho^2)^{\frac{3}{2}}}+\frac{\delta'(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}+H(ct-\rho)\frac{(2c^2t^2+\rho^2)}{(c^2t^2-\rho^2)^{\frac{5}{2}}}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\nabla^2-\frac{1}{c^2}\frac{\partial}{\partial t^2}\right)\hat{g}(x,y,t) =& \frac{2A}{(c^2t^2-\rho^2)^{\frac{3}{2}}}\left(\frac{-c^2t^2-\rho^2}{\rho}+2ct\right)\delta(ct-\rho) \\
=& -\frac{2A\delta(ct-\rho)}{\rho}\frac{(ct-\rho)^2}{(ct+\rho)^{\frac{3}{2}}(ct-\rho)^{\frac{3}{2}}}=\frac{-2A\delta(ct-\rho)}{\rho}\cdot\frac{\sqrt{ct-\rho}}{(ct+\rho)^{\frac{3}{2}}}.
\end{aligned}$$

(b) From the sifting property of the delta function,

$$
\begin{aligned}
&-2A\lim_{\alpha\to 0}\int_0^\infty d\rho\int_{-\infty}^{\infty} dt\delta(ct-\rho)\frac{\sqrt{ct-\rho+\alpha}}{(ct+\rho+\alpha)^{\frac{3}{2}}}\\
&=-2A\lim_{\alpha\to 0}\int_0^\infty d\rho\int_{-\infty}^{\infty}\frac{du}{c}\,\frac{\delta(u-\rho)\sqrt{u-\rho+\alpha}}{(u+\rho+\alpha)^{\frac{3}{2}}}\quad (u=ct,\, du=cdt),\\
&=-2A\lim_{\alpha\to 0}\frac{\sqrt{\alpha}}{c}\int_0^\infty d\rho\frac{1}{(2\rho+\alpha)^{\frac{3}{2}}}=-2A\lim_{\alpha\to 0}\frac{\sqrt{\alpha}}{c}\cdot\frac{1}{\sqrt{\alpha}}\\
&=-\frac{2A}{c}.
\end{aligned}
$$

(c) Thus,

$$
\begin{aligned}
\left(\nabla_s^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\hat{g}(x,y,t)&=-2A\frac{\delta(ct-\rho)\sqrt{ct-\rho}}{\rho(ct+\rho)^{\frac{3}{2}}}\\
&=\frac{-2A}{c}\,\frac{\delta(t)\delta(\rho)}{\rho},\quad \text{in a distributional sense.}
\end{aligned}
$$

But from (4.5.5),

$$\left(\nabla_s^2-\frac{1}{c^2}\frac{\partial}{\partial t^2}\right)\hat{g}(x,y,t)=-\delta(x)\delta(y)\delta(t)=\frac{-\delta(\rho)\delta(t)}{2\pi\rho}.$$

So,

$$\frac{-2A}{c}\,\frac{\delta(\rho)\delta(t)}{\rho}=\frac{-\delta(\rho)\delta(t)}{2\pi\rho}\Rightarrow A=\frac{c}{4\pi}.$$

(d) Consider the general solution for $\hat{g}(x,y,t)$,

$$\hat{g}(x,y,t)=H(ct-\rho)2A\frac{\cosh\left(\sqrt{c^2t^2-\rho^2}/2c\tau\right)}{\sqrt{c^2t^2-\rho^2}}. \tag{4.5.9}$$

$\hat{g}$ is strongly peaked around $\rho=ct$, then tails off smoothly, as shown in the figure. Now, look at the Fourier transform of $\hat{g}(x,y,t)$,

$$\tilde{\hat{g}}(x,y,\omega)=\int_{\frac{\rho}{c}}^{\infty}\hat{g}(x,y,t)e^{i\omega t}\,dt.$$

For large ω, $e^{i\omega t}$ is highly oscillatory, so the only contribution to $\tilde{\hat{g}}(x,y,\omega)$ will come from $t\sim\frac{\rho}{c}$, where $\hat{g}(x,y,t)$ is rapidly varying. For t very close to $\frac{\rho}{c}$, $\hat{g}(x,y,t)$ reduces to

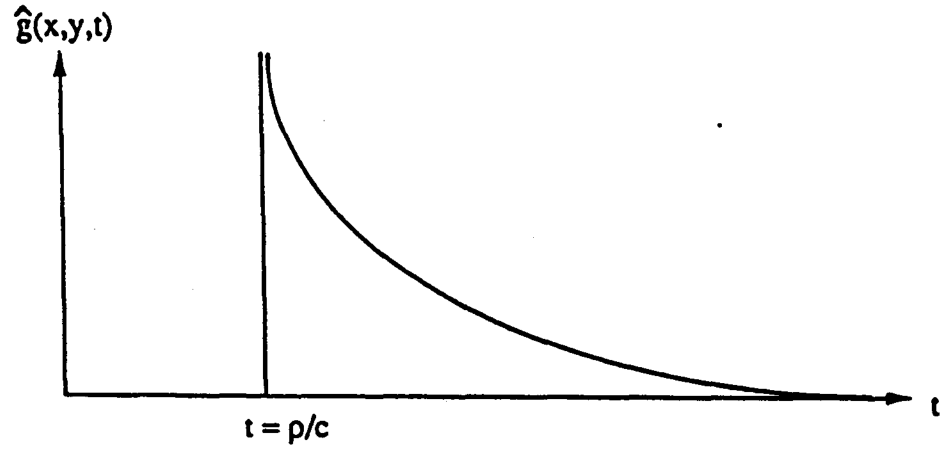

Figure for Exercise Solution 4.13

$\hat{g}(x,y,t) \simeq H(ct-\rho)\cdot\dfrac{2A}{\sqrt{c^2t^2-\rho^2}}$. Thus, for very high ω,

$\tilde{\hat{g}}(x,y,\omega)$ should be linearly proportional to A.

Also, for a single frequency component ω, Equation (4.5.5) reduces to

$$\left(\nabla_s^2+\frac{\omega^2}{c^2}+\frac{1}{4c^2\tau^2}\right)\tilde{\hat{g}}(x,y,\omega)=-\delta(x)\delta(y),$$

or

$$(\nabla_s^2+k^2)\tilde{\hat{g}}(x,y,\omega)=-\delta(x)\delta(y),$$

where

$$k^2=\frac{\omega^2}{c^2}+\frac{1}{4c^2\tau^2}.$$

Thus,

$$\tilde{\hat{g}}(x,y,\omega)=\frac{i}{4}H_0^{(1)}(k\rho).$$

Note that $k=\sqrt{\dfrac{\omega^2}{c^2}+\dfrac{1}{4c^2\tau^2}}=\dfrac{\omega}{c}\sqrt{1+\dfrac{1}{4\omega^2\tau^2}}\simeq\dfrac{\omega}{c}\left(1-\dfrac{1}{8\omega^2\tau^2}\right)$

$$=\frac{\omega}{c}-\frac{1}{8\omega c\tau}\longrightarrow\frac{\omega}{c},\quad\text{as}\quad\omega\to\infty.$$

Thus, for large ω, $\tilde{\hat{g}}(x,y,\omega)$ should be independent of τ.

So, for large ω, $\tilde{\hat{g}}$ is both proportional to A and independent of τ. This implies that A must be independent of τ.

§4.14

In four dimensions, $g(r)$ satisfies

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+\frac{\partial^2}{\partial x_4^2}\right)g(r)=-\delta(\mathbf{r})=-\delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4).$$

Taking a Fourier transform of this equation with respect to x_4,

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+k^2-k_{x_4}^2\right)\tilde{g}=-\delta(x_1)\delta(x_2)\delta(x_3),$$

where

$$\tilde{g}(x_1,x_2,x_3,k_{x_4})=\int_{-\infty}^{\infty} g(r)e^{-ik_{x_4}x_4}dx_4.$$

The solution for $\tilde{g}$ is the 3-D Green's function,

$$\tilde{g}=\frac{e^{\pm i\sqrt{k^2-k_{x_4}^2}\,r_3}}{4\pi r_3},$$
$$r_3\equiv\sqrt{x_1^2+x_2^2+x_3^2}=\sqrt{r^2-x_4^2},$$
$$r^2=x_1^2+x_2^2+x_3^2+x_4^2.$$

Assuming $e^{-i\omega t}$ time dependence, the '+' sign in the exponent above is the correct choice to yield an outgoing wave solution. Then,

$$g(r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(x_1,x_2,x_3,k_{x_4})\cdot e^{ik_{x_4}x_4}dk_{x_4}$$
$$=\frac{1}{8\pi^2 r_3}\int_{-\infty}^{\infty}e^{i\left(k_{x_4}x_4+\sqrt{k^2-k_{x_4}^4}\,r_3\right)}dk_{x_4}.$$

From (2.2.11),

$$H_0^{(1)}(kr)=\frac{1}{\pi}\int_{-\infty}^{\infty}dk_{x_4}\frac{e^{i\left(k_{x_4}x_4+\sqrt{k^2-k_{x_4}^2}\,r_3\right)}}{\sqrt{k^2-k_{x_4}^2}},$$

so, $$\frac{\partial}{\partial r_3}H_0^{(1)}(kr)=\frac{i}{\pi}\int_{-\infty}^{\infty}dk_{x_4}e^{i\left(k_{x_4}x_4+\sqrt{k^2-k_{x_4}^2}\,r_3\right)}.$$

Thus,

$$g(r)=\frac{-i}{8\pi r_3}\frac{\partial}{\partial r_3}H_0^{(1)}(kr)=\frac{-i}{8\pi r_3}\frac{\partial}{\partial kr}H_0^{(1)}(kr)\cdot\frac{\partial kr}{\partial r_3}.$$

Since

$$\frac{\partial}{\partial x}H_0^{(1)}(x) = -H_1^{(1)}(x) \text{ and } \frac{\partial r}{\partial r_3} = \frac{\partial}{\partial r_3}\sqrt{r_3^2 + x_4^2} = \frac{r_3}{r},$$

$$g(r) = \frac{ik}{8\pi r}H_1^{(1)}(kr).$$

§4.15

(a) Taylor expanding $\phi(\mathbf{r}, t)$,

$$\phi(\mathbf{r}, t + \Delta t) = \phi(\mathbf{r}, t) + \frac{\partial \phi}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 \phi}{\partial t^2}\Delta t^2 + \cdots .$$

Forward difference approximation:

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} \approx \frac{\phi(\mathbf{r}, t + \Delta t) - \phi(\mathbf{r}, t)}{\Delta t} = \frac{\frac{\partial \phi}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 \phi}{\partial t^2}\Delta t^2 + \cdots}{\Delta t}$$
$$= \frac{\partial \phi}{\partial t} + \frac{1}{2!}\frac{\partial^2 \phi}{\partial t^2}\Delta t + \cdots = \frac{\partial \phi}{\partial t}(\mathbf{r}, t) + O(\Delta t).$$

Backward difference:

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} \approx \frac{\phi(\mathbf{r}, t) - \phi(\mathbf{r}, t - \Delta t)}{\Delta t} = \frac{\frac{\partial \phi}{\partial t}\Delta t - \frac{1}{2!}\frac{\partial^2 \phi}{\partial t^2}\Delta t^2 + \cdots}{\Delta t}$$
$$= \frac{\partial \phi}{\partial t} - \frac{1}{2!}\frac{\partial^2 \phi}{\partial t^2}\Delta t + \cdots = \frac{\partial \phi}{\partial t} + O(\Delta t).$$

Central difference:

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} \approx \frac{\phi\left(\mathbf{r}, t + \frac{\Delta t}{2}\right) - \phi\left(\mathbf{r}, t - \frac{\Delta t}{2}\right)}{\Delta t}$$
$$= \frac{\left[\phi(\mathbf{r}, t) + \frac{\partial \phi}{\partial t}\left(\frac{\Delta t}{2}\right) + \frac{1}{2!}\frac{\partial^2 \phi}{\partial t^2}\left(\frac{\Delta t}{2}\right)^2 + \cdots\right]}{\Delta t}$$
$$- \frac{\left[\phi(\mathbf{r}, t) - \frac{\partial \phi}{\partial t}\left(\frac{\Delta t}{2}\right) + \frac{1}{2!}\frac{\partial^2 \phi}{\partial t^2}\left(\frac{\Delta t}{2}\right)^2 - \cdots\right]}{\Delta t}$$
$$= \frac{\frac{\partial \phi}{\partial t}\Delta t + \frac{2}{3!}\frac{\partial^3 \phi}{\partial t^3}\left(\frac{\Delta t}{2}\right)^3 + \cdots}{\Delta t} = \frac{\partial \phi}{\partial t} + \frac{2}{3!}\frac{\partial^3 \phi}{\partial t^3}\frac{\Delta t^2}{8} + \cdots = \frac{\partial \phi}{\partial t} + O[(\Delta t)^2].$$

(b) To get an error of $O[(\Delta t)^4]$, we can combine two central-difference approximations with different widths in order to cancel the first error terms.

$$\frac{\phi\left(t + \frac{\Delta t}{2}\right) - \phi\left(t - \frac{\Delta t}{2}\right)}{\Delta t} = \frac{\partial \phi}{\partial t} + \frac{2}{3!}\frac{\partial^3 \phi}{\partial t^3}\frac{\Delta t^2}{8} + O[(\Delta t)^4]$$
$$\frac{\phi\left(t + \frac{\Delta t}{4}\right) - \phi\left(t - \frac{\Delta t}{4}\right)}{\frac{\Delta t}{2}} = \frac{\partial \phi}{\partial t} + \frac{2}{3!}\frac{\partial^3 \phi}{\partial t^3}\frac{\Delta t^2}{4 \cdot 8} + O[(\Delta t)^4].$$

Thus,

$$\frac{8\left[\phi\left(t+\frac{\Delta t}{4}\right)-\phi\left(t-\frac{\Delta t}{4}\right)\right]-\left[\phi\left(t+\frac{\Delta t}{2}\right)-\phi\left(t-\frac{\Delta t}{2}\right)\right]}{\Delta t}$$
$$=3\,\frac{\partial\phi}{\partial t}+O[(\Delta t)^4].$$

So, a suitable differencing scheme is

$$\frac{\partial\phi(t)}{\partial t}\approx\frac{-\frac{1}{3}\phi\left(t+\frac{\Delta t}{2}\right)+\frac{8}{3}\phi\left(t+\frac{\Delta t}{4}\right)-\frac{8}{3}\phi\left(t-\frac{\Delta t}{4}\right)+\frac{1}{3}\phi\left(t-\frac{\Delta t}{2}\right)}{\Delta t}.$$

§4.16

For a homogeneous medium with $\Delta x\neq\Delta y\neq\Delta z$, (4.6.10) becomes

$$\begin{aligned}\phi^{l+1}_{m,n,p}-2\phi^{l}_{m,n,p}+\phi^{l-1}_{m,n,p}&\\=(\Delta t)^2c^2\Bigg\{&\frac{1}{(\Delta x)^2}\left[\phi^{l}_{m+1,n,p}-2\phi^{l}_{m,n,p}+\phi^{l}_{m-1,n,p}\right]\\&+\frac{1}{(\Delta y)^2}\left[\phi^{l}_{m,n+1,p}-2\phi^{l}_{m,n,p}+\phi^{l}_{m,n-1,p}\right]\\&+\frac{1}{(\Delta z)^2}\left[\phi^{l}_{m,n,p+1}-2\phi^{l}_{m,n,p}+\phi^{l}_{m,n,p-1}\right]\Bigg\}.\end{aligned}$$

Inserting the discrete plane wave $\phi^l_{m,n,p}=g^le^{ik_xm\Delta x+ik_yn\Delta y+ik_zp\Delta z}$ into this equation, one obtains

$$\begin{aligned}&\left(g-2+g^{-1}\right)\phi^l_{m,n,p}\\&=-4(\Delta t)^2c^2\left[\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)+\frac{1}{(\Delta y)^2}\right.\\&\quad\left.\sin^2\left(\frac{k_y\Delta y}{2}\right)+\frac{1}{(\Delta z)^2}\sin^2\left(\frac{k_z\Delta z}{2}\right)\right]\phi^l_{m,n,p}.\\&=-4r^2s^2\phi^l_{m,n,p},\end{aligned}$$

where

$r=\Delta t\cdot c$, and

$$s^2=\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right)+\frac{1}{(\Delta y)^2}\sin^2\left(\frac{k_y\Delta y}{2}\right)+\frac{1}{(\Delta z)^2}\sin^2\left(\frac{k_z\Delta z}{2}\right).$$

This is the same as (4.6.16), but with slightly different definitions for r and s.

As before, the condition $r^2s^2<1$ ensures stability.

Since $s^2\leq\left(\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}+\frac{1}{(\Delta z)^2}\right)$ in this case, the condition for stability becomes

$$\Delta t<\frac{1}{c\sqrt{\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}+\frac{1}{(\Delta z)^2}}}.$$

The generalization for n-dimensions is $\Delta t < \frac{1}{c\sqrt{\frac{1}{(\Delta x_1)^2}+\frac{1}{(\Delta x_2)^2}+\cdots+\frac{1}{(\Delta x_n)^2}}}$.

§4.17

The lossy wave equation is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\mu\sigma\frac{\partial}{\partial t}\right)\phi(\mathbf{r},t)=\mu\nabla\cdot\mu^{-1}\nabla\phi(\mathbf{r},t). \tag{4.6.23}$$

For a homogeneous medium, using central differencing, this is approximated as

$$\begin{aligned}&\frac{1}{c^2(\Delta t)^2}\left[\phi^{l+1}_{m,n,p}-2\phi^{l}_{m,n,p}+\phi^{l-1}_{m,n,p}\right]+\frac{\mu\sigma}{2\Delta t}\left[\phi^{l+1}_{m,n,p}-\phi^{l-1}_{m,n,p}\right]\\ =&\frac{1}{(\Delta s)^2}\left[(\phi^{l}_{m+1,n,p}-2\phi^{l}_{m,n,p}+\phi^{l}_{m-1,n,p})+(\phi^{l}_{m,n+1,p}\right.\\ &\left.-2\phi^{l}_{m,n,p}+\phi^{l}_{m,n-1,p})+(\phi^{l}_{m,n,p+1}-2\phi^{l}_{m,n,p}+\phi^{l}_{m,n,p-1})\right].\end{aligned}$$

Assuming $\phi^l_{m,n,p}=g^l e^{im\Delta s k_x+in\Delta s k_y+ip\Delta s k_z}$, then

$$g-2+g^{-1}+\alpha(g-g^{-1})=-4r^2s^2,$$

where

$$\alpha=\frac{\mu\sigma c^2}{2}\Delta t,\qquad r=\left(\frac{\Delta t}{\Delta s}\right)c,$$

and

$$s^2=\sin^2\left(\frac{k_x\Delta s}{2}\right)+\sin^2\left(\frac{k_y\Delta s}{2}\right)+\sin^2\left(\frac{k_z\Delta s}{2}\right).$$

Multiplying by g,

$$g^2(1+\alpha)+g(4r^2s^2-2)+(1-\alpha)=0.$$

Solving for g,

$$\begin{aligned}g&=\frac{1-2r^2s^2\pm\sqrt{\alpha^2-4r^2s^2(1-r^2s^2)}}{1+\alpha}\\ &=\frac{1-2r^2s^2\pm i\sqrt{4r^2s^2(1-r^2s^2)-\alpha^2}}{1+\alpha}.\end{aligned}$$

Suppose that $r^2s^2<1$, which is the stability criterion for the lossless wave equation. We want to show that $|g|\le 1$ if this condition holds.

There are two cases to consider. First, suppose that $\alpha^2<4r^2s^2(1-r^2s^2)$. Then

$$\begin{aligned}|g|^2&=\frac{(1-2r^2s^2)^2+(4r^2s^2(1-r^2s^2)-\alpha^2)^2}{(1+\alpha)^2}\\ &=\frac{1-\alpha^2}{(1+\alpha)^2}=\frac{1-\alpha}{1+\alpha}<1.\end{aligned}$$

Next, consider the case where $\alpha^2 > 4r^2s^2(1-r^2s^2)$.

Then,

$$g = \frac{1-2r^2s^2 \pm \sqrt{\alpha^2 - 4r^2s^2(1-r^2s^2)}}{1+\alpha}.$$

Since $0 \le r^2s^2 < 1$, $-1 < 1-2r^2s^2 \le 1$.
Also, $0 < \sqrt{\alpha^2 - 4r^2s^2(1-r^2s^2)} \le \alpha$ is this case. Therefore,

$$\frac{-1-\alpha}{1+\alpha} < g \le \frac{1+\alpha}{1+\alpha}, \quad \text{or } |g| \le 1.$$

Thus, if $r^2s^2 < 1$, then $|g| \le 1$ for all α. That is, the lossy wave equation has the same stability criterion as the lossless wave equation.

§4.18

(a) Applying central differencing in time,

$$\frac{\partial \phi^l_{m,n,p}}{\partial t} \longrightarrow \frac{1}{2\Delta t}\left(\phi^{l+1}_{m,n,p} - \phi^{l-1}_{m,n,p}\right).$$

Assume a Fourier mode $\phi^l_{m,n,p} = g^l e^{im\Delta s k_x + in\Delta s k_y + ip\Delta s k_z}$. Then using central differencing in space, $\nabla^2 \phi^l_{m,n,p} \longrightarrow \frac{-4}{(\Delta s)^2} s^2 \phi^l_{m,n,p}$, where

$$s^2 = \sin^2\left(\frac{k_x \Delta s}{2}\right) + \sin^2\left(\frac{k_y \Delta s}{2}\right) + \sin^2\left(\frac{k_z \Delta s}{2}\right).$$

Then the diffusion equation becomes

$$\frac{-4s^2}{(\Delta s)^2}\phi^l_{m,n,p} - \frac{\mu\sigma}{2\Delta t}(g - g^{-1})\phi^l_{m,n,p} = 0$$

or

$$g^2 + \gamma g - 1 = 0, \quad \gamma = \frac{8s^2\Delta t}{\mu\sigma(\Delta s)^2}.$$

Solving for g,

$$g = \frac{-\gamma \pm \sqrt{\gamma^2 + 4}}{2}.$$

Since $\gamma > 0$ for most (k_x, k_y, k_z), $\sqrt{\gamma^2+4} > 2$, and $-\gamma - \sqrt{\gamma^2+4} < -\gamma - 2 < -2$. So, there is a solution

$$g = \frac{-\gamma - \sqrt{\gamma^2+4}}{2} < \frac{-2}{2} = -1.$$

Thus, there is always an unstable mode with $|g| > 1$.

(b) Using forward differencing in time,

$$\frac{\partial}{\partial t}\phi^l_{m,n,p} \longrightarrow \frac{1}{\Delta t}\left(\phi^{l+1}_{m,n,p} - \phi^l_{m,n,p}\right) = \frac{1}{\Delta t}(g-1)\phi^l_{m,n,p},$$

for a Fourier mode.

Then, with central differencing in space, the diffusion equation becomes

$$\frac{\mu\sigma}{\Delta t}(g-1)\phi^l_{m,n,p} = \frac{-4}{(\Delta s)^2}s^2\phi^l_{m,n,p}.$$

The amplification factor g is

$$g = 1 - \frac{4s^2\Delta t}{\mu\sigma(\Delta s)^2}.$$

To have $|g| < 1$, we must have $\frac{4s^2\Delta t}{\mu\sigma(\Delta s)^2} < 2$, or $\Delta t < \frac{\mu\sigma(\Delta s)^2}{2s^2}$. Since $s^2 \leq n$ for an n-dimensional problem, the condition for stability is

$$\Delta t < \frac{\mu\sigma(\Delta s)^2}{2n}. \tag{4.6.25}$$

(c) The given equation is a linear equation relating ϕ^{l+1}_m, ϕ^{l+1}_{m-1}, and ϕ^{l+1}_{m+1} to ϕ^l_m, ϕ^l_{m-1}, andϕ^l_{m+1}. Since this holds for all m, with appropriate boundary conditions at the ends of the computation region, this can be written as a matrix equation

$$\overline{\mathbf{A}}\cdot\phi^{l+1} = \overline{\mathbf{B}}\cdot\phi^l,$$

where $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ are tridiagonal matrices.

To step forward in time, $\overline{\mathbf{A}}$ must be inverted to compute

$$\phi^{l+1} = \overline{\mathbf{A}}^{-1}\cdot\overline{\mathbf{B}}\cdot\phi^l.$$

(d) Let $\phi^l_m = g^l e^{im\Delta sk}$.

Then the diffusion equation becomes

$$(g-1)\phi^l_m = \frac{\Delta t}{2(\Delta s)^2\mu\sigma}\left[g\left(e^{i\Delta sk} - 2 + e^{i\Delta sk}\right) + \left(e^{i\Delta sk} - 2 + e^{-i\Delta sk}\right)\right]\phi^l_m$$

or

$$(g-1) = -4Ls^2(g+1), \quad L = \frac{\Delta t}{2(\Delta s)^2\mu\sigma}, \quad s^2 = \sin^2\left(\frac{\Delta s\cdot k}{2}\right).$$

Solving for g,

$$g = \frac{1-4Ls^2}{1+4Ls^2}.$$

Since $Ls^2 \geq 0$, $|g| \leq 1$, so the Crank-Nicholson method is unconditionally stable.

§4.19

Equation (4.6.26) is the lossy wave equation (4.6.23) with $c = \frac{\Delta s}{\sqrt{n\Delta t}}$. As shown in problem (4.17), the stability criterion for the lossy wave equation (using central-differencing) is the CFL condition (4.6.22), $\Delta t < \frac{\Delta s}{c\sqrt{n}}$.

With $c = \frac{\Delta s}{\sqrt{n}\Delta t}$, the quantity $\frac{\Delta s}{c\sqrt{n}}$ becomes Δt.

Thus, the requirement for stability is $\Delta t < \Delta t$. This would seem to indicate a situation of marginal stability for any Δt. To resolve this difficulty, consider $c = \frac{\Delta s}{\sqrt{n}\Delta t(1+\epsilon)}$, where ϵ is a small positive number. Then $\frac{\Delta s}{c\sqrt{n}} = \Delta t(1+\epsilon)$, and the stability criterion is $\Delta t < \Delta t(1+\epsilon)$, which is satisfied for any positive Δt. Thus, the central-difference approximation to (4.6.26) will be unconditionally stable as long as the leading term is multiplied by a constant factor $(1+\epsilon)^2$, $\quad \epsilon > 0$.

§4.20

(a) The finite-difference approximations of Maxwell's equations (4.6.35a) to (4.6.35c) on the six faces of a cube are

$$\frac{1}{\Delta t}\left[B^{l+\frac{1}{2}}_{x\ m,n+\frac{1}{2},p+\frac{1}{2}} - B^{l-\frac{1}{2}}_{x\ m,n+\frac{1}{2},p+\frac{1}{2}}\right] = \frac{1}{\Delta z}\left[E^{l}_{y\ m,n+\frac{1}{2},p+1} - E^{l}_{y\ m,n+\frac{1}{2},p}\right] - \frac{1}{\Delta y}\left[E^{l}_{z\ m,n+1,p+\frac{1}{2}} - E^{l}_{z\ m,n,p+\frac{1}{2}}\right] \quad (1)$$

$$\frac{1}{\Delta t}\left[B^{l+\frac{1}{2}}_{y\ m+\frac{1}{2},n,p+\frac{1}{2}} - B^{l-\frac{1}{2}}_{y\ m+\frac{1}{2},n,p+\frac{1}{2}}\right] = \frac{1}{\Delta x}\left[E^{l}_{z\ m+1,n,p+\frac{1}{2}} - E^{l}_{z\ m,n,p+\frac{1}{2}}\right] - \frac{1}{\Delta z}\left[E^{l}_{x\ m+\frac{1}{2},n,p+1} - E^{l}_{x\ m+\frac{1}{2},n,p}\right] \quad (2)$$

$$\frac{1}{\Delta t}\left[B^{l+\frac{1}{2}}_{z\ m+\frac{1}{2},n+\frac{1}{2},p} - B^{l-\frac{1}{2}}_{z\ m+\frac{1}{2},n+\frac{1}{2},p}\right] = \frac{1}{\Delta y}\left[E^{l}_{x\ m+\frac{1}{2},n+1,p} - E^{l}_{x\ m+\frac{1}{2},n,p}\right] - \frac{1}{\Delta x}\left[E^{l}_{y\ m+1,n+\frac{1}{2},p} - E^{l}_{y\ m,n+\frac{1}{2},p}\right] \quad (3)$$

$$\frac{1}{\Delta t}\left[B^{l+\frac{1}{2}}_{x\ m+1,n+\frac{1}{2},p+\frac{1}{2}} - B^{l-\frac{1}{2}}_{x\ m+1,n+\frac{1}{2},p+\frac{1}{2}}\right] = \frac{1}{\Delta z}\left[E^{l}_{y\ m+1,n+\frac{1}{2},p+1} - E^{l}_{y\ m+1,n+\frac{1}{2},p}\right] - \frac{1}{\Delta y}\left[E^{l}_{z\ m+1,n+1,p+\frac{1}{2}} - E^{l}_{z\ m+1,n,p+\frac{1}{2}}\right] \quad (4)$$

$$\frac{1}{\Delta t}\left[B^{l+\frac{1}{2}}_{y\ m+\frac{1}{2},n+1,p+\frac{1}{2}} - B^{l-\frac{1}{2}}_{y\ m+\frac{1}{2},n+1,p+\frac{1}{2}}\right] = \frac{1}{\Delta x}\left[E^{l}_{z\ m+1,n+1,p+\frac{1}{2}} - E^{l}_{z\ m,n+1,p+\frac{1}{2}}\right] - \frac{1}{\Delta z}\left[E^{l}_{x\ m+\frac{1}{2},n+1,p+1} - E^{l}_{x\ m+\frac{1}{2},n+1,p}\right] \quad (5)$$

$$\frac{1}{\Delta t}\left[B^{l+\frac{1}{2}}_{z\ m+\frac{1}{2},n+\frac{1}{2},p+1} - B^{l-\frac{1}{2}}_{z\ m+\frac{1}{2},n+\frac{1}{2},p+1}\right] = \frac{1}{\Delta y}\left[E^{l}_{x\ m+\frac{1}{2},n+1,p+1} - E^{l}_{x\ m+\frac{1}{2},n,p+1}\right] - \frac{1}{\Delta x}\left[E^{l}_{y\ m+1,n+\frac{1}{2},p+1} - E^{l}_{y\ m,n+\frac{1}{2},p+1}\right]. \quad (6)$$

The finite-difference approximation for the time derivative of the total flux passing through the cube is

$$\frac{\partial}{\partial t}\int_s ds\hat{n}\cdot\mathbf{B} \simeq \Delta y\Delta z\,[LHS(4) - LHS(1)] + \Delta x\Delta z$$
$$[LHS(5) - LHS(2)] + \Delta x\Delta y\,[LHS(6) - LHS(3)]\,,$$

where LHS (i) indicates the left-hand-side of finite difference equation (i).

Replacing the left-hand-sides with their equivalent right-hand-sides in the equation above, we find that every term is cancelled, so the sum is zero.

So, the Yee algorithm is consistent with

$$\frac{\partial}{\partial t}\int_s ds\hat{n}\cdot\mathbf{B} = 0 \quad \text{or} \quad \int_s ds\hat{n}\cdot\mathbf{B} = c, \quad \text{a constant.}$$

From causality, there is a time before the source is turned on when $\mathbf{B} = 0$, which implies that the constant C must be zero.

(b) Using the Yee grid, the total magnetic flux passing through an elemental cube is

$$\int_s ds\hat{n}\cdot\mathbf{B} = \Delta x\Delta y\left(B^l_{z\ m+\frac{1}{2},n+\frac{1}{2},p+1} - B^l_{z\ m+\frac{1}{2},n+\frac{1}{2},p}\right)$$
$$+ \Delta y\Delta z\left(B^l_{x\ m+1,n+\frac{1}{2},p+\frac{1}{2}} - B^l_{x\ m,n+\frac{1}{2},p+\frac{1}{2}}\right)$$
$$+ \Delta x\Delta z\left(B^l_{y\ m+\frac{1}{2},n+1,p+\frac{1}{2}} - B^l_{y\ m+\frac{1}{2},n,p+\frac{1}{2}}\right) = 0.$$

So,

$$B^l_{z\ m+\frac{1}{2},n+\frac{1}{2},p+1} - B^l_{z\ m+\frac{1}{2},n+\frac{1}{2},p}$$
$$= -\frac{\Delta z}{\Delta x}\left(B^l_{x\ m+1,n+\frac{1}{2},p+\frac{1}{2}} - B^l_{x\ m,n+\frac{1}{2},p+\frac{1}{2}}\right)$$
$$- \frac{\Delta z}{\Delta y}\left(B^l_{y\ m+\frac{1}{2},n+1,p+\frac{1}{2}} - B^l_{y\ m+\frac{1}{2},n,p+\frac{1}{2}}\right).$$

The above is a finite-difference equation which can be solved for B_z by space-stepping, if B_z is known for some value of z such as a boundary point.

§4.21

The finite-difference approximations of Maxwell's equations (4.6.35d) to (4.6.35f) on the six faces of a cube are

$$\frac{1}{\Delta t}\left[D^l_{x\ m-\frac{1}{2},n,p} - D^{l-1}_{x\ m-\frac{1}{2},n,p}\right] = \frac{1}{\Delta y}\left[H^{l-\frac{1}{2}}_{z\ m-\frac{1}{2},n+\frac{1}{2},p} - H^{l-\frac{1}{2}}_{z\ m-\frac{1}{2},n-\frac{1}{2},p}\right]$$
$$- \frac{1}{\Delta z}\left[H^{l-\frac{1}{2}}_{y\ m-\frac{1}{2},n,p+\frac{1}{2}} - H^{l-\frac{1}{2}}_{y\ m-\frac{1}{2},n,p-\frac{1}{2}}\right] J^{l-\frac{1}{2}}_{x\ m-\frac{1}{2},n,p} \quad (1)$$

$$\frac{1}{\Delta t}\left[D^{l}_{y\ m,n-\frac12,p} - D^{l-1}_{y\ m,n-\frac12,p}\right] = \frac{1}{\Delta z}\left[H^{l-\frac12}_{x\ m,n-\frac12,p+\frac12} - H^{l-\frac12}_{x\ m,n-\frac12,p-\frac12}\right]$$
$$- \frac{1}{\Delta x}\left[H^{l-\frac12}_{z\ m+\frac12,n-\frac12,p} - H^{l-\frac12}_{z\ m-\frac12,n-\frac12,p}\right] - J^{l-\frac12}_{y\ m,n-\frac12,p} \quad (2)$$

$$\frac{1}{\Delta t}\left[D^{l}_{z\ m,n,p-\frac12} - D^{l-1}_{z\ m,n,p-\frac12}\right] = \frac{1}{\Delta x}\left[H^{l-\frac12}_{y\ m+\frac12,n,p-\frac12} - H^{l-\frac12}_{y\ m-\frac12,n,p-\frac12}\right]$$
$$- \frac{1}{\Delta y}\left[H^{l-\frac12}_{x\ m,n+\frac12,p-\frac12} - H^{l-\frac12}_{x\ m,n-\frac12,p-\frac12}\right] - J^{l-\frac12}_{z\ m,n,p-\frac12} \quad (3)$$

$$\frac{1}{\Delta t}\left[D^{l}_{x\ m+\frac12,n,p} - D^{l-1}_{x\ m+\frac12,n,p}\right] = \frac{1}{\Delta y}\left[H^{l-\frac12}_{z\ m+\frac12,n+\frac12,p} - H^{l-\frac12}_{z\ m+\frac12,n-\frac12,p}\right]$$
$$- \frac{1}{\Delta z}\left[H^{l-\frac12}_{y\ m+\frac12,n,p+\frac12} - H^{l-\frac12}_{y\ m+\frac12,n,p-\frac12}\right] - J^{l-\frac12}_{x\ m+\frac12,n,p} \quad (4)$$

$$\frac{1}{\Delta t}\left[D^{l}_{y\ m,n+\frac12,p} - D^{l-1}_{y\ m,n+\frac12,p}\right] = \frac{1}{\Delta z}\left[H^{l-\frac12}_{x\ m,n+\frac12,p+\frac12} - H^{l-\frac12}_{x\ m,n+\frac12,p-\frac12}\right]$$
$$- \frac{1}{\Delta x}\left[H^{l-\frac12}_{z\ m+\frac12,n+\frac12,p} - H^{l-\frac12}_{z\ m-\frac12,n+\frac12,p}\right] - J^{l-\frac12}_{y\ m,n+\frac12,p} \quad (5)$$

$$\frac{1}{\Delta t}\left[D^{l}_{z\ m,n,p+\frac12} - D^{l-1}_{z\ m,n,p+\frac12}\right] = \frac{1}{\Delta x}\left[H^{l-\frac12}_{y\ m+\frac12,n,p+\frac12} - H^{l-\frac12}_{y\ m-\frac12,n,p+\frac12}\right]$$
$$- \frac{1}{\Delta y}\left[H^{l-\frac12}_{x\ m,n+\frac12,p+\frac12} - H^{l-\frac12}_{x\ m,n-\frac12,p+\frac12}\right] - J^{l-\frac12}_{z\ m,n,p+\frac12}\ . \quad (6)$$

The finite-difference approximation for $\frac{\partial}{\partial t}\int_s ds\hat{n}\cdot\mathbf{D}$, where S is the cube surface, is

$$\frac{\partial}{\partial t}\int_s ds\hat{n}\cdot\mathbf{D} \simeq \Delta y\Delta z[LHS(4) - LHS(1)]$$
$$+ \Delta x\Delta z[LHS(5) - LHS(2)] + \Delta x\Delta y[LHS(6) - LHS(3)],$$

where LHS (i) is the left-hand-side of finite-difference equation (i). Replacing the left-hand-sides above with the right-hand-sides of equations (1) through (6), the terms involving H all cancel, leaving

$$\frac{\partial}{\partial t}\int_s ds\hat{n}\cdot\mathbf{D} \simeq -\left\{\Delta y\Delta z\left[J^{l-\frac12}_{x\ m+\frac12,n,p} - J^{l-\frac12}_{x\ m-\frac12,n,p}\right]\right.$$
$$+ \Delta x\Delta z\left[J^{l-\frac12}_{y\ m,n+\frac12,p} - J^{l-\frac12}_{y\ m,n-\frac12,p}\right]$$
$$\left. + \Delta x\Delta y\left[J^{l-\frac12}_{z\ m,n,p+\frac12} - J^{l-\frac12}_{z\ m,n,p-\frac12}\right]\right\}.$$

The right-hand-side above is the finite-difference approximation for

$$-\int_s ds\hat{n}\cdot\mathbf{J},$$

where S is the cube surface.

So, the Yee algorithm is consistent with

$$\frac{\partial}{\partial t}\int_s ds\hat{n}\cdot\mathbf{D} = -\int_s ds\hat{n}\cdot\mathbf{J},$$

which is the integral form of $\frac{\partial}{\partial t}\nabla\cdot\mathbf{D} = -\nabla\cdot\mathbf{J}$, the continuity equation.

§4.22

(a) In three dimensions, the central difference approximation to the wave equation is

$$\frac{(\mathbf{H}^l_{m+1,n,p} - 2\mathbf{H}^l_{m,n,p} + \mathbf{H}^l_{m-1,n,p})}{(\Delta x)^2} + \frac{(\mathbf{H}^l_{m,n+1,p} - 2\mathbf{H}^l_{m,n,p} + \mathbf{H}^l_{m,n-1,p})}{(\Delta y)^2}$$
$$+\frac{(\mathbf{H}^l_{m,n,p+1} - \mathbf{H}^l_{m,n,p} + \mathbf{H}^l_{m,n,p-1})}{(\Delta z)^2} - \frac{1}{c^2}\frac{(\mathbf{H}^{l+1}_{m,n,p} - 2\mathbf{H}^l_{m,n,p} + \mathbf{H}^{l-1}_{m,n,p})}{(\Delta t)^2} = 0.$$

(b) To make the necessary manipulations more transparent, it is helpful to use an operator notation for the central differencing. Let δ_x, δ_y, δ_z, and δ_t denote the central difference operators on the space and time variables. Then, for example,

$$\delta_x\phi_{m,n,p} = \frac{1}{\Delta x}\left(\phi^l_{m+\frac{1}{2},n,p} - \phi^l_{m-\frac{1}{2},n,p}\right),$$
$$\delta_x^2\phi_{m,n,p} = \frac{1}{(\Delta x)^2}\left(\phi^l_{m+1,n,p} - 2\phi^l_{m,n,p} + \phi^l_{m-1,n,p}\right).$$

The wave equation can then be written

$$\left(\delta_x^2 + \delta_y^2 + \delta_z^2 - \frac{1}{c^2}\delta_t^2\right)\mathbf{H}^l_{m,n,p} = 0.$$

In this notation, (4.6.35a) becomes

$$-\delta_t B^l_{x\ m,n+\frac{1}{2},p+\frac{1}{2}} = \delta_y E^l_{z\ m,n+\frac{1}{2},p+\frac{1}{2}} - \delta_z E^l_{y\ m,n+\frac{1}{2},p+\frac{1}{2}}.$$

Then, applying δ_t to the above equation,

$$\mu\delta_t^2 H^l_{x\,m,n+\frac{1}{2}p+\frac{1}{2}} = -\delta_y\delta_t E^l_{z\,m,n+\frac{1}{2},p+\frac{1}{2}} + \delta_z\delta_t E^l_{y\,m,n+\frac{1}{2},p+\frac{1}{2}}.$$

To eliminate $\mathbf{E}$, use (4.6.35 e-f):

$$\delta_t D_{y\ m,n+\frac{1}{2},p+\frac{1}{2}} = \delta_z H^l_{x\ m,n+\frac{1}{2},p+\frac{1}{2}} - \delta_x H^l_{z\ m,n+\frac{1}{2},p+\frac{1}{2}}$$
$$\delta_t D_{z\ m,n+\frac{1}{2},p+\frac{1}{2}} = \delta_x H^l_{y\ m,n+\frac{1}{2},p+\frac{1}{2}} - \delta_y H_{x\ m,n+\frac{1}{2},p+\frac{1}{2}}.$$

Then (dropping the $m, n+\frac{1}{2}, p+\frac{1}{2}$ subscript, since it is constant throughout),

$$\mu\delta_t^2 H^l_x = \frac{1}{\epsilon}\left[-\delta_y\delta_x H^l_y + \delta_y\delta_y H^l_x + \delta_z\delta_z H^l_x - \delta_z\delta_x H^l_z\right]$$
$$= \frac{1}{\epsilon}\left[\delta_y^2 H^l_x + \delta_z^2 H^l_x - \delta_x(\delta_y H^l_y + \delta_z H^l_z)\right].$$

Multiplying through by ϵ and adding and subtracting $\delta_x^2 H^l_x$,

$$\mu\epsilon\delta_t^2 H^l_x = (\delta_x^2 + \delta_y^2 + \delta_z^2)H^l_x - \delta_x(\delta_x H^l_y + \delta_y H^l_y + \delta_z H^l_z).$$

The last term on the right is just the finite-difference approximation of $\nabla \cdot \mathbf{H}$, and is zero in the Yee algorithm (see problem 4.20).

Therefore, we have

$$\frac{1}{c^2}\delta_t^2 H^l_x = (\delta_x^2 + \delta_y^2 + \delta_z^2)H^l_x, \tag{4.6.22}$$

which is the finite-difference approximation of the wave equation for H_x.

(c) Since the Yee algorithm is equivalent to finite-differencing the vector wave equation directly, the CFL stability criterion applies for the Yee algorithm. For stability,

$$\Delta t < \frac{\Delta s}{c\sqrt{n}}. \tag{4.6.22}$$

§4.23

(a) Equation (4.7.4) is

$$\phi_0^{l+1} = \phi_0^l\left(1 - \frac{c\Delta t}{\Delta x}\right) + \frac{c\Delta t}{\Delta x}\phi_1^l \tag{4.7.4}$$

Substituting a Fourier mode, $\phi^l_m = g^l e^{ik_x m\Delta x}$, gives

$$g\phi_0^l = \left[\left(1 - \frac{c\Delta t}{\Delta x}\right) + \frac{c\Delta t}{\Delta x}e^{ik_x\Delta x}\right]\phi_0^l$$
$$\Rightarrow \quad g = 1 - \frac{c\Delta t}{\Delta x}(1 - e^{ik_x\Delta x}) = 1 - \alpha(1 - e^{ik_x\Delta x}),$$

where $\alpha \equiv \frac{c\Delta t}{\Delta x}$.

Then, $|g|^2 = gg^* = 1 - 4(\alpha - \alpha^2)\sin^2\left(\frac{k_x\Delta x}{2}\right)$

For stability, $|g|^2 \leq 1$.

So, require

$$1 - 4(\alpha - \alpha^2)\sin^2\left(\frac{k_x \Delta x}{2}\right) \leq 1$$

or

$$-4(\alpha - \alpha^2)\sin^2\left(\frac{k_x \Delta x}{2}\right) \leq 0$$

or

$$(\alpha - \alpha^2)\sin^2\left(\frac{k_x \Delta x}{2}\right) \geq 0.$$

The inequality will be satisfied if $\alpha - \alpha^2 \geq 0$, or $1 \geq \alpha$.

Thus, (4.7.4) will be stable when $\frac{c\Delta t}{\Delta x} \leq 1$.

(b) Equation (4.7.7) is

$$\phi_0^{n+1} = \phi_1^n + \left(\frac{c\Delta t - \Delta x}{c\Delta t + \Delta x}\right)(\phi_1^{n+1} - \phi_0^n) \qquad (4.7.7)$$

Substituting $\phi_m^n = g^n e^{ik_x m\Delta x}$,

$$g\phi_0^n = \left[e^{ik_x\Delta x} + \beta(g e^{ik_x\Delta x} - 1)\right]\phi_0^n,$$

where $\beta \equiv \frac{c\Delta t - \Delta x}{c\Delta t + \Delta x}$.

Solving for g,

$$g = \frac{e^{ik_x\Delta x} - \beta}{1 - \beta e^{ik_x\Delta x}} = e^{ik_x\Delta x} \cdot \frac{1 - \beta e^{-ik_x\Delta x}}{1 - \beta e^{+ik_x\Delta x}}.$$

Since β is real, the last fraction is a complex number divided by its conjugate, so the ratio has magnitude one.

Thus,

$$|g| = 1,$$

so (4.7.7) is always stable.

§4.24

Expanding $\phi(x,t)$ in t about $t_0 = (n + \frac{1}{2})\Delta t$, $x_0 = m\Delta x$,

$$\phi(x_0, t) = \phi(x_0, t_0) + \frac{\partial\phi}{\partial t}(t - t_0) + \frac{1}{2}\frac{\partial^2\phi}{\partial t^2}(t - t_0)^2 + \cdots.$$

So,

$$\frac{1}{2}\left(\phi_m^{n+1}+\phi_m^n\right)=\frac{1}{2}\left[\phi\left(x_0,t_0+\frac{\Delta t}{2}\right)+\phi\left(x_0,t_0-\frac{\Delta t}{2}\right)\right]$$

$$=\frac{1}{2}\left[\left(\phi(x_0,t_0)+\frac{\partial\phi}{\partial t}\cdot\left(\frac{\Delta t}{2}\right)+\frac{1}{2}\frac{\partial^2\phi}{\partial t^2}\cdot\left(\frac{\Delta t}{2}\right)^2+\cdots\right)\right.$$

$$\left.+\left(\phi(x_0,t_0)+\frac{\partial\phi}{\partial t}\cdot\left(\frac{-\Delta t}{2}\right)+\frac{1}{2}\frac{\partial^2\phi}{\partial t^2}\cdot\left(\frac{-\Delta t}{2}\right)^2+\cdots\right)\right]$$

$$=\phi(x_0,t_0)+\frac{\partial^2\phi}{\partial t^2}\cdot\left(\frac{\Delta t}{2}\right)^2+\cdots=\phi_m^{n+\frac{1}{2}}+O[(\Delta t)^2].$$

Thus, (4.7.6a) is second-order accurate for smooth functions of t.

Clearly a similar result holds for (4.6.11a) and (4.7.6b), for smooth functions of x.

§4.25

The Lindman absorbing boundary condition requires solution of two equations,

$$\left[\frac{\partial}{\partial x}-\frac{1}{c}\frac{\partial}{\partial t}\right]\phi(\mathbf{r},t)=-\sum_{m=1}^{M}h_m(\mathbf{r},t) \qquad (4.7.19)$$

and

$$\frac{\partial^2}{\partial t^2}h_m(\mathbf{r},t)-\beta_m c^2\frac{\partial^2}{\partial y^2}h_m(\mathbf{r},t)=\alpha_m c^2\frac{\partial^3}{\partial y^2\partial x}\phi(\mathbf{r},t). \qquad (4.7.20)$$

Let $h_m(\mathbf{r},t)$ be defined one-half cell inside the boundary located at $x=0$. Then using central finite-differencing, the above equations can be approximated as

$$\frac{\left(\phi_{1,n}^l-\phi_{0,n}^l\right)}{\Delta x}-\frac{1}{c}\frac{\left[\frac{\left(\phi_{1,n}^{l+1}+\phi_{0,n}^{l+1}\right)}{2}-\frac{\left(\phi_{1,n}^{l-1}+\phi_{0,n}^{l-1}\right)}{2}\right]}{2\Delta t}=-\sum_{m=1}^{M}h_{m_{\frac{1}{2},n}}^l$$

and

$$\frac{\left(h_{m_{\frac{1}{2},n}}^{l+1}-2h_{m_{\frac{1}{2},n}}^{l}+h_{m_{\frac{1}{2},n}}^{l-1}\right)}{(\Delta t)^2}-\beta_m c^2\frac{\left(h_{m_{\frac{1}{2},n+1}}^{l}-2h_{m_{\frac{1}{2},n}}^{l}+h_{m_{\frac{1}{2},n-1}}^{l}\right)}{(\Delta y)^2}$$

$$=\alpha_m c^2\frac{\left[\frac{\left(\phi_{1,n+1}^l-\phi_{0,n+1}^l\right)}{\Delta x}-2\frac{\left(\phi_{1,n}^l-\phi_{0,n}^l\right)}{\Delta x}+\frac{\left(\phi_{1,n-1}^l-\phi_{0,n-1}^l\right)}{\Delta x}\right]}{(\Delta y)^2}.$$

In the first equation above, the approximation

$$\phi_{\frac{1}{2},n} \simeq \frac{(\phi_{1,n} + \phi_{0,n})}{2} \quad \text{has been used.}$$

The first equation above can be solved for $\phi_{0,n}^{l+1}$, while the second can be used to time-step $h^l_{m_{\frac{1}{2},n}}$.

§4.26

(a) The second order Bayliss-Turkel absorbing boundary condition is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{c}\frac{\partial^2}{\partial r \partial t} + \frac{4}{r}\frac{\partial}{\partial r} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{4}{rc}\frac{\partial}{\partial t} + \frac{3}{r^2}\right)\phi(\mathbf{r}, t) = 0. \quad (4.7.29)$$

To remove the $\frac{\partial^2}{\partial r^2}$ term, we may use the fact that

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = 0.$$

In spherical coordinates,

$$\begin{aligned}\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.\end{aligned}$$

So,

$$\frac{\partial^2\phi}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right).$$

Thus, the boundary condition can be written

$$\begin{aligned}\Bigg(\frac{2}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}&\\ + \frac{2}{c}\frac{\partial^2}{\partial r\partial t} + \frac{4}{rc}\frac{\partial}{\partial t} + \frac{3}{r^2}\Bigg)\phi(\mathbf{r}, t) &= 0.\end{aligned}$$

(b) To implement this boundary condition in spherical coordinates, make a finite-difference approximation of the above equation on a spherical grid.

§4.27

(a) Equation (4.7.34) states

$$u(\eta + \epsilon) = \sum_{m=1}^{N} \Delta^{m-1}u(\eta) + \Delta^N u(\eta + \epsilon). \quad (4.7.34)$$

By definition,

$$\Delta u(\eta+\epsilon) = u(\eta+\epsilon) - u(\eta). \tag{4.7.33}$$

Therefore,

$$\begin{aligned}
\Delta u(\eta+\epsilon) - u(\eta+\epsilon) + u(\eta) &= 0 \\
\Delta^2 u(\eta+\epsilon) - \Delta u(\eta+\epsilon) + \Delta u(\eta) &= 0 \\
\Delta^3 u(\eta+\epsilon) - \Delta^2 u(\eta+\epsilon) + \Delta^2 u(\eta) &= 0 \\
&\vdots \\
\Delta^N u(\eta+\epsilon) - \Delta^{N-1} u(\eta+\epsilon) + \Delta^{N-1} u(\eta) &= 0.
\end{aligned}$$

Adding these equations,

$$\Delta^N u(\eta+\epsilon) - u(\eta+\epsilon) + \sum_{m=1}^{N} \Delta^{m-1} u(\eta) = 0,$$

or

$$u(\eta+\epsilon) = \sum_{m=1}^{N} \Delta^{m-1} u(\eta) + \Delta^N u(\eta+\epsilon). \tag{4.7.34}$$

(b) Assume that

$$\phi(x,t) = \sum_i \sum_l u_{il}(c_l t - x\cos\theta_i).$$

$$\text{Then} \quad \phi(x,t+\Delta t) = \sum_{i,l} u_{il}(\eta_{il} + \epsilon_{il}),$$

$$\eta_{il} = c_l t - (x - \alpha_l c_l \Delta t)\cos\theta_i, \quad \epsilon_{il} = c_l \Delta t(1 - \alpha_l \cos\theta_i).$$

As before, we now approximate

$$u(\eta+\epsilon) \simeq \sum_{m=1}^{N} \Delta^{m-1} u(\eta) = \sum_{j=1}^{N} (-1)^{j+1} C_j^N u\left[\eta - (j-1)\epsilon\right],$$

where $C_j^N = \frac{N!}{j!(N-j)!}$.

So, the boundary condition becomes

$$\begin{aligned}
&\phi(x,t+\Delta t) \\
&= \sum_{i,l} \sum_{j=1}^{N} (-1)^{j+1} C_j^N u_{il}[\eta_{il} - (j-1)\epsilon_{il}] \\
&= \sum_{j=1}^{N} (-1)^{j+1} C_j^N \sum_{i,l} u_{il}[\eta_{il} - (j-1)\epsilon_{il}] \\
&= \sum_{j=1}^{N} (-1)^{j+1} C_j^N \sum_{i,l} u_{il}[c_l[t - (j-1)\Delta t] - (x - j\alpha_l c_l \Delta t)\cos\theta_i].
\end{aligned}$$

If we choose the α_l such that $\alpha_l c_l = a$, a constant for all l, then the inner summation above becomes

$$\sum_{i,l} u_{il}\left[c_l[t-(j-1)\Delta t]-(x-ja\Delta t)\cos\theta_i\right],$$

which by comparison with the assumed form for $\phi(x,t)$ is equal to

$$\phi[(x-ja\Delta t),(t-(j-1)\Delta t)].$$

So, the absorbing boundary condition is

$$\phi(x,t+\Delta t)=\sum_{j=1}^{N}(-1)^{j+1}C_j^N\phi[(x-ja\Delta t),(t-(j-1)\Delta t)].$$

§4.28

(a) We are given three values of ϕ,

$$\begin{aligned}\phi_1^1&=\phi(x,t)\\ \phi_2^1&=\phi(x-\Delta x,t)\\ \phi_3^1&=\phi(x-2\Delta x,t).\end{aligned}$$

To derive a quadratic interpolation formula, assume that $\phi(x+\alpha,t)=a+b\alpha+c\alpha^2$. Then from the data above, we have

$$\begin{pmatrix}1&0&0\\1&-\Delta x&(\Delta x)^2\\1&-2\Delta x&4(\Delta x)^2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}\phi_1^1\\ \phi_2^1\\ \phi_3^1\end{pmatrix}.$$

Solving for a,b, and c,

$$\begin{aligned}a&=\phi_1^1\\ b&=\frac{1}{2\Delta x}\left(3\phi_1^1-4\phi_2^1+\phi_3^1\right)\\ c&=\frac{1}{2(\Delta x)^2}\left(\phi_1^1-2\phi_2^1+\phi_3^1\right).\end{aligned}$$

Using these coefficients to compute $\phi(x-c\Delta t,t)$ yields the interpolation formula

$$\phi(x-c\Delta t,t)=\frac{(2-s)(1-s)}{2}\phi_1^1+s(2-s)\phi_2^1+\frac{s(s-1)}{2}\phi_3^1,\quad(4.7.43)$$

where $s=\frac{c\Delta t}{\Delta x}$.

Equation (4.7.47) follows immediately from applying the above formula with $x\rightarrow x-c\Delta t,\ t\rightarrow t-\Delta t$.

(b) By definition,

$$\boldsymbol{\Phi}_3^2(x - c\Delta t) = [\phi_1^2(x - c\Delta t),\ \phi_2^2(x - c\Delta t),\ \phi_3^2(x - c\Delta t)]^t,$$

where

$$\phi_i^m(x) = \phi[x - (i-1)\Delta x, t - (m-1)\Delta t].$$

From (4.7.48a-c),

$$\boldsymbol{\Phi}_3^2(x - c\Delta t) = \left[\mathbf{T}^1 \cdot \boldsymbol{\Phi}_3^2(x), \mathbf{T}^1 \cdot \boldsymbol{\Phi}_3^2(x - \Delta x), \mathbf{T}^1 \cdot \boldsymbol{\Phi}_3^2(x - 2\Delta x)\right]^t.$$

Since

$$\begin{aligned} \boldsymbol{\Phi}_3^2(x) &= \left[\phi_1^2(x), \phi_2^2(x), \phi_3^2(x)\right]^t, \\ \boldsymbol{\Phi}_3^2(x - \Delta x) &= \left[\phi_2^2(x), \phi_3^2(x), \phi_4^2(x)\right]^t, \\ \text{and} \quad \boldsymbol{\Phi}_3^2(x - 2\Delta x) &= \left[\phi_3^2(x), \phi_4^2(x), \phi_5^2(x)\right]^t, \end{aligned}$$

this can be written as

$$\boldsymbol{\Phi}_3^2(x - c\Delta t) = \begin{bmatrix} \mathbf{T}^1 & 0 & 0 \\ 0 & \mathbf{T}^1 & 0 \\ 0 & 0 & T^1 \end{bmatrix} \cdot \boldsymbol{\Phi}_5^2(x), \tag{4.7.49}$$

where

$$\boldsymbol{\Phi}_5^2(x) = \left[\phi_1^2(x), \phi_2^2(x), \phi_3^2(x), \phi_4^2(x), \phi_5^2(x)\right]^t. \tag{4.7.50}$$

Hence,

$$\phi(x - 2c\Delta t, t - \Delta t) = \mathbf{T}^1 \cdot \begin{bmatrix} \mathbf{T}^1 & 0 & 0 \\ 0 & \mathbf{T}^1 & 0 \\ 0 & 0 & \mathbf{T}^1 \end{bmatrix} \cdot \boldsymbol{\Phi}_5^2(x) = \mathbf{T}^2 \cdot \boldsymbol{\Phi}_5^2(x). \tag{4.7.51}$$

The general formula (4.7.52) can be established by induction. Suppose it is true for some j, i.e. $\phi(x - jc\Delta t, t - (j-1)\Delta t) = \mathbf{T}^j \cdot \boldsymbol{\Phi}_{2j+1}^j(x)$. Then for $j+1$, applying (4.7.45),

$$\begin{aligned} &\phi(x - jc\Delta t - c\Delta t, t - j\Delta t) = \mathbf{T}^1 \cdot \boldsymbol{\Phi}_3^{j+1}(x - jc\Delta t) \\ &= \mathbf{T}^1 \cdot \left[\phi_1^{j+1}(x - jc\Delta t), \phi_2^{j+1}(x - jc\Delta t), \phi_3^{j+1}(x - jc\Delta t)\right]^t, \end{aligned}$$

$$\begin{aligned} \text{where} \quad &\phi_n^{j+1}(x - jc\Delta t) = \phi(x - jc\Delta t - (n-1)\Delta x, t - j\Delta t) \\ &= \mathbf{T}^j \cdot \boldsymbol{\Phi}_{2j+1}^{j+1}(x - (n-1)\Delta x), \end{aligned}$$

by the induction hypothesis.

Thus,

$$\begin{aligned}
\phi(x-(j+1)c\Delta t, t-j\Delta t) &= \mathbf{T}^1 \cdot \begin{bmatrix} \mathbf{T}^j \cdot \boldsymbol{\Phi}_{2j+1}^{j+1}(x) \\ \mathbf{T}^j \cdot \boldsymbol{\Phi}_{2j+1}^{j+1}(x-\Delta x) \\ \mathbf{T}^j \cdot \boldsymbol{\Phi}_{2j+1}^{j+1}(x-2\Delta x) \end{bmatrix} \\
&= \mathbf{T}^1 \cdot \begin{bmatrix} \mathbf{T}^j & 0 & 0 \\ 0 & \mathbf{T}^j & 0 \\ 0 & 0 & \mathbf{T}^j \end{bmatrix} \cdot \boldsymbol{\Phi}_{2(j+1)+1}^{j+1}(x) \\
&= \mathbf{T}^{j+1} \cdot \boldsymbol{\Phi}_{2(j+1)+1}^{j+1}(x),
\end{aligned}$$

where $\boldsymbol{\Phi}_{2(j+1)+1}^{j+1}(x) = \left[\phi_1^{j+1}(x), \phi_2^{j+1}(x), \cdots, \phi_{2(j+1)+1}^{j+1}(x)\right]^t,$

and $$\mathbf{T}^{j+1} = \mathbf{T}^1 \begin{bmatrix} \mathbf{T}^j & 0 & 0 \\ 0 & \mathbf{T}^j & 0 \\ 0 & 0 & \mathbf{T}^j \end{bmatrix}.$$

CHAPTER 5

EXERCISE SOLUTIONS

by J.H. Lin

§5.1

Assume $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. It is obvious that (5.1.1a) and (5.1.1b) are satisfied.

For any two numbers α and β, (5.1.1c)-(5.1.1e) follow easily.

These relations also hold for real and complex numbers.

§5.2

$$\begin{aligned} \|f+g\| &= \langle f^* + g^*, f+g\rangle^{\frac{1}{2}} \\ &= \left[\langle f^*, f\rangle + \langle f^*, g\rangle + \langle g^*, f\rangle + \langle g^*, g\rangle\right]^{\frac{1}{2}} \\ &= \left[\|f\|^2 + \langle f^*, g\rangle + \langle g^*, f\rangle + \|g\|^2\right]^{\frac{1}{2}} \\ &\neq \|f\| + \|g\| \quad \text{in general} \end{aligned}$$

So, $\|f\|$ is a nonlinear real functional.

§5.3

$\{\hat{x}, \hat{y}, \hat{z}\}$ is an orthonormal basis set. From (5.1.26), the identity operator can be expressed as:

$$\bar{\mathbf{I}} = (\hat{x}\,)\langle\,\hat{x}) + (\hat{y}\,)\langle\,\hat{y}) + (\hat{z}\,)\langle\,\hat{z}).$$

Now we have two vectors $\mathbf{a}$ and $\mathbf{b}$

$$\begin{aligned} \mathbf{a}\cdot\bar{\mathbf{I}}\cdot\mathbf{b} &= \langle\mathbf{a},\hat{x}\rangle\,\langle\hat{x},\mathbf{b}\rangle + \langle\mathbf{a},\hat{y}\rangle\,\langle\hat{y},\mathbf{b}\rangle + \langle\mathbf{a},\hat{z}\rangle\,\langle\hat{z},\mathbf{b}\rangle \\ &= a_x b_x + a_y b_y + a_z b_z \\ &= \mathbf{a}\cdot\mathbf{b}. \end{aligned}$$

So, the identity operator can be inserted between an inner product and yet leaves its value unchanged.

§5.4

There are two different orthonormal basis sets: $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ and $\{\hat{x}'_1, \hat{x}'_2, \hat{x}'_3\}$. One vector $\mathbf{a}$ can be expressed in terms of either of the two basis as:

$$\mathbf{a} = a_i\hat{x}_i = a'_i\hat{x}'_i.$$

The a_i and a_i' can be related by a transformation matrix, namely,

$$a_i' = T_{ij}a_j.$$

Consider another vector **b** and form the inner product of **a** and **b**.

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b}\rangle &= a_i'b_i' = T_{ij}a_jT_{ik}b_k \\ &= a_jT_{ij}T_{ik}b_k = a_j\delta_{jk}b_k \\ &= a_jb_j.\end{aligned}$$

The above equation is just the analogue of Parseval's Theorem in 3D vector space and also states that an inner product between two vectors is invariant with respect to their representation.

§5.5

(a)

$$\left\langle e^{\frac{-im\pi x}{a}}, e^{\frac{in\pi x}{a}}\right\rangle = \int_{-a}^{a} dx e^{\frac{i(n-m)\pi x}{a}}.$$

If $n = m$, then the inner product $= 2a$.
If $n \neq m$, inner product $= \frac{a}{i(n-m)\pi}e^{\frac{i(n-m)\pi x}{a}}|_{-a}^{a} = 0$.
Hence, $e^{\frac{in\pi x}{a}}$ is orthogonal.
Let the basis be $\frac{1}{\sqrt{2a}}e^{\frac{in\pi x}{a}}$, then we have an orthonormal basis.

(b) If

$$f(x) = \sum_{n=-\infty}^{\infty} b_n \frac{1}{\sqrt{2a}}e^{\frac{in\pi x}{a}},$$

then

$$\begin{aligned}\left\langle \frac{1}{\sqrt{2a}}e^{\frac{-im\pi x}{a}}, f\right\rangle &= \sum_{n=-\infty}^{\infty} b_n \left\langle \frac{1}{\sqrt{2a}}e^{\frac{-im\pi x}{a}}, \frac{1}{\sqrt{2a}}e^{\frac{in\pi x}{a}}\right\rangle \\ &= \sum_{n=-\infty}^{\infty} b_n\delta(n-m) = b_m.\end{aligned}$$

Therfore,

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2a}}e^{\frac{in\pi x}{a}} \left\langle \frac{1}{\sqrt{2a}}e^{\frac{-in\pi x}{a}}, f\right\rangle.$$

Assume that there is another square integrable function $g(x)$ for $-a < x < a$. Then,

$$\langle g^*, f\rangle = \sum_{n=-\infty}^{\infty} \left\langle g^*, \frac{1}{\sqrt{2a}}e^{\frac{in\pi x}{a}}\right\rangle \left\langle \frac{1}{\sqrt{2a}}e^{\frac{-in\pi x}{a}}, f\right\rangle.$$

We can define an identity operator as

$$I = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2a}} e^{\frac{in\pi x}{a}} \rangle\langle \frac{1}{\sqrt{2a}} e^{\frac{-in\pi x}{a}}$$
$$= \sum_{n=-\infty}^{\infty} f_n \rangle\langle f_n^*.$$

Now

$$\langle x^*, I x' \rangle = \sum_{n=-\infty}^{\infty} \langle x^*, f_n \rangle \langle f_n^*, x' \rangle$$
$$= \sum_{n=-\infty}^{\infty} f_n(x)\, f_n^*(x') = \frac{1}{2a} \sum_{n=-\infty}^{\infty} e^{\frac{in\pi(x-x')}{a}}$$
$$= \delta(x - x').$$

So $\delta(x - x')$ is the coordinate space of the representation of I.

(c)

$$\langle f^*, g \rangle = \int_{-a}^{a} dx f^*(x)\, g(x)$$
$$= \int_{-a}^{a} dx \sum_{n=-\infty}^{\infty} \left\langle \frac{1}{\sqrt{2a}} e^{\frac{-in\pi x}{a}}, f \right\rangle^* \frac{1}{\sqrt{2a}} e^{\frac{-in\pi x}{a}}$$
$$\sum_{n=-\infty}^{\infty} \left\langle \frac{1}{\sqrt{2a}} e^{\frac{-im\pi x}{a}}, g \right\rangle \cdot \frac{1}{\sqrt{2a}} e^{\frac{im\pi x}{a}}$$
$$= \sum_{n,m=-\infty}^{\infty} \left\langle \frac{1}{\sqrt{2a}} e^{\frac{-in\pi x}{a}}, f \right\rangle^* \left\langle \frac{1}{\sqrt{2a}} e^{\frac{-im\pi x}{a}}, g \right\rangle \int_{-a}^{a} \frac{1}{2a} e^{\frac{i(m-n)\pi x}{a}} dx$$
$$= \sum_{n,m=-\infty}^{\infty} \left\langle \frac{1}{\sqrt{2a}} e^{\frac{-in\pi x}{a}}, f \right\rangle^* \left\langle \frac{1}{\sqrt{2a}} e^{\frac{-im\pi x}{a}}, g \right\rangle \delta(m - n)$$
$$= \sum_{n=-\infty}^{\infty} \left\langle \frac{1}{\sqrt{2a}} e^{\frac{-in\pi x}{a}}, f \right\rangle^* \left\langle \frac{1}{\sqrt{2a}} e^{\frac{-in\pi x}{a}}, g \right\rangle$$
$$\triangleq \sum_{n=-\infty}^{\infty} f^*(n)\, g(n).$$

§5.6

Assume that $\mathbf{f}$ and $\mathbf{g}$ are vectors and $\overline{G}$ a matrix. Then,

$$\langle \mathbf{f}, \overline{G}\mathbf{g} \rangle = f_i G_{ij} g_j.$$

If $\overline{G}$ is symmetric, i.e. $G_{ij} = G_{ji}$, then

$$\langle \mathbf{f}, \overline{G}\mathbf{g} \rangle = f_i G_{ji} g_j = g_j G_{ji} f_i = \langle \mathbf{g}, \overline{G}\mathbf{f} \rangle,$$

which is the same definition as that of symmetric operator.

(b)

$$\langle \mathbf{f}^*, \overline{G}\mathbf{g} \rangle = f_i^* G_{ij} g_j$$

If $\overline{G}$ is Hermitian, i.e. $G_{ij} = G_{ji}^*$, then

$$\langle \mathbf{f}^*, \overline{G}\mathbf{g} \rangle = f_i^* G_{ji}^* g_j = g_j G_{ji}^* f_i^* = \left(g_j^* G_{ji} f_i\right)^* = \langle \mathbf{g}^*, \overline{G}\mathbf{f} \rangle^*,$$

which is the same definition as that of Hermitian operator.

§5.7

(a) The matrix representation of $\mathcal{D}$ is

$$\begin{aligned}
&\langle f_m, \mathcal{D} f_n \rangle \\
&= \int_0^a f_m(x) \left[\frac{d^2}{dx^2} + k^2(x)\right] f_n(x) dx \\
&= \int_0^a f_m(x) \frac{d^2}{dx^2} f_n(x) dx + \int_0^a k^2(x) f_m(x) f_n(x) dx \\
&= \int_0^a \left\{\frac{d}{dx}\left[f_m(x) + \frac{df_n(x)}{dx}\right] - \frac{df_m}{dx}\frac{df_n}{dx}\right\} dx + \int_0^a k^2(x) f_m(x) f_n(x) dx \\
&= -\int_0^a \frac{df_m}{dx}\frac{df_n}{dx} dx + \int_0^a k^2(x) f_m(x) f_n(x) dx \\
&\left(\text{since} \quad f_m(0) = f_m(a), \quad f_m(x)\frac{df_n(x)}{dx}\Big|_0^a = 0\right) \\
&= -\int_0^a \left[\frac{d}{dx}\left(\frac{df_m}{dx} f_n\right) - \frac{d^2 f_m}{dx^2} f_n\right] dx + \int_0^a k^2(x) f_m(x) f_n(x) dx \\
&= \int_0^a \frac{d^2 f_m(x)}{dx^2} f_n(x) dx + \int_0^a k^2(x) f_m(x) f_n(x) dx \\
&= \langle f_n, \mathcal{D} f_m \rangle .
\end{aligned}$$

Thus, the matrix is symmetric.

Case I. $|m - n| \geq 2$. Then there is no overlapping part of $f_m(x)$ and $f_n(x)$. So $\langle f_m, \mathcal{D} f_n \rangle = 0$.

Case II. $|m-n|=1$. Without loss of generality, we assume $m = n+1$.

$$\text{Note: } \frac{df_n(x)}{dx} = \begin{cases} 1/\Delta, & (n-1)\Delta < x < n\Delta, \\ -1/\Delta, & n\Delta < x < (n+1)\Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} &\langle f_m, \mathcal{D} f_n \rangle \\ &= -\int_{n\Delta}^{(n+1)\Delta} \left(\frac{1}{\Delta}\right)\left(\frac{-1}{\Delta}\right) dx + \int_{n\Delta}^{(n+1)\Delta} k^2(x) \frac{(n+1)\Delta - x}{\Delta} \cdot \frac{x - n\Delta}{\Delta} dx \\ &= \frac{1}{\Delta} + \int_{n\Delta}^{(n+1)\Delta} k^2(x) \left[1 - \left(\frac{x-n\Delta}{\Delta}\right)^2\right] dx. \end{aligned}$$

Case III. $m = n$.

$$\begin{aligned} &\langle f_n, \mathcal{D} f_n \rangle \\ &= -2 \int_{(n-1)\Delta}^{n\Delta} (1/\Delta)^2 dx + \int_{(n-1)\Delta}^{n\Delta} k^2(x) \left[\frac{x-(n-1)\Delta}{\Delta}\right]^2 dx \\ &\qquad + \int_{n\Delta}^{(n+1)\Delta} k^2(x) \left[\frac{(n+1)\Delta - x}{\Delta}\right]^2 dx \\ &= \frac{-2}{\Delta} + \int_{(n-1)\Delta}^{n\Delta} k^2(x) \left[\frac{x-(n-1)\Delta}{\Delta}\right]^2 dx \\ &\qquad + \int_{n\Delta}^{(n+1)\Delta} k^2(x) \left[\frac{x-(n+1)\Delta}{\Delta}\right]^2 dx. \end{aligned}$$

Therefore the matrix assumes the following form

$$\begin{bmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} & 0 & 0 & \cdots & \cdots \\ \mathcal{D}_{21} & \mathcal{D}_{22} & \mathcal{D}_{23} & 0 & \cdots & \cdots \\ 0 & \mathcal{D}_{32} & \mathcal{D}_{33} & \mathcal{D}_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \mathcal{D}_{N-1,N-2} & \mathcal{D}_{N-1,N-1} \end{bmatrix},$$

where $\mathcal{D}_{mn} = \mathcal{D}_{nm}$. State vector $g\rangle$ can also be represented in this space.

If

$$g\rangle = \sum_{n=1}^{N-1} b_n\, f_n\rangle,$$

then

$$\langle f_m, g\rangle = \sum_{n=1}^{N-1} \langle f_m, f_n\rangle\, b_n.$$

We can solve for b_n to get this representation. Let $f\rangle = \sum_{n=1}^{N-1} a_n\, f_n\rangle$. Then the differential equation becomes

$$\mathcal{D}f\rangle = \sum_{n=1}^{N-1} a_n \mathcal{D} f_n\rangle = g\rangle. \tag{1}$$

Forming the inner product of (1) with f_m, we have

$$\sum_{n=1}^{N-1} \langle f_m, \mathcal{D} f_n\rangle\, a_n = \langle f_m, g\rangle\,.$$

This is a matrix equation:

$$\overline{\boldsymbol{D}} \cdot \mathbf{a} = \mathbf{g},$$

where $\overline{\boldsymbol{D}}$ is the matrix representation of $\mathcal{D}$ we have just found, and $\mathbf{g} = [g_1, \cdots g_{N-1}]^t$, $g_i \triangleq \langle f_i, g\rangle$.

(b) Assume $k(x)$ is a constant. From (a) when $|m-n| = 1$,

$$\begin{aligned}
\mathcal{D}_{mn} &= \frac{1}{\Delta} + k^2 \int_{n\Delta}^{(n+1)\Delta} \left[1 - \left(\frac{x - n\Delta}{\Delta}\right)^2\right] dx \\
&= \frac{1}{\Delta} + k^2 \left[\Delta - \frac{1}{\Delta^2} \frac{(x-n\Delta)^3}{3}\Big|_{n\Delta}^{(n+1)\Delta}\right] \\
&= \frac{1}{\Delta} + k^2 \left[\Delta - \frac{\Delta}{3}\right] = \frac{1}{\Delta} + \frac{2k^2\Delta}{3}.
\end{aligned}$$

When $m = n$,

$$\begin{aligned}
\mathcal{D}_{nn} &= \frac{-2}{\Delta} + k^2 \left[\int_{(n-1)\Delta}^{n\Delta} \left(\frac{x-(n-1)\Delta}{\Delta}\right)^2 dx \right. \\
&\qquad \left. + \int_{n\Delta}^{(n+1)\Delta} \left(\frac{x-(n+1)\Delta}{\Delta}\right)^2 dx\right] \\
&= \frac{-2}{\Delta} + \frac{2k^2\Delta}{3}.
\end{aligned}$$

So

$$\overline{\mathcal{D}} = \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 & 0 & 0 & \cdots & \cdots \\ \mathcal{D}_2 & \mathcal{D}_1 & \mathcal{D}_2 & 0 & \cdots & \cdots \\ 0 & \mathcal{D}_2 & \mathcal{D}_1 & \mathcal{D}_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \mathcal{D}_2 & \mathcal{D}_1 \end{bmatrix}_{(N-1)\times(N-1)},$$

where $\mathcal{D}_1 = -2 + \frac{2k^2\Delta^2}{3}$, $\mathcal{D}_2 = 1 + \frac{2k^2\Delta^2}{3}$. The matrix equation becomes $\overline{\mathcal{D}} \cdot \mathbf{a} = \Delta \mathbf{g}$. Using finite difference

$$\frac{d^2 f(x)}{dx^2}\Big|_{n\Delta} \simeq \frac{f(n\Delta + \Delta) - 2f(n\Delta) + f(n\Delta - \Delta)}{\Delta^2},$$

then we have

$$\frac{f((n+1)\Delta) - 2f(n\Delta) + f((n-1)\Delta)}{\Delta^2} + k^2 f(n\Delta) = g(n\Delta),$$

or

$$f((n+1)\Delta) + (\Delta^2 k^2 - 2) f(n\Delta) + f((n-1)\Delta) = \Delta^2 g(n\Delta).$$

Expressed in matrix form, this is

$$\begin{bmatrix} -2+\Delta^2k^2 & 1 & 0 & \cdots & \cdots & \cdots \\ 1 & -2+\Delta^2k^2 & 1 & 0 & \cdots & \cdots \\ 0 & 1 & -2+\Delta^2k^2 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ \cdots & \cdots & \cdots & 0 & 1 & -2+\Delta^2k^2 \end{bmatrix}_{(N-1)\times(N-1)}$$

$$\begin{bmatrix} f(\Delta) \\ \vdots \\ \vdots \\ f((N-1)\Delta) \end{bmatrix} = \Delta^2 \begin{bmatrix} g(\Delta) \\ \vdots \\ \vdots \\ g((N-1)\Delta) \end{bmatrix}.$$

These two matrix equations from two different schemes are very similar. The bandwidth of two coefficient matrices is 3.

The only difference is that the right-hand side vector $\mathbf{g}$ of the finite-element matrix equation has to be calculated by the inner product (i.e. involving some integration); but for finite-difference, we obtain the vector by having $g(s)$ evaluated at the discretization points. But the latter one may cause some problem if $g(x)$ has some singularities or is not continuous at those points.

(c) Judging form the band structure of these matrix equation, it exhibits the local nature of the differential operator.

(d) When $f(0) \neq 0$ and $f(a) \neq 0$,

$$f(x) = \sum_{n=0}^{N} a_n f_n(x),$$

$$f_0(x) = \begin{cases} \frac{\Delta - x}{\Delta}, & 0 < x < \Delta, \\ 0, & \text{otherwise}, \end{cases}$$

$$f_N(x) \triangleq \begin{cases} \frac{x-(N-1)\Delta}{\Delta}, & (N-1)\Delta < x < N\Delta, \\ 0, & \text{otherwise}. \end{cases}$$

Note that in order to satisfy boundary conditions, $a_0 = f(0)$ and $a_N = f(N\Delta)$.

The matrix equation becomes

$$\begin{bmatrix} \mathcal{D}_3 & \mathcal{D}_2 & 0 & \cdots & \cdots & \cdots \\ \mathcal{D}_2 & \mathcal{D}_1 & \mathcal{D}_2 & 0 & \cdots & \cdots \\ 0 & \mathcal{D}_2 & \mathcal{D}_1 & \mathcal{D}_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \mathcal{D}_\epsilon & \mathcal{D}_1 & \mathcal{D}_2 \\ \cdots & \cdots & \cdots & 0 & \mathcal{D}_2 & \mathcal{D}_3 \end{bmatrix}_{N\times N} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_{N-1} \\ a_N \end{bmatrix} = \begin{bmatrix} g(0) \\ g(\Delta) \\ \vdots \\ \vdots \\ g((N-1)\Delta) \\ g(N\Delta) \end{bmatrix}$$

where $\mathcal{D}_3 = -1 + \frac{k^2\Delta^2}{3}$.

Although $g(0)$ and $g(N\Delta)$ are not defined in our case, it does not matter since a_0 and a_1 are known from the boundary condition. So we can rearrange the above matrix equation as follows:

$$\begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 & 0 & \cdots & \cdots \\ \mathcal{D}_2 & \mathcal{D}_1 & \mathcal{D}_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \cdots & \cdots & \cdots & \mathcal{D}_2 & \mathcal{D}_1 \end{bmatrix}_{(N-1)\times(N-1)} \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ \vdots \\ a_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} g(\Delta) - \mathcal{D}_2 a_0 \\ g(2\Delta) \\ \vdots \\ g((N-2)\Delta) \\ g((N-1)\Delta) - \mathcal{D}_2 a_N \end{bmatrix}.$$

So the solution depends on the specified boundary conditions.

§5.8

(a) We already know that

$$\langle f_i, f_j \rangle = 0, \quad i \neq j,$$

and

$$|\langle f_i, f_i\rangle| > 0.$$

Trying to prove $\sum_{i=1}^{N} a_i\, f_i\rangle = 0$ implies that $a_i = 0$, $i = 1, \ldots, N$. Forming the inner product of $\sum_{i=1}^{N} a_i\, f_i\rangle = 0$ with f_j, we have

$$\left\langle f_j, \sum_{i=1}^{N} a_i\, f_i \right\rangle = 0 \rightarrow \sum_{i=1}^{N} a_i \langle f_j, f_i\rangle = 0$$

$$\rightarrow \quad a_j \langle f_j, f_j\rangle + \sum_{\substack{j \neq 1 \\ i=1}}^{N} a_i \langle f_j, f_i\rangle = 0.$$

The second term is zero since $\langle f_j, f_i\rangle = 0$, when $i \neq j$. Also owing to $|\langle f_j, f_j\rangle| \neq 0$, a_j has to be zero for any j. So, $f_i\rangle$ is linearly independent.

(b) Let

$$\mathbf{a} = \hat{x} + i\hat{y} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Then,

$$\mathbf{a}^\dagger = (1 \quad -i). \qquad \mathbf{a}^t = (1 \quad i),$$

and

$$\mathbf{a}^\dagger \cdot \mathbf{a} = (1 \quad -i)\begin{pmatrix} 1 \\ i \end{pmatrix} = 1 + 1 = 2 \neq 0,$$

$$\mathbf{a}^t \cdot \mathbf{a} = (1 \quad i)\begin{pmatrix} 1 \\ i \end{pmatrix} = 1 - 1 = 0.$$

Therefore, for a noncomplex inner product, a nonzero length vector may have zero self inner product

(c) Hermitian operator is defined by introducing the complex inner product. Since $\langle f^*, f\rangle \geq 0$, the equality holds only when $f = 0$. Hence, the problem that exists in symmetric operator does not occur here.

(d) (1) If $e_i\rangle, i = 1, \cdots, N$, is a set of mutually orthonormal vectors that spans an N dimensinal space, then by definition, any vector $g\rangle$ in the space can be expressed as

$$g\rangle = \sum_{i=1}^{N} g_i e_i\rangle.$$

where $g_i = \langle e_i^*, g\rangle$. If $e_i\rangle$ is not complete, then there exists a vector $h\rangle$ in the space such that $\langle e_i, h\rangle = 0, i = 1, \cdots, N$. Such a vector can also be expressed as

$$h\rangle = \sum_{i=1}^{N} h_j e_j\rangle,$$

where $h_j = \langle e_j^*, h\rangle$. But $h_j = 0$, implying that $h\rangle = 0$.

(2) Since the eigenvectors of a Hermitian matrix are linearly independent and orthogonal, from (1), they form a complete set.

(3) A set of orthogonal vectors can be obtained by Gram-Schmidt procedure. First we find out $f_i\rangle$'s which satisfy $\mathcal{G}f_i\rangle = \lambda f_i\rangle$. Then, let $g_1\rangle = f_1\rangle$. Next,

$$g_2\rangle = f_2\rangle - g_1\rangle\langle g_i^*, f_2\rangle / \langle g_1^*, g_1\rangle.$$

Continuing this process, we have

$$g_n\rangle = f_n\rangle - \sum_{i=1}^{n-1} g_i\rangle\langle g_i^*, f_n\rangle / \langle g_i^*, g_i\rangle.$$

Thus, these $g_n\rangle$ vectors are orthogonal to each other and they are also eigenvectors of a Hermitian operator.

§5.9

Wave equation in an inhomogeneous and unbounded medium can be expressed as

$$\nabla \times \overline{\mu}^{-1} \nabla \times \mathbf{E} - \omega^2 \epsilon \mathbf{E} = i\omega \mathbf{J},$$

where $\mathbf{E}$ satisfies the radiation condition. Expressed in terms of an operator, we have $\mathcal{L}\mathbf{E} = i\omega\mathbf{J}$.

(a)

$$\begin{aligned}
&\langle \mathbf{E}_1, \mathcal{L}\mathbf{E}_2\rangle \\
&= \int_V d\mathbf{r}\mathbf{E}_1 \cdot (\nabla \times \mu^{-1} \nabla \times \mathbf{E}_2 - \omega^2 \epsilon\, \mathbf{E}_2) \\
&= \int_V d\mathbf{r}\mathbf{E}_1 \cdot \nabla \times \mu^{-1} \nabla \times \mathbf{E}_2 - \int_V d\mathbf{r}\mathbf{E}_2 \cdot \omega^2 \epsilon\, \mathbf{E}_1.
\end{aligned}$$

The first term of the above equation is

$$\begin{aligned}
&\int_V d\mathbf{r}\mathbf{E}_1 \cdot \nabla \times \mu^{-1} \nabla \times \mathbf{E}_2 \\
&= \int_V d\mathbf{r}\nabla \cdot (-\mathbf{E}_1 \times \nabla \times \mu^{-1} \nabla \times \mathbf{E}_2) + \mu^{-1} \nabla \times \mathbf{E}_2 \cdot \nabla \times \mathbf{E}_1 \\
&= -\oint_S \mathbf{E}_1 \times \mu^{-1} \nabla \times \mathbf{E}_2 \cdot \hat{n} dS + \int_V d\mathbf{r}\mu^{-1} \nabla \times \mathbf{E}_2 \cdot \nabla \times \mathbf{E}_1.
\end{aligned}$$

When $S \to \infty$, the fields, which are produced by sources of finite extent, become plane wave in the far field. Hence, $\nabla \to i\mathbf{k}$, which is the case for plane waves. Consequently, the integrand of the first integral in the above can be written as

$$\mu^{-1}\mathbf{E}_1 \times (i\mathbf{k} \times \mathbf{E}_2).$$

Using the fact that $\mathbf{k}\cdot\mathbf{E}_1 = 0$, the above can be reduced to $i\mu^{-1}\mathbf{k}(\mathbf{E}_1 - \mathbf{E}_2)$. Hence, the integral is symmetric about $\mathbf{E}_1$ and $\mathbf{E}_2$. So, $\mathcal{L}$ is a symmetric operator.

If $\mathbf{E}_n$'s are independent vectors with respect to the inner product, then the matrix representation of $\mathcal{L}$ can be expressed as

$$\overline{\mathbf{L}} = \begin{bmatrix} \langle \mathbf{E}_1, \mathcal{L}\mathbf{E}_1 \rangle & \langle \mathbf{E}_1, \mathcal{L}\mathbf{E}_2 \rangle & \cdots & \langle \mathbf{E}_1, \mathcal{L}\mathbf{E}_n \rangle \\ \langle \mathbf{E}_2, \mathcal{L}\mathbf{E}_1 \rangle & \langle \mathbf{E}_2, \mathcal{L}\mathbf{E}_2 \rangle & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{E}_n, \mathcal{L}\mathbf{E}_1 \rangle & \cdots & \cdots & \langle \mathbf{E}_n, \mathcal{L}\mathbf{E}_n \rangle \end{bmatrix}$$

Since $\langle \mathbf{E}_i, \mathcal{L}\mathbf{E}_j \rangle = \langle \mathbf{E}_j, \mathcal{L}\mathbf{E}_i \rangle$, $\overline{\mathbf{L}} = \overline{\mathbf{L}}^t$, which is a symmetric matrix.

(b) Since

$$\langle \mathbf{E}_m, \mathcal{L}\mathbf{E}_n \rangle = \langle \mathbf{E}_m, i\omega\mathbf{J}_n \rangle ,$$
$$\langle \mathbf{E}_n, \mathcal{L}\mathbf{E}_m \rangle = \langle \mathbf{E}_n, i\omega\mathbf{J}_m \rangle ,$$

and in (a), we have already proved that

$$\langle \mathbf{E}_m, \mathcal{L}\mathbf{E}_n \rangle = \langle \mathbf{E}_n, \mathcal{L}\mathbf{E}_m \rangle ,$$

the following equality holds:

$$\langle \mathbf{E}_m, \mathbf{J}_n \rangle = \langle \mathbf{E}_n, \mathbf{J}_m \rangle$$

which means that the medium is reciprocal.

(c) When the medium is lossless and bounded, $\mu = \mu^*$ and $\epsilon = \epsilon^*$.

$$\begin{aligned} &\langle \mathbf{E}_1, \mathcal{L}\mathbf{E}_2 \rangle \\ &= \int_V d\mathbf{r}\mathbf{E}_1^* \cdot (\nabla \times \mu^{-1}\nabla \times \mathbf{E}_2 - \omega^2\epsilon\,\mathbf{E}_2) \\ &= \left[\int_V d\mathbf{r}\mathbf{E}_1 \cdot (\nabla \times \mu^{-1}\nabla \times \mathbf{E}_2^* - \omega^2\epsilon\,\mathbf{E}_2^*) \right]^* \\ &= \left[\int_V \mathbf{E}_2^* \cdot (\nabla \times \mu^{-1}\nabla \times \mathbf{E}_1 - \omega^2\epsilon\,\mathbf{E}_1) \right]^* \\ &= \langle \mathbf{E}_2^*, \mathcal{L}\mathbf{E}_1 \rangle^* , \quad \text{so } \mathcal{L} \text{ is Hermitian.} \end{aligned}$$

When the medium is lossy, $\mu \neq \mu^*$ and $\epsilon \neq \epsilon^*$. We are not able to get $\langle \mathbf{E}_1^*, \mathcal{L}\mathbf{E}_2\rangle = \langle \mathbf{E}_2^*, \mathcal{L}\mathbf{E}_1\rangle^*$ since $\langle \mathbf{E}_1^*, \mathcal{L}\mathbf{E}_2\rangle = \langle \mathbf{E}_2^*, \mathcal{L}^*\mathbf{E}_1\rangle^*$, but $\mathcal{L} \neq \mathcal{L}^*$. So $\mathcal{L}$ is not Hermitian when medium is lossy.

§5.10

Given the functional

$$I = \langle f^*, \mathcal{L}f\rangle - \langle f^*, g\rangle - \langle g^*, f\rangle ,$$

its complex conjugate is

$$I^* = \langle f^*, \mathcal{L}f\rangle^* - \langle f^*, g\rangle^* - \langle g^*, f\rangle^* .$$

If $\mathcal{L}$ is self-adjoint, then $\langle f^*, \mathcal{L}f\rangle^* = \langle f^*, \mathcal{L}f\rangle$.
Therefore,

$$I^* = \langle f^*, \mathcal{L}f\rangle - \langle g^*, f\rangle - \langle f^*, g\rangle = I.$$

So, I is a real-valued functional.

§5.11

Equation (5.2.10) is

$$I = \mathbf{a}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a} - \mathbf{a}^\dagger \cdot \mathbf{g} - \mathbf{g}^\dagger \cdot \mathbf{a}.$$

Let $\mathbf{a} = \mathbf{a}_e + \delta\mathbf{a}.$, where $\mathbf{a}_e$ is the solution to $\overline{\mathbf{L}} \cdot \mathbf{a}_e = \mathbf{g}$.
Then,

$$\begin{aligned} I =&(\mathbf{a}_e + \delta\mathbf{a})^\dagger \cdot \overline{\mathbf{L}} \cdot (\mathbf{a}_e + \delta\mathbf{a}) - (\mathbf{a}_e + \delta\mathbf{a})^\dagger \cdot \mathbf{g} - \mathbf{g}^\dagger \cdot (\mathbf{a}_e + \delta\mathbf{a}) \\ =&[\mathbf{a}_e^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}_e - \mathbf{a}\mathbf{e}^\dagger \cdot \mathbf{g} - \mathbf{g}^\dagger \cdot \mathbf{a}_e] \\ &+ [\mathbf{a}_e^\dagger \cdot \overline{\mathbf{L}} \cdot \delta\mathbf{a} + \delta\mathbf{a}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}_e - \delta\mathbf{a}^\dagger \cdot \mathbf{g} - \mathbf{g}^\dagger \cdot \delta\mathbf{a} \\ &+ [\delta\mathbf{a}^\dagger \cdot \overline{\mathbf{L}} \cdot \delta\mathbf{a}]. \end{aligned}$$

Since $\overline{\mathbf{L}}$ is self-adjoint, the second term of the above equation becomes

$$\begin{aligned} &(\delta\mathbf{a}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}_e)^\dagger + \delta\mathbf{a}^\dagger \cdot \mathbf{g} - \delta\mathbf{a}^\dagger \cdot \mathbf{g} - \mathbf{g}^\dagger \cdot \delta\mathbf{a} \\ &= (\delta\mathbf{a}^\dagger \cdot \mathbf{g})^\dagger - \mathbf{g}^\dagger \cdot \delta\mathbf{a} = 0. \end{aligned}$$

Hence,

$$I = (\mathbf{a}_e^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}_e - \mathbf{a}_e^\dagger \cdot \mathbf{g} - \mathbf{g}^\dagger \cdot \mathbf{a}_e) + \delta\mathbf{a}^\dagger \cdot \overline{\mathbf{L}} \cdot \delta\mathbf{a} \tag{1}$$

for any $\mathbf{a} = \mathbf{a}_e + \delta\mathbf{a}$.

(a) If $\overline{\mathbf{L}}$ is a positive definite matrix, then $\delta\mathbf{a}^\dagger \cdot \overline{\mathbf{L}} \cdot \delta\mathbf{a} > 0$. So from (1), I has a minimum at $\mathbf{a}_e$.

(b) If $\overline{\mathbf{L}}$ is a negative definite matrix, then $\delta\mathbf{a}^\dagger \cdot \overline{\mathbf{L}} \cdot \delta\mathbf{a} < 0$. So I has a maximum at $\mathbf{a}_e$.

(c) If $\overline{\mathbf{L}}$ is an indefinite matrix, then $\delta\mathbf{a}^\dagger \cdot \overline{\mathbf{L}} \cdot \delta\mathbf{a}$ may be positive, negative or zero. So, for some $\mathbf{a}_m$, I is a minimum but for some $\mathbf{a}_M$, I a maximum. The picture of I in 3D is like a saddle and $\mathbf{a}_e$ is the saddle point.

§5.12

A linear functional $u = \langle h, f\rangle$. Consider an expression of $u = \frac{\langle f_a^*, g\rangle\langle h^*, f\rangle}{\langle f_a^*, \mathcal{L}f\rangle}$, where $\mathcal{L}$ is a symmetric operator and $\mathcal{L}f = g$, $\mathcal{L}f_a = h$. Now let $f = f_e + \delta f$, $f_a = f_{ae} + \delta f_a$, where f_e and f_{ae} are exact solutions to $\mathcal{L}f = g$ and $\mathcal{L}f_a = h$ respectively. Then,

$$\begin{aligned} u &= \frac{\langle (f_{ae} + \delta f_a), g\rangle\langle (f_e + \delta f), h\rangle}{\langle (f_{ae} + \delta f_a), \mathcal{L}(fe + \delta f)\rangle} \\ &= \frac{[\langle f_{ae}, g\rangle + \langle \delta f_a, g\rangle][\langle f_e,\ h\rangle + \langle \delta f, h\rangle]}{\langle f_{ae}, \mathcal{L}f_e\rangle + \langle f_{ae}, \mathcal{L}\delta f\rangle + \langle \delta f_a, \mathcal{L}f_e\rangle + \langle \delta f_a, \mathcal{L}\delta f\rangle}. \end{aligned}$$

u can be expressed as $u = u_e + \delta u + \delta^2 u + \cdots$, where $u_e = \langle h, f_e\rangle$

Thus,

$$\begin{aligned} &(u_e + \delta u + \delta^2 u + \cdots)\,[\langle f_{ae}, \mathcal{L}f_e\rangle + \langle f_{ae}, \mathcal{L}\delta f\rangle + \langle \delta f_a, \mathcal{L}f_e\rangle + \langle \delta f_a, \mathcal{L}\delta f\rangle] \\ &= \langle f_{ae}, g\rangle\langle f_e, h\rangle + \langle f_{ae}, g\rangle\langle \delta f, h\rangle + \langle \delta f_a, g\rangle\langle f_e, h\rangle + \langle \delta f_a, g\rangle \cdot \langle \delta f, h\rangle. \end{aligned}$$

Considering only the first variation, we have

$$\begin{aligned} &u_e\,(\langle f_{ae}, \mathcal{L}\delta f\rangle + \langle \delta f_a, \mathcal{L}f_e\rangle) + \delta u\langle f_{ae}, \mathcal{L}f_e\rangle \\ &= \langle f_{ae}, g\rangle\langle \delta f, h\rangle + \langle \delta f_a, g\rangle\langle f_e, h\rangle. \end{aligned} \tag{1}$$

Note that

$$\langle f_{ae}, g\rangle = \langle f_{ae}, \mathcal{L}f_e\rangle = \langle \mathcal{L}f_{ae}, f_e\rangle = \langle h, f_e\rangle = u_e.$$

and

$$\langle f_{ae}, \mathcal{L}\delta f\rangle = \langle \mathcal{L}f_{ae}, \delta f\rangle = \langle h, \delta f\rangle.$$

Hence, (1) becomes

$$u_e\,(\langle h, \delta f\rangle + \langle \delta f_a, g\rangle) + \delta u\ u_e = u_e\,(\langle \delta f, h\rangle + \langle \delta f_a, g\rangle)\,.$$

It follows that $u_e \delta u = 0$ or $\delta u = 0$. So $u = \frac{\langle f_a^*, g\rangle\langle h^*, f\rangle}{\langle f_a^*, \mathcal{L}f\rangle}$ is a variational expression.

§5.13

(a)

$$I = -\int_V d\mathbf{r}\, p|\nabla\phi(\mathbf{r})|^2 + \int_V d\mathbf{r}\, k^2 p|\phi(\mathbf{r})|^2 - 2\Re e \int_V d\mathbf{r}\, \phi^*(\mathbf{r})\, s(\mathbf{r}). \tag{5.2.43}$$

is a variational expression.

Let $\phi(\mathbf{r}) = \sum_{n=1}^{N} a_n \phi_n(\mathbf{r})$, where $\phi_n(\mathbf{r})$'s are known basis functions and a_n's are unknowns to be determined.

Then, I can be expressed as

$$I = \mathbf{a}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a} - 2\Re[\mathbf{a}^\dagger \cdot \mathbf{g}],$$

where $\mathbf{a}$ is the column vector which contains a_n's,

$$[\overline{\mathbf{L}}]_{mn} = -\int_V d\mathbf{r} p \nabla \phi_m^* \cdot \nabla \phi_n + \int_V d\mathbf{r} k^2 p \phi_m^* \phi_n$$

and

$$[\mathbf{g}]_n = \int_V d\mathbf{r} \phi_n^*(\mathbf{r}) S(\mathbf{r}).$$

In order for $\delta I = 0$, from (3.2.26), one requires

$$\overline{\mathbf{L}} \cdot \mathbf{a} = \mathbf{g}.$$

Consequently, the solution $\mathbf{a}$ to the above matrix equation is the optimal values of a_n's in the expansion formula, $\phi(\mathbf{r}) = \sum_{n=1}^{N} a_n \phi_n(\mathbf{r})$.

(b) The variational expression for $u = \hat{\phi}(\mathbf{r}_0)$ is

$$u = \hat{\phi}(\mathbf{r}_0) = \frac{\phi(\mathbf{r}_0) \int_V d\mathbf{r} \phi_a^* S}{-\int_V d\mathbf{r} p \nabla \phi_a^* \cdot \nabla \phi + \int_V d\mathbf{r} k^2 p \phi_a^* \phi}.$$

Let

$$\phi(\mathbf{r}) = \sum_{n=1}^{N} a_n \phi_n(\mathbf{r}),$$

$$\phi_a(\mathbf{r}) = \sum_{m=1}^{N} b_m \phi_{am}(\mathbf{r}).$$

Then u can be expressed as

$$u = \frac{\mathbf{b}^\dagger \cdot \mathbf{g}\mathbf{h}^\dagger \cdot \mathbf{a}}{\mathbf{b}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}},$$

where $\mathbf{b}$ and $\mathbf{a}$ are column vectors containing $\mathbf{b}_m$'s and a_n's respectively,

$$[\mathbf{g}]_m = \int_V d\mathbf{r} \phi_{am}^*(\mathbf{r}) S(\mathbf{r}),$$

$$[\mathbf{h}]_n = \phi_n^*(\mathbf{r}_0),$$

$$\text{and} \quad [\overline{\mathbf{L}}]_{mn} = -\int_V d\mathbf{r} p \nabla \phi_{am}^* \nabla \phi_n + \int_V d\mathbf{r} k^2 p \phi_{am}^* \phi.$$

In order for $\delta u = 0$, (5.2.33) to (5.2.36) shows that **a** and **b** must satisfy

$$\overline{\mathbf{L}} \cdot \mathbf{a} = \mathbf{g},$$
$$\overline{\mathbf{L}} \cdot \mathbf{b} = \mathbf{h},$$

and after solving for **a** or **b**, the optimal approximation can be expressed as

$$u = \mathbf{h}^{\dagger} \cdot \mathbf{a} \quad \text{or} \quad u = \mathbf{b} \cdot \mathbf{g}.$$

§5.14

Let the inner product be expressed explicitly as

$$\langle \mathbf{E}_1^*(\mathbf{r}), \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2(\mathbf{r}) \rangle = \int_V d\mathbf{r} \mathbf{E}_1^*(\mathbf{r}) \cdot \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2(\mathbf{r}).$$

Applying the vector identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = B \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$, the equation above can be written as

$$- \int_V d\mathbf{r} \nabla \cdot [\mathbf{E}_1^* \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2] + \int_V d\mathbf{r} \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2 \cdot \nabla \times \mathbf{E}_1^*$$
$$= - \int_S dS \hat{n} \cdot \mathbf{E}_1^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2 + \int_V d\mathbf{r} \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2 \cdot \nabla \times \mathbf{E}_1^*$$

In order for the first term to be zero, $\hat{n} \times \mathbf{E}_1 = 0$ on S or

$$\hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2 = 0 \quad \rightarrow \quad \hat{n} \times i\omega \mathbf{H}_2 = 0 \quad \rightarrow \quad \nabla \times \mathbf{H}_2.$$

So,

$$\langle \mathbf{E}_1^*(\mathbf{r}), \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2(\mathbf{r}) \rangle = \int_V d\mathbf{r} \nabla \times \mathbf{E}_1^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2$$

if $\hat{n} \times \mathbf{E} = 0$ or $\hat{n} \times \mathbf{H} = 0$ on S (a surface enclosing V).

§5.15

(a) Original equation is

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) = i\omega \mathbf{J}(\mathbf{r}).$$

The linear operator in this equation can be identified as

$$\mathcal{L} = (\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times) - \omega^2 \overline{\boldsymbol{\epsilon}}.$$

The adjoint operator $\mathcal{L}^a$ satisfies

$$\langle \mathbf{E}_1^* \cdot \mathcal{L} \mathbf{E}_2 \rangle = \langle (\mathcal{L}^a \mathbf{E}_1)^*, \mathbf{E}_2 \rangle.$$

Now,

$$
\begin{aligned}
&\langle \mathbf{E}_1^*, \mathcal{L}\mathbf{E}_2\rangle \\
&= \int_V d\mathbf{r}\mathbf{E}_1^* \cdot \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2 - \omega^2 \int_V d\mathbf{r}\mathbf{E}_1^* \cdot \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_2 \\
&= \int_V d\mathbf{r}\nabla \times \mathbf{E}_1^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2 - \omega^2 \int_V d\mathbf{r}(\overline{\boldsymbol{\epsilon}}^\dagger \cdot \mathbf{E}_1)^* \cdot \mathbf{E}_2.
\end{aligned}
$$

The reason why we obtain the first term as such has been discussed in 5.14 and appropriate boundary conditions, $n \times \mathbf{E} = 0$ or $\hat{n} \times \mathbf{H} = 0$ on S, are satisfied.

The first term in the above equation can be written as

$$
\begin{aligned}
&\int_V d\mathbf{r}(\overline{\boldsymbol{\mu}}^{\dagger^{-1}} \cdot \nabla \times \mathbf{E}_1)^* \cdot \nabla \times \mathbf{E}_2 \\
&= \int_S dS\hat{n} \cdot \mathbf{E}_2 \times (\overline{\boldsymbol{\mu}}^{/dagger^{-1}} \cdot \nabla \times \mathbf{E}_1)^* + \int_V d\mathbf{r}\nabla \times (\overline{\boldsymbol{\mu}}^{/dagger^{-1}} \cdot \nabla \times \mathbf{E}_1)^* \cdot \mathbf{E}_2.
\end{aligned}
$$

Vector identity again is invoked to derived the above equation and recalling those boundary conditions mentioned above, the first term is zero.

Hence,

$$
\begin{aligned}
&\langle \mathbf{E}_1^*, \mathcal{L}\mathbf{E}_2\rangle \\
&= \int_V d\mathbf{r}(\nabla \times \overline{\boldsymbol{\mu}}^{\dagger^{-1}} \cdot \nabla \times \mathbf{E}_1)^* \cdot \mathbf{E}_2 - \omega^2 \int_V d\mathbf{r}(\overline{\boldsymbol{\epsilon}}^\dagger \cdot \mathbf{E}_1)^* \cdot \mathbf{E}_2 \\
&= \langle (\mathcal{L}^a \mathbf{E}_1)^*, \mathbf{E}_2\rangle,
\end{aligned}
$$

where $\mathcal{L}^a = \nabla \times \overline{\boldsymbol{\mu}}^{\dagger^{-1}} \cdot \nabla \times -\omega^2 \overline{\boldsymbol{\epsilon}}^\dagger$.

So the auxiliary equations is

$$\nabla \times (\overline{\boldsymbol{\mu}}^\dagger)^{-1} \cdot \nabla \times \mathbf{E}_a(\mathbf{r}) - \omega^2 \overline{\boldsymbol{\epsilon}}^\dagger \cdot \mathbf{E}_a(\mathbf{r}) = i\omega \mathbf{J}_a(\mathbf{r}).$$

(b) We want to find a variational expression for $u = \hat{a} \cdot \mathbf{E}(\mathbf{r}_0)$.

From (5.3.8) - (5.3.10),

$$
\begin{aligned}
\hat{a} \cdot \hat{\mathbf{E}}(\mathbf{r}_0) &= \frac{\langle \mathbf{E}_a^*, i\omega \mathbf{J}\rangle \langle \mathbf{a}\delta(\mathbf{r} - \mathbf{r}_0), \mathbf{E}\rangle}{\langle \mathbf{E}_a^*, \mathcal{L}\mathbf{E}\rangle} \\
&= \frac{\left(i\omega \int_V d\mathbf{r}\mathbf{E}_a^* \cdot \mathbf{J}\right)(\hat{a} \cdot \mathbf{E}(\mathbf{r}_0))}{\int_V d\mathbf{r}\mathbf{E}_a^* \cdot \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} - \int_V d\mathbf{r}\mathbf{E}_a^* \cdot \omega^2 \overline{\boldsymbol{\epsilon}}^\dagger \cdot \mathbf{E}_a} \\
&= \frac{i\omega \hat{a} \cdot \mathbf{E}(\mathbf{r}_0) \int_V d\mathbf{r}\mathbf{E}_a^* \cdot \mathbf{J}}{\int_V d\mathbf{r}\nabla \times \mathbf{E}_a^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} - \omega^2 \int_V d\mathbf{r}\mathbf{E}_a^* \cdot \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r})}
\end{aligned}
$$

where the boundary condition $\hat{n}\times\mathbf{E}_a = 0$ or $\hat{n}\times\mathbf{H} = 0$ on S are imposed.

§5.16

A partial differental equation is given by

$$\nabla^2\phi + \mathbf{a}\cdot\nabla\phi + k^2\phi = S(\mathbf{r}),$$

where $\mathbf{a}$ is a constant real vector and k^2 is a constant real scalar.

(a) Linear operator is recognized as

$$\mathcal{L} = \nabla^2 + \mathbf{a}\cdot\nabla + k^2.$$

If $\langle\phi_1^*, \mathcal{L}\phi_2\rangle = \langle(\mathcal{L}\phi_1)^*, \phi_2\rangle$, then $\mathcal{L}$ is self-adjoint.

Now,

$$\langle\phi_1^*, \mathcal{L}\phi_2\rangle = \int_V d\mathbf{r}\phi_1^*\nabla^2\phi_2 + \int_V d\mathbf{r}\phi_1^*\mathbf{a}\cdot\nabla\phi_2 + \int_V d\mathbf{r}\phi_1^* k^2\phi_2.$$

Consider the second term in the right-hand side:

$$\int_V d\mathbf{r}\phi_1^*\mathbf{a}\cdot\nabla\phi_2 = \mathbf{a}\cdot\int_V d\mathbf{r}\phi_1^*\nabla\phi_2$$

$$= \mathbf{a}\cdot\left[\oint_S \phi_1^*\phi_2\hat{n}dS - \int_V d\mathbf{r}\phi_2\nabla\phi_1^*\right].$$

If $\phi = 0$ on S, then

$$\mathbf{a}\cdot\int_V d\mathbf{r}\phi_1^*\nabla\phi_2 = -\mathbf{a}\cdot\int_V d\mathbf{r}\phi_2\nabla\phi_1^*.$$

Recognizing that there is a minus sign in the right-hand side, this $\mathcal{L}$ cannot satisfy the condition of self-adjointness. So, $\mathcal{L}$ is not self-adjoint.

(b) We want to find a $\mathcal{L}^a$ to satisfy

$$\langle\phi_1^*, \mathcal{L}\phi_2\rangle = \langle(\mathcal{L}^a\phi_1)^*, \phi_2\rangle.$$

It is well known that $(\nabla^2 + k^2)$ is a self-adjoint operator, where boundary conditions $\phi = 0$ or $\frac{\partial\phi}{\partial n} = 0$ are imposed on S.

So from the fact above and (a), we can directly write down $\mathcal{L}^a$ as

$$\mathcal{L}^a = \nabla^2 - \mathbf{a}\cdot\nabla + k^2.$$

Therefore, the adjoint equation is

$$\nabla^2\phi_a - \mathbf{a}\cdot\nabla\phi_a + k^2\phi_a = S_a(\mathbf{r}).$$

(c) For a non self-adjoint problem, a variational functional I can be expressed as (from (5.3.3))

$$I = \langle \phi_a^*, \mathcal{L}\phi \rangle - \langle \phi_a^*, S \rangle - \langle S_a^*, \phi \rangle$$
$$= \int_V d\mathbf{r}\phi_a^*(\nabla^2\phi + \mathbf{a}\cdot\nabla\phi + k^2\phi) - \int_V d\mathbf{r}\phi_a^* S - \int_V d\mathbf{r} S_a^*\phi.$$

§5.17

(a) Maxwell's equations can be shown as follows:

$$\begin{cases} \nabla\times\mathbf{E} - i\omega\overline{\mu}\cdot\mathbf{H} = 0 \\ \nabla\times\mathbf{H} + i\omega\overline{\epsilon}\cdot\mathbf{E} = \mathbf{J}. \end{cases}$$

These two equations can be written in the matrix form:

$$\begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix}\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} + i\omega\begin{bmatrix} \overline{\epsilon} & 0 \\ 0 & -\overline{\mu} \end{bmatrix}\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathbf{J} \\ 0 \end{bmatrix}.$$

Let $\mathcal{L}$ be defined as $\begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix} + i\omega\begin{bmatrix} \overline{\epsilon} & 0 \\ 0 & -\overline{\mu} \end{bmatrix}$. Then, the matrix equation is of the form $\mathcal{L}f = S$, where $f = \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}$ and $S = \begin{bmatrix} \mathbf{J} \\ 0 \end{bmatrix}$.

(b) If $\mathcal{L}$ is self-adjoint, $\langle f_1^*, \mathcal{L}f_2 \rangle = \langle f_2^*, \mathcal{L}f_1 \rangle^*$. Now,

$$\langle f_1^*, \mathcal{L}f_2 \rangle = \int_V d\mathbf{r}[\mathbf{E}_1^*\,\mathbf{H}_1^*]\cdot\begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix}\begin{bmatrix} \mathbf{E}_2 \\ \mathbf{H}_2 \end{bmatrix}$$
$$+ i\omega\int_V d\mathbf{r}[\mathbf{E}_1^*\,\mathbf{H}_1^*]\cdot\begin{bmatrix} \overline{\epsilon} & 0 \\ 0 & -\overline{\mu} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{E}_2 \\ \mathbf{H}_2 \end{bmatrix} \quad (1)$$

The first term of the above equation can be written using vector identity as:

$$\int_V d\mathbf{r}[\mathbf{E}_1^*\,\mathbf{H}_1^*]\cdot\begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix}\begin{bmatrix} \mathbf{E}_2 \\ \mathbf{H}_2 \end{bmatrix}$$
$$= -\int_V d\mathbf{r}[\nabla\cdot, \nabla\cdot]\begin{bmatrix} 0 & \mathbf{E}_1^*\times \\ \mathbf{H}_1^*\times & 0 \end{bmatrix}\begin{bmatrix} \mathbf{E}_2 \\ \mathbf{H}_2 \end{bmatrix}$$
$$+ \int_V d\mathbf{r}[\mathbf{E}_2\,\mathbf{H}_2]\cdot\begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix}\begin{bmatrix} \mathbf{E}_1^* \\ \mathbf{H}_1^* \end{bmatrix}$$
$$= -\int_S dS\hat{n}\cdot\begin{bmatrix} 0 & \mathbf{E}_1^*\times \\ \mathbf{H}_1^*\times & 0 \end{bmatrix}\begin{bmatrix} \mathbf{E}_2 \\ \mathbf{H}_2 \end{bmatrix} + \int_V d\mathbf{r}[\mathbf{E}_2\,\mathbf{H}_2]\cdot\begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix}\begin{bmatrix} \mathbf{E}_1^* \\ \mathbf{H}_1^* \end{bmatrix}.$$

If $\mathbf{E}=0$ or $\hat{n}\times\mathbf{H}=0$ on S (or vice versa), then the first integral is zero. Therefore, we are left with

$$\int_V d\mathbf{r}[\mathbf{E}_1^*\,\mathbf{H}_1^*]\cdot\begin{bmatrix}0 & \nabla\times\\ \nabla\times & 0\end{bmatrix}\begin{bmatrix}\mathbf{E}_2\\ \mathbf{H}_2\end{bmatrix}=$$

$$\left\{\int_V d\mathbf{r}[\mathbf{E}_2^*\,\mathbf{H}_2^*]\begin{bmatrix}0 & \nabla\times\\ \nabla\times & 0\end{bmatrix}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{H}_1\end{bmatrix}\right\}^*.$$

Next, considering the second integral of (1),

$$i\omega\int_V d\mathbf{r}[\mathbf{E}_1^*\,\mathbf{H}_1^*]\cdot\begin{bmatrix}\overline{\epsilon} & 0\\ 0 & -\overline{\mu}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{E}_2\\ \mathbf{H}_2\end{bmatrix}$$

$$=\left\{-i\omega\int_V d\mathbf{r}[\mathbf{E}_2^*\,\mathbf{H}_2^*]\cdot\begin{bmatrix}\overline{\epsilon}^\dagger & 0\\ 0 & -\overline{\mu}^\dagger\end{bmatrix}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{H}_1\end{bmatrix}\right\}^*.$$

If $\overline{\epsilon}^\dagger=-\overline{\epsilon}$ and $\overline{\mu}^\dagger=-\overline{\mu}$, then the operator in braces is the same as $i\omega\begin{bmatrix}\overline{\epsilon} & 0\\ 0 & -\overline{\mu}\end{bmatrix}$.

So, $\mathcal{L}$ is a self-adjoint operator if $\mathbf{E}=0$ or $\hat{n}\times\mathbf{H}=0$ on S (or vice versa) and $\overline{\epsilon}^\dagger=-\overline{\epsilon}$, $\overline{\mu}^\dagger=-\overline{\mu}$.

In order for $\mathcal{L}$ to be symmetric, $\langle f_1,\mathcal{L}f_2\rangle=\langle f_2,\mathcal{L}f_1\rangle$,

$$\langle f_1,\mathcal{L}f_2\rangle$$
$$=\int_V d\mathbf{r}[\mathbf{E}_1\,\mathbf{H}_1]\cdot\begin{bmatrix}0 & \nabla\times\\ \nabla\times & 0\end{bmatrix}\begin{bmatrix}\mathbf{E}_2\\ \mathbf{H}_2\end{bmatrix}+i\omega\int_V d\mathbf{r}[\mathbf{E}_1\,\mathbf{H}_1]\cdot\begin{bmatrix}\overline{\epsilon} & 0\\ 0 & -\overline{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{E}_2\\ \mathbf{H}_2\end{bmatrix}.$$

Similar manipulation as above applied here, we obtain

$$\langle f_1,\ \mathcal{L}f_2\rangle$$
$$=-\int_S dS\hat{n}\cdot\begin{bmatrix}0 & \mathbf{E}_1\times\\ \mathbf{H}_1\times & 0\end{bmatrix}\begin{bmatrix}\mathbf{E}_2\\ \mathbf{H}_2\end{bmatrix}+\int_V d\mathbf{r}[\mathbf{E}_2\,\mathbf{H}_2]\cdot\begin{bmatrix}0 & \nabla\times\\ \nabla\times & 0\end{bmatrix}\begin{bmatrix}\mathbf{E}_1\\ \mathbf{H}_1\end{bmatrix}$$
$$+i\omega\int_V d\mathbf{r}[\mathbf{E}_2\,\mathbf{H}_2]\begin{bmatrix}\overline{\epsilon}^t & 0\\ 0 & -\overline{\mu}^t\end{bmatrix}\cdot\begin{bmatrix}\mathbf{E}_1\\ \mathbf{H}_1\end{bmatrix}.$$

Therefore, if $\mathbf{E}=0$, or $\hat{n}\times\mathbf{H}=0$ on S (or vice versa), and $\overline{\epsilon}^t=\overline{\epsilon}$, $\overline{\mu}^t=\overline{\mu}$, then $\mathcal{L}$ is a symmetric operator.

(c) When $\mathcal{L}$ is non-self-adjoint, we first have to find out its adjoint operator $\mathcal{L}^a$ such that

$$\langle f_1^*,\ \mathcal{L}f_2\rangle=\langle f_2^*,\ \mathcal{L}^a f_1\rangle^*.$$

From (a),

$$\langle f_1^*, \mathcal{L} f_2\rangle = \left\{ \int_V d\mathbf{r}[\mathbf{E}_2^* \mathbf{H}_2^*] \cdot \begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{H}_1 \end{bmatrix} \right.$$

$$\left. -i\omega \int_V d\mathbf{r}[\mathbf{E}_2^* \mathbf{H}_2^*] \cdot \begin{bmatrix} \overline{\epsilon}^\dagger & 0 \\ 0 & -\overline{\mu}^\dagger \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{H}_1 \end{bmatrix} \right\}^* = \langle f_2^*, \mathcal{L}^a f_1\rangle^*,$$

where we assume $\mathbf{E} = 0$ or $\hat{n} \times \mathbf{H} = 0$ on S (or vice versa)

Thus,

$$\mathcal{L}^a = \begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix} - i\omega \begin{bmatrix} \overline{\epsilon}^\dagger & 0 \\ 0 & -\overline{\mu}^\dagger \end{bmatrix}.$$

The adjoint equation $\mathcal{L}^a f_a = g_a$ can be written as

$$\begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_a \\ \mathbf{H}_a \end{bmatrix} - i\omega \begin{bmatrix} \overline{\epsilon}^\dagger & 0 \\ 0 & -\overline{\mu}^\dagger \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E}_a \\ \mathbf{H}_a \end{bmatrix} = \begin{bmatrix} \mathbf{J}_a \\ 0 \end{bmatrix}.$$

A variational expression (5.3.3) is then

$$\begin{aligned} I &= \langle f_a^*, \mathcal{L} f\rangle - \langle f_a^*, g\rangle - \langle g_a^*, f\rangle \\ &= \int_V d\mathbf{r}[\mathbf{E}_a^* \mathbf{H}_a^*] \cdot \left\{ \begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix} + i\omega \begin{bmatrix} \overline{\epsilon} & 0 \\ 0 & -\overline{\mu} \end{bmatrix} \right\} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} \\ &\quad - \int_V d\mathbf{r}[\mathbf{E}_a^* \mathbf{H}_a^*] \cdot \begin{bmatrix} \mathbf{J} \\ 0 \end{bmatrix} - \int_V d\mathbf{r}[\mathbf{J}_a^* \, 0] \cdot \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}. \end{aligned}$$

(d) Assume that $\mathbf{J} = 0$ and $\mathcal{L}$ is symmetric. Then, Maxwell's equations becomes

$$\begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \omega \begin{bmatrix} -i\overline{\epsilon} & 0 \\ 0 & i\overline{\mu} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix},$$

which is of the form $\mathcal{L} f = \lambda B f$, where the eigenvalues λ is the resonant frequencies ω in our cavity case.

Hence from (5.4.4a), a variation expression for ω can be expressed as

$$\omega = \frac{\langle f, \mathcal{L} f\rangle}{\langle f, B f\rangle} = \frac{\int_V d\mathbf{r}[\mathbf{E}\,\mathbf{H}] \cdot \begin{bmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}}{-i \int_V [\mathbf{E}\,\mathbf{H}] \cdot \begin{bmatrix} \overline{\epsilon} & 0 \\ 0 & -\overline{\mu} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}}.$$

§5.18

(a) (5.4.2a) is $\lambda = \frac{\langle f^*, \mathcal{L} f\rangle}{\langle f^*, B f\rangle}$.

Let $f=\sum_{n=1}^{N} a_n f_n$, then the equation above can be represented by

$$\lambda=\frac{\langle \sum_{m=1}^{N} a_m^* f_m^*, \mathcal{L} \sum_{n=1}^{N} a_n f_n\rangle}{\langle \sum_{m=1}^{N} a_m^* f_m^*, \mathcal{B} \sum_{n=1}^{N} a_n f_n\rangle}$$
$$=\frac{\sum_{m,n=1}^{N} a_m^* \langle f_m^*, \mathcal{L} f_n\rangle a_n}{\sum_{m,n=1}^{N} a_m^* \langle f_m^*, \mathcal{B} f_n\rangle a_n}=\frac{\mathbf{a}^\dagger\cdot\overline{\mathbf{L}}\cdot\mathbf{a}}{\mathbf{a}^\dagger\cdot\overline{\mathbf{B}}\cdot\mathbf{a}}.$$

Consequently, it follows that

$$\lambda\mathbf{a}^\dagger\cdot\overline{\mathbf{B}}\cdot\mathbf{a}=\mathbf{a}^\dagger\cdot\overline{\mathbf{L}}\cdot\mathbf{a}.$$

Taking the first variation of the above yields

$$\delta\lambda\mathbf{a}_e^\dagger\cdot\overline{\mathbf{B}}\cdot\mathbf{a}_e+\lambda\left[\delta\mathbf{a}^\dagger\cdot\overline{\mathbf{B}}\cdot\mathbf{a}_e+\mathbf{a}_e^\dagger\cdot\overline{\mathbf{B}}\cdot\delta\mathbf{a}\right]$$
$$=\left[\delta\mathbf{a}^\dagger\cdot\overline{\mathbf{L}}\cdot\mathbf{a}_e+\mathbf{a}_e^\dagger\cdot\overline{\mathbf{L}}\cdot\delta\mathbf{a}\right].$$

In order for $\delta\lambda=0$,

$$\delta\mathbf{a}^\dagger\cdot[\lambda\overline{\mathbf{B}}\cdot\mathbf{a}_e]=\delta\mathbf{a}^\dagger\cdot[\overline{\mathbf{L}}\cdot\mathbf{a}_e],$$
$$[\mathbf{a}_e^\dagger\cdot\lambda\overline{\mathbf{B}}]\cdot\delta\mathbf{a}=[\mathbf{a}_e^\dagger\cdot\overline{\mathbf{L}}]\cdot\delta\mathbf{a}.$$

Since $\delta\mathbf{a}$ is arbitrary, we have

$$\lambda\overline{\mathbf{B}}\cdot\mathbf{a}_e=\overline{\mathbf{L}}\cdot\mathbf{a}_e \quad\text{and}\quad \lambda\mathbf{a}_e^\dagger\cdot\overline{\mathbf{B}}=\mathbf{a}_e^\dagger\cdot\overline{\mathbf{L}}.$$

Also invoking the fact that $\overline{\mathbf{B}}$ and $\overline{\mathbf{L}}$ are Hermitian, the above two equations are equivalent.

So, the matrix eigenvalue equation is of the form

$$\overline{\mathbf{L}}\cdot\mathbf{a}=\lambda\overline{\mathbf{B}}\cdot\mathbf{a},$$

where

$$[\overline{\mathbf{L}}]_{mn}=\langle f_m^*, \mathcal{L} f_n\rangle \quad\text{and}\quad [\overline{\mathbf{B}}]_{mn}=\langle f_m^*, \mathcal{B} f_n\rangle.$$

As to (5.4.4a), $\lambda=\frac{\langle f,\mathcal{L}f\rangle}{\langle f,\mathcal{B}f\rangle}$, the same procedure as above can be applied here to get

$$\overline{\mathbf{L}}\cdot\mathbf{a}=\lambda\overline{\mathbf{B}}\cdot\mathbf{a},$$

where $[\overline{\mathbf{L}}]_{mn}=\langle f_m, \mathcal{L} f_n\rangle$ and $[\overline{\mathbf{B}}]_{mn}=\langle f_m, \mathcal{B} f_n\rangle$.

(b) When $\mathcal{L}$ and $\mathcal{B}$ are non-self-adjoint, a variational expression for λ is

$$\lambda=\frac{\langle f_a^*, \mathcal{L} f\rangle}{\langle f_a^*, \mathcal{B} f\rangle}, \tag{5.4.6}$$

with an auxiliary equation

$$\mathcal{L}^a f_a = \lambda^* \mathcal{B}^a f_a.$$

Let

$$f = \sum_{n=1}^{N} a_n f_n \quad \text{and} \quad f_a = \sum_{n=1}^{N} b_n g_n.$$

Then (5.4.6) becomes

$$\begin{aligned}\lambda &= \frac{\langle \sum_{m=1}^{N} b_m^* g_m^*, \mathcal{L} \sum_{n=1}^{N} a_n f_n \rangle}{\langle \sum_{m=1}^{N} b_m^* g_m^*, \mathcal{B} \sum_{n=1}^{N} a_n f_n \rangle} \\ &= \frac{\sum_{m,n=1}^{N} b_m^* \langle g_m^*, \mathcal{L} f_n \rangle a_n}{\sum_{m,n=1}^{N} b_m^* \langle g_m^*, \mathcal{B} f_n \rangle a_n} = \frac{\mathbf{b}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}}{\mathbf{b}^\dagger \cdot \overline{\mathbf{B}} \cdot \mathbf{a}}, \\ \longrightarrow \quad & \lambda(\mathbf{b}^\dagger \cdot \overline{\mathbf{B}} \cdot \mathbf{a}) = \mathbf{b}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}.\end{aligned}$$

Taking the first variation of the above yields

$$\begin{aligned}\delta\lambda \mathbf{b}_e^\dagger \cdot \overline{\mathbf{B}} \cdot \mathbf{a}_e + \lambda \left[\delta \mathbf{b}^\dagger \cdot \overline{\mathbf{B}} \cdot \mathbf{a}_e + \mathbf{b}_e^\dagger \cdot \mathbf{B} \cdot \delta \mathbf{a} \right] \\ = \delta \mathbf{b}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}_e + \mathbf{b}_e^\dagger \cdot \overline{\mathbf{L}} \cdot \delta \mathbf{a}.\end{aligned}$$

In order for $\delta\lambda = 0$ and since $\delta\mathbf{b}$ and $\delta\mathbf{a}$ are arbitrary, we require that

$$\begin{aligned}\lambda \overline{\mathbf{B}} \cdot \mathbf{a}_e &= \overline{\mathbf{L}} \cdot \mathbf{a}_e, \\ \mathbf{b}_e^\dagger \cdot \lambda \overline{\mathbf{B}} &= \mathbf{b}_e^\dagger \cdot \overline{\mathbf{L}}.\end{aligned}$$

Therefore, we have

$$\overline{\mathbf{L}} \cdot \mathbf{a} = \lambda \overline{\mathbf{B}} \cdot \mathbf{a},$$

and

$$\overline{\mathbf{L}}^\dagger \cdot \mathbf{b} = \lambda^* \overline{\mathbf{B}}^\dagger \cdot \mathbf{b},$$

where

$$[\overline{\mathbf{L}}]_{mn} = \langle g_m^*, \mathcal{L} f_n \rangle \quad \text{and} \quad [\overline{\mathbf{B}}]_{mn} = \langle g_m^*, \mathcal{B} f_n \rangle.$$

A completely analogous procedure as above applied to (5.4.9) leads to

$$\overline{\mathbf{L}} \cdot \mathbf{a} = \lambda \overline{\mathbf{B}} \cdot \mathbf{a},$$

and

$$\overline{\mathbf{L}}^t \cdot \mathbf{b} = \lambda \overline{\mathbf{B}}^t \cdot \mathbf{b},$$

where

$$[\overline{\mathbf{L}}]_{mn} = \langle g_m, \mathcal{L} f_n \rangle \quad \text{and} \quad [\overline{\mathbf{B}}]_{mn} = \langle g_m, \mathcal{B} f_n \rangle.$$

(c) Yes, the above matrix equation can be obtained using the method of weighted residuals.

For the eigenvalue problem

$$\mathcal{L}f = \lambda \mathcal{B} f,$$

where $\mathcal{L}$ and $\mathcal{B}$ are self-adjoint operator, we choose basis functions f_n's and approximate f by

$$f = \sum_{n=1}^{N} a_n f_n.$$

Then,

$$\sum_{n=1}^{N} a_n \mathcal{L} f_n = \lambda \sum_{n=1}^{N} a_n \mathcal{B} f_n.$$

Forming the inner product with testing functions g_m's, we obtain

$$\sum_{n=1}^{N} a_n \langle g_m^*, \mathcal{L} f_n \rangle = \lambda \sum_{n=1}^{N} \langle g_m^*, \mathcal{B} f_n \rangle a_n.$$

Written in the matrix equation form, the above becomes

$$\overline{\mathbf{L}} \cdot \mathbf{a} = \lambda \overline{\mathbf{B}} \cdot \mathbf{a}.$$

But we do not have the knowledge of how to choose the testing functions if we use this scheme. On the other hand, when using variational method, it is guaranteed that choosing testing function g_n as f_n will yield a stationary value for λ whose error is second order.

Same reasoning applied to non-self-adjoint problem give us the best choice of g_n's, which turns out to be the basis functions of its auxiliary adjoint problem. But for the method of weighted residuals, we are given no hint to this choice.

§5.19

For a uniform waveguide loaded with inhomogeneity, ϵ and μ are functions of $\boldsymbol{\rho}$. In this structure, the field can be assumed to have $e^{ik_z z}$ dependence.

First, consider $\nabla \cdot \epsilon \mathbf{E} = 0$, and let $\nabla = \nabla_s + \hat{z}\frac{\partial}{\partial z}$. We have

$$\left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right) \cdot \epsilon(\mathbf{E}_s + \hat{z}E_z) = 0$$

$$\rightarrow \quad \nabla_s \cdot \epsilon \mathbf{E}_s + \underbrace{\nabla_s \cdot \epsilon \hat{z} E_z}_{0} + \underbrace{\epsilon \hat{z} \cdot \frac{\partial}{\partial z}\mathbf{E}_s}_{0} + \epsilon \frac{\partial}{\partial z} E_z = 0$$

$$\rightarrow \quad \nabla_s \cdot \epsilon \mathbf{E}_s + i\epsilon k_z E_z = 0.$$

Therefore,

$$E_z = \frac{-1}{i\epsilon k_z}\nabla_s \cdot \epsilon \mathbf{E}_s.$$

Next, the wave equation governing electric field is as follows:

$$\nabla \times \mu^{-1}\nabla \times \mathbf{E} - \omega^2\epsilon\mathbf{E} = 0$$

$$\rightarrow \quad \left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right) \times \mu^{-1}\left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right) \times (\mathbf{E}_s + \hat{z}E_z) - \omega^2\epsilon(\mathbf{E}_s + \hat{z}E_z) = 0$$

$$\rightarrow \quad \left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right) \times \mu^{-1}\left(\nabla_s \times \mathbf{E}_s + \nabla_s \times \hat{z}E_z + \hat{z} \times \frac{\partial}{\partial z}\mathbf{E}_s\right) - \omega^2\epsilon(\mathbf{E}_s + \hat{z}E_z) = 0$$

$$\rightarrow \quad \nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_s + \overbrace{\nabla_s \times \mu^{-1}\nabla_s \times \hat{z}E_z}^{\hat{z}\ \text{component}} + \overbrace{\nabla_s \times \mu^{-1}\hat{z} \times \frac{\partial}{\partial z}}^{\hat{z}\ \text{component}} + \underbrace{\mu^{-1}\hat{z} \times \frac{\partial}{\partial z}\nabla_s \times \mathbf{E}_s}_{0} + \mu^{-1}\hat{z} \times \frac{\partial}{\partial z}\nabla_s \times \hat{z}E_z + \mu^{-1}\hat{z} \times \frac{\partial}{\partial z}\,\hat{z} \times \frac{\partial}{\partial z}\mathbf{E}_s - \omega^2\epsilon(\mathbf{E}_s + \hat{z}E_z) = 0.$$

Applying $\mu\hat{z}\times$ to the above equation, we have

$$\mu\hat{z} \times \nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_s + \hat{z} \times \hat{z} \times \frac{\partial}{\partial z}\nabla_s \times \hat{z}E_z + \hat{z} \times \hat{z} \times \hat{z} \times \frac{\partial^2}{\partial z^2}\mathbf{E}_s - \omega^2\mu\epsilon\hat{z} \times \mathbf{E}_s = 0.$$

Using vector identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$, we obtain

$$\mu\hat{z} \times \nabla s \times \mu^{-1}\nabla \times \mathbf{E}_s - \frac{\partial}{\partial z}\nabla_s \times \hat{z}E_z - \hat{z} \times \frac{\partial^2}{\partial z^2}\mathbf{E}_s - k^2\hat{z} \times \mathbf{E}_s = 0,$$

where $k^2 = \omega^2\mu\epsilon$.
Substituting $E_z = \frac{-1}{i\epsilon_z k^2}\nabla_s \cdot \epsilon\mathbf{E}_s$ in the above yields

$$\mu\hat{z} \times \nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_s - ik_z\nabla_s \times \hat{z}\frac{1}{-i\epsilon k_z}\nabla_s \cdot \epsilon\mathbf{E}_s + k_z^2\hat{z} \times \mathbf{E}_s - k^2\hat{z} \times \mathbf{E}_s = 0.$$

So,

$$\mu\,\hat{z} \times \nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_s - \hat{z} \times \nabla_s\epsilon^{-1}\nabla_s \cdot \epsilon\,\mathbf{E}_s - k^2\hat{z} \times \mathbf{E}_s + k_z^2\,\hat{z} \times \mathbf{E}_s = 0. \qquad (5.4.19)$$

§5.20

(a) An extended operator is defined by

$$\mathcal{L} = \nabla \cdot p\nabla - \Delta(S)p\,\hat{n}\cdot\nabla + k^2 p. \qquad (5.5.9)$$

Then,

$$\begin{aligned}
&\langle \phi_1, \mathcal{L}\phi_2\rangle \\
&= \int_V d\mathbf{r}\, \phi_1 \nabla\cdot p\nabla\phi_2 - \int_S dS\phi_1 p\hat{n}\cdot\nabla\phi_2 + \int_V d\mathbf{r}\phi_1 k^2 p\phi_2 \\
&= \int_V d\mathbf{r}\,[\nabla\cdot(\phi p\nabla\phi_2) - p\nabla\phi_2\cdot\nabla\phi_1] - \int_S dS\phi_1 p\hat{n}\cdot\nabla\phi_2 + \int_V d\mathbf{r}\phi_1 k^2 p\phi_2 \\
&\int_S dS\hat{n}\cdot\phi_1 p\nabla\phi_2 - \int_V d\mathbf{r}p\nabla\phi_1 - \int_S dS\phi_1 p\hat{n}\cdot\nabla\phi_2 + \int_V d\mathbf{r}\phi_1 k^2 p\phi_2 \\
&= -\int_V d\mathbf{r}\,(\nabla\phi_1)\cdot p(\nabla\phi_2) + \int_V d\mathbf{r}\, k^2 p\,\phi_1\phi_2, \qquad (5.5.10)
\end{aligned}$$

without imposing any boundary conditions on ϕ_1 or ϕ_2.

(b) Another extended operator can be defined by

$$\mathcal{L} = \nabla\cdot p\nabla - \nabla\cdot p\,\hat{n}\Delta(S) + k^2 p. \qquad (5.5.13)$$

Then, $\langle \phi_1, \mathcal{L}\phi_2\rangle$

$$\begin{aligned}
&= \int_V d\mathbf{r}\phi_1\nabla\cdot p\nabla\phi_2 - \int_S dS\phi_1\nabla\cdot p\hat{n}\phi_2 + \int_V d\mathbf{r}\phi_1 k^2 p\phi_2 \\
&= \int_V d\mathbf{r}\,[\nabla\cdot(\phi_1 p\nabla\phi_2) - (\nabla\phi_1)\cdot p\nabla\phi_2] \\
&\qquad - \int_S dS\phi_1\nabla\cdot p\hat{n}\phi_2 + \int_V d\mathbf{r}\phi_1 k^2 p\phi_2 \\
&= \int_S dS\hat{n}\cdot\phi_1 p\nabla\phi_2 - \int_V d\mathbf{r}(\nabla\phi_1)\cdot p\nabla\phi_2 \\
&\qquad - \int_S dS[\nabla\cdot(\phi_1 p\hat{n}\phi_2) - \nabla\phi_1\cdot\hat{n}p\phi_2] + \int_V d\mathbf{r}\phi_1 k^2 p\phi_2 \\
&= -\int_V d\mathbf{r}(\nabla\phi_1)\cdot p\nabla\phi_2 + \int_S dS\hat{n}\cdot(\phi_1 p\nabla\phi_2) + \int_S dS\nabla\phi_1\cdot p\hat{n}\phi_2 \\
&\qquad + \int d\mathbf{r}k^2 p\phi_1\phi_2 - \int_S dS\nabla\cdot(\phi_1 p\hat{n}\phi_2).
\end{aligned}$$

The last term can be expressed as

$$\lim_{\Delta t \to 0} \int\limits_{dV} d\mathbf{r} \nabla \cdot (\phi_1 p \hat{n} \phi_2),$$

where dV is the volume of a shell along S whose thickness is Δt. Invoking Gauss theorem, we have

$$\lim_{\Delta t \to 0} \int\limits_{dV} d\mathbf{r} \nabla \cdot (\phi_1 p \hat{n} \phi_2)$$
$$= \hat{n} \cdot \phi_1 p \hat{n} \phi_2 |_{\mathbf{r} \text{ on } S} + (-\hat{n}) \cdot \phi_1 p \hat{n} \phi_2 |_{\mathbf{r} \text{ on } S - \hat{n}\Delta t}.$$

Since $\phi_1 p \hat{n} \phi_2$ has to be continuous in V and on S, the above is zero. So,

$$\begin{aligned} \langle \phi_1, \mathcal{L}\phi_2 \rangle = &- \int\limits_V d\mathbf{r} (\nabla \phi_1) \cdot p (\nabla \phi_2) + \int\limits_A dS \hat{n} \cdot (\phi_1 p \nabla \phi_2) \\ &+ \int\limits_S dS (\nabla \phi_1) \cdot p \hat{n} \phi_2 + \int\limits_V d\mathbf{r} k^2 p \phi_1 \phi_2. \end{aligned} \tag{5.5.14}$$

§5.21

(a)

$$I = - \int\limits_V d\mathbf{r} p (\nabla \phi)^2 + 2 \int\limits_S dS \beta \phi + \int d\mathbf{r} k^2 p \phi^2 - 2 \int\limits_V d\mathbf{r} \phi s. \tag{5.5.17}$$

Taking the first variation of the above yields

$$\begin{aligned} \delta I = &- 2 \int\limits_V d\mathbf{r} p \nabla \phi_e \cdot \nabla \delta \phi + 2 \int\limits_S dS \beta \delta \phi + 2 \int\limits_V d\mathbf{r} k^2 p \phi_e \delta \phi - 2 \int\limits_V d\mathbf{r} s \delta \phi \\ = &- 2 \int\limits_V d\mathbf{r} [\nabla \cdot (\delta \phi p \nabla \phi_e) - \delta \phi \nabla \cdot (p \nabla \phi_e)] + 2 \int\limits_S dS \beta \delta \phi \\ &+ 2 \int\limits_V k^2 p \phi_e \delta \phi - 2 \int\limits_V d\mathbf{r} s \delta \phi \\ = &2 \int\limits_S dS [\beta \delta \phi - \hat{n} \cdot (p \nabla \phi_e) \delta \phi] + 2 \int\limits_V d\mathbf{r} [\nabla \cdot (p \nabla \phi_e) + k^2 p \phi_e - s] \delta \phi. \end{aligned}$$

Therefore, in order for $\delta I = 0$, the optimal solution must satisfy

$$\hat{n} \cdot (p \nabla \phi_e) = \beta \quad \text{on} \quad S$$

and the wave equation $\nabla \cdot (p \nabla \phi_e) + k^2 p \phi_e = s$.

(b)

$$I = -\int_V d\mathbf{r}\, p(\nabla\phi)^2 + 2\int_S dS\,\hat{n}\cdot(\phi-\gamma)p\nabla\phi$$
$$+\int_V d\mathbf{r}\, k^2 p\,\phi^2 - 2\int_V d\mathbf{r}\,\phi\, s. \qquad (5.5.18)$$

The first variation of the above can be written as

$$\delta I = -2\int d\mathbf{r} p\nabla\phi_e\cdot\nabla\delta\phi + 2\int_S dS\hat{n}\cdot(\phi_e - r)p\nabla\delta\phi$$
$$2\int_S dS\hat{n}\cdot\delta\phi p\nabla\phi^e + 2\int_V d\mathbf{r} k^2 p\phi_e\delta\phi - 2\int_V d\mathbf{r} s\delta\phi.$$

By vector identity and Gauss theorem, the first term of the above equation can be expressed as

$$-2\int_S dS\hat{n}\cdot(p\nabla\phi_e)\delta\phi + 2\int_V d\mathbf{r}\delta\phi\nabla\cdot(p\nabla\phi_e).$$

Then, δI becomes

$$\delta I = 2\int_S dS\hat{n}\cdot(\phi_e - r)p\nabla\delta\phi + 2\int_V d\mathbf{r}\left[\nabla\cdot(p\nabla\phi_e) + k^2 p\phi_e - s\right]\delta\phi.$$

Since $\delta\phi$ is arbitrary, the conditions for $\delta I = 0$ require that

$$\phi_e = r \quad \text{on} \quad S,$$

and ϕ_e is the solution to $\nabla\cdot(p\nabla\phi_e) + k^2 p\phi_e = s$.

(c)

$$I = -\int_V d\mathbf{r}\, p(\nabla\phi)^2 + \int_S dS\,\alpha\phi^2 + \int_V d\mathbf{r}\, k^2 p\,\phi^2 - 2\int_V d\mathbf{r}\,\phi\, s. \qquad (5.5.19)$$

δI, the first variation can be expressed as

$$\delta I = -2\int_V d\mathbf{r} p\nabla\phi_e\cdot\nabla\delta\phi + 2\int_S dS\alpha\phi_e\delta\phi + 2\int_V d\mathbf{r} k^2 p\phi_e\delta\phi - 2\int_V d\mathbf{r} s\delta\phi$$
$$= -2\int_S dS\hat{n}\cdot(p\nabla\phi_e)\delta\phi + 2\int_V d\mathbf{r}\nabla\cdot(p\nabla\phi_e)\delta\phi + 2\int S dS\alpha\phi_e\delta\phi$$
$$+2\int_V d\mathbf{r} k^2 p\phi_e\delta\phi - 2\int_V d\mathbf{r} s\delta\phi$$
$$=2\int_S dS\left[\alpha\phi_e - \hat{n}\cdot(p\nabla\phi_e)\right]\delta\phi + 2\int_V d\mathbf{r}\left[\nabla\cdot(p\nabla\phi_e) + k^2 p\phi_e - s\right]\delta\phi.$$

So, the conditions that

$$\hat{n}\cdot p\phi_e = \alpha\phi^e \quad \text{on} \quad S$$

and

$$\nabla\cdot(p\nabla\phi_e) + k^2 p\phi_e = S$$

make

$$\delta I = 0.$$

§5.22

(a)

$$I = \langle \nabla\times\mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}\rangle - \omega^2\langle\mathbf{E}_a^*, \overline{\epsilon}\cdot\mathbf{E}\rangle - i\omega\langle\mathbf{E}_a^*, \mathbf{J}\rangle + i\omega\langle\mathbf{J}_a^*, \mathbf{E}\rangle, \qquad (5.5.24)$$

where $\mathbf{E}$ and $\mathbf{E}_a$ satisfy

$$\mathcal{L}\mathbf{E} = i\omega\mathbf{J}$$

and

$$\mathcal{L}^a\mathbf{E}_a = i\omega\mathbf{J}_a$$

respectively, and

$$\mathcal{L} = (\nabla\times\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times) - \left[\Delta(S)\hat{n}\times\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\right] - \omega^2\overline{\epsilon}.$$

Therefore,

$$\mathcal{L}^a = \left[\nabla\times(\overline{\boldsymbol{\mu}}^\dagger)^{-1}\cdot\nabla\times\right] - \left[\Delta(S)\hat{n}\times(\overline{\boldsymbol{\mu}}^\dagger)^{-1}\cdot\nabla\times\right] - \omega^2\overline{\epsilon}^\dagger.$$

The first variation of I can be expressed as

$$\begin{aligned}\delta I =& \langle\nabla\times\delta\mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}_e\rangle + \langle\nabla\times\mathbf{E}_{ae}^*, \overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\delta\mathbf{E}\rangle \\ &- \omega^2\langle\delta\mathbf{E}_a^*, \overline{\epsilon}\cdot\mathbf{E}_e\rangle - \omega^2\langle\mathbf{E}_{ae}^*, \overline{\epsilon}\cdot\delta\mathbf{E}\rangle \\ &- i\omega\langle\delta\mathbf{E}_a^*, \mathbf{J}\rangle + i\omega\langle\mathbf{J}_a^*, \delta\mathbf{E}\rangle.\end{aligned}$$

The first term of the above is written explicitly as

$$\begin{aligned}&\int_V d\mathbf{r}\nabla\times\delta\mathbf{E}_a^*\cdot\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}_e \\ &= \int_V d\mathbf{r}\nabla\cdot\left(\delta\mathbf{E}_a^*\times\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}_e\right) + \int_V d\mathbf{r}\delta\mathbf{E}_a^*\cdot\nabla\times\left(\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}_e\right) \\ &= \int_S dS\hat{n}\cdot\delta\mathbf{E}_a^*\times\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}_e + \int_V d\mathbf{r}\delta\mathbf{E}_a^*\cdot\nabla\times\left(\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}_e\right).\end{aligned}$$

And the second term is

$$\int_V d\mathbf{r}\nabla\times\mathbf{E}_{ae}^*\cdot\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\delta\mathbf{E}=\left[\int_V d\mathbf{r}\nabla\times\delta\mathbf{E}^*\cdot\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_{ae}\right]^*$$

$$=\left[\int_V d\mathbf{r}\nabla\cdot\left(\delta\mathbf{E}^*\times\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_{ae}\right)\right.$$

$$\left.+\int_V d\mathbf{r}\delta\mathbf{E}^*\cdot\nabla\times\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_{ae}\right]^*$$

$$=\left[\int_S dS\hat{n}\cdot\delta\mathbf{E}^*\times\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_{ae}\right.$$

$$\left.+\int_V d\mathbf{r}\delta\mathbf{E}^*\cdot\nabla\times\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_{ae}\right]^*.$$

Thus, δI becomes

$$\delta I=\int_V d\mathbf{r}\delta\mathbf{E}_a^*\cdot\left[\nabla\times\left(\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}_e\right)-\omega^2\overline{\boldsymbol{\epsilon}}\cdot\mathbf{E}_e-i\omega\mathbf{J}\right]$$

$$+\left\{\int_V d\mathbf{r}\delta\mathbf{E}^*\cdot\left[\nabla\times\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_{ae}-\omega^2\overline{\boldsymbol{\epsilon}}^{\dagger}\cdot\mathbf{E}_{ae}-i\omega\mathbf{J}_a\right]\right\}^*$$

$$-\int_S dS\delta\mathbf{E}_a^*\cdot\hat{n}\times\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}_e-\left[\int_S dS\delta\mathbf{E}^*\cdot\hat{n}\times\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_{ae}\right]^*.$$

So, the conditions for $\delta I=0$ require that the optimal solutions satisfy

$$\nabla\times\left(\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}_e\right)-\omega^2\overline{\boldsymbol{\epsilon}}\cdot\mathbf{E}_e=i\omega\mathbf{J},$$
$$\nabla\times\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_{ae}-\omega^2\overline{\boldsymbol{\epsilon}}^{\dagger}\cdot\mathbf{E}_{ae}=i\omega\mathbf{J}_a,$$

and

$$\hat{n}\times\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}_e=0\quad\text{on}\quad S,$$
$$\hat{n}\times\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_{ae}=0\quad\text{on}\quad S.$$

(b)

$$I=\langle\nabla\times\mathbf{E}_a^*,\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}\rangle+\langle\mathbf{E}_a^*,\Delta(S)\hat{n}\times\overline{\boldsymbol{\mu}}^{-1}\cdot\nabla\times\mathbf{E}\rangle$$
$$+\langle\hat{n}\times\nabla\times\mathbf{E}_a^*\cdot\overline{\boldsymbol{\mu}}^{-1},\Delta(S)\mathbf{E}\rangle-\omega^2\langle\mathbf{E}_a^*,\overline{\boldsymbol{\epsilon}}\cdot\mathbf{E}\rangle-i\omega\langle\mathbf{E}_a^*,\mathbf{J}\rangle$$
$$+i\omega\langle\mathbf{J}_a^*,\mathbf{E}\rangle.\tag{5.5.39}$$

Then,

$$\begin{aligned}\delta I = &\left\langle \nabla \times \delta \mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e \right\rangle + \left\langle \nabla \times \mathbf{E}_{ae}^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta \mathbf{E} \right\rangle \\ &+ \left\langle \delta \mathbf{E}_a^*, \Delta(S)\hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \mathbf{E}_e \right\rangle + \left\langle \mathbf{E}_{ae}^*, \Delta(S)\hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta \mathbf{E} \right\rangle \\ &+ \left\langle \hat{n} \times \nabla \times \delta \mathbf{E}_a^* \cdot \overline{\boldsymbol{\mu}}^{-1}, \Delta(S)\mathbf{E}_e \right\rangle + \left\langle \hat{n} \times \nabla \times \mathbf{E}_{ae} \cdot \overline{\boldsymbol{\mu}}^{-1}, \Delta(S)\delta \mathbf{E} \right\rangle \\ &- \omega^2 \left\langle \delta \mathbf{E}_a^*, \overline{\epsilon} \mathbf{E}_e \right\rangle - \omega^2 \left\langle \mathbf{E}_{ae}^*, \overline{\epsilon} \cdot \delta \mathbf{E} \right\rangle - i\omega \left\langle \delta \mathbf{E}_a^*, \mathbf{J} \right\rangle + i\omega \left\langle \mathbf{J}_a^*, \delta \mathbf{E} \right\rangle .\end{aligned}$$

Expressing the above explicitly and using some vector identities and Gauss theorem, we have

$$\begin{aligned}\delta I = &\int S dS \hat{n} \cdot \delta \mathbf{E}_a^* \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e + \int_V d\mathbf{r} \delta \mathbf{E}_a^* \cdot \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e \\ &+ \Bigg[\int_S dS \hat{n} \cdot \delta \mathbf{E}^* \times \left(\overline{\boldsymbol{\mu}}^{\dagger} \right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} \\ &+ \int_V d\mathbf{r} \delta \mathbf{E}^* \cdot \nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger} \right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} \Bigg]^* \\ &+ \int_S dS \delta \mathbf{E}_a^* \cdot \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e + \int_S dS \mathbf{E}_{ae}^* \cdot \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta \mathbf{E} \\ &+ \int_S dS \hat{n} \times \nabla \times \delta \mathbf{E}_a^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \mathbf{E}_e + \int_S dS \hat{n} \times \nabla \times \mathbf{E}_{ae}^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \delta \mathbf{E} \\ &- \omega^2 \int_V d\mathbf{r} \delta \mathbf{E}_a^* \cdot \overline{\epsilon} \cdot \mathbf{E}_e - \omega^2 \left[\int_V d\mathbf{r} \delta \mathbf{E}^* \cdot \overline{\epsilon}^{\dagger} \cdot \mathbf{E}_{ae} \right]^* \\ &- i\omega \int_V d\mathbf{r} \delta \mathbf{E}_a^* \cdot \mathbf{J} + i\omega \left[\int_V d\mathbf{r} \delta \mathbf{E}^* \cdot \mathbf{J}_a \right]^* .\end{aligned}$$

Notice that the first and the fifth terms cancel with each other and so do the third and the eighth terms.

The sixth term can be written as

$$-\int_S dS \hat{n} \times \mathbf{E}_{ae}^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta \mathbf{E} = - \left[\int_S \nabla \times \delta \mathbf{E}^* \cdot \left(\overline{\boldsymbol{\mu}}^{\dagger} \right)^{-1} \cdot \hat{n} \times \mathbf{E}_{ae} \right]^* ,$$

and the seventh term as

$$-\int_S dS \nabla \times \delta \mathbf{E}_a^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \hat{n} \times \mathbf{E}_e .$$

Hence,

$$\delta I = \int_V d\mathbf{r}\delta\mathbf{E}_a^* \cdot \left[\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e - \omega^2 \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_e - i\omega\mathbf{J}\right]$$
$$+ \left\{ \int_V d\mathbf{r}\delta\mathbf{E}^* \cdot \left[\nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} - \omega^2 \overline{\boldsymbol{\epsilon}}^{\dagger} \cdot \mathbf{E}_{ae} - i\omega\mathbf{J}_a\right] \right\}^*$$
$$- \int_S dS \nabla \times \delta\mathbf{E}_a^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \hat{n} \times \mathbf{E}_e - \left[\int_S dS \nabla \times \delta\mathbf{E}^* \cdot \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \hat{n} \times \mathbf{E}_{ae} \right]^* .$$

Consequently, the optimal solutions which make $\delta I = 0$ satisfy the following conditions:

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e - \omega^2 \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_e = i\omega\mathbf{J},$$
$$\nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} - \omega^2 \overline{\boldsymbol{\epsilon}}^{\dagger} \cdot \mathbf{E}_{ae} = i\omega\mathbf{J}_a,$$

and

$$\hat{n} \times \mathbf{E}_e = 0 \quad \text{on} \quad S,$$
$$\hat{n} \times \mathbf{E}_{ae} = 0 \quad \text{on} \quad S.$$

§5.23

(a)

$$I = \langle \nabla \times \mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \rangle - \omega^2 \langle \mathbf{E}_a^*, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} \rangle$$
$$- i\omega \langle \mathbf{E}_a^*, [\mathbf{J} - \Delta(S)\boldsymbol{\alpha}] \rangle + i\omega \langle [\mathbf{J}_a^* - \Delta(S)\boldsymbol{\alpha}_a^*], \mathbf{E} \rangle. \quad (5.5.41)$$

The first variation of the above is

$$\delta I = \langle \nabla \times \delta\mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla\mathbf{E}_e \rangle + \langle \nabla \times \mathbf{E}_{ae}^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta\mathbf{E} \rangle - \omega^2 \langle \delta\mathbf{E}_a^*, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_e \rangle$$
$$- \omega^2 \langle \mathbf{E}_{ae}^*, \overline{\boldsymbol{\epsilon}}\delta\mathbf{E} \rangle - i\omega \langle \delta\mathbf{E}_a^*, [\mathbf{J}_a^* - \Delta(S)\boldsymbol{\alpha}_a^*], \delta\mathbf{E} \rangle$$
$$i\omega \langle [\mathbf{J}_a^* - \Delta(S)\boldsymbol{\alpha}_a^*] \delta\mathbf{E} \rangle$$
$$= \int S dS \hat{n} \cdot \delta\mathbf{E}_a^* \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e + \int_V d\mathbf{r}\delta\mathbf{E}_a^* \cdot \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e$$
$$+ \left\{ \int_S dS \hat{n} \cdot \delta\mathbf{E}^* \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} \right.$$
$$\left. + \int_V d\mathbf{r}\delta\mathbf{E}^* \cdot \nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} \right\}^*$$

$$-\omega^2 \int_V d\mathbf{r}\delta\mathbf{E}_a^* \cdot \overline{\epsilon} \cdot \mathbf{E}_e - \omega^2 \left[\int_V d\mathbf{r}\delta\mathbf{E} \cdot \overline{\epsilon}^\dagger \cdot \mathbf{E}_{ae} \right]^*$$
$$- i\omega \int_V d\mathbf{r}\delta\mathbf{E}_a^* \cdot \mathbf{J} + i\omega \int_S dS\delta\mathbf{E}_a^* \cdot \boldsymbol{\alpha}$$
$$- \left[i\omega \int_V d\mathbf{r}\delta\mathbf{E}^* \cdot \mathbf{J}_a \right]^* + \left[i\omega \int_S dS\delta\mathbf{E}^* \cdot \boldsymbol{\alpha}_a \right]^*$$
$$= \int_V d\mathbf{r}\delta\mathbf{E}_a^* \cdot \left[\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e - \omega^2 \overline{\epsilon} \cdot \mathbf{E}_e - i\omega \mathbf{J} \right]$$
$$+ \left\{ \int_V d\mathbf{r}\delta\mathbf{E}^* \cdot \left[\nabla \times \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} - \omega^2 \overline{\epsilon}^\dagger \cdot \mathbf{E}_{ae} - i\omega \mathbf{J}_a \right] \right\}^*$$
$$+ \int_S dS\delta\mathbf{E}_a^* \cdot \left[-\hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e + i\omega \boldsymbol{\alpha} \right]$$
$$+ \left\{ \int_S d\hat{S}\delta\mathbf{E}^* \left[-\hat{n} \times \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} + i\omega \boldsymbol{\alpha}_a \right] \right\}^* .$$

Therefore, $\delta I = 0$ requires that $\mathbf{E}_e$ and $\mathbf{E}_{ae}$ satisfy the wave equations respectively, and the boundary conditions:

$$\boldsymbol{\alpha} = \frac{1}{i\omega} \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e = \hat{n} \times \mathbf{H} \qquad \text{on} \quad S,$$

and

$$\boldsymbol{\alpha}_a = \frac{1}{i\omega} \hat{n} \times \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} = \hat{n} \times \mathbf{H}_a \qquad \text{on} \quad S.$$

(b) (5.5.43) is

$$\begin{aligned} I =& \langle \mathbf{E}_a^*, \mathcal{L}\mathbf{E} \rangle - i\omega \langle \mathbf{E}_a^*, \mathbf{J} \rangle + \left\langle \nabla \times \mathbf{E}_a^*, \Delta(S)\, \overline{\boldsymbol{\mu}}^{-1} \cdot \boldsymbol{\beta} \right\rangle \\ &+ i\omega \langle \mathbf{J}_a^*, \mathbf{E} \rangle + \left\langle \Delta(S) \left(\overline{\boldsymbol{\mu}}^t\right)^{-1} \cdot \boldsymbol{\beta}_a^*, \nabla \times \mathbf{E} \right\rangle \\ =& \langle \nabla \times \mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \rangle + \langle \mathbf{E}_a^*, \Delta(S) \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \rangle - i\omega \langle \mathbf{E}_a^*, \mathbf{J} \rangle \\ &+ \langle \nabla \times \mathbf{E}_a^*, \Delta(S) \overline{\boldsymbol{\mu}}^{-1} \cdot \boldsymbol{\beta} \rangle + i\omega \langle \mathbf{J}_a^*, \mathbf{E} \rangle \\ &+ \langle \Delta(s) \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \boldsymbol{\beta}_a^*, \nabla \times \mathbf{E} \rangle . \end{aligned}$$

Taking the first variation of the above yields

$$\begin{aligned}
\delta I =& \langle \nabla \times \delta \mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e \rangle + \langle \nabla \times \mathbf{E}_{ae}^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta \mathbf{E} \rangle \\
&+ \langle \delta \mathbf{E}_a^*, \Delta(S) \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e \rangle + \langle \mathbf{E}_{ae}^*, \Delta(S) \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta \mathbf{E} \rangle \\
&+ \langle \hat{n} \times \nabla \delta \mathbf{E}_a^* \cdot \overline{\boldsymbol{\mu}}^{-1}, \Delta(S) \mathbf{E}_e \rangle + \langle \hat{n} \times \nabla \times \mathbf{E}_{ae}^* \cdot \overline{\boldsymbol{\mu}}^{-1}, \Delta(S) \delta \mathbf{E} \rangle \\
&- \omega^2 \langle \delta \mathbf{E}_a^*, \overline{\epsilon} \cdot \mathbf{E}_e \rangle - \omega^2 \langle \mathbf{E}_{ae}^*, \overline{\epsilon} \cdot \delta \mathbf{E} \rangle - i\omega \langle \delta \mathbf{E}_a^*, \mathbf{J} \rangle \\
&+ \langle \nabla \times \delta \mathbf{E}_a^*, \Delta(S) \overline{\boldsymbol{\mu}}^{-1} \cdot \boldsymbol{\beta} \rangle + i\omega \langle \mathbf{J}_a^*, \delta \mathbf{E} \rangle \\
&+ \langle \Delta(S) \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \boldsymbol{\beta}_a^*, \nabla \times \delta \mathbf{E} \rangle \\
=& \int_S dS \hat{n} \cdot \delta \mathbf{E}_a^* \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e + \int_V d\mathbf{r} \delta \mathbf{E}_a^* \cdot \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e \\
&+ \left\{ \int_S dS \hat{n} \delta \mathbf{E}_a^* \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \mathbf{E}_{ae} + \int_V d\mathbf{r} \delta \mathbf{E}_a^* \cdot \nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} \right\}^* \\
&+ \int_S dS \delta \mathbf{E}_a^* \cdot \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e + \int_S dS \hat{n} \times \nabla \times \mathbf{E}_{ae}^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \delta \mathbf{E} \\
&+ \int_V d\mathbf{r} \mathbf{E}_{ae}^* \cdot \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta \mathbf{E} + \int_S dS \hat{n} \times \nabla \times \delta \mathbf{E}_a^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \mathbf{E}_e \\
&- \omega^2 \int_V d\mathbf{r} \delta \mathbf{E}_a^* \cdot \overline{\epsilon} \cdot \mathbf{E}_e - \omega^2 \left\{ \int_V d\mathbf{r} \delta \mathbf{E}^* \cdot \overline{\epsilon}^{\dagger} \cdot \mathbf{E}_{ae} \right\}^* - i\omega \int_V d\mathbf{r} \delta \mathbf{E}_a^* \cdot \mathbf{J} \\
&+ \int_S dS \nabla \times \delta \mathbf{E}_a^* \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \boldsymbol{\beta} - \left\{ i\omega \int_V d\mathbf{r} \delta \mathbf{E}^* \cdot \mathbf{J}_a \right\}^* \\
&+ \left\{ \int_S dS \Delta \times \delta \mathbf{E}^* \cdot \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \boldsymbol{\beta}_a \right\}^* .
\end{aligned}$$

Notice that the first and the fifth terms cancel with each other and so do the third and the sixth terms. Thus, δI becomes

$$\begin{aligned}
\delta I =& \int_V d\mathbf{r} \delta \mathbf{E}_a^* \cdot \left[\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \mathbf{E}_e - \omega^2 \overline{\epsilon} \cdot \mathbf{E}_e - i\omega \mathbf{J}\right] \\
&+ \left\{ \int_V d\mathbf{r} \delta \mathbf{E}^* \cdot \left[\nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} - \omega^2 \overline{\epsilon}^{\dagger} \cdot \mathbf{E}_{ae} - i\omega \mathbf{J}_a\right] \right\}^* \\
&+ \int_S dS \nabla \times \delta \mathbf{E}_a^* \cdot \left[\overline{\boldsymbol{\mu}}^{-1} \cdot \boldsymbol{\beta} - \overline{\boldsymbol{\mu}}^{-1} \cdot \hat{n} \times \mathbf{E}_e\right] \\
&+ \left\{ \int_S dS \nabla \times \delta \mathbf{E}^* \cdot \left[\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \boldsymbol{\beta}_a - \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \hat{n} \times \mathbf{E}_{ae}\right] \right\}^* .
\end{aligned}$$

Consequently, the optimal solutions satisfy the boundary conditions:

$$\hat{n} \times \mathbf{E}_e = \boldsymbol{\beta} \qquad \text{on} \quad S,$$

$$\hat{n} \times \mathbf{E}_{ae} = \boldsymbol{\beta}_a \qquad \text{on} \quad S,$$

as well as the wave equations.

(c)

$$I = \langle \nabla \times \mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \rangle - \omega^2 \langle \mathbf{E}_a^*, [\overline{\epsilon} - \Delta(S)\overline{\mathbf{r}}] \cdot \mathbf{E} \rangle - i\omega \langle \mathbf{E}_a^*, \mathbf{J} \rangle + i\omega \langle \mathbf{J}_a^*, \mathbf{E} \rangle. \quad (5.5.44)$$

Then,

$$\begin{aligned}
\delta I &= \langle \nabla \times \delta\mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e \rangle + \langle \nabla \times \mathbf{E}_{ae}^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta\mathbf{E} \rangle \\
&\quad - \omega^2 \langle \delta\mathbf{E}_a^*, [\overline{\epsilon} + \Delta(S)\overline{\mathbf{r}}]\, \mathbf{E}_e \rangle - \omega^2 \langle \mathbf{E}_{ae}^*, [\overline{\epsilon} + \Delta(S)\overline{\mathbf{r}}] \cdot \delta\mathbf{E} \rangle \\
&\quad - i\omega \langle \delta\mathbf{E}_a^*, \mathbf{J} \rangle + i\omega \langle \mathbf{J}_a^*, \delta\mathbf{E} \rangle \\
&= \int_S dS\hat{n} \cdot \delta\mathbf{E}_a^* \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e + \int_V d\mathbf{r}\delta\mathbf{E}_a^* \cdot \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e \\
&\quad + \left\{ \int_S dS\hat{n} \cdot \delta\mathbf{E}^* \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} \right. \\
&\quad \left. + \int_V d\mathbf{r}\delta\mathbf{E}^* \cdot \nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} \right\}^* \\
&\quad - \omega^2 \int_V d\mathbf{r}\delta\mathbf{E}_a^* \cdot \overline{\epsilon} \cdot \mathbf{E}_e + \omega^2 \int_S dS\delta\mathbf{E}_a^* \cdot \overline{\mathbf{r}} \cdot \mathbf{E}_e \\
&\quad - \omega^2 \left[\int_V d\mathbf{r}\delta\mathbf{E}^* \cdot \overline{\epsilon}^{\dagger} \cdot \mathbf{E}_{ae} \right]^* + \omega^2 \left[\int_S dS\delta\mathbf{E}^* \cdot \overline{\mathbf{r}}^{\dagger} \cdot \mathbf{E}_{ae} \right]^* \\
&\quad - i\omega \int_V d\mathbf{r}\delta\mathbf{E}_a^* \cdot \mathbf{J} - \left[i\omega \int_V d\mathbf{r}\delta\mathbf{E}^* \cdot \mathbf{J}_a \right]^* \\
&= \int_V d\mathbf{r}\delta\mathbf{E}^* \cdot \left[\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e - \omega^2 \overline{\epsilon} \cdot \mathbf{E}_e - i\omega \mathbf{J}_a \right] \\
&\quad + \left\{ \int_V d\mathbf{r}\delta\mathbf{E}^* \cdot \left[\nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} - \omega^2 \overline{\epsilon}^{\dagger} \cdot \mathbf{E}_{ae} - i\omega \mathbf{J}_a \right] \right\}^* \\
&\quad + \int_S dS\delta\mathbf{E}_a^* \cdot \left[-\hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e + \omega^2 \overline{\mathbf{r}} \cdot \mathbf{E}_e \right] \\
&\quad + \left\{ \int_S dS\delta\mathbf{E}^* \cdot \left[-\hat{n} \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} + \omega^2 \overline{\mathbf{r}}^{\dagger} \cdot \mathbf{E}_{ae} \right] \right\}^* .
\end{aligned}$$

Therefore, $\delta I = 0$ requires that in addition to the wave equation being satisfied, the boundary conditions are as follows:

$$\hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_e = \omega^2 \overline{\mathbf{r}} \cdot \mathbf{E}_e \quad \rightarrow \hat{n} \times \mathbf{H}_e = -i\omega \overline{\mathbf{r}} \cdot \mathbf{E}_e \quad \text{on} \quad S,$$
$$\hat{n} \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_{ae} = \omega^2 \overline{\mathbf{r}}^{\dagger} \cdot \mathbf{E}_{ae} \rightarrow \hat{n} \times \mathbf{H}_{ae} = -i\omega \overline{\mathbf{r}}^{\dagger} \cdot \mathbf{E}_{ae} \quad \text{on} \quad S.$$

CHAPTER 6

EXERCISES SOLUTIONS

by C.C. Lu

§6.1

(a) Let $\phi_1(z)$ and $\phi_2(z)$ be two different functions. By definition,

$$\begin{aligned}\langle\phi_1,\mathcal{D}\phi_2\rangle &= \int_\Gamma \phi_1\left[\frac{d}{dz}p^{-1}\frac{d}{dz}+k^2p^{-1}\right]\phi_2 dz\\ &= \left[\phi_1 p^{-1}\frac{d\phi_2}{dz}\right]_{\Gamma_0} - \int_\Gamma\left[p^{-1}\frac{d\phi_2}{dz}\frac{d\phi_1}{dz}-k^2p^{-1}\phi_1\phi_2\right]dz \qquad (1)\\ \langle\phi_2,\mathcal{D}\phi_1\rangle &= \left[\phi_2 p^{-1}\frac{d\phi_2}{dz}\right]_{\Gamma_0} - \int_\Gamma\left[p^{-1}\frac{d\phi_1}{dz}\frac{d\phi_2}{dz}-k^2p^{-1}\phi_2\phi_1\right]dz \qquad (2)\end{aligned}$$

If $\mathcal{D}$ is a symmetric operator, then $\langle\phi_1,\mathcal{D}\phi_2\rangle = \langle\phi_2,\mathcal{D}\phi_1\rangle$. From the above equations, it is easily seen that the condition for $\mathcal{D}$ to be symmetry is

$$\left[\phi_1 p^{-1}\frac{d\phi_2}{dz}-\phi_2 p^{-1}\frac{d\phi_1}{dz}\right]_{\Gamma_0}=0, \qquad (3)$$

where Γ_0 stands for the end points on Γ. This is the general case. In particular, we have

$$\begin{aligned} \phi\Big|_{\Gamma_0} &= 0 \qquad \text{(Dirichlet)},\\ \text{or}\quad \frac{d\phi}{dz}\Big|_{\Gamma_0} &= 0 \qquad \text{(Neumann)}.\end{aligned}$$

In the same way, we can show that $\mathcal{B}$ is always symmetric.

(b)

$$\begin{aligned}\langle\phi_1^*,\mathcal{D}\phi_2\rangle &= \phi_1^* p^{-1}\frac{d\phi_2}{dz}\Big]_{\Gamma_0} - \int_\Gamma\left[p^{-1}\frac{d\phi_1^*}{dz}\frac{d\phi_2}{dz}-k^2p^{-1}\phi_1^*\phi_2\right]dz,\\ \langle\phi_2^*,\mathcal{D}\phi_1\rangle &= \phi_2^* p^{-1}\frac{d\phi_1}{dz}\Big]_{\Gamma_0} - \int_\Gamma\left[p^{-1}\frac{d\phi_1}{dz}\frac{d\phi_2^*}{dz}-k^2p^{-1}\phi_1\phi_2^*\right]dz,\\ \langle\phi_2^*,\mathcal{D}\phi_1\rangle^* &= \phi_2\left[p^{-1}\frac{d\phi_1}{dz}\right]^*_{\Gamma_0} - \int_\Gamma\left[(p^{-1})^*\frac{d\phi_1^*}{dz}\frac{d\phi_2}{dz}-(k^2p^{-1})^*\phi_1^*\phi_2\right]dz.\end{aligned}$$

From the above, we can see that $\mathcal{D}$ will be Hermitian if ϕ satisfies

$$\phi_1^* p^{-1}\tfrac{d\phi_2}{dz} = 0, \quad \phi_2\left[p^{-1}\tfrac{d\phi_1}{dz}\right]^* = 0,$$
$$p^{-1} \text{ is real,} \qquad k^2 p^{-1} \text{ is real.}$$

§6.2

Suppose ϕ_1 and ϕ_2 are two different eigensolutions of Equation (4) [Chew, p. 328]. The corresponding eigenvalues are k_1^2 and k_2^2, respectively. Their inner product

$$\langle \phi_1, \mathcal{B}\phi_2 \rangle = \int \phi_1 \mathcal{B}\phi_2 dz. \tag{1}$$

Since ϕ_1, ϕ_2 satisfy (4) we have

$$\mathcal{D}\phi_1 = k_1^2 \mathcal{B}\phi_1, \tag{2}$$
$$\mathcal{D}\phi_2 = k_2^2 \mathcal{B}\phi_2. \tag{3}$$

Multipling (1) by ϕ_2 and (2) by ϕ_1, and subtracting the resultant equations, give

$$\phi_1\mathcal{D}\phi_2 - \phi_2\mathcal{D}\phi_1 = k_2^2\phi_1\mathcal{B}\phi_2 - k_1^2\phi_2\mathcal{B}\phi_1. \tag{4}$$

From Exercise (1), we know that $\mathcal{B}$ is always symmetric, when we integrate (4) over z. Then using the relation

$$\int \phi_1\mathcal{B}\phi_2 dz = \int \phi_2\mathcal{B}\phi_1 dz,$$

the result is

$$\int \left(\phi_1\mathcal{D}\phi_2 - \phi_2\mathcal{D}\phi_1\right) dz = \left(k_2^2 - k_1^2\right) \int \phi_1\mathcal{B}\phi_2 dz. \tag{5}$$

Since $\mathcal{D}$ is also symmetric, the left hand side of the above equation vanishes, resulting in

$$\left(k_2^2 - k_1^2\right) \int \phi_2\mathcal{B}\phi_2 dz = 0.$$

Since $k_1^2 \neq k_2^2$, we have

$$\int \phi_1\mathcal{B}\phi_2 dz = 0.$$

This means that ϕ is $\mathcal{B}$ orthogonal.

§6.3

(a) Total field consists of a source term $e^{ik_{1z}|z-z'|}/k_{1z}$, an upgoing wave $Ae^{ik_{1z}z}$, and a downgoing wave $Be^{-ik_{1z}z}$. Therefore,

$$F(z) = \frac{1}{k_{1z}} e^{ik_{1z}|z-z'|} + Ae^{ik_{1z}z} + Be^{-ik_{1z}z}. \tag{1}$$

The constraint condition at $z = z_{max}$ is

$$Be^{-ik_{1z}z_{max}} = \tilde{R}_{1max}\left(\frac{1}{k_{1z}}e^{ik_{1z}|z_{max}-z'|} + Ae^{ik_{1z}z_{max}}\right). \tag{2}$$

The constraint condition at $z = -d_1$

$$Ae^{-ik_{1z}d_1} = \tilde{R}_{12}\left(\frac{1}{k_{1z}}e^{ik_{1z}|d_1+z'|} + Be^{ik_{1z}d_1}\right). \tag{3}$$

Solving for B and A from (2) and (3) yields

$$Be^{-ik_{1z}z_{max}} = \tilde{R}_{1max}\left[e^{ik_{1z}|z_{max}-z'|} + e^{ik_{1z}(d_1+z_{max})}\tilde{R}_{12}e^{ik_{1z}|d_1+z'|}\right]\frac{\tilde{M}_1}{k_{1z}}, \tag{4}$$

$$Ae^{-ik_{1z}d_1} = \tilde{R}_{12}\left[e^{ik_{1z}|d_1+z'|} + e^{-ik_{1z}(d_1+z_{max})}\tilde{R}_{1max}e^{ik_{1z}|z_{max}-z'|}\right]\frac{\tilde{M}_1}{k_{1z}}. \tag{5}$$

$$\tilde{M}_1 = \left[1 - \tilde{R}_{12}\tilde{R}_{1max}e^{2ik_{1z}(z_{max}+d_1)}\right]^{-1}$$

After substituting (4) and (5) back into (1), we have

$$F_+(z,z') = \left[e^{-ik_{1z}z'} + e^{ik_{1z}(z'+2d_1)}\tilde{R}_{12}\right] \cdot \left[e^{ik_{1z}z} + e^{-ik_{1z}(z-2z_{max})}\tilde{R}_{1max}\right]\frac{\tilde{M}_1}{k_{1z}}, \quad z > z', \tag{6}$$

$$F_-(z,z') = \left[e^{ik_{1z}z'} + e^{-ik_{1z}(z'-z_{max})}\tilde{R}_{1max}\right] \cdot \left[e^{-ik_{1z}z} + e^{ik_{1z}(z+2d_1)}\tilde{R}_{12}\right]\frac{\tilde{M}_1}{k_{1z}}, \quad z < z'. \tag{7}$$

In this problem, $z_{max} \to d_1$, $d_1 \to 0$.

(b) In the above, the total field consists of the source term, the upgoing wave and downgoing wave. When we change the sign of k_{1z}, it follows that the upgoing wave will change to the downgoing wave, and the downgoing wave will change to the upgoing wave. From the uniqueness theorem, the reflection coefficient will be changed to its reciprocal. That is

$$\tilde{R}(-k_{1z}) = 1/\tilde{R}(k_{1z}) \tag{8}$$

Using this formula, we can easily show that

$$F_\pm(-k_{1z}) = F_\pm(k_{1z}) \tag{9}$$

Hence, $F_\pm(z,z')$ are branch-point free functions. In fact, from a mathematical point of view, we can also show that (8) is true.

(c) From (9), when $z = z_{max}, R_{max} = -1$, and $F_+(z, z') = 0$,

$$\frac{1 + \tilde{R}_{12} e^{2ik_{1z}(z_{max}+d_1)}}{1 + \tilde{R}_{12} e^{2ik_{1z}(z_{max}-d_1)}} = 0.$$

§6.4

(a) Since $\phi(z) = A\left(e^{-ik_{1z}z} + \tilde{R}_{12}e^{ik_{1z}z}\right)$, the guidance condition is

$$1 + \tilde{R}_{12}e^{2ik_{1z}d_1} = 0.$$

(b) From Equation (2.1.21) [Chew, p. 51],

$$f(k_s) = \frac{1 + \tilde{R}_{12}e^{2ik_{1z}d_1}}{\tilde{R}_{12} + e^{2ik_{1z}d_1}},$$

where

$$\tilde{R}_{12} = R_{12} + \frac{T_{12}R_{23}T_{21}e^{2ik_{1z}d_1}}{1 - R_{21}R_{23}e^{2ik_{1z}d_1}}, \tag{2.1.21}$$

$R_{23} = -1, \quad T_{12} = 1 + R_{12}$, and $T_{21} = 1 + R_{21}$.

$$R_{12} = \frac{p_2k_{1z} - p_1k_{1z}}{p_2k_{1z} + p_1k_{1z}} \Rightarrow R_{12}(-k_{1z}) = \frac{1}{R_{12}}.$$

Through algebraic manipulation, we can obtain

$$\tilde{R}_{12} = \frac{R_{12} - e^{2ik_{2z}d_2}}{1 - R_{12}e^{2ik_{2z}d_2}} \Rightarrow \tilde{R}_{12}(-k_{1z}) = \frac{1/R_{12} - e^{2ik_{2z}d_2}}{1 - \frac{1}{R_{12}}e^{2ik_{2z}d_2}} = \frac{1}{\tilde{R}_{12}},$$

$$f(-k_{1z}) = \frac{1 + \tilde{R}_{12}(-k_{1z})e^{-2ik_{1z}d_1}}{\tilde{R}_{12}(-k_{1z}) + e^{-2ik_{1z}d_1}} = \frac{\tilde{R}_{12}(k_{1z})e^{2ik_{1z}d_1} + 1}{e^{2ik_{1z}d_1} + \tilde{R}_{12}(k_{1z})} = f(k_{1z}).$$

This demonstrates that $f(k_{1z})$ is a branch-point free function.

(c) The zeros of $f(k_s)$ is the root of the following equation

$$1 + \tilde{R}_{12}e^{2ik_{1z}d_1} = 0.$$

This is also the guidance condition in (a). Since there is the periodic function $e^{2ik_{1z}d_1}$ involved, the root of the above equation is discrete. Also, from Exercise 6.3, we can see that the above proof can be generalized to any enclosed layered slabs.

§6.5

(a) Wave expressions in region 1 can be solved to be

$$\phi(z) = a_1\left(e^{-ik_{1z}z} + \tilde{R}_{12}e^{ik_{1z}z}\right) \tag{1}$$

where, a_1 is the amplitude for the downgoing wave

$$\tilde{R}_{12} = R_{12} + \frac{T_{12}T_{23}T_{21}e^{2ik_{2z}d_2}}{1 - R_{21}R_{23}e^{2ik_{2z}d_2}}. \tag{2}$$

At boundary of $z = d_1$,

$$\phi(z) = 0, \quad \text{(for TE wave)},$$
$$\phi'(z) = 0, \quad \text{(for TM wave)}.$$

This is only possible when

$$1 + \tilde{R}_{12}^{TE} e^{2ik_{1z}d_1} = 0 \quad \text{(TE)}, \tag{3}$$
$$1 - \tilde{R}_{12}^{TM} e^{2ik_{1z}d_1} = 0 \quad \text{(TM)}. \tag{4}$$

The above are the guidance conditions for TE and TM waves respectively.

(b) In order to solve for the location of the modes, we first simplify (2). Using relations between T_{12}, R_{12}, T_{21} and R_{21}, we can arrive at

$$\tilde{R}_{12} = \frac{R_{12} + R_{23}e^{2ik_{2z}d_2}}{1 + R_{23}R_{12}e^{2ik_{2z}d_2}} \tag{5}$$

$$R_{23} = \begin{cases} -1, & \text{For TE wave} \\ 1, & \text{For TM wave} \end{cases}$$

$$R_{12} = \frac{p_2k_{1z} - p_1k_{2z}}{p_2k_{1z} + p_1k_{2z}}, \quad p_i = \begin{cases} \mu_i, & \text{for TE} \\ \epsilon_i, & \text{for TM} \end{cases}$$

Substitute $\tilde{R}_{12}$ of (2) into (3) and (4), we have

$$\left(1 - R_{12}e^{2ik_{2z}d_2}\right) + \left(R_{12} - e^{2ik_{2z}d_2}\right) e^{2ik_{1z}d_1} = 0, \qquad \text{(TE)}, \tag{6}$$
$$\left(1 + R_{12}e^{2ik_{2z}d_2}\right) - \left(R_{12} + e^{2ik_{2z}d_2}\right) e^{2ik_{1z}d_1} = 0, \qquad \text{(TM)}. \tag{7}$$

In the above, $k_{iz}^2 = k_i^2 - k_x^2, \ i = 1,2$. To solve (6) and (7) for k_x, we note that k_x must satisfy

$$0 < k_x < \min(k_1, k_2)$$

Therefore, the guided modes k_{xj} are located on the real k_x axis and limited by $0 < k_{xj} < \min(k_1, k_2), j = 1, 2, \ldots$. The number of guided mode will increase when d_1 or d_2 increases, as shown in the Table ($\epsilon_{r1} = 1, \epsilon_{r2} = 2.6$, TE).

k_x/k_0 table

	$d_1 = 0.5\lambda_0$	$d_1 = 1.0\lambda_0$	$d_1 = 2.0\lambda_0$
$d_2 = 1\lambda_0$	0.976 0.390	0.9962 0.8310 0.317	0.954 0.848 0.653 0.246
$d_2 = 2\lambda_0$	0.847 0.463	0.907 0.783 0.400	0.970 0.894 0.647 0.334
$d_2 = 3\lambda_0$	0.786 0.494	0.878 0.744 0.446	0.964 0.873 0.774 0.391

Table for Exercise Solution 6.5

(c) When the medium has some small loss, then, k_x should have a small imaginary part which corresponds to wave decay. Hence, the location of the modes will be slightly above the read k_x–axis. For TM cases, we have a similar result.

§6.6

(a) Eigensolution are of the form

$$\phi^{TE}(z) = A_m \left(e^{ik_{1z}z} + \tilde{R}_{12}^{TE} e^{ik_{1z}z}\right),$$
$$\phi^{TM}(z) = A_m \left(e^{ik_{1z}z} + \tilde{R}_{12}^{TM} e^{ik_{1z}z}\right),$$

where k_{1z} satisfies

$$1 + \tilde{R}_{12}^{TE} e^{2ik_{1z}d_1} = 0, \quad \text{TE},$$
$$1 - \tilde{R}_{12}^{TM} e^{2ik_{1z}d_1} = 0, \quad \text{TM}.$$

(b)

$$N_{mm} = \int_0^{d_1} A_m^2 \left(e^{-ik_{1z}z} + \tilde{R}_{12} e^{ik_{1z}z}\right)\left(e^{-ik_{1z}^* z} + \tilde{R}_{12}^* e^{ik_{1z}^* z}\right) p^{-1} dz$$
$$= A_m^2 \int_0^{d1} \left[1 + |\tilde{R}_{12}|^2 + \tilde{R}_{12} e^{2ik_{1z}z} + \tilde{R}_{12}^* e^{-2ik_{1z}z}\right] p^{-1} dz$$
$$= A_m^2 p^{-1} \left\{ d_1 \left(1 + |\tilde{R}_{12}|^2\right) + \tilde{R}_{12}^* \frac{e^{2ik_{1z}zd_1} - 1}{2ik_{1z}} \right.$$
$$\left. + \tilde{R}_{12}^* \frac{1 - e^{-2ik_{1z}zd_1}}{2ik_{1z}} \right\} = 1,$$
$$A_m = \left\{ p^{-1} \left[d_1 \left(1 + |\tilde{R}_{12}|^2\right) + \frac{\sin k_{1z}d_1}{k_{1z}} \cdot \Re e \left(\tilde{R}_{12} e^{ik_{1z}d_1}\right)\right]\right\}^{-\frac{1}{2}}$$

(c) When $\mu_1 = \mu_2$, $p_1 = p_2$, $\tilde{R}_{12} = \pm e^{2ik_{2z}d_2}$ (TE, TM), and

$$A_m = \sqrt{p}\left\{2d_1 + \frac{\sin k_{12}d_1}{k_{1z}}\left[\pm\cos 2\left(k_{1z}d_1 + k_{2z}d_2\right)\right]\right\}^{\frac{1}{2}}$$

Note that $A_m \to \infty$, when $d_1 \to \infty$ for the ideal case without loss.

§6.7

(a) Maxwell's equations are

$$\nabla \times \mathbf{E} = i\omega\mathbf{B} = i\omega\mu\mathbf{H}, \tag{1}$$
$$\nabla \times \mathbf{H} = -i\omega\mathbf{D} + \mathbf{J}, \tag{2}$$
$$\nabla \cdot \epsilon\mathbf{E} = \rho, \tag{3}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$
$$\nabla \cdot \mathbf{J} - i\omega\rho = 0. \tag{5}$$

$$\begin{aligned}\nabla \times [\text{Equation (1)}] \Rightarrow\ & \nabla \times \nabla \times \mathbf{E}\\ &= i\omega\nabla \times \mu\mathbf{H}\\ &= i\omega\nabla\mu \times \mathbf{H} + i\omega\mu\nabla \times \mathbf{H}\\ &= i\omega\frac{\partial\mu}{\partial z}\,\hat{z} \times \mathbf{H} + i\omega\mu \underbrace{\left[-i\omega\mathbf{D} + \mathbf{J}\right]}_{\text{[from Equation(2)]}}\end{aligned} \tag{6}$$

$$\text{Equation (3)} \Rightarrow \nabla\epsilon \cdot \mathbf{E} + \epsilon\nabla \cdot \mathbf{E} = \rho \Rightarrow \nabla \cdot \mathbf{E} = \epsilon^{-1}\rho - \epsilon^{-1}\frac{\partial\epsilon}{\partial z}E_z. \tag{7}$$

$$\hat{z} \cdot [\nabla\nabla \cdot \mathbf{E}] = \frac{\partial}{\partial z}(\nabla \cdot \mathbf{E}) = \frac{\partial}{\partial z}\left(\epsilon^{-1}\rho - \epsilon^{-1}\frac{\partial\epsilon}{\partial z}E_z\right), \tag{8}$$

$$\begin{aligned}\hat{z} \cdot \nabla^2\mathbf{E} &= \nabla_s^2 E_z + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\,\epsilon^{-1}D_z\right)\\ &= \epsilon^{-1}\nabla_s^2 D_z + \frac{\partial}{\partial z}\epsilon^{-1}\frac{\partial}{\partial z}D_z + \frac{\partial}{\partial z}\left(D_z\frac{\partial\epsilon^{-1}}{\partial z}\right),\end{aligned} \tag{9}$$

$\epsilon\hat{z}\cdot$ [both sides of Equation (6)] $\Rightarrow$

$$\epsilon\underbrace{\hat{z} \cdot (\nabla\nabla \cdot \mathbf{E})}_{\text{[Equation (8)]}} - \epsilon\underbrace{\hat{z} \cdot \nabla^2\mathbf{E}}_{\text{[Equation (9)]}} = \omega^2\mu\epsilon D_z + i\omega\mu\epsilon J_z \tag{10}$$

$$\begin{aligned}&\epsilon\hat{z} \cdot (\nabla\nabla \cdot \mathbf{E}) - \epsilon\hat{z} \cdot \nabla^2\mathbf{E}\\ &= \left[\epsilon\frac{\partial}{\partial z}\epsilon^{-1}\rho - \epsilon\frac{\partial}{\partial z}\left(\epsilon^{-1}E_z\frac{\partial\epsilon}{\partial z}\right)\right]\\ &\quad - \left[\nabla_s^2 D_z + \epsilon\frac{\partial}{\partial z}\left(\epsilon^{-1}\frac{\partial}{\partial z}D_z\right) + \epsilon\frac{\partial}{\partial z}\left(D_z\frac{\partial\epsilon^{-1}}{\partial z}\right)\right],\end{aligned}$$

Since

$$\epsilon^{-1}E_z\frac{\partial\epsilon}{\partial z}+D_z\frac{\partial\epsilon^{-1}}{\partial z}=\epsilon^{-1}E_z\frac{\partial\epsilon}{\partial z}-D_z\epsilon^{-2}\frac{\partial\epsilon}{\partial z}\equiv 0, \tag{11}$$

and

$$\nabla_s^2 D_2=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)D_z=\left(\frac{\partial^2}{\partial y^2}-k_x^2\right)D_z, \tag{12}$$

Equation (10) becomes

$$\epsilon\frac{\partial}{\partial z}\epsilon^{-1}\rho-\left[\frac{\partial^2}{\partial y^2}+\epsilon\frac{\partial}{\partial z}\epsilon^{-1}\frac{\partial}{\partial z}-k_x^2\right]D_z=k^2D_z+i\omega\mu\epsilon J_z.$$

It equivalent to

$$\left(\frac{\partial^2}{\partial y^2}+\epsilon\frac{\partial}{\partial z}\epsilon^{-1}\frac{\partial}{\partial z}+k^2-k_x^2\right)D_z=-i\omega\mu\epsilon J_z+\epsilon\frac{\partial}{\partial z}(\epsilon^{-1}\rho). \tag{13}$$

This is Equation (1) in the book (page 335).

(b)

$$\begin{aligned}\nabla\times\quad\text{Equation (2)}\quad\Rightarrow\quad \nabla\times\nabla\times\mathbf{H}&=\nabla\nabla\cdot\mathbf{H}-\nabla^2\mathbf{H}\\ &=-i\omega\nabla\times\mathbf{D}+\nabla\times\mathbf{J},\end{aligned} \tag{14}$$

$$\begin{aligned}\nabla\times\mathbf{D}=\nabla\times(\epsilon\mathbf{E})&=\nabla\epsilon\times E+\epsilon\nabla\times\mathbf{E}\\ &=\nabla\epsilon\times\mathbf{E}+\epsilon[i\omega B],\\ \hat{z}\cdot\nabla\times\mathbf{D}&=i\omega\epsilon B_z.\end{aligned} \tag{15}$$

From Equation (3),

$$0=\nabla\cdot\mathbf{B}=\nabla\cdot\mu\mathbf{H}=\nabla\mu\cdot\mathbf{H}+\mu\nabla\cdot\mathbf{H}\quad\Rightarrow\quad\nabla\cdot\mathbf{H}=-\mu^{-1}\frac{\partial\mu}{\partial z}H_z, \tag{16}$$

$$\hat{z}\cdot(\nabla\nabla\cdot\mathbf{H})=\frac{\partial}{\partial z}(\nabla\cdot\mathbf{H})=-\frac{\partial}{\partial z}\left(\mu^{-1}H_z\frac{\partial\mu}{\partial z}\right), \tag{17}$$

$$\begin{aligned}\hat{z}\cdot\nabla^2\mathbf{H}&=\mu^{-1}\nabla_s^2B_z+\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\,\mu^{-1}B_z\right)\\ &=\mu^{-1}\left(\frac{d^2}{dy^2}-k_x^2\right)B_z+\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}B_z+\frac{\partial}{\partial z}\left(B_z\frac{\partial\mu^{-1}}{\partial z}\right),\end{aligned} \tag{18}$$

$$\hat{z}\cdot\nabla\times\mathbf{J}=\nabla_s\times\mathbf{J}. \tag{19}$$

Since

$$B_z\frac{\partial\mu^{-1}}{\partial z}+\mu^{-1}H_z\frac{\partial\mu}{\partial z}=0,$$

the $\hat{z}$ component of Equation (14) becomes

$$-\mu^{-1}\left(\frac{d^2}{dy^2}-k_x^2\right)B_z-\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}B_z=\omega^2\epsilon B_z+\nabla_s\times\mathbf{J}.$$

Multiplying both sides of the above equation by $(-\mu)$ yields

$$\left(\frac{d^2}{dy^2}+\mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}+k^2-k_x^2\right)B_z=-\mu\nabla_s\times\mathbf{J}. \tag{20}$$

This is Equation (6.2.2) in the book (p. 335).

§6.8

(a)

$$\mathbf{J}=\hat{z}I\ell\delta(y)\delta(z), \tag{1}$$

$$\rho=\frac{1}{i\omega}\nabla\cdot\mathbf{J}=\frac{I\ell}{i\omega}\delta(y)\delta'(z), \tag{2}$$

$$\epsilon\frac{\partial}{\partial z}\epsilon^{-1}\rho=\epsilon\rho\frac{d\epsilon^{-1}}{dz}+\frac{d\rho}{dz}=\epsilon\frac{d\epsilon^{-1}}{dz}\left[\frac{I\ell}{i\omega}\delta(y)\delta'(z)\right]+\frac{I\ell}{i\omega}\delta(y)\delta''(z). \tag{3}$$

From the above equation, we can see that the source is composed of three terms. Thus, Equation (6.2.1) becomes

$$\left[\frac{d^2}{dy^2}+\epsilon\frac{\partial}{\partial z}\epsilon^{-1}\frac{\partial}{\partial z}+k^2-k_x^2\right]D_z(y,\ z)$$
$$=-i\omega\mu\epsilon I\ell\delta(y)\delta(z)+\epsilon\frac{d\epsilon^{-1}}{dz}\frac{I\ell}{i\omega}\delta(y)\delta'(z)+\frac{I\ell}{i\omega}\delta(y)\delta''(z). \tag{4}$$

Suppose, D_z has an eigenmode expansion

$$D_z(y,\ z)=\sum_{n=1}^{\infty}a_n(y)\phi_\epsilon(n,z). \tag{5}$$

On substituting (5) into (4), one obtains

$$\sum_{n=1}^{\infty}\left[\frac{d^2}{dy^2}+k_{nz\epsilon}^2-k_x^2\right]a_n(y)\phi_\epsilon(n,\ z)=\text{Source}. \tag{6}$$

In the above, we have used the fact that $\phi_\epsilon(z)$ satisfies

$$\left(\epsilon\frac{d}{dz}\epsilon^{-1}\frac{d}{dz}+k^2\right)\phi_\epsilon(n,\ z)=k_{nz\epsilon}^2\phi_\epsilon(n,\ z). \tag{7}$$

Also, we know from exercise (2) that $\phi_\epsilon(n,\ z)$ are ϵ^{-1} orthogonal. On multiplying (6) by $\epsilon^{-1}\phi_\epsilon(m,\ z)$, and integrating, we have

$$\left(\frac{d^2}{dy^2}+k_{my\epsilon}^2\right)a_m(y)=f_m\delta(y), \tag{8}$$

where,

$$\begin{aligned} f_m &= \int dz \epsilon^{-1}\phi_e(m,z)\left[-i\omega\mu\epsilon I\ell\delta(z) + \epsilon\frac{d\epsilon^{-1}}{dz}\frac{I\ell}{i\omega}\delta'(z) + \frac{I\ell}{i\omega}\delta''(z)\right] \\ &= -i\omega\mu I\ell\phi_e(m,\ 0) + \int \epsilon^{-1}\phi_e(m,\ z)\cdot\left[\epsilon\frac{\partial}{\partial z}\epsilon^{-1}\frac{I\ell}{i\omega}\delta'(z)\right] dz \\ &= -i\omega\mu I\ell\phi_e(m,\ 0) + \frac{I\ell}{i\omega\epsilon}(k_{mz\epsilon}^2 - k^2)\phi_e(m,\ 0) \\ &= \frac{I\ell}{i\omega\epsilon}k_{mz\epsilon}^2\phi_e(m,\ 0). \end{aligned}$$

The solution to Equation (8) is as follows:

$$a_m(y) = \frac{1}{2k_{my\epsilon}}f_m \cdot e^{ik_{my\epsilon}|y|} = \frac{I\ell\phi_e(m,\ 0)}{2i\omega\epsilon k_{my\epsilon}}e^{ik_{my\epsilon}|y|}. \tag{9}$$

Substitute (9) into (5), we get the final solution to (4):

$$D(y,\ z) = \sum_{n=1}^{\infty}\left[\frac{I\ell}{2i\omega\epsilon k_{my\epsilon}}\right] e^{ik_{my\epsilon}|y|}\phi_e(n,\ 0)\phi_e(n,\ z). \tag{10}$$

As for B_z, its source $= -\mu(\nabla_s \times \mathbf{J})_z = 0$, implying no B_z component.

(b) To use the Fourier transform method to solve for $a_m(y)$ we multiply (8) by $e^{-i\alpha y}$ and integrate over y from $-\infty$ to ∞ to obtain

$$\left(k_{my\epsilon}^2 - \alpha^2\right)\tilde{a}_m(\alpha) = f_m, \tag{11}$$

$$\tilde{a}_m(\alpha) = \frac{1}{k_{my\epsilon}^2 - \alpha^2}f_m(\alpha), \tag{12}$$

where,

$$a_m(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{a}_m(\alpha)e^{i\alpha y}d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_m}{k_{my\epsilon}^2 - \alpha^2}e^{i\alpha y}, \tag{13}$$

and f_m is as before.

(c) The integrand in (13) has two poles located at $\alpha = \pm k_{my\epsilon}$. Since the medium has a small loss, so that $k_{my\epsilon}$ has a small imaginary part to shift the poles off the $\Re e(\alpha)$ axis. On the other hand, in Equation (13) the integrand satisfies Jordan's Lemma. We can close the integral path in the upper half plane when $y > 0$. Then, using Cauchy's formula, we have

$$a_m(y) = 2\pi i \cdot \Re es\left[\frac{1}{2\pi}\frac{f_m}{k_{my\epsilon}^2 - \alpha^2}e^{i\alpha|y|}\right]_{\alpha=-k_{my\epsilon}} = \frac{if_m}{2k_{my\epsilon}}e^{ik_{my\epsilon}|y|}.$$

Substituting this into (5), we get exactly the same result as (10).

§6.9

By definition

$$e^{i\overline{\mathbf{K}}_M z} = \sum_{n=1}^{\infty} \frac{1}{n\,!}(i\overline{\mathbf{K}}_M z)^n = \sum_{n=1}^{\infty} \frac{(iz)^n}{n\,!}(\overline{\mathbf{K}}_M)^n$$

$$= \sum_{n=1}^{\infty} \frac{(iz)^n}{n\,!} \begin{bmatrix} k_1^n & & & \\ & k_2^n & & \\ & & k_3^n & \\ & & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{n=1}^{\infty} \frac{1}{n\,!}(ik_1 z)^n & & & \\ & \sum_{n=1}^{\infty} \frac{1}{n\,!}(ik_2 z)^n & & \\ & & \ddots & \\ & & & \sum \frac{(ik_{mn} z)^n}{n\,!} \end{bmatrix}$$

$$= \begin{bmatrix} e^{ik_1 z} & & & \\ & e^{ik_2 z} & & \\ & & \ddots & \\ & & & e^{ik_M z} \end{bmatrix}.$$

Therefore, $e^{i\overline{\mathbf{K}}_M z}$ is a diagonal matrix. Its elements are

$$e^{ik_1 z},\ e^{ik_2 z},\ \cdots,\ e^{ik_n z},\ \cdots\cdot e^{ik_M z}.$$

In the above, $\overline{\mathbf{K}}_M$ is a diagonal matrix of order $M \times M$, since each elements in the above will converge into a finite number, it is valid for any M.

§6.10

From Maxwell's equations in source free region with time dependence of $e^{-i\omega t}$,

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}, \tag{1}$$
$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E}. \tag{2}$$

Since $\mathbf{E} = \epsilon^{-1}\mathbf{D}$, $\mathbf{H} = \mu^{-1}\mathbf{B}$, the above equations can also be written as

$$\nabla \times \epsilon^{-1}\mathbf{D} = i\omega\mu\mathbf{H}, \tag{3}$$
$$\nabla \times \mu^{-1}\mathbf{B} - i\omega\epsilon\mathbf{E}. \tag{4}$$

To extract transverse component of the equation, we let $\nabla = \nabla_s + \nabla_z$, and $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_z$, where $\mathbf{A}$ stands for one of $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$, $\mathbf{D}$. Then we have for (3)

$$(\nabla_s + \nabla_z) \times (\epsilon^{-1}\mathbf{D}_s + \epsilon^{-1}\mathbf{D}_z) = i\omega\mu\mathbf{H}_s + i\omega\mu\mathbf{H}_z. \tag{5}$$

Extract the $\hat{s}$ component of the above, we have

$$\nabla_s \times \epsilon^{-1}\mathbf{D}_z + \nabla_z \times \epsilon^{-1}\mathbf{D}_s = i\omega\mu\mathbf{H}_s. \tag{6}$$

In the same way, we can derive from (4),

$$\nabla_s \times \mu^{-1}\mathbf{B}_z + \nabla_z \times \mu^{-1}\mathbf{B}_s = -i\omega\epsilon\mathbf{E}_s. \tag{7}$$

Multiplying (7) by $-i\omega\epsilon$ and making use of $k^2 = \omega^2\mu\epsilon$, we have

$$\begin{aligned} k^2\mathbf{H}_s &= -i\omega\nabla_s \times \mathbf{D}_z - i\omega\epsilon\nabla_z \times \mathbf{E}_s \\ &= -i\omega\nabla_s \times \mathbf{D}_z - i\omega\nabla_z \times \epsilon\mathbf{E}_s, \text{ (piecewise homogeneous).} \end{aligned}$$

Substituting (7) into the above for $-i\omega\epsilon\mathbf{E}_s$, we have

$$\begin{aligned} k^2\mathbf{H}_s &= -i\omega\nabla_s \times \mathbf{D}_z + \nabla_z \times \nabla_s \times \mu^{-1}\mathbf{B}_z + \nabla_z \times \nabla_z \times \mathbf{H}_s \\ &= -i\omega\nabla_s \times \mathbf{D}_z + \mu^{-1}\frac{\partial}{\partial z}\nabla_s B_z - \frac{\partial^2}{\partial z^2}\mathbf{H}_s. \end{aligned} \tag{8}$$

Substituting $\left(k^2 - \frac{\partial^2}{\partial z^2}\right)\mathbf{H}_s = k_{s\epsilon}^2\mathbf{H}_s$ into the above, and dividing both sides by $k_{s\epsilon}^2$, yield

$$\mathbf{H}_s = \frac{1}{k_{s\epsilon}^2}\mu^{-1}\frac{\partial}{\partial z}\nabla_s B_z - \frac{i\omega}{k_{s\epsilon}^2}\nabla_s \times \mathbf{D}_z. \tag{9}$$

In the same way as above,

$$\mathbf{E}_s = \frac{1}{k_{s\mu}^2}\epsilon^{-1}\frac{\partial}{\partial z}\nabla_s D_z + \frac{i\omega}{k_{s\mu}^2}\nabla_s \times \mathbf{B}_z. \tag{10}$$

§6.11

(a) We first write field solutions for both regions, and match them at the discontinuities. Finally, we use the orthogonal relation to solve for the reflection and transmission coefficients. The solutions are:

Region one:

$$\mathbf{A}_{1z} = \overline{\boldsymbol{\Phi}}_1^t(z) \cdot \left[e^{i\overline{\mathbf{k}}_1|y|} \cdot \boldsymbol{\phi}_{s\pm}(0) + e^{i\overline{\mathbf{k}}_1(y+d)} \cdot \overline{\mathbf{R}}_{12} \cdot e^{i\overline{\mathbf{k}}_1 d} \boldsymbol{\phi}_{s-}(0) \right], \tag{1}$$

$$\mathbf{A}_{1x} = \left[\overline{\mathbf{N}}_{1\pm}(z) \cdot e^{i\overline{\mathbf{k}}_1|y|} \cdot \boldsymbol{\phi}_{s\pm}(0) + \mathbf{N}_{1-} \cdot e^{i\overline{\mathbf{k}}_1(y+d)} \cdot \overline{\mathbf{R}}_{12} \cdot e^{i\overline{\mathbf{k}}_1 d} \boldsymbol{\phi}_{s-}(0) \right]. \tag{2}$$

Region two:

$$\mathbf{A}_{2z} = \boldsymbol{\Phi}_2^t(z) \cdot e^{-i\overline{\mathbf{k}}_2(y+d)} \cdot \overline{\mathbf{T}}_{12} \cdot e^{i\overline{\mathbf{k}}_2 d} \cdot \boldsymbol{\phi}_{s-}(0), \tag{3}$$

$$\mathbf{A}_{2x} = \overline{\mathbf{N}}_{2-} \cdot e^{-i\overline{\mathbf{k}}_2(y+d)} \cdot \overline{\mathbf{T}}_{12} \cdot e^{i\overline{\mathbf{k}}_2 d} \cdot \boldsymbol{\phi}_{s-}(0). \tag{4}$$

Tangential components of the field are continuous at $y = -d$, that is

$$\overline{\mathbf{E}}_1^{-1}(z) \cdot \boldsymbol{\Phi}_1^t(z) \cdot [\overline{\mathbf{I}} + \overline{\mathbf{R}}_{12}] = \overline{\mathbf{E}}_2^{-1}(z) \cdot \overline{\boldsymbol{\Phi}}_2^t(z) \cdot \overline{\mathbf{T}}_{12}, \tag{5}$$

$$\overline{\mathbf{N}}_{1-} + \overline{\mathbf{N}}_{1+} \cdot \overline{\mathbf{R}}_{12} = \overline{\mathbf{N}}_{2-} \cdot \overline{\mathbf{T}}_{12}, \tag{6}$$

where

$$\overline{\mathbf{E}}_i(z) = \begin{bmatrix} \epsilon_i(z) 0 & \\ 0 & \mu_i(z) \end{bmatrix}. \tag{7}$$

Since $\overline{\boldsymbol{\Phi}}_i^t$ are ϵ_i^{-1} orthogonal, multiplying (5) by $\overline{\boldsymbol{\Phi}}_1(z)$ and integrating give

$$\overline{\mathbf{I}} + \overline{\mathbf{R}}_{12} = \left\langle \overline{\boldsymbol{\Phi}}_1(z), \overline{\mathbf{E}}_2^{-1}(z) \cdot \overline{\boldsymbol{\Phi}}_2^t(z) \right\rangle \cdot \overline{\mathbf{T}}_{12} = \overline{\mathbf{B}}_{12} \cdot \overline{\mathbf{T}}_{12}. \tag{8}$$

Multiplying (6) by $\overline{\boldsymbol{\Phi}}_1(z) \cdot \overline{\mathbf{E}}_1^t$, and then integrating over z give

$$\overline{\mathbf{H}}_{-1} + \overline{\mathbf{H}}_{1+} \cdot \overline{\mathbf{R}}_{12} = \overline{\mathbf{H}}_{2-} \cdot \overline{\mathbf{T}}_{12}, \tag{9}$$

where $\overline{\mathbf{B}}_{12} = \left\langle \overline{\boldsymbol{\Phi}}_1(z), \overline{\mathbf{E}}_2^t(z) \cdot \overline{\boldsymbol{\Phi}}_2^t(z) \right\rangle$, $\overline{\mathbf{H}}_{i\pm} = \langle \overline{\boldsymbol{\Phi}}_1(z) \mathbf{E}_1^\epsilon, \overline{\mathbf{N}}_{i\pm} \rangle$. Solving (8) and (9) for $\mathbf{R}_{12}$ and $\mathbf{T}_{12}$, we have

$$\overline{\mathbf{R}}_{12} = -(\overline{\mathbf{B}}_{12}^{-1} - \overline{\mathbf{H}}_{2-}^{-1} \cdot \overline{\mathbf{H}}_{1+})^{-1} \cdot (\overline{\mathbf{B}}_{12}^{-1} - \mathbf{H}_{2-}^{-1} \cdot \overline{\mathbf{H}}_{1-}),$$

$$\overline{\mathbf{T}}_{12} = \overline{\mathbf{B}}_{12}^{-1} \cdot (\overline{\mathbf{I}} + \overline{\mathbf{R}}_{12}).$$

(b) This solution is not identical to that when source is on the left of the junction, the reason is that the field is not the same on different sides of the source.

§6.12

Since $\{S_i(z)\}$ is a complete set with orthogonal property, a function in the same space can be expanded by $\{S_i\}$ such that

$$f(z) = \sum_n a_n S_n(z) = \mathbf{S}^t(z) \cdot \mathbf{a}. \tag{1}$$

Multiplying the above equation by $S_m(z)$, and integrating over z give

$$\int f(z)S_m(z)dz = \sum_n a_n \int S_n(z)S_m(z)dz = a_m\delta_{mn}.$$

Therefore,

$$a_m = \int f(z)\cdot S_m(z)dz = \langle S_m(z), f(z)\rangle,$$

or

$$\mathbf{a} = \langle \mathbf{S}(z),\ f(z)\rangle.$$

Substituting $\mathbf{a}$ into (1) gives $f(z) = \mathbf{S}^t(z)\cdot\langle\mathbf{S}(z),\ f(z)\rangle$. As for Equation (23), let

$$f(z) = p^{-1}(z)\frac{\partial}{\partial z}S_m(z),$$

we have

$$p^{-1}(z)\frac{\partial}{\partial z}S_m(z) = \mathbf{S}^t(z)\cdot\langle\mathbf{S}(z),\ p^{-1}(z)\frac{\partial}{\partial z}S_m(z)\rangle.$$

In vector form, the above is

$$p^{-1}(z)\frac{\partial}{\partial z}\mathbf{S}^t(z) = \mathbf{S}^t(z)\cdot\langle\mathbf{S}(z),\ p^{-1}\frac{\partial}{\partial z}\mathbf{S}^t(z)\rangle.$$

§6.13

From Equation (6.3.33) we have

$$\bar{\mathbf{r}}_{12} = \overline{\mathbf{B}}_1^t : \overline{\mathbf{R}}_{12} : \overline{\mathbf{B}}_1, \tag{1}$$

$$\bar{\mathbf{t}}_{12} = \overline{\mathbf{B}}_2^t : \overline{\mathbf{T}}_{12} : \overline{\mathbf{B}}_2. \tag{2}$$

Left-multiplying (1) by $\overline{\mathbf{B}}\cdot\overline{\mathbf{P}}^I$, and using Equation (6.3.29) where

$$\overline{\mathbf{B}}_1\cdot\overline{\mathbf{p}}_1^I\cdot\overline{\mathbf{B}}_1^t = \delta(k_z - k_z'),$$

and the relation

$$\delta(k_z,\ k_z') : f(k_z') = f(k_z),$$

we have

$$\overline{\mathbf{B}}_1\cdot\overline{\mathbf{p}}_1^I\cdot\bar{\mathbf{r}}_{12} = \overline{\mathbf{R}}_{12} : \overline{\mathbf{B}}_1. \tag{3}$$

Again, right multiplying (3) by $\overline{\mathbf{p}}_1^I\cdot\overline{\mathbf{B}}_1^t$ yields

$$\overline{\mathbf{R}}_{12} = \overline{\mathbf{B}}_1\cdot\overline{\mathbf{p}}_1^I\cdot\bar{\mathbf{r}}_{12}\cdot\overline{\mathbf{p}}_1^I\cdot\overline{\mathbf{B}}_1^t. \tag{4}$$

In the same way, we can derive the following from (2):

$$\overline{\mathbf{T}}_{12} = \overline{\mathbf{B}}_2\cdot\overline{\mathbf{p}}_2^I\cdot\bar{\mathbf{t}}_{12}\cdot\overline{\mathbf{p}}_2^I\cdot\overline{\mathbf{B}}_2^t. \tag{5}$$

§6.14

(a)

$$\phi(z) = \sum_{n=1}^{N} a_n S_n(z) = (S_1(z),\ S_2(z),\ \cdots,\ S_N(z)) \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} = \mathbf{S}^t(z) \cdot \mathbf{a}$$

where

$$\mathbf{S}(z) = \begin{pmatrix} S_1(z) \\ S_2(z) \\ \vdots \\ S_N(z) \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}.$$

(b) From (6.7.6), we have

$$L_{mn} = -\langle S'_m(z),\ p^{-1}S'_n(z)\rangle + \langle S_m(z),\ k^2p^{-1}S_n(z)\rangle. \tag{1}$$

Using the result of (a), we have

$$(L_{m1},\ L_{m2}\ \cdots\cdots L_{mn}) = -\langle S'_m,\ p^{-1}\mathbf{S}^{t\prime}\rangle + \langle S_m,\ k^2p^{-1}\mathbf{S}^t\rangle.$$

$$\begin{aligned}
\overline{\mathbf{L}} = (L_{mn}) &= -\langle \mathbf{S}',\ p^{-1}\mathbf{S}^{t\prime}\rangle + \langle \mathbf{S},\ k^2p^{-1}\mathbf{S}^t\rangle \\
&= -\int \mathbf{S}' \cdot p^{-1}(\mathbf{S}^t)'dz + \int \mathbf{S} \cdot k^2p^{-1}\mathbf{S}^t dz \\
&= -\left[\mathbf{S} \cdot p^{-1}(\mathbf{S}^t)'\right]_a^b + \int \mathbf{S} \cdot \left[\frac{d}{dz}p^{-1}\frac{d\mathbf{S}^t}{dz} + k^2p^{-1}\mathbf{S}^t\right] dz \\
&= \int \mathbf{S} \cdot \mathcal{L}(z)\mathbf{S}^t dz \\
&= \langle \mathbf{S}(z),\ \mathcal{L}(z)\mathbf{S}^t\rangle, \\
\overline{\mathbf{P}}^I &= (p^I_{mn}) = \langle \mathbf{S},\ p^{-1}\mathbf{S}^t\rangle.
\end{aligned}$$

(c) By direct integration, we can see from (1) that

$$\begin{aligned}
\mathcal{L}_{mn} &= -\int S'_m p^{-1} S'_n dz + \int S_m k^2 p^{-1} S_n dz \\
&= -\langle S'_n,\ p^{-1}S'_m\rangle + \langle S_n,\ k^2p^{-1}S_m\rangle = \mathcal{L}_{nm}.
\end{aligned}$$

Therefore,

$$(\overline{\mathbf{L}})^t = \overline{\mathbf{L}}.$$

In the same way, $(\overline{\mathbf{p}}^I)^t = \overline{\mathbf{p}}^I$, so that $\overline{\mathbf{L}}$ and $\overline{\mathbf{p}}^I$ are symmetric matrices.

§6.15

(a) Assume that b_i and b_j are eigenvectors of (6.4.7a); that is

$$\overline{\mathbf{L}} \cdot \mathbf{b}_i = k_i^2 \overline{\mathbf{p}}^I \cdot \mathbf{b}_i, \tag{1}$$

$$\overline{\mathbf{L}} \cdot \mathbf{b}_j = k_j^2 \overline{\mathbf{p}}^I \cdot \mathbf{b}_j. \tag{2}$$

Multiplying (1) by $\mathbf{b}_j^t$, and (2) by $\mathbf{b}_i^t$, respectively, and then subtracting the resulting equations, we have

$$k_i^2\mathbf{b}_j^t \cdot \overline{\mathbf{p}}^t \cdot \mathbf{b}_i - k_j^2 \cdot \mathbf{b}_i^t \cdot \overline{\mathbf{p}}^I \cdot \mathbf{b}_j = \mathbf{b}_j^t \cdot \overline{\mathbf{L}} \cdot \mathbf{b}_i - \mathbf{b}_i^t \cdot \overline{\mathbf{L}} \cdot \mathbf{b}_j \tag{3}$$

Since $\overline{\mathbf{p}}^I$ and $\overline{\mathbf{L}}$ are symmetric, we have

$$\begin{aligned} \mathbf{b}_j^t \cdot \overline{\mathbf{p}}^I \cdot \mathbf{b}_i &= \mathbf{b}_i^t \cdot \overline{\mathbf{p}}^I \cdot \mathbf{b}_j, \\ \mathbf{b}_j^t \cdot \overline{\mathbf{L}} \cdot \mathbf{b}_i &= \mathbf{b}_i^t \cdot \overline{\mathbf{L}} \cdot \mathbf{b}_j. \end{aligned} \tag{4}$$

Substituting the above into (3), we can find that the right hand side of (3) equals to zero, so that

$$(k_i^2 - k_j^2)\mathbf{b}_i^t \cdot \overline{\mathbf{p}}^I \cdot \overline{\mathbf{b}}_j = 0.$$

If $i \neq j$, then $k_i^2 \neq k_j^2$, then $\mathbf{b}_i^t \cdot \overline{\mathbf{p}}^I \cdot \mathbf{b}_j = 0$. That is to say, $\mathbf{b}_i$ are $\overline{\mathbf{p}}^I$ orthogonal.

(b)

$$\begin{aligned} &\langle \tilde{\phi}(i,z),\ p^{-1}\tilde{\phi}(j,z)\rangle = \int \tilde{\phi}(i,z)p^{-1}\phi(j,z)dz \\ =&\int \mathbf{S}^t(z) \cdot \mathbf{b}_i p^{-1}\mathbf{S}^t \cdot \mathbf{b}_j dz = \int \mathbf{b}_i^t \cdot \mathbf{S}(z)p^{-1}\mathbf{S}^t(z) \cdot \mathbf{b}_j dz \\ =&\mathbf{b}_i^t \cdot \left\{ \int \mathbf{S}^t p^{-1} S dz \right\} \cdot \mathbf{b}_j = \mathbf{b}_i^t \cdot \overline{\mathbf{p}}^I \cdot \mathbf{b}_j = 0, \quad (i \neq j). \end{aligned}$$

§6.16

$$\tilde{\phi}(i,z) = \sum_{n=1}^{N} b_{in}S_n(z), \quad i = 1,2,\cdots,N.$$

$$\begin{aligned} &\langle \tilde{\phi}_\epsilon(m,z), \left(\frac{\partial}{\partial z}\epsilon^{-1}\frac{\partial}{\partial z} + \epsilon^{-1}k^2\right)\tilde{\phi}_\epsilon(n,\ z)\rangle \\ =&\langle \sum_{i=1}^{N} b_{mi}S_i(z),\ \mathcal{L}(z)\sum_{j=1}^{N} b_{nj}S_j(z)\rangle \\ =&\sum_{j=1}^{N}\sum_{i=1}^{N} b_{mi}\langle S_i(z),\ \mathcal{L}(z)S_j(z)\rangle b_{nj} \\ =&\sum_{j=1}^{N}\sum_{i=1}^{N} b_{mi}\mathcal{L}_{ij}b_{nj} = \mathbf{b}_m^t \cdot \overline{\mathbf{L}} \cdot \mathbf{b}_n. \end{aligned} \tag{1}$$

Also, $\overline{\mathbf{L}} \cdot \mathbf{b}_n = k_{ns}^2 \cdot \overline{\mathbf{p}}^I \cdot \mathbf{b}_n$ from (6.4.7a). Substituting into (1), we have

$$\mathbf{b}_m^t \cdot \overline{\mathbf{L}} \cdot \overline{\mathbf{b}}_n = \mathbf{b}_m^t \cdot k_{ns}^2 \overline{\mathbf{p}}^I \cdot \mathbf{b}_n$$
$$= k_{ns}^2 \underbrace{\mathbf{b}_m^t \cdot \overline{\mathbf{p}}^I \cdot \mathbf{b}_n}_{\text{orthogonal property of } \mathbf{b}_n} = k_{ms}^2 \delta_{mn}. \qquad (2)$$

§6.17

For metallic boundary of symmetric cylinder, there is no TE and TM coupling, so that $R_{12} = R_{21} = 0$. Therefore,

$$E_{Nz} = b_{N1}\left[H_n^{(1)}(k_{N\rho}\rho) + R_{11} J_n(k_{N\rho}\rho)\right], \qquad (1)$$
$$H_{Nz} = b_{N2}\left[H_n^{(1)}(k_{N\rho}\rho) + R_{22} J_n(k_{N\rho}\rho)\right]. \qquad (2)$$

For TM to z wave,

$$E_{Nz}\big|_{\rho=\rho_{\max}} = 0. \qquad (3)$$

For TE to z wave

$$\left.\frac{\partial H_{Nz}}{\partial \rho}\right|_{\rho=\rho_{\max}} = 0. \qquad (4)$$

From (3),

$$H_n^{(1)}(k_{N\rho}\rho_{\max}) + R_{11} J_n(k_{N\rho}\rho_{\max}) = 0,$$
$$R_{11} = \frac{-H_n^{(1)}(k_{N\rho}\rho_{max})}{J_n(k_{N\rho}\rho_{\max})}.$$

From (4),

$$R_{22} = \frac{-H_n^{(1)'}(k_{N\rho}\rho_{\max})}{J_n'(k_{N\rho}\rho_{\max})}.$$

Therefore,

$$\overline{\mathbf{R}}_{N,N+1} = \begin{bmatrix} \frac{-H_n^{(1)}(k_{N\rho}\rho_{\max})}{J_n(k_{N\rho}\rho_{\max})} & 0 \\ 0 & \frac{-H_n^{(1)'}(k_{N\rho}\rho_{\max})}{J_n'(k_{N\rho}\rho_{\max})} \end{bmatrix}.$$

§6.18

(a) Maxwell's equations in a source region are

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} - \mathbf{M}, \qquad (1)$$
$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} + \mathbf{J}. \qquad (2)$$

$\mu\nabla \times \mu^{-1}$ operating on (1) results in

$$\mu\nabla \times \mu^{-1}\nabla \times \mathbf{E} = i\omega\mu\nabla \times \mathbf{H} - \mu\nabla \times \mu^{-1}\mathbf{M}. \qquad (3)$$

Substituting (2) into (3) yields

$$\mu\nabla \times \mu^{-1} \times \nabla \times \mathbf{E} = i\omega\mu(-i\omega\epsilon\mathbf{E} + \mathbf{J}) - \mu\nabla \times \mu^{-1}\mathbf{M}.$$

Since $k^2 = \omega^2\mu\epsilon$, the above becomes

$$\mu\nabla \times \mu^{-1}\nabla \times \mathbf{E} - k^2\mathbf{E} = i\omega\mu\mathbf{J} - \mu\nabla \times \mu^{-1}\mathbf{M}. \tag{4}$$

To extract transverse components of the above, we let $\mathbf{E} = \mathbf{E}_s + \mathbf{E}_z$, and $\nabla = \nabla_s + \nabla_z$. Substituting into (4), the left-hand side becomes

$$\begin{aligned}
&\mu\nabla \times \mu^{-1}(\nabla_s + \nabla_z) \times (\mathbf{E}_s + \mathbf{E}_z) - k^2(\mathbf{E}_s + \mathbf{E}_z)\\
=&\mu(\nabla_s + \nabla_z) \times \mu^{-1}(\nabla_s \times \mathbf{E}_s + \nabla_s \times \mathbf{E}_z + \nabla_z \times \mathbf{E}_s\\
&+\nabla_z \times \mathbf{E}_z) - k^2(\mathbf{E}_s + \mathbf{E}_z)\\
=&\mu\nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_s + \mu\nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_z + \mu\nabla_s \times \mu^{-1}\nabla_z \times \mathbf{E}_s\\
&- k^2(\mathbf{E}_s + \mathbf{E}_z) + \mu\nabla_z \times \mu^{-1}\nabla_s \times \mathbf{E}_s\\
&+ \mu\nabla_z \times \mu^{-1}\nabla_s \times \mathbf{E}_z + \mu\nabla_z \times \mu^{-1}\nabla_z \times \mathbf{E}_s.
\end{aligned} \tag{5}$$

Extracting the $\hat{s}$-component, we have

$$\begin{aligned}
\mu\nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_s + \mu\nabla_z \times \mu^{-1}\nabla_s \times \mathbf{E}_z + \mu\nabla_z \times \mu^{-1}\nabla_z \times \mathbf{E}_s - k^2\mathbf{E}_s&\\
= i\omega\mu\mathbf{J}_s - \mu(\nabla \times \mu^{-1}\mathbf{M})_s.&
\end{aligned} \tag{6}$$

Since μ is a function of $\mathbf{r}_s$, we have

$$\mu\nabla_z \times \mu^{-1}\nabla_z \times \mathbf{E}_s = \nabla_z \times \nabla_z \times \mathbf{E}_s = -\frac{\partial^2}{\partial z^2}\mathbf{E}_s. \tag{7}$$

Also, we have $\nabla \cdot \epsilon\mathbf{E} = \rho$. From this equation, we can relate $\mathbf{E}_z$ with $\mathbf{E}_s$ yielding

$$(\nabla_s + \nabla_z) \cdot (\epsilon\mathbf{E}_s + \epsilon E_z\hat{z}) = \nabla_s \cdot \epsilon\mathbf{E}_s + \epsilon\frac{\partial E_z}{\partial z} = \rho.$$

Multiplying the above by $\nabla_s\epsilon^{-1}$, we obtain

$$\frac{\partial}{\partial z}\nabla_s\mathbf{E}_z = \nabla_s\frac{\rho}{\epsilon} - \nabla_s\epsilon^{-1}\nabla_s \cdot \epsilon\mathbf{E}_s. \tag{8}$$

The second term of the left hand side in (7) can be written as

$$\mu\nabla_z \times \mu^{-1}\nabla_s \times \mathbf{E}_z = \nabla_z \times \nabla_s \times \mathbf{E}_z = \frac{\partial}{\partial z}\nabla_s E_z. \tag{9}$$

Substituting (8) and (9) into (6), we finally get

$$\begin{aligned}
\mu\nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_s - \nabla_s\epsilon^{-1}\nabla_s \cdot \epsilon\mathbf{E}_s - \left(k^2 + \frac{\partial^2}{\partial z^2}\right)\mathbf{E}_s&\\
= i\omega\mu\mathbf{J}_s - \mu\nabla \times \mu^{-1}\mathbf{M})_s - \nabla_s\frac{\rho}{\epsilon}.&
\end{aligned} \tag{10}$$

Follow the same steps as above, we can derive that

$$\begin{aligned}
&\epsilon\nabla_s \times \epsilon^{-1}\nabla_s \times \mathbf{H}_s - \nabla_s\mu^{-1}\nabla_s \cdot \mu\mathbf{H}_s - \left(k^2 + \frac{\partial^2}{\partial z^2}\right)\mathbf{H}_s \\
&= i\omega\epsilon\mathbf{M}_s + \epsilon\left(\nabla \times \epsilon^{-1}\mathbf{J}\right)_s - \nabla_s\frac{\rho_m}{\mu}.
\end{aligned} \tag{11}$$

(b) If μ and ϵ are functions of ρ only, and $\frac{\partial}{\partial\phi} = in$, n is an integer, we can further expand (10) or (11) into two scalar equations. To do so, we first write

$$\nabla_s \cdot \mathbf{E}_s = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_\rho + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_\rho + \frac{in}{\rho}e_\phi, \tag{12}$$

$$\nabla_s \times \mathbf{E}_s = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_\phi - \frac{in}{\rho}E_\rho\right)\hat{z}, \tag{13}$$

$$\nabla_s \times [f(\rho,\phi)\hat{z}] = \frac{in}{\rho}f\hat{\rho} - \frac{\partial f}{\partial\rho}\hat{\phi}. \tag{14}$$

We consider Equation (10) first. Its left hand side has three terms, two of which need to be simplified. The first one is

$$\begin{aligned}
&\mu\nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_s \\
&= \mu\nabla_s \times \mu^{-1}\left[\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_\phi - \frac{in}{\rho}E_\rho\right)\hat{z}\right] \\
&= \mu\cdot\frac{in}{\rho}\cdot\left(\mu^{-1}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_\phi - \mu^{-1}\frac{in}{\rho}E_\rho\right)\hat{\rho} \\
&\quad - \left(\mu\frac{\partial}{\partial\rho}\frac{1}{\rho\mu}\frac{\partial}{\partial\rho}\rho E_\phi - \mu\frac{\partial}{\partial\rho}\frac{in}{\rho\mu}E_\rho\right)\hat{\phi} \\
&= \left(\frac{in}{\rho^2}\frac{\partial}{\partial\rho}\rho E_\phi + \frac{n^2}{\rho^2}E_\rho\right)\hat{\rho} \\
&\quad - \left(\mu\frac{\partial}{\partial\rho}\cdot\frac{1}{\rho\mu}\frac{\partial}{\partial\rho}\rho E_\phi - \mu\frac{\partial}{\partial\rho}\frac{in}{\rho\mu}E_\rho\right)\hat{\phi}.
\end{aligned} \tag{15}$$

The second one is

$$\begin{aligned}
&-\nabla_s\epsilon^{-1}\nabla_s \cdot \epsilon\mathbf{E}_s \\
&= -\left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{\partial}{\rho\partial\phi}\right)\epsilon^{-1}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\epsilon E_\rho + \frac{in}{\rho}\epsilon E_\phi\right) \\
&= -\hat{\rho}\left(\frac{\partial}{\partial\rho}\frac{1}{\rho\epsilon}\frac{\partial}{\partial\rho}\rho\epsilon E_\rho + \frac{\partial}{\partial\rho}\cdot\frac{in}{\rho}E_\phi\right) - \hat{\phi}\left(\frac{in}{\rho^2\epsilon}\frac{\partial}{\partial\rho}\rho\epsilon E_\rho - \frac{n^2}{\rho^2}E_\rho\right).
\end{aligned} \tag{16}$$

Substituting (15) and (16) into (10), we get a vector equation, which can be sorted into two scalar equations: one is for $\hat{\phi}$ component, another is for $\hat{\phi}$ component, i.e.,

$$-\left(\frac{\partial}{\partial\rho}\frac{1}{\rho\epsilon}\frac{\partial}{\partial\rho}\rho\epsilon E_\rho - \frac{n^2}{\rho^2}E_\rho + k^2E_\rho + \frac{\partial^2 E_\rho}{\partial z^2}\right) + \frac{2in}{\rho^2}E_\phi = 0, \tag{17}$$

$$\left(\mu\frac{\partial}{\partial\rho}\frac{in}{\rho\mu} - \frac{in}{\rho^2\epsilon}\frac{\partial}{\partial\rho}\rho\epsilon\right)E_\rho - \left(\mu\frac{\partial}{\partial\rho}\frac{1}{\rho\mu}\frac{\partial}{\partial\rho}\rho - \frac{n^2}{\rho^2} + k^2 + \frac{\partial^2}{\partial z^2}\right)E_\phi = 0. \tag{18}$$

Substituting $\epsilon = \epsilon_0\epsilon_r$, $\mu = \mu_0\mu_r$ into the above two equations, and making some mathematical changes in the form, we get

$$-\frac{1}{\rho\epsilon_r}\left(\rho\epsilon_r\frac{\partial}{\partial\rho}\frac{1}{\rho\epsilon_r\rho} - \frac{n^2}{\rho^2} + k^2 + \frac{\partial^2}{\partial z^2}\right)\rho\epsilon_r E_\rho + \frac{2in}{\rho^2}E_\phi = 0, \tag{19}$$

$$\left[in\frac{\partial}{\partial\rho}\left(\frac{1}{\rho^2\mu_r\epsilon_r}\right)\right]\rho\epsilon_r E_\rho - \frac{1}{\rho\mu_r}\left(\rho\mu_r\frac{\partial}{\partial\rho}\frac{1}{\rho\mu_r}\frac{\partial}{\partial\rho} - \frac{n^2}{\rho^2} + k^2 + \frac{\partial^2}{\partial z^2}\right)\rho E_\phi = 0. \tag{20}$$

In the same way as above, we can derive equations for H_ρ and H_ϕ. They are

$$-\frac{1}{\rho\epsilon_r}\left(\rho\epsilon_r\frac{\partial}{\partial\rho}\frac{1}{\rho\epsilon_r}\frac{\partial}{\partial\rho} - \frac{n^2}{\rho^2} + k^2 + \frac{\partial^2}{\partial z^2}\right)\rho H_\phi + \left[in\frac{\partial}{\partial\rho}\left(\frac{1}{\rho^2\mu_r\epsilon_r}\right)\right]\rho\mu_r H_\rho = 0, \tag{21}$$

$$\frac{2in}{\rho^2}H_\phi - \frac{1}{\rho\mu_r}\left(\rho\mu_r\frac{\partial}{\partial\rho}\frac{1}{\rho\mu_r}\frac{\partial}{\partial\rho} - \frac{n^2}{\rho^2} + k^2 + \frac{\partial^2}{\partial z^2}\right)\rho\mu_r H_\rho = 0. \tag{22}$$

§6.19

(a) Let $\overline{\mathbf{P}} = (\overline{\mathbf{A}} + \lambda\overline{\mathbf{B}})$. The eigenvalue of $\overline{\mathbf{P}}$ is the solution of the following algebraic equation.

$$\left|\overline{\mathbf{P}} - \bar{\mathbf{I}}\lambda\right| = 0.$$

This equation can be expanded as

$$\left|\overline{\mathbf{P}} - \bar{\mathbf{I}}\lambda\right| = \lambda^n + S_1\lambda^{n-1} + S_2\lambda^{n-2}\cdots + S_{n-1}\lambda + (-1)^n\left|\overline{\mathbf{P}}\right| = 0$$

where $S_m = (-1)^m \cdot \sum_{\text{all}} M_m^{(1)}, M_m^{(i)}$ is $\overline{\mathbf{P}}$'s m-order principle minor.

Since all kind of principle minors of $\overline{\mathbf{P}}$ and $\overline{\mathbf{P}}^t$ are the same, so that

$$|\overline{\mathbf{P}} - I\lambda| = |\overline{\mathbf{P}}^t - \bar{\mathbf{I}}\lambda|.$$

This means $\overline{\mathbf{P}}$ and $\overline{\mathbf{P}}^t$ share the same set of eigenvalues.

(b) Let λ_i and λ_j be defined as

$$(\overline{\mathbf{A}} + \lambda_j \overline{\mathbf{B}}) \cdot \mathbf{a}_j = 0, \tag{1}$$

$$(\overline{\mathbf{A}}^t + \lambda_i \overline{\mathbf{B}}^t) \cdot \overline{\mathbf{b}}_i = 0. \tag{2}$$

Multiplying (1) by $\mathbf{b}_i^t$ and (2) by $\mathbf{a}_j^t$, and then subtracting the resultant equations, we obtain

$$(\mathbf{b}_i^t \cdot \overline{\mathbf{A}} \cdot \mathbf{a}_j + \lambda_j \mathbf{b}_i^t \cdot \overline{\mathbf{B}} \cdot \mathbf{a}_j) - (\mathbf{a}_i^t \cdot \overline{\mathbf{A}}^t \cdot \mathbf{b}_i + \lambda_i \mathbf{a}_j^t \cdot \overline{\mathbf{B}} \cdot \mathbf{b}_i) = 0.$$

This can be simplified to

$$(\lambda_i - \lambda_j)\mathbf{b}_i^t \cdot \overline{\mathbf{B}} \cdot \mathbf{a}_j = 0.$$

Since λ_i and λ_j are not identical, we have

$$\mathbf{b}_i^t \cdot \mathbf{B} \cdot \mathbf{a}_j = 0.$$

(c) To show the orthogonality, we first consider two sets of solutions to Maxwell's curl equations:

$$\nabla \times \mathbf{E}_i = i\omega\mu \mathbf{H}_i, \tag{3}$$

$$\nabla \times \mathbf{E}_j = i\omega\mu \mathbf{H}_j. \tag{4}$$

Dot multiplying (3) by $\mathbf{H}_j$ and (4) by $\mathbf{H}_i$, and subtracting the resultant equations, we get

$$\mathbf{H}_j \cdot \nabla \times \mathbf{E}_i - \mathbf{H}_i \cdot \nabla \times \mathbf{E}_j = 0, \tag{5}$$

Likewise,

$$\mathbf{E}_j \cdot \nabla \times \mathbf{H}_i - \mathbf{E}_i \cdot \nabla \times \mathbf{H}_j = 0. \tag{6}$$

Adding (5) and (6), and using the following identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

we have

$$\nabla \cdot (\mathbf{E}_i \times \mathbf{A}_j - \mathbf{E}_j \times \mathbf{H}_i) = 0. \tag{7}$$

The $\hat{z}$-dependence of $\mathbf{E}_i$ and $\mathbf{H}_i$ are $\exp(-ik_{zi}z)$. Thus, Equation (7) can be written as

$$0 = \nabla_s \cdot (\mathbf{E}_i \times \mathbf{H}_j - \mathbf{E}_j \times \mathbf{H}_i) + \hat{z}\frac{\partial}{\partial z} \cdot (\mathbf{E}_i \times \mathbf{H}_j - \mathbf{E}_j \times \mathbf{H}_i),$$

or

$$\nabla_s \cdot (\mathbf{E}_i \times \mathbf{H}_j - \mathbf{E}_j \times \mathbf{H}_i) = i(k_{zi} + k_{zj})\hat{z} \cdot (\mathbf{E}_i \times \mathbf{H}_j - \mathbf{E}_j \times \mathbf{H}_i) \quad (8)$$

$$= i(k_{zi} + k_{zj})\hat{z} \cdot (\mathbf{E}_{si} \times \mathbf{H}_{sj} - \mathbf{E}_{sj} \times \mathbf{H}_{si}). \quad (9)$$

In deriving (8) and (9), we have made use of the fact that

$$\begin{aligned} \hat{z} \cdot (\mathbf{E} \times \mathbf{H}) &= \hat{z} \cdot [(\mathbf{E}_s + \mathbf{E}_z) \times (\mathbf{H}_s + \mathbf{H}_z)] \\ &= \hat{z} \cdot [\mathbf{E}_s \times \mathbf{H}_s + \mathbf{E}_s \times \mathbf{H}_z + \mathbf{E}_z \times \mathbf{H}_s + \mathbf{E}_z \times \mathbf{H}_z] \\ &= \hat{z} \cdot \mathbf{E}_s \times \mathbf{H}_s. \end{aligned} \quad (10)$$

Applying two-dimensional form of the divergence theorem to (9), we get

$$\iint_S \nabla_s \cdot (\mathbf{E}_i \times \mathbf{H}_j - \mathbf{E}_j \times \mathbf{H}_i) dS = \oint_C \hat{n} \cdot (\mathbf{E}_i \times \mathbf{H}_j - \mathbf{E}_j \times \mathbf{H}_i) dl$$

$$= i(k_{zi} + k_{zj}) \iint_S \hat{z} \cdot (\mathbf{E}_{si} \times \mathbf{H}_{sj} - \mathbf{E}_{sj} \times H_{si}) dS \quad (11)$$

Extending the contour C to infinity, since $\mathbf{E}$ and $\mathbf{H}$ satisfy radiation condition, the contour integral vanishes. Hence,

$$(k_{zi} + k_{zj}) \iint_S \hat{z} \cdot [\mathbf{E}_{si} \times \mathbf{H}_{sj} - \mathbf{E}_{sj} \times \mathbf{H}_{si}] dS = 0. \quad (12)$$

Now, reversing the propagation direction of mode j to $-\hat{z}$, by the same way as the above, we have another equation

$$(k_{zi} - k_{zj}) \iint_S \hat{z} \cdot [\mathbf{E}_{si} \times \mathbf{H}_{sj} - \mathbf{E}_{sj} \times \mathbf{H}_{si}] = 0. \quad (13)$$

Adding (12) and (13), we get

$$\iint_S \hat{z} \cdot \mathbf{E}_{si} \times \mathbf{H}_{sj} ds = 0, \qquad (i \neq j), \quad (14)$$

thus proving the orthogonality of $(\mathbf{E}_i, \mathbf{H}_j)$. Since the above process does not depend on $k(\mathbf{r})$, (14) is valid for inhomogeneous media.

§6.20

(a) Matrix $\overline{\mathbf{L}}$ is defined by (6.5.20a) as

$$\overline{\mathbf{L}} = \langle \overline{\mathbf{S}}, \mathcal{L} \cdot \overline{\mathbf{S}}^t \rangle,$$

where

$$\overline{\mathbf{S}} = \begin{bmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix}, \qquad \mathcal{L}(\rho) = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}.$$

l_{ij} can be found from (6.5.13). They are

$$l_u = \frac{1}{\rho\epsilon_r}\left(\frac{\partial}{\partial\rho}\frac{1}{\rho\epsilon}\frac{\partial}{\partial\rho} - \frac{n^2}{\rho^2} + k^2\right), \qquad l_{12} = -\frac{2in}{\rho^2},$$

$$l_u = -in\left(\frac{\partial}{\partial\rho}\cdot\frac{1}{\rho^2\epsilon_r\mu_r}\right), \qquad l_{22} = \frac{1}{\rho\mu_r}\left(\rho\mu_r\frac{\partial}{\partial\rho}\cdot\frac{1}{\rho\mu_1}\frac{\partial}{\partial\rho} - \frac{n^2}{\rho^2} + k^2\right).$$

Hence,

$$\overline{\mathbf{L}} = \int_0^{\rho_{max}} \begin{bmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix} \cdot \begin{bmatrix} l_{11} & l_{22} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} \mathbf{S}_1^t & 0 \\ 0 & \mathbf{S}_2^t \end{bmatrix} d\rho$$

$$= \int_0^{\rho_{max}} \begin{bmatrix} \mathbf{S}_1 \cdot l_{11}\mathbf{S}_1^t & \mathbf{S}_1 \cdot l_{12}\mathbf{S}_2^t \\ \mathbf{S}_2 \cdot l_{21}\mathbf{S}_1^t & \mathbf{S}_2 \cdot l_{22}\mathbf{S}_2^t \end{bmatrix} d\rho = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix},$$

where, $\overline{\mathbf{L}}_{ij}$ are block matrices of $\overline{\mathbf{L}}$. They are defined by

$$\left(\overline{\mathbf{L}}_{12}\right)_{mn} = \int_0^{\rho_{max}} (\mathbf{S}_1)_m l_{12} (\mathbf{S}_2)_n d\rho \tag{1}$$

while its symmetric element is $(\overline{\mathbf{L}}_{21})_{nm}$

$$(\overline{\mathbf{L}}_{21})_{nm} = \int_0^{\rho_{max}} (\mathbf{S}_2)_n l_{21} (\mathbf{S}_1)_m d\rho. \tag{2}$$

From (1) and (2) we can see that for $\overline{\mathbf{L}}$ to be symmetric, we require l_{ij} to be a symmetric matrix.

As for $\overline{\mathbf{R}}$, it is a block diagonal matrix, its two blocks are

$$\overline{\mathbf{R}}_{11} = \int_0^{\rho_{max}} \mathbf{S}_1 \cdot r_{11}\mathbf{S}_1^t d\rho, \quad \overline{\mathbf{R}}_{22} = \int_0^{\rho_{max}} \mathbf{S}_2 \cdot r_{22}\mathbf{S}_2^t d\rho$$

Since $r_{11} = 1/\rho\epsilon_r$, $r_{22} = 1/\rho\mu_r$, they are symmetrical operators. Hence, $\overline{\mathbf{R}}_{11}$, $\overline{\mathbf{R}}_{22}$ are symmetric matrices, therefore $\overline{\mathbf{R}} = \begin{bmatrix} \overline{\mathbf{R}}_{11} & 0 \\ 0 & \overline{\mathbf{R}}_{22} \end{bmatrix}$ is also a symmetric matrix.

(b) The operator $\partial^2/\partial z^2$ in (6.5.14) will yield a solution of z-dependence of $e^{ik_z z}$. Hence, (6.5.14) becomes

$$\bar{\mathcal{L}}^t(\rho) \cdot \mathbf{H}(\rho) - k_z^2 \bar{\mathcal{R}} \cdot \mathbf{H}(\rho) = 0, \tag{3}$$

where

$$\mathbf{H} = \begin{bmatrix} \rho H_\phi \\ \rho\mu_r H_\rho \end{bmatrix}. \tag{4}$$

In order to solve (3) for **H**, we expand **H** in a basis set, such that

$$\mathbf{H}(\rho) = \begin{bmatrix} \rho H_\phi \\ \rho\mu_r H_\rho \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} S_{i1}(\rho)\cdot b_{i1} \\ \sum_{i=1}^{N} S_{i2}(\rho)\cdot b_{i2} \end{bmatrix} = \sum_{i=1}^{N} \begin{bmatrix} \mathbf{S}_{i1} & 0 \\ 0 & \mathbf{S}_{i2} \end{bmatrix} \begin{bmatrix} b_{i1} \\ b_{i2} \end{bmatrix} = \overline{\mathbf{S}}^t \cdot \mathbf{b} \tag{5}$$

where S_{i1} and S_{i2} are basis functions that can approximate ρH_ϕ and $\rho\mu_r H_\rho$ respectively in the region of $0 < \rho < \rho_{max}$. Using (5) in (3), we get

$$\bar{\mathcal{L}}^t(\rho)\cdot\mathbf{S}^t(\rho)\cdot\mathbf{b} - k_z^2\bar{\mathcal{R}}\cdot\mathbf{S}^t\cdot\mathbf{b} = 0. \tag{6}$$

Multiplying the above by $\mathbf{S}(\rho)$, and integrate over ρ from 0 to ρ_{max}, we have

$$\langle\mathbf{S}, \bar{\mathcal{L}}^t\mathbf{S}^t\rangle\cdot\mathbf{b} - k_z^2\langle\mathbf{S}, \bar{\mathcal{R}}\cdot\mathbf{S}^t\rangle\cdot\mathbf{b} = 0.$$

The short notation of the above is

$$\overline{\mathbf{L}}^t\cdot\mathbf{b} - k_z^2\overline{\mathbf{R}}\cdot\mathbf{b} = 0 \tag{7}$$

where,

$$\overline{\mathbf{L}}^t = \langle\mathbf{S}, \bar{\mathcal{L}}\cdot\mathbf{S}^t\rangle^t. \tag{8}$$

We can see from the above process that Equation (7) is just the transpose of Equation (6.5.19) in page 356.

To derive Equation (6.5.22), we first choose two eigenvectors for (6.5.19) and (6.5.21). One is $\mathbf{a}_i$ for (6.5.19) with eigenvalues k_{zi}^2, another is $\mathbf{b}_j$ for (6.5.21) with eigenvalues k_{zj}^2. Hence, we have two equations

$$\overline{\mathbf{L}}\cdot\mathbf{a}_i - k_{zi}^2\overline{\mathbf{R}}\cdot\mathbf{a}_i = 0, \tag{9}$$

$$\overline{\mathbf{L}}^t\cdot\mathbf{b}_j - k_{zj}^2\overline{\mathbf{R}}\cdot\mathbf{b}_j = 0. \tag{10}$$

Left-multiplying (9) by $\mathbf{b}_j^t$ and left multiplying (10) by $\mathbf{a}_i^t$, we have

$$\mathbf{b}_j^t\cdot\overline{\mathbf{L}}\cdot\mathbf{a}_i - k_{zi}^2\mathbf{b}_j\cdot\overline{\mathbf{R}}\cdot\mathbf{a}_i = 0, \tag{11}$$

$$\mathbf{a}_i^t\cdot\overline{\mathbf{L}}^t\cdot\mathbf{b}_j - k_{zj}^2\mathbf{a}_i^t\cdot\overline{\mathbf{R}}\cdot\mathbf{b}_j = 0. \tag{12}$$

Transposing (11) and then subtracting (12), and using the symmetry property of $\overline{\mathbf{R}}$, i.e., $\overline{\mathbf{R}}^t = \overline{\mathbf{R}}$, we find

$$(k_{zi}^2 - k_{zj}^2)\mathbf{a}_i^t\cdot\overline{\mathbf{R}}\cdot\mathbf{b}_j = 0. \tag{13}$$

If $i \neq j$, then $k_{zi}^2 \neq k_{zj}^2$, we have $\mathbf{a}_i^t\cdot\overline{\mathbf{R}}\cdot\mathbf{b}_j = 0$. If $i = j$ then $\mathbf{a}_i^t\cdot\overline{\mathbf{R}}\cdot\mathbf{b}_i =$ constant (related to i). To summarize, we have

$$\mathbf{a}_i^t\cdot\overline{\mathbf{R}}\cdot\mathbf{b}_j = \delta_{ij}D_i. \tag{14}$$

§6.21

When the source is a vertical electrical dipole, such that

$$\mathbf{J} = \hat{z} I\ell \delta(\mathbf{r}) = \hat{z} I\ell \frac{\delta(\rho - \rho')}{\rho'} \delta(\phi - \phi')\delta(z - z'), \tag{1}$$

then,

$$(\nabla \times \mathbf{J})_s = -\hat{\phi} I\ell \frac{\delta'(\rho - \rho')}{\rho'} \delta(\phi - \phi')\delta(z - z') \\ + \hat{\rho} I\ell \frac{\partial(\rho - \rho')}{(\rho')^2} \delta'(\phi - \phi')\delta(z - z'). \tag{2}$$

The ϕ variation of the source could be expanded using Fourier series as

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\phi - \phi')}. \tag{3}$$

Hence, the source term in (2) can be expanded in terms of $e^{in\phi}$harmonics. By linearity of the field equation, each of these harmonics will excite a field with $e^{in\phi}$ dependence. For the n-th harmonics, Equation (6.5.8) can be written as

$$\bar{\mathcal{L}}^t(\rho) \cdot \mathbf{H}(\rho, z) + \frac{\partial^2}{\partial z^2} \bar{\mathcal{R}}(\rho) \cdot \mathbf{H}(\rho, z) = \mathbf{S}(\rho, z) = \begin{Bmatrix} S_1(\rho, x) \\ S_2(\rho, z) \end{Bmatrix} \tag{4}$$

where

$$S_1(\rho, z) = \frac{inI\ell}{\omega\pi(\rho')^2} \delta(\rho - \rho')\delta(z - z') \cdot e^{-in\phi'}, \tag{5}$$

$$S_2(\rho, z) = -\frac{I\ell}{2\pi\rho'} \delta'(\rho - \rho')\delta(z - z') e^{-in\phi'}, \tag{6}$$

and

$$\mathbf{H} = \begin{Bmatrix} \rho H_\phi \\ \rho\mu_r H_\rho \end{Bmatrix}. \tag{7}$$

The unknown field, $\mathbf{H}(\rho, z)$, can be expanded in terms of the eigenfunctions $\mathbf{H}_\beta(\rho)$ such that

$$\mathbf{H}(\rho, z) = \sum_{\beta=1}^{2N} \mathbf{H}_\beta(\rho) b_\beta(z). \tag{8}$$

Substituting (8) into (4), we have

$$\sum_{\beta=1}^{2N} \bar{\mathcal{L}}^t(\rho) \cdot \mathbf{H}_\beta(\rho) \cdot b_\beta(z) + \frac{\partial^2}{\partial z^2} \sum_{\beta=1}^{2N} \bar{\mathcal{R}}(\rho) \cdot \mathbf{H}_\beta(\rho) \cdot b_\beta(z) = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}. \tag{9}$$

Multiplying the above by $\mathbf{E}^t_\alpha(\rho)$ and integrating over ρ, we have

$$\sum_{\beta=1}^{2N}\langle \mathbf{E}^t_\alpha, \bar{\mathcal{L}}^t \cdot \mathbf{H}_\beta\rangle b_\beta(z) + \frac{\partial^2}{\partial z^2}\sum_{\beta=1}^{2N}\langle \mathbf{E}^t_\alpha, \bar{\mathcal{R}}(\rho)\cdot \mathbf{H}_\beta\rangle b_\beta(z) = B_\alpha \delta(z-z'), \quad (9a)$$

where

$$B_\alpha = \frac{I\ell}{2\pi\rho'}e^{-in\phi'}\left[\frac{in}{\rho'}E_{\alpha 1}(\rho') + E'_{\alpha 2}(\rho')\right]. \quad (10)$$

Since

$$\mathbf{E}_\alpha(\rho) = \mathbf{S}^t(\rho)\cdot \mathbf{a}_\alpha,$$
$$\mathbf{H}_\beta(\rho) = \mathbf{S}^t(\rho)\cdot \mathbf{b}_\beta,$$

we can show that

$$\begin{aligned}\langle \mathbf{E}^t_\alpha, \bar{\mathcal{L}}^t\cdot \mathbf{H}_\beta\rangle &= \mathbf{a}^t_\alpha \cdot \langle \mathbf{S}, \bar{\mathcal{L}}^t \cdot \mathbf{S}^t\rangle \cdot \mathbf{b}_\beta = \mathbf{a}^t_\alpha \cdot \langle \mathbf{S}^t, \bar{\mathcal{L}}\cdot \mathbf{S}\rangle^t \cdot \mathbf{b}_\beta \\ &= [\mathbf{b}^t_\beta \cdot \langle \mathbf{S}^t, \bar{\mathcal{L}}\cdot \mathbf{S}\rangle \cdot \mathbf{a}_\alpha]^t = [\mathbf{b}^t_\beta \cdot \overline{\mathbf{L}} \cdot \mathbf{a}_\alpha]^t \\ &= [k^2_{\alpha z}\mathbf{b}^t_\beta \cdot \overline{\mathbf{R}}\cdot \mathbf{a}_\alpha]^t = k^2_{\alpha z}\delta_{\alpha\beta}. \end{aligned} \quad (11)$$

In this way, (9a) can be transformed to

$$\frac{d^2}{dz^2}b_\beta(z) + k^2_\beta(z) b_\beta(z) = B_\alpha \delta(z-z'). \quad (12)$$

Solving (12) yields

$$b_\alpha(z) = B_\alpha \frac{e^{ik_{\alpha z}|z-z'|}}{2ik_{\alpha z}}. \quad (13)$$

Hence, (8) can be written as

$$\mathbf{H}(\rho, z) = \sum_{\beta=1}^{2N}\mathbf{H}_\beta(\rho)\cdot B_\beta \frac{1}{2ik_\beta z}e^{ik_{\beta z}|z-z'|}. \quad (14)$$

Using vector notation, (14) becomes

$$\mathbf{H}(\rho, z) = \mathbf{H}^t(\rho)\cdot e^{i\overline{\mathbf{K}}_z|z-z'|}\cdot \mathbf{S}(\rho'), \quad (15)$$

where $\mathbf{H}^t(\rho) = [\mathbf{H}_1(\rho), \mathbf{H}_2(\rho), \ldots, \mathbf{H}_{2N}(\rho')]$ is a $2\times 2N$ matrix, $\overline{\mathbf{K}}_z$ is a diagonal matrix containing $k_{\beta z}$, $\mathbf{S}(\rho')$ is a column vector containing B_β which embodies the characteristics of the source alone. For sources that assume different value for $z \lessgtr z'$, we have general solution

$$\mathbf{H}(\rho, z) = \mathbf{H}^t(\rho)\cdot e^{i\overline{\mathbf{K}}_z|z-z'|}\cdot \mathbf{S}_\pm(\rho'), \quad z \lessgtr z'. \quad (16)$$

§6.22

In source free region, Maxwell's equations are

$$\nabla \times \mathbf{E} = i\omega\mu \mathbf{H}, \quad (1)$$
$$\nabla \times \mathbf{H} = -i\omega\epsilon \mathbf{E}. \quad (2)$$

Decomposing ∇ and **E**, **H** into the sum of transverse and $\hat{z}$ components, i.e.,

$$\nabla = \nabla_s + \nabla_z, , \qquad \mathbf{E} + \mathbf{E}_s + \mathbf{E}_z, \qquad \mathbf{H} + \mathbf{H}_s + \mathbf{H}_z.$$

Substituting the above into (1) and (2), we have

$$(\nabla_s + \nabla_z) \times (\mathbf{E}_s + \mathbf{E}_z) = i\omega\mu(\mathbf{H}_s + \mathbf{H}_z), \tag{3}$$

$$(\nabla_s + \nabla_z) \times (\mathbf{H}_s + \mathbf{H}_z) = -i\omega\mu\epsilon(\mathbf{E}_s + \mathbf{E}_z). \tag{4}$$

Making the $\hat{s}$ component of both sides equal, we get

$$\nabla_s \times \mathbf{E}_z + \nabla_z \times E_s = i\omega\mu\mathbf{H}_s, \tag{5}$$

$$\nabla_s \times \mathbf{H}_z + \nabla_z \times \mathbf{H}_s = -i\omega\epsilon\mathbf{E}_s. \tag{6}$$

Again, making the $\hat{z}$ component of both sides in (3) and (4) equal, we have

$$\nabla_s \times \mathbf{E}_s = i\omega\mu\mathbf{H}_z, \tag{7}$$

$$\nabla_s \times \mathbf{H}_s = -i\omega\epsilon\mathbf{E}_z. \tag{8}$$

Now, multiplying both sides of (7) by $(i\omega\mu)^{-1}$, we have $\mathbf{H}_z = (i\omega\mu)^{-1}\nabla_s \times \mathbf{E}_s$ and substituting this relation into (6), we arrive at

$$(i\omega)^{-1}\nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_s + \nabla_z \times \mathbf{H}_s = -i\omega\epsilon\mathbf{E}_s. \tag{9}$$

Since $\nabla_z = \hat{z}\frac{\partial}{\partial z}$, the above can be rewritten as

$$-i\omega\hat{z} \times \frac{\partial}{\partial z}\mathbf{H}_s = \nabla_s \times \mu^{-1}\nabla_s \times \mathbf{E}_s - \omega^2\epsilon\mathbf{E}_s. \tag{10}$$

This is (6.5.39) on page 359.

We can further simplify (10) into a matrix form which relates $\mathbf{H}_s$ and $\mathbf{E}_s$. To do so, we make use of the following identity in cylindrical coordinates:

$$\nabla_s \times \mathbf{E}_s = \hat{z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_\phi - \frac{in}{\rho}E_\rho\right), \tag{11}$$

$$\hat{z} \times \mathbf{H}_s = \hat{z} \times \left(\hat{\rho}H_\rho + \hat{\phi}H_\rho\right) = \hat{\phi}H_\rho - \hat{\rho}H_\phi, \tag{12}$$

where we have used $\frac{\partial}{\partial\phi} = in$.

In this way, (10) becomes

$$-i\omega\frac{\partial}{\partial z}\begin{bmatrix} -\frac{1}{\rho} & 0 \\ 0 & \frac{1}{\rho\mu_{r_2}} \end{bmatrix}\begin{bmatrix} \rho H_\phi \\ \rho\mu_r H_\rho \end{bmatrix}$$

$$= \begin{bmatrix} \frac{in}{\mu\rho^2}\frac{\partial}{\partial\rho}\rho E_\phi + \frac{n^2}{\mu\rho^2}E_\rho \\ -\frac{\partial}{\partial\rho}\frac{1}{\rho\mu}\frac{\partial}{\partial\rho}\rho E_\phi + \frac{\partial}{\partial\rho}\frac{in}{\rho\mu}E_\rho \end{bmatrix} - \begin{bmatrix} \omega^2\epsilon & E_\rho \\ \omega^2\epsilon & E_\epsilon \end{bmatrix}.$$

In short, the above matrix equation can be written as

$$\frac{\partial}{\partial z}\mathbf{R}(\rho) \cdot \mathbf{H}(\rho, z) = \mathbf{M}(\rho) \cdot \mathbf{E}(\rho, z),$$

where,

$$\mathbf{R}(\rho)=\begin{bmatrix}\frac{1}{\rho\epsilon_r} & 0\\ 0 & \frac{1}{\rho\mu_r}\end{bmatrix},\quad \mathbf{H}=\begin{bmatrix}\rho H_\phi\\ \rho\mu_r H_\rho\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}\rho\epsilon_r E_\rho\\ \rho E_\rho\end{bmatrix},$$

$$\mathbf{M}(\rho)=\begin{bmatrix}\frac{n^2}{\rho^3\mu_r\epsilon_r^2}-\frac{k_0^2}{\rho\mu_r\epsilon_r}, & \frac{in}{\rho^2\mu_r\epsilon_r}\frac{\partial}{\partial\rho}\\ -\frac{\partial}{\partial\rho}\frac{in}{\rho^2\mu\epsilon_r}, & \frac{\partial}{\partial\rho}\frac{1}{\rho\mu_r}\frac{\partial}{\partial\rho}+\frac{k^2}{\rho\mu_r}\end{bmatrix}\cdot\frac{1}{i\omega\mu_0}.$$

§6.23

Equation (5a) is

$$\overline{\mathbf{D}}_{2-}=\left[\overline{\mathbf{I}}-\overline{\mathbf{R}}_{21}\cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}\cdot\overline{\mathbf{R}}_{23}\cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}\right]^{-1}. \qquad (1)$$

For the sake of clarity, we denote

$$\overline{\mathbf{A}}=\overline{\mathbf{R}}_{21}\cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}\cdot\overline{\mathbf{R}}_{23}\cdot e^{i\overline{\mathbf{K}}_{2z}\cdot(d_1-d_2)}. \qquad (2)$$

Then, we can set

$$\overline{\mathbf{D}}_{2-}=(\overline{\mathbf{I}}-\overline{\mathbf{A}})^{-1}=\overline{\mathbf{I}}+a_1\overline{\mathbf{A}}+a_2\overline{\mathbf{A}}^2+a_3\overline{\mathbf{A}}^3+a_n\overline{\mathbf{A}}^n+\dots \qquad (3)$$

where $a_1, a_2, \cdots, a_n, \cdots$ are expansion coefficients.

Multiplying (4) by $(\overline{\mathbf{I}}-\overline{\mathbf{A}})$, we have

$$\begin{aligned}&(\overline{\mathbf{I}}-\overline{\mathbf{A}})\cdot\overline{\mathbf{D}}_{2-}\\ =&\overline{\mathbf{I}}+a_1\overline{\mathbf{A}}+a_2\overline{\mathbf{A}}^2+a_3A^3+\cdots+a_n\overline{\mathbf{A}}^n+\dots.\\ &-\overline{\mathbf{A}}-a_1\overline{\mathbf{A}}^2-a_2a^3-\cdots-a_{n-1}\overline{\mathbf{A}}^n-\cdots\\ =&\overline{\mathbf{I}}+(a_1-1)\overline{\mathbf{A}}+(a_2-a_1)\overline{\mathbf{A}}^2+(a_3-a_2)\overline{\mathbf{A}}^3+\cdots+(a_n-a_{n-1})\overline{\mathbf{A}}^n+\cdots.\end{aligned}$$

The above results in

$$(a_1-1)\overline{\mathbf{A}}+(a_2-a_1)\overline{\mathbf{A}}^2+(a_3-a_2)\overline{\mathbf{A}}^3+\cdots+(a_n-a_{n-1})\overline{\mathbf{A}}^n+\cdots=0$$

This equation is valid for arbitrary full rank $\overline{\mathbf{A}}$, that is $|\overline{\mathbf{A}}|\neq 0$, it is only possible when

$$a_1-1=0,\quad a_2-a_1=0,\quad a_3-a_2=0,\quad\cdots\quad a_n-a_{n-1}=0,\quad\cdots$$

This means $a_1=a_2=a_3=\cdots a_n=1$, hence

$$\overline{\mathbf{D}}_{2-}=\overline{\mathbf{I}}+\overline{\mathbf{A}}+\overline{\mathbf{A}}^2+\overline{\mathbf{A}}^3+\cdots+\overline{\mathbf{A}}^n+\cdots. \qquad (4)$$

Substitute (5) into (6.6.8), we arrive at

$$\begin{aligned}\tilde{\overline{\mathbf{R}}}_{12}=\overline{\mathbf{R}}_{12}&+\overline{\mathbf{T}}_{21}\cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}\cdot\overline{\mathbf{R}}_{23}\cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}\cdot\overline{\mathbf{T}}_{12}\\ &+\overline{\mathbf{T}}_{21}\cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}\cdot\overline{\mathbf{R}}_{23}\cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}\cdot\overline{\mathbf{A}}\cdot\overline{\mathbf{T}}_{12}\\ &+\overline{\mathbf{T}}_{21}\cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}\cdot\overline{\mathbf{R}}_{23}\cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}\cdot\overline{\mathbf{A}}^2\cdot\overline{\mathbf{T}}_{12}\\ &+\cdots.\end{aligned} \qquad (5)$$

This is the series expansion for $\overline{\mathbf{R}}_{12}$. It has a very clear physical meaning as it stands for multiple reflection. We give explanation in the following. The incident wave meets the first boundary. Then it reflects and transmits. The reflected part is $\overline{\mathbf{R}}_{12}$ [this is the first term in (5)]; the transmitted part is $\overline{\mathbf{T}}_{21}$. This part will go on travelling until it meets the second boundary. After this travelling, it has phase shift of exp $[i\overline{\mathbf{K}}_{2z}(d_1 - d_2)]$ at the second boundary. It also reflects and transmits (we will omit the transmitted part, because it will never come back again). The reflected part is

$$\overline{\mathbf{T}}_{21} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \cdot \overline{\mathbf{R}}_{23}.$$

This part will go upward until it meets the first boundary after a phase shift of $e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}$, after which it reflects and transmits. Consequently, we have

$$\text{Transmitted part:}\quad \overline{\mathbf{T}}_{21} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \cdot \overline{\mathbf{R}}_{23} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \cdot \overline{\mathbf{T}}_{12},$$

$$\text{Reflected part:}\quad \overline{\mathbf{T}}_{21} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \cdot \overline{\mathbf{R}}_{23} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \cdot \overline{\mathbf{R}}_{21}.$$

The transmitted part will add to the field in region 1, forming the second term in (5). The reflected part will repeat the above process.

§6.24

In Figure 6.6.2, when each region is planarly layered, we can find the generalized reflection operator in the same way as §§6.6.1. In this case, the field in region 1, using the similar compact notation of (**29a**) but changing the coordinates to that in Figure 6.6.2, can be expressed as

$$\mathbf{A}_{1y} = \overline{\boldsymbol{\Phi}}_1^t(y)\cdot\left[e^{i\overline{\mathbf{K}}_{1z}(z-z')} \cdot \phi_{s\pm}(y') + e^{i\overline{\mathbf{K}}_{1z}(z-d_1)} \cdot \tilde{\overline{\mathbf{R}}}_{12} \cdot e^{i\overline{\mathbf{K}}_{1z}|d_1-z'|} \cdot \phi_{s-}(y')\right]. \tag{1}$$

In region 2, the field is

$$\mathbf{A}_{2y} = \overline{\boldsymbol{\Phi}}_2^t(y) \cdot \left[e^{-i\overline{\mathbf{K}}_{2z}(z-d_2)} + e^{i\overline{\mathbf{K}}_{2z}(z-d_2)} \cdot \overline{\mathbf{R}}_{23}\right] \cdot \mathbf{B}_2. \tag{2}$$

In region 3, we have only downgoing waves, or

$$\mathbf{A}_{3y} = \overline{\boldsymbol{\Phi}}_3^t(y) \cdot e^{-i\overline{\mathbf{K}}_{3z}(z-d_2)} \cdot \mathbf{B}_3 \tag{3}$$

$\mathbf{B}_2$ and $\mathbf{B}_3$ are constant vectors, which can be found by using constraint conditions. That is, the downgoing wave in region 2 is a consequence of the transmission of the downgoing wave in region 1 plus the reflection of the upgoing wave in region 2, or

$$\begin{aligned} &e^{-i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \cdot \mathbf{B}_2 \\ =&\overline{\mathbf{T}}_{12} \cdot e^{i\overline{\mathbf{K}}_{1z}|d_1-z'|} \cdot \phi_{s-}(y') + \overline{\mathbf{R}}_{21} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_2-d_1)} \cdot \overline{\mathbf{R}}_{23} \cdot \mathbf{B}_2. \end{aligned} \tag{4}$$

Solving the above for $\mathbf{B}_2$ we find,

$$\mathbf{B}_2 = e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \cdot \overline{\mathbf{D}}_{2-} \cdot \overline{\mathbf{T}}_{12} \cdot e^{i\overline{\mathbf{K}}_{1z}|d_1-z'|} \cdot \phi_{s-}(y'), \tag{5}$$

where

$$\overline{\mathbf{D}}_{2-} = \left[\overline{\mathbf{I}} - \overline{\mathbf{R}}_{2-} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \cdot \overline{\mathbf{R}}_{23} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)}\right]^{-1}. \tag{6}$$

Similarly, $\mathbf{B}_3$ is just the transmission of the downgoing wave in region 2. Hence,

$$\mathbf{B}_3 = \overline{\mathbf{T}}_{23} \cdot \mathbf{B}_2. \tag{7}$$

In region 1, the reflected wave is the reflection of downgoing wave in region 1 due to the discontinuity at $z = d_1$ plus a transmission of the upgoing wave in region 2. Therefore, at $z = d_1$, we have

$$\tilde{\overline{\mathbf{R}}}_{12} e^{i\overline{\mathbf{K}}_{1z}|d_1-z'|} \cdot \phi_{s-}(y') = \overline{\mathbf{R}}_{12} \cdot e^{i\overline{\mathbf{K}}_{1z}|d_1-z'|} \cdot \phi_{s-}(y') + \overline{\mathbf{T}}_{21} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \cdot \overline{\mathbf{R}}_{23} \cdot \mathbf{B}_2.$$

Substituting $\mathbf{B}_2$ of (5) into the above yields

$$\tilde{\overline{\mathbf{R}}}_{12} = \overline{\mathbf{R}}_{12} + \overline{\mathbf{T}}_{21} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \overline{\mathbf{R}}_{23} \cdot e^{i\overline{\mathbf{K}}_{2z}(d_1-d_2)} \cdot \overline{\mathbf{D}}_{2-} \cdot \overline{\mathbf{T}}_{12}. \tag{8}$$

This is the generalized reflection operator as it incorporates subsurface reflections.

§6.25

(6.6.13) and (6.6.14) can be rewritten in the following form:

$$\begin{aligned} & e^{i\overline{\mathbf{K}}_{mz}d_m} \cdot \mathbf{C}_m - \tilde{\overline{\mathbf{R}}}_{m,m+1} \cdot e^{-i\overline{\mathbf{K}}_{mz}d_m} \cdot \overline{\mathbf{D}}_m \\ &= \tilde{\overline{\mathbf{R}}}_{m,m+1} \cdot e^{i\overline{\mathbf{K}}_{mz}|d_m-z'|} \cdot \mathbf{S}_{m-}(\rho'), && (1) \\ & -\tilde{\overline{\mathbf{R}}}_{m,m-1} \cdot e^{i\overline{\mathbf{K}}_{mz}d_{m-1}} \cdot \mathbf{C}_m + e^{-i\overline{\mathbf{K}}_{mz}d_{m-1}} \cdot \overline{\mathbf{D}}_m \\ &= \tilde{\overline{\mathbf{R}}}_{m,m-1} \cdot e^{i\overline{\mathbf{K}}_{mz}|d_{m-1}-z'|} \cdot \mathbf{S}_{m+}(\rho'). && (2) \end{aligned}$$

Left-multiplying (1) by $\tilde{\overline{\mathbf{R}}}_{m,m-1} \cdot e^{i\overline{\mathbf{K}}_{mz}d_{m-1}}$ and left-multiplying (2) by $e^{i\overline{\mathbf{K}}_{mz}d_m}$ and then add the resultant equations, and making use of the fact that $e^{i\overline{\mathbf{K}}d}$ is a diagonal matrix, we have

$$\begin{aligned} & \left[e^{i\overline{\mathbf{K}}_{mz}(d_m-d_{m-1})} - \tilde{\overline{\mathbf{R}}}_{m,m-1} \cdot \tilde{\overline{\mathbf{R}}}_{m,m+1} e^{-i\overline{\mathbf{K}}_{mz}(d_m-d_{m-1})}\right] \cdot \mathbf{D}_m \\ &= \tilde{\overline{\mathbf{R}}}_{m,m-1} \cdot \tilde{\overline{\mathbf{R}}}_{m,m+1} \cdot e^{i\overline{\mathbf{K}}_m(d_{m-1}+|d_m-z'|)} \cdot \mathbf{S}_{m-}(\rho') \\ & \quad + \tilde{\overline{\mathbf{R}}}_{m,m-1} e^{i\overline{\mathbf{K}}_m(d_m+|d_{m-1}-z'|)} \cdot S_{m+}(\rho'). \end{aligned}$$

Multiplying both sides of the above equation by $e^{-i\overline{\mathbf{K}}_{mz}d_m}$ and defining

$$\tilde{\overline{\mathbf{M}}}_{m-} = \left[\overline{\mathbf{I}} - \tilde{\overline{\mathbf{R}}}_{m,m-1} \cdot e^{i\overline{\mathbf{K}}_{mz}(d_{m-1}-d_m)} \cdot \tilde{\overline{\mathbf{R}}}_{m,m+1} \cdot e^{i\overline{\mathbf{K}}_{mz}(d_{m-1}-d_m)}\right]^{-1}, \tag{3}$$

we get

$$e^{-i\overline{\mathbf{K}}_{mz}d_{m-1}}\mathbf{D}_m = \widetilde{\overline{\mathbf{M}}}_{m-} \cdot \widetilde{\overline{\mathbf{R}}}_{m,m-1} \cdot \Big[e^{i\overline{\mathbf{K}}_{mz}|d_{m-1}-z'|} \cdot \mathbf{S}_{m+}(\rho') + e^{i\overline{\mathbf{K}}_{mz}(d_{m-1}-d_m)} \cdot \widetilde{\overline{\mathbf{R}}}_{m,m+1} \cdot e^{i\overline{\mathbf{K}}_{mz}|d_m-z'|} \cdot \mathbf{S}_{m-}(\rho')\Big] .$$

This is (6.6.16b). In the same way as above, we can derive (16a) to be

$$e^{i\overline{\mathbf{K}}_{mz}d_m}\mathbf{C}_m = \widetilde{\overline{\mathbf{M}}}_{m+} \cdot \widetilde{\overline{\mathbf{R}}}_{m,m+1} \cdot \Big[e^{i\overline{\mathbf{K}}_{mz}|d_m-z'|} \cdot \mathbf{S}_{m-}(\rho') + e^{i\overline{\mathbf{K}}_{mz}(d_{m-1}-d_m)} \cdot \widetilde{\overline{\mathbf{R}}}_{m,m-1} \cdot e^{i\overline{\mathbf{K}}_{mz}|d_{m-1}-z'|} \cdot \mathbf{S}_{m+}(\rho')\Big] .$$

CHAPTER 7

EXERCISE SOLUTIONS

by Irsadi Aksun[1], Caicheng Lu[2], Greg Otto[3] and Rob Wagner[4]

§7.1

$$\left(\nabla^2 + k^2\right) g(\mathbf{r}) = -\delta(\mathbf{r}), \tag{1}$$
$$g(\mathbf{r}) = A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r}.$$

since only outgoing wave is possible, $B = 0$, and

$$g(\mathbf{r}) = A\frac{e^{ikr}}{r}. \tag{2}$$

Substitute (2) into (1) and integrate around the origon,

$$\int_{\delta V} \nabla^2 A\frac{e^{ikr}}{r} dV + \int_{\delta V} k^2 A\frac{e^{ikr}}{r} dV = -\int \delta(\mathbf{r}) dV$$

Second term on the left handside goes to zero for $r \to 0$. Hence,

$$\lim_{r\to 0} \int_{\delta V} \nabla^2 A\frac{e^{ikr}}{r} dV = -1.$$

But

$$\lim_{r\to 0} \int_{\delta V} \nabla \cdot \nabla A\frac{e^{ikr}}{r} dV = \lim_{r\to 0} \int_{\delta S} \nabla A\frac{e^{ikr}}{r} \cdot d\mathbf{S},$$

or

$$\lim_{r\to 0} A\ 4\pi r^2 \frac{ikr\, e^{ikr} - e^{ikr}}{r^2} = -1,$$

[1] Exercises 1-12

[2] Exercises 13-19

[3] Exercises 20-26

[4] Exercises 27-30

or

$$A = \frac{1}{4\pi}.$$

Hence,

$$g(\mathbf{r}) = \frac{1}{4\pi}\frac{e^{ikr}}{r} \Rightarrow g(\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}.$$

§7.2

$$\nabla^2\mathbf{A} + k_0^2\mathbf{A} - \nabla(\nabla\cdot\mathbf{A}) = i\omega\mu\epsilon\nabla\phi - \mu\mathbf{J}, \qquad (7.1.9)$$

$$\nabla\cdot\mathbf{A} = 0, \quad \text{(coulomb's gauge)}.$$

By using the definition of the electric field given in (7.1.7)

$$\mathbf{E} = i\omega\mathbf{A} - \nabla\phi, \qquad (7.1.7)$$

then

$$\nabla\cdot\mathbf{E} = i\omega\nabla\cdot\mathbf{A} - \nabla\cdot\nabla\phi = \frac{\rho}{\epsilon},$$

and

$$\nabla^2\phi = -\frac{\rho}{\epsilon}$$

is obtained, which corresponds to (7.1.12).
From Equation (7.1.9),

$$\nabla^2\mathbf{A} + k_0^2\mathbf{A} = i\omega\mu\epsilon\nabla\phi - \mu\mathbf{J}. \qquad (1)$$

Since $\nabla\phi$ is an irrotational field, it may be canceled by the irrotational component of $\mathbf{J}$:

$$\mathbf{J} = \mathbf{J}_{ir} + \mathbf{J}_s$$

where

$$\nabla\times\mathbf{J}_{ir} = 0; \quad \nabla\cdot\mathbf{J}_s = 0,$$
$$\nabla\times\nabla\times\mathbf{J} = \nabla(\nabla\cdot\mathbf{J}) - \nabla^2\mathbf{J},$$
$$\nabla^2\mathbf{J} = \nabla(\nabla\cdot\mathbf{J}) - \nabla\times\nabla\times\mathbf{J}.$$

First term on the right hand side is irrotational part while the second term is the solenoidal part of the current density. By using the fact that

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')$$

has the solution of

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|r-r'|},$$

the irrotational and solenoidal components of $\mathbf{J}$ is obtained as

$$\mathbf{J}_{ir} = G(\mathbf{r},\mathbf{r}') * [-\nabla(\nabla\cdot\mathbf{J})] = -\int_V d\mathbf{r}' \frac{\nabla'(\nabla'\cdot\mathbf{J}(\mathbf{r}'))}{4\pi|\mathbf{r}-\mathbf{r}'|},$$

$$\mathbf{J}_s = G(\mathbf{r},\mathbf{r}') * [\nabla\times(\nabla\times\mathbf{J})] = \int_V d\mathbf{r}' \frac{\nabla'\times(\nabla'\times\mathbf{J}(\mathbf{r}'))}{4\pi|\mathbf{r}-\mathbf{r}'|},$$

$$\mathbf{J}_{ir} = -\int_V d\mathbf{r}'\nabla' \left[\frac{\nabla'\cdot\mathbf{J}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right] + \int_V d\mathbf{r}' \left[\nabla' \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right](\nabla'\cdot\mathbf{J}),$$

where the vector identity $g\nabla f = \nabla(fg) - f\nabla g$ is used. Moreover, by the use of the gradient identity for the first term on the right hand side

$$\mathbf{J}_{ir} = -\oint_S d\mathbf{r}' \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} - \frac{1}{4\pi}\int_V d\mathbf{r}' \left[\nabla \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right](\nabla'\cdot\mathbf{J}).$$

The first term is zero because the source $\mathbf{J}$ is only supported in V. Therefore

$$\mathbf{J}_{ir} = -\nabla \int_V d\mathbf{r}' \frac{i\omega\rho(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} = -i\omega\epsilon\nabla\phi.$$

Hence, Equation (1) is written as

$$\nabla^2\mathbf{A} + k_0^2\mathbf{A} = -\mu\mathbf{J}_s$$

which corresponds to (7.1.11), and the final electromagnetic field would not change with this gauge.

§7.3

For

$$f(x) = \begin{cases} \frac{1}{\log\frac{1}{|x|}} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0$ continuous at $x = 0$. Holder condition requires,

$$|f(x) - f(0)| \leq A|x|^\alpha, \quad \forall x, \quad x \leq c$$

where c, A and $\alpha > 0$. Therefore, we require

$$\left|\frac{1}{\log\frac{1}{|x|}}\right| \leq A|x|^\alpha,$$

or

$$\left|\log\frac{1}{|x|}\right| \geq \frac{1}{A}|x|^{-\alpha},$$

or

$$|x|^{\alpha}|\log|x|| \geq \frac{1}{A}.$$

This is not true for $x \to 0$, because

$$\lim_{x\to 0}\{|x|^{\alpha}|\log|x||\} \to 0 \quad \text{for} \quad \alpha > 0.$$

Therefore, for any c, we cannot find A and α satisfying the Holder condition, that is, $f(x)$ doesn't satisfy the Holder's condition.

§7.4

$$\mathbf{E}(r) = i\omega\mu \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}')\mathbf{J}(r') - \frac{\nabla\nabla}{i\omega\epsilon} \cdot \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}').$$

The second term can be written as

$$= \lim_{V_\delta \to 0}\left[\nabla\nabla \cdot \int_{V-V_\delta} d\mathbf{r}' g(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}') + \nabla\nabla \cdot \int_{V_\delta} d\mathbf{r}' g(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')\right]$$

$$= \lim_{V_\delta \to 0}\left[\int_{V-V_\delta} d\mathbf{r}' \nabla\nabla g(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') - \int_{V_\delta} d\mathbf{r}' \nabla' g(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\right]$$

$$= \lim_{V_\delta \to 0} \int_{V-V_\delta} d\mathbf{r}' \nabla\nabla g(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') - \overline{\mathbf{L}} \cdot \mathbf{J}(\mathbf{r}),$$

where

$$\begin{aligned}
\overline{\mathbf{L}} \cdot \mathbf{J}(\mathbf{r}) &= -\nabla \int_{V_\delta \to 0} d\mathbf{r}' \nabla' g(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') \\
&= -\nabla \int_{V_\delta \to 0} d\mathbf{r}' [\nabla' \cdot g(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}') - g(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{J}(\mathbf{r}')] \\
&= -\nabla \int_{\delta S} g(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r},\mathbf{r}') \cdot \hat{n}dS' + \nabla \int_{\delta V} g(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{J}(\mathbf{r}')d\mathbf{r}' \\
&= \int_{\delta S} [\nabla' g(\mathbf{r},\mathbf{r}')]\mathbf{J}(\mathbf{r}') \cdot \hat{n}dS' - \int_{\delta V} [\nabla' g(\mathbf{r},\mathbf{r}')]\nabla' \cdot \mathbf{J}(\mathbf{r}')d\mathbf{r}',
\end{aligned}$$

and

$$\begin{aligned}
\nabla' g(\mathbf{r},\mathbf{r}') &= \nabla'\left(\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) \\
&= \left(ik - \frac{1}{R}\right)\frac{e^{ikR}}{r\pi R}\hat{R} \cong -\frac{\hat{R}}{4\pi R^2}, \qquad \delta V \to 0, \qquad \mathbf{R} = \mathbf{r}' - \mathbf{r},
\end{aligned}$$

because the fields become static when the exclusion volume goes to zero. Therefore,

$$\overline{\mathbf{L}} \cdot \mathbf{J}(\mathbf{r}) = \lim_{R \to 0} \int_{\delta S} \frac{-\hat{R}}{4\pi R^2} \mathbf{J}(\mathbf{r}') \cdot \hat{n} dS' - \lim_{R \to 0} \int_{\delta V} \frac{-\hat{R}}{4\pi R^2} \nabla' \cdot \mathbf{J}(\mathbf{r}') d\mathbf{r}'.$$

The volume integral vanishes with $R \to 0$ provided that $\mathbf{J}(\mathbf{r}')$ is piecewise differentiable, because

$$\lim_{R \to 0} \frac{\hat{R}}{4\pi R^2} \left[\nabla' \cdot \mathbf{J}(\mathbf{r}') \right]_{\mathbf{r}'=\mathbf{r}} 4\pi R^2 dR \to 0.$$

Hence

$$\overline{\mathbf{L}} \cdot \mathbf{J}(\mathbf{r}) = -\mathbf{J}(\mathbf{r}) \cdot \int_{\delta S} \frac{\hat{n}\hat{R}}{4\pi R^2} dS$$

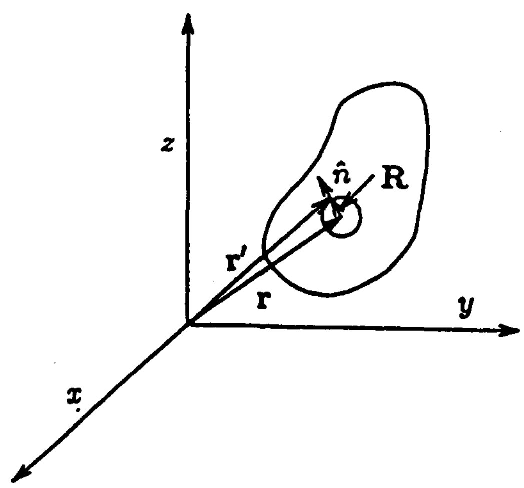

Figure for Exercise Solution 7.4

In the figure,

$$\hat{n} = \hat{R} = \hat{x} \sin\theta \cos\phi + \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta$$

$$\hat{n}\hat{R} = \hat{x}\hat{x} \sin^2\theta \cos^2\phi + \hat{y}\hat{y} \sin^2\theta \sin^2\phi + \hat{z}\hat{z} \cos^2\theta + (\hat{x}\hat{y} + \hat{y}\hat{x}) \sin^2\theta \sin\phi \cos\phi + (\hat{x}\hat{z} + \hat{z}\hat{x}) \sin\theta \cos\theta \cos\phi + (\hat{y}\hat{z} + \hat{z}\hat{y}) \sin\theta \cos\theta \sin\phi.$$

Therefore,

$$\int_{\delta S} \frac{\hat{n}\hat{R}}{4\pi R^2} dS = \int_0^{\pi}\int_0^{2\pi} \frac{R^2 \sin\theta \hat{R}\hat{R}}{4\pi R^2} d\phi d\theta$$

Note that the vector components $\hat{x}\hat{y}$, $\hat{y}\hat{x}$, $\hat{x}\hat{z}$, $\hat{z}\hat{x}$, and $\hat{y}\hat{z}$, $\hat{z}\hat{y}$ are zero. Then,

$$\int_0^{\pi}\int_0^{2\pi} \frac{\sin^3\theta\cos^2\phi}{4\pi} d\phi d\theta = \int_0^{\pi}\int_0^{2\pi} \frac{\sin^3\theta\sin^2\phi}{4\pi} d\phi d\theta$$
$$= \int_0^{\pi}\int_0^{2\pi} \frac{\sin\theta\cos^2\theta}{4\pi} d\phi d\theta = \frac{1}{3}.$$

Therefore,

$$\overline{\mathbf{L}} \cdot \mathbf{J}(\mathbf{r}) = -\frac{\overline{\mathbf{I}}}{3} \cdot \mathbf{J}(\mathbf{r}) \quad \Rightarrow \quad \overline{\mathbf{L}} = \frac{\overline{\mathbf{I}}}{3}.$$

For other exclusion volumes, the same procedure is applied to find the corresponding $\overline{\mathbf{L}}$.

§7.5

$$-\mathbf{k}\times\mathbf{k}\times\tilde{\overline{\mathbf{G}}}(k,\mathbf{r}') - k_0^2\tilde{\overline{\mathbf{G}}}(k,\mathbf{r}') = \overline{\mathbf{I}}\, e^{-i\mathbf{k}\cdot\mathbf{r}'}, \tag{1}$$

$$\tilde{\overline{\mathbf{G}}}(\mathbf{k},\mathbf{r}') = \frac{\overline{\mathbf{I}}k_0^2 - \mathbf{k}\mathbf{k}}{k_0^2(k^2-k_0^2)} e^{-i\mathbf{k}\cdot\mathbf{r}'}, \tag{2}$$

$$\nabla\times\nabla\times\mathbf{E} - k_0^2\mathbf{E} = i\omega\mu\mathbf{J}(\mathbf{r}), \tag{3}$$

$$\nabla\times\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') - k_0^2\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'). \tag{4}$$

For (3)

$$\frac{\nabla\cdot\mathbf{J}}{i\omega} = \rho, \qquad \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon}, \qquad \nabla\cdot\mathbf{E} = \frac{\nabla\cdot\mathbf{J}}{i\omega\epsilon}.$$

For (4)

$$\nabla\cdot\overline{\mathbf{G}} = \frac{\nabla\cdot\overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}')}{i\omega\mu(i\omega\epsilon)},$$

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{1}{8\pi^3}\int d^3k\,\tilde{\overline{\mathbf{G}}}(k,\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}},$$

$$\nabla\cdot\overline{\mathbf{G}} = \frac{1}{8\pi^3}\int d^3k\left[\mathbf{k}\cdot\tilde{\overline{\mathbf{G}}}(\mathbf{k},\mathbf{r}')\right]e^{i\mathbf{k}\cdot\mathbf{r}},$$

$$-\frac{\nabla\cdot\overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}')}{k_0^2} = \frac{1}{8\pi^3}\int d^3k\left[\frac{-\mathbf{k}\cdot\overline{\mathbf{I}}}{k_0^2}e^{i\mathbf{k}\cdot\mathbf{r}'}\right]e^{i\mathbf{k}\cdot\mathbf{r}},$$

$$\mathbf{k}\cdot\tilde{\overline{\mathbf{G}}}(\mathbf{k},\mathbf{r}') = -\frac{\mathbf{k}}{k_0^2}e^{-i\mathbf{k}\cdot\mathbf{r}'}.$$

From (1),

$$\left(-\mathbf{k}\cdot\tilde{\overline{\mathbf{G}}}\right)\mathbf{k} - (-\mathbf{k}\cdot\mathbf{k})\tilde{\overline{\mathbf{G}}} - k_0^2\tilde{\overline{\mathbf{G}}} = \overline{\mathbf{I}}e^{-i\mathbf{k}\cdot\mathbf{r}'},$$

$$\frac{\mathbf{k}\mathbf{k}}{k_0^2}e^{-i\mathbf{k}\cdot\mathbf{r}'} + (k^2-k_0^2)\tilde{\overline{\mathbf{G}}} = \overline{\mathbf{I}}e^{-i\mathbf{k}\cdot\mathbf{r}'},$$

$$\Longrightarrow\quad \tilde{\overline{\mathbf{G}}}(k,\mathbf{r}') = \frac{\overline{\mathbf{I}}k_0^2 - \mathbf{k}\mathbf{k}}{k_0^2(k^2-k_0^2)}e^{-i\mathbf{k}\cdot\mathbf{r}'}.$$

§7.6

$$F(x) = \int_0^\infty dx f(x)e^{-ik_x x}.$$

When $k_x \to \infty$, most of the contribution of the above integral will come from around $x \to 0$. Therefore, we have

$$F(k_x) \sim \int_0^\infty dx x^\alpha e^{-ik_x x}$$

where we have replaced $f(x)$ by the approximation x^α. Letting $ik_x x = s$, the above is

$$F(k_x) \sim \frac{1}{(ik_x)^{1+\alpha}}\int_0^\infty ds s^\alpha e^{-s} = \frac{\Gamma(1+\alpha)}{(ik_x)^{1+\alpha}}, \quad k_x \to \infty.$$

The singularity at the origin of $f(x)$ is integrable if $\alpha > -1$. It is more singular when α is smaller. But for smaller α, the more slowly would $F(k_x)$ decay when $k_x \to \infty$.

§7.7

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{i}{8\pi^2}\iint_{-\infty}^{\infty} d\mathbf{k}_s\, e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}_s')+ik_{0z}|z-z'|}\left[\frac{\overline{\mathbf{I}}k_0^2 - \mathbf{k}_0\mathbf{k}_0}{k_0^2 k_{0z}}\right] - \frac{\hat{z}\hat{z}}{k_0^2}\delta(\mathbf{r}-\mathbf{r}'). \tag{7.1.34}$$

For $z = z'$, the integral becomes

$$\iint_{-\infty}^{+\infty} d\mathbf{k}_s e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}_s')}\left[\frac{\overline{\mathbf{I}}k_0^2 - \mathbf{k}_0\mathbf{k}_0}{k_0^2(k_0^2-k_s^2)^{\frac{1}{2}}}\right],$$

$$\text{integrand} \cong \frac{\bar{\mathbf{I}}k_0^2 - \mathbf{k}_0\mathbf{k}_0}{ik_0^2}\frac{e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)}}{k_s}, \qquad \text{for} \quad k_s \gg k_0.$$

$$\iint_{-\infty}^{+\infty} \frac{e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)}}{k_s} d\mathbf{k}_s$$

is divergant because the integrand $> A\frac{1}{k_s^{2+\epsilon}}$ for $k_s \to \infty$. (7.1.34) is devergent for $z = z'$ because the integrand is the Fourier transform of the dyadic Green's function

$$\overline{\mathbf{G}} = \left(\mathbf{I} + \frac{\nabla\nabla}{k^2}\right) g_0$$

which is not the Fourier transformable as it is as it is not absolutely integrable.

§7.8

$$\mathbf{G}(\mathbf{r},\mathbf{r}') = \left[\mathbf{I} + \frac{\nabla\nabla}{k_0^2}\right] g(\mathbf{r},\mathbf{r}')$$

$$= \left[\mathbf{I} + \frac{\nabla\nabla}{k_0^2}\right] \frac{i}{8\pi^2} \iint_{-\infty}^{\infty} d\mathbf{k}_s \frac{e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|}}{k_z},$$

$$\nabla\nabla \iint_{-\infty}^{\infty} d\mathbf{k}_s \frac{e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|}}{k_z} = \nabla \iint_{-\infty}^{\infty} d\mathbf{k}_s \frac{i\mathbf{k}_s \pm ik_z\hat{z}}{k_z} e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|}$$

$$= \nabla \iint_{-\infty}^{\infty} d\mathbf{k}_s \frac{i\mathbf{k}_s}{k_z} e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|} + \nabla \iint_{-\infty}^{\infty} d\mathbf{k}_s i\,\mathrm{sgn}(z-z') e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|}$$

$$= \iint_{-\infty}^{\infty} d\mathbf{k}_s \frac{i\mathbf{k}_s}{k_z}(i\mathbf{k}_s \pm ik_z\hat{z}) e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|} + \iint d\mathbf{k}_s(\mp\hat{z})\mathbf{k}_s e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|}$$

$$+ \iint d\mathbf{k}_s(-k_z\hat{z}\hat{z}) e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|} + i \iint_{-\infty}^{\infty} d\mathbf{k}_s \hat{z}\hat{z} 2\delta(z-z') e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|}$$

$$= \iint_{-\infty}^{\infty} d\mathbf{k}_s \frac{-\mathbf{k}\mathbf{k}}{k_z} e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|} + \hat{z}\hat{z}\delta(z-z')\, i2 \underbrace{\iint_{-\infty}^{\infty} d\mathbf{k}_s e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|}}_{i8\pi^2\delta(\mathbf{r}_s-\mathbf{r}'_s)}$$

$$= \frac{i}{8\pi^2} \iint_{-\infty}^{\infty} d\mathbf{k}_s e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)+ik_z|z-z'|} \cdot \frac{\mathbf{I}k_0^2 - \mathbf{k}\mathbf{k}}{k_0^2 k_z} - \frac{\hat{z}\hat{z}}{k_0^2}\delta(\mathbf{r}-\mathbf{r}').$$

§7.9

$$P.V.\frac{2i}{y} = \int_{-\infty}^{\infty} dx \mathrm{sgn}(x) e^{ixy} = f(y)$$

$$\tilde{f}(x) = \mathrm{sgn}(x), \qquad \tilde{f}_n(x) = \mathrm{sgn}(x) e^{-\frac{|x|}{n}},$$

$$\begin{aligned} f_n(y) &= \int_{-\infty}^{\infty} dx \mathrm{sgn}(x) e^{-\frac{|x|}{n}} e^{ixy} \\ &= -\int_{-\infty}^{0} dx e^{\frac{x}{n}} e^{ixy} + \int_{0}^{\infty} dx e^{-\frac{x}{n}} e^{ixy} \\ &= -\frac{1}{\frac{1}{n}+iy} - \frac{1}{-\frac{1}{n}+iy} = \frac{i2y}{\frac{1}{n^2}+y^2} = f_n(y). \end{aligned}$$

$\lim\limits_{n\to\infty} f_n(y)$ is a generalized function. Choose a testing function $\phi(y)$

$$\lim_{n\to\infty} \langle f_n, \phi \rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} dy \frac{2iy}{\frac{1}{n^2}+y^2} \phi(y)$$

$$= \lim_{n\to\infty} \lim_{\epsilon\to 0} \left\{ \int_{-\infty}^{-\epsilon} dy \frac{2iy}{\frac{1}{n^2}+y^2} \phi(y) + \int_{\epsilon}^{\infty} dy \frac{2iy}{\frac{1}{n^2}+y^2} \phi(y) + \underbrace{\int_{-\epsilon}^{\epsilon} dy \frac{2iy}{\frac{1}{n^2}+y^2} \phi(y)}_{\phi(0) \int_{-\epsilon}^{\epsilon} \frac{2iy}{\frac{1}{n^2}+y^2} = 0} \right\}$$

$$= \lim_{n\to\infty} \lim_{\epsilon\to 0} \left\{ \int_{-\infty}^{-\epsilon} dy \frac{2iy}{\frac{1}{n^2}+y^2} \phi(y) + \int_{\epsilon}^{\infty} dy \frac{2iy}{\frac{1}{n^2}+y^2} \phi(y) \right\}$$

$$= \lim_{n\to\infty} \langle P.V.\frac{2iy}{\frac{1}{n^2}+y^2}, \phi \rangle = \langle P.V.\frac{2i}{y}, \phi \rangle.$$

Therefore,

$$P.V.\frac{2i}{y} = \lim_{n\to\infty} f_n(y) = \int_{-\infty}^{\infty} dx \mathrm{sgn}(x) e^{ixy}.$$

§7.10

$$\nabla \times \nabla \times \mathbf{F} - k^2\mathbf{F} = 0, \tag{7.2.1}$$

$$(\nabla^2 + k^2)\psi = 0, \tag{7.2.2}$$

$$\mathbf{M}(\mathbf{r}) = \nabla \times \mathbf{c}\psi, \tag{7.2.3}$$

$$\mathbf{N}(\mathbf{r}) = \frac{1}{k}\nabla \times \mathbf{M}(\mathbf{r}). \tag{7.2.4}$$

From (7.2.1), $\nabla\nabla \cdot \mathbf{F} - \nabla^2\mathbf{F} - k^2\mathbf{F} = 0$, and substitute with (7.2.3), we have

$$\nabla\nabla \cdot (\nabla \times \mathbf{c}\psi) - \nabla^2(\nabla \times \mathbf{c}\psi) - k^2(\nabla \times \mathbf{c}\psi) = 0$$
$$-\nabla \times \mathbf{c}\underbrace{\left[\nabla^2\psi - k^2\psi\right]}_{=0} = 0.$$

The above implies that $\mathbf{M}(\mathbf{r})$ is the solution of (7.2.1). Furthermore,

$$-\nabla^2\left(\frac{1}{k}\nabla \times \mathbf{M}\right) - k^2\left(\frac{1}{k}\nabla \times M\right) = 0$$
$$-\nabla \times \underbrace{(\nabla^2\mathbf{M} + k^2\mathbf{M})}_{=0} = 0,$$

which implies that $\mathbf{N}(\mathbf{r})$ is the solution of (7.2.1).

§7.11

$\psi = 0$ on S implies that $\hat{n} \cdot \mathbf{M} = \mathbf{n} \times \mathbf{N} = 0$ on the sidewall, and $\hat{n} \times \mathbf{M} = \hat{n} \cdot \mathbf{N} = 0$ on the end caps.

$$\begin{aligned}
\mathbf{M} &= \nabla \times \mathbf{c}\psi, \\
\hat{n} \cdot \mathbf{M} &= \hat{n} \cdot (\nabla \times \mathbf{c}\psi) \\
&= \nabla \cdot (\mathbf{c}\psi \times \hat{n}) \\
&= \nabla \cdot (\psi\hat{\tau})
\end{aligned}$$

$\hat{n} \cdot \mathbf{M} = 0$ on S in because $\psi = 0$ on S . Furthermore,

$$\begin{aligned}
\hat{n} \times \mathbf{N} = \hat{n} \times \left[\frac{1}{k}\nabla \times \mathbf{M}\right] &= \frac{1}{k}\hat{n} \times [\nabla \times \nabla \times \mathbf{c}\psi] \\
&= \frac{1}{k}\hat{n} \times \left[\nabla\nabla \cdot \mathbf{c}\psi - \nabla^2\mathbf{c}\psi\right], \qquad \text{where} \qquad \nabla^2\psi = -k^2\psi \\
&= \frac{1}{k}\hat{n} \times \left[\nabla\nabla \cdot \mathbf{c}\psi + \mathbf{c}k^2\psi\right] \\
&= \left\{\frac{1}{k}\hat{n} \times \left[\nabla\frac{\partial\psi}{\partial \mathbf{c}}\right] + k\psi\hat{\tau}\right\}.
\end{aligned}$$

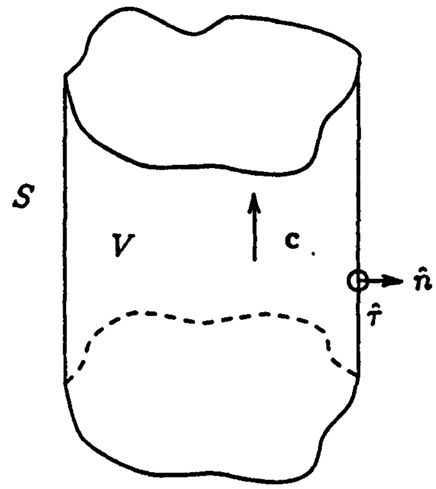

Figure 1 for Exercise Solution 7.11

Since $\psi = 0$ on S, $\frac{\partial\psi}{\partial c} = 0$ on S, then

$$\hat{n} \times \mathbf{N} = 0,$$

$$\begin{aligned}
\hat{n} \times \mathbf{M} &= \hat{n} \times (\nabla \times \mathbf{c}\psi) = \hat{n} \times (\nabla\psi \times \mathbf{c}),\\
\nabla\psi &= \frac{\partial\psi}{\partial n}\hat{n}_\Omega + \left(\frac{\partial\psi}{\partial n_s}\hat{n}_s\right)_{r\epsilon\Gamma},\\
\hat{n}_\Omega &= \frac{\mathbf{c}}{|\mathbf{c}|}\\
&= \hat{n} \times \left[\left(\frac{\partial\psi}{\partial n}\frac{\mathbf{c}}{|\mathbf{c}|} + \left.\frac{\partial\psi}{\partial n_s}\right|_{r\epsilon\Gamma}\hat{n}_s\right) \times \mathbf{c}\right]
\end{aligned}$$

$$= \hat{n} \times \left[\left.\frac{\partial\psi}{\partial n_s}\right|_{r\epsilon\Gamma}\hat{n}_s \times \mathbf{c}\right] = \left.\frac{\partial\psi}{\partial n_s}\right|_{r\epsilon\Gamma}\hat{n}_s(\hat{n}\cdot\mathbf{c}) - \mathbf{c}\left(\hat{n}\cdot\hat{n}_s\left.\frac{\partial\psi}{\partial n_s}\right|_{r\epsilon\Gamma}\right).$$

Therefore,

$$\hat{n} \times \mathbf{M} = 0 \quad \text{on} \quad \Omega \quad (\text{excluding } \Gamma),$$

$$\begin{aligned}
\hat{n} \cdot \mathbf{N} &= \hat{n} \cdot \left[\frac{1}{k}\nabla \times \mathbf{M}\right]\\
&= \nabla \cdot \left[\frac{1}{k}\mathbf{M} \times \hat{n}\right] + \frac{1}{k}\mathbf{M} \cdot \nabla \times \hat{n} = 0,
\end{aligned}$$

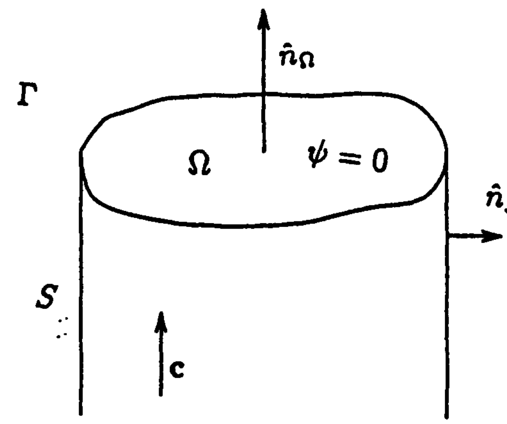

Figure 2 for Exercise Solution 7.11

where the following vector identity is used:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}.$$

§7.12

If $\psi(\mathbf{k}, \mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$, then

$$\begin{aligned} \mathbf{M}(\mathbf{k}, \mathbf{r}) i\mathbf{k} \times \hat{z} e^{i\mathbf{k}\cdot\mathbf{r}}, \\ \mathbf{N}(\mathbf{k}, \mathbf{r}) = -\frac{1}{k}\mathbf{k} \times \mathbf{k} \times \hat{z} e^{i\mathbf{k}\cdot\mathbf{r}}, \\ \mathbf{L}(\mathbf{k}, \mathbf{r}) = i\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}. \end{aligned}$$

$$\begin{aligned} (7.2.28) &= \iiint_{-\infty}^{\infty} dxdydz\, \psi(\mathbf{k}, \mathbf{r})\psi(-\mathbf{k}', \mathbf{r}) = \iiint_{-\infty}^{\infty} dxdydz\, e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{k}'\cdot\mathbf{r}} \\ &= \iint_{-\infty}^{\infty} dxdydz\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'). \qquad (7.2.28) \end{aligned}$$

$$
\begin{aligned}
(7.2.29) &= \iiint_{-\infty}^{\infty} d\mathbf{r}\, \mathbf{M}(\mathbf{k},\mathbf{r}) \cdot \mathbf{M}(-\mathbf{k}',\mathbf{r}) \\
&= \iiint_{-\infty}^{\infty} d\mathbf{r} \left[i\mathbf{k} \times \hat{z}\, e^{i\mathbf{k}\cdot\mathbf{r}}\right] \cdot \left[-i\mathbf{k}' \times \hat{z}\, e^{-i\mathbf{k}'\cdot\mathbf{r}}\right] \\
&= \iiint_{-\infty}^{\infty} d\mathbf{r}\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} (\mathbf{k} \times \hat{z}) \cdot (\mathbf{k}' \times \hat{z}) \\
&= \iiint_{-\infty}^{\infty} d\mathbf{r}\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} [\hat{z} \times (\mathbf{k} \times \hat{z})] \cdot \mathbf{k}' \\
&= \iiint_{-\infty}^{\infty} d\mathbf{r}\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} [\mathbf{k}_s \cdot \mathbf{k}_s'] = k_s^2 (2\pi)^3\, \delta(\mathbf{k} - \mathbf{k}').
\end{aligned} \tag{7.2.29}
$$

$$
\begin{aligned}
(7.2.30) &= \iiint_{-\infty}^{\infty} d\mathbf{r}\, \mathbf{N}(\mathbf{k},\mathbf{r}) \cdot \mathbf{N}(-\mathbf{k}',\mathbf{r}) \\
&= \iiint_{-\infty}^{\infty} d\mathbf{r} \left[-\frac{1}{k}\mathbf{k} \times \mathbf{k} \times \hat{z}\, e^{i\mathbf{k}\cdot\mathbf{r}}\right] \cdot \left[-\frac{1}{k'}(-\mathbf{k}' \times (-\mathbf{k}' \times \hat{z}))e^{-i\mathbf{k}'\cdot\mathbf{r}}\right] \\
&= \iiint_{-\infty}^{\infty} d\mathbf{r}\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \frac{[\mathbf{k} \times (\mathbf{k} \times \hat{z})] \cdot [\mathbf{k}' \times (\mathbf{k}' \times \hat{z})]}{kk'} \\
&= \iiint_{-\infty}^{\infty} d\mathbf{r}\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \frac{(\mathbf{k}k_z - \hat{z}k^2) \cdot (\mathbf{k}'k_z' - \hat{z}k'^2)}{kk'} \\
&= \iiint_{-\infty}^{\infty} d\mathbf{r}\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \frac{k_z k_z' \mathbf{k} \cdot \mathbf{k}' - k_z^2 k'^2 - k^2 k_z'^2 + k^2 k'^2}{kk'} \\
&= k_s^2 (2\pi)^3\, \delta(\mathbf{k} - \mathbf{k}').
\end{aligned} \tag{7.2.30}
$$

$$
\begin{aligned}
(7.2.31) &= \iiint_{-\infty}^{\infty} d\mathbf{r} \mathbf{L}(\mathbf{k},\mathbf{r}) \cdot \mathbf{L}(-\mathbf{k}',\mathbf{r}) \\
&= \iiint_{-\infty}^{\infty} d\mathbf{r}\, i\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} \cdot (-i\mathbf{k}') e^{-i\mathbf{k}'\cdot\mathbf{r}'} = k^2 (2\pi)^3\, \delta(\mathbf{k} - \mathbf{k}').
\end{aligned} \tag{7.2.31}
$$

§7.13

(a) The n-th order Bessel function $J_n(k\rho)$ has the following integral represectation [p. 82, Eq. (2.2.18)].

$$J_n(k\rho) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha e^{ik\rho\cos\alpha + in(\alpha - \frac{\pi}{2})}. \tag{1}$$

Thus

$$J_{-n}(-k\rho) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha e^{-ik\rho\cos\alpha + in(-\alpha + \frac{\pi}{2})}. \tag{2}$$

Let $\alpha = \pi - \alpha'$, then $\alpha' = \pi - \alpha, d\alpha = -d\alpha'$, the integration range of α' is $[+\pi, -\pi]$. Hence

$$J_{-n}(-k\rho) = \frac{1}{2\pi} \int_{+\pi}^{-\pi} (-d\alpha') e^{-ik\rho\cos(\pi - \alpha') + in(\alpha' - \frac{\pi}{2})}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\alpha' e^{ik\rho\cos\alpha' + in(\alpha' - \frac{\pi}{2})}. \tag{3}$$

Since $ik\rho\cos\alpha' + in(\alpha' - \frac{\pi}{2})$ is a periodic function of α' with period $T = 2\pi$, its integration over $[-\pi, \pi]$ is the same as over $[0, 2\pi]$, i.e.

$$J_{-n}(-kp) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha' e^{ikp\cos\alpha' + in(\alpha' - \frac{\pi}{2})}. \tag{4}$$

Comparing (1) and (4), we conclude that

$$J_{-n}(-x) = J_n(x). \tag{5}$$

(b) From (7.3.32)

$$\psi_n(k_\rho, k_z, \mathbf{r}) = J_n(k_\rho\rho) e^{ik_z z + in\phi}. \tag{6}$$

Likewise,

$$\psi_{-n'}(-k'_\rho, -k'_z, \mathbf{r}) = J_n(-k_\rho\rho) e^{-ik_z z - in\phi}. \tag{7}$$

Using the conclusion we have proved in part (a) of this problem, we have

$$J_{-n'}(-k_\rho\rho) = J_{n'}(k'_\rho\rho).$$

Therefore,

$$\psi_n(k_\rho, k_z, \mathbf{r})\psi_{-n'}(-k_\rho, -k_z, \mathbf{r}) = J_n(k_\rho\rho) J_{n'}(k'_\rho\rho) e^{i(k_z - k'_z)z} e^{i(n-n')\phi}. \tag{8}$$

Since

$$\int^{2\pi} e^{i(n-n')\phi} d\phi = 2\pi\delta_{nn'}, \tag{9}$$

$$\int_{-\infty}^{\infty} e^{i(k_z - k'_z)z} dz = 2\pi\delta(k_z - k'_z), \tag{10}$$

and

$$\int_0^\infty \rho \quad d\rho J_n(k_\rho \rho) J_{n'}(k'_\rho \rho) = \frac{1}{k_\rho}\delta(k_\rho - k'_\rho), \tag{11}$$

we obtain

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \int_0^\infty \rho \quad d\rho \quad \psi_n(k_\rho, k_z, \mathbf{r})\psi_{n'}(-k'_\rho, -k'_z, \mathbf{r}) =$$

$$\int_0^{2\pi} e^{i(n-n')\phi} d\phi \int_{-\infty}^{\infty} e^{i(k_z - k'_z)z} dz \cdot \int_0^\infty \rho \quad d\rho \quad J_n(k_\rho \rho) J_{n'}(k'_\rho \rho)$$

$$=(2\pi)^2 \delta_{nn'} \delta(k_z - k'_z) \frac{\delta(k_\rho - k'_\rho)}{k_\rho}. \tag{12}$$

This is equation (7.3.37). It is the basic orthogonal relation which can be used to prove the other three orthogonal relations for the vector wave functions **M**, **N**, and **L**. We shall prove them as follows.

By definition, $\mathbf{M} = \nabla \times (\hat{z}\psi_1)$. Hence,

$$\begin{aligned}\mathbf{M}_n(k_\rho, k_z, \mathbf{r}) =& \nabla \times [\hat{z}\psi_n(k_\rho, k_z, \mathbf{r})] \\ =& [\hat{\rho}\frac{in}{\rho} J_n(k_\rho \rho) - \hat{\phi} k_\rho J'_n(k_\rho \rho)] e^{ik_z z + in\phi}.\end{aligned} \tag{13}$$

$$\begin{aligned}\mathbf{M}_{-n'}(-k'_\rho, -k'_z, \mathbf{r}) =& [\hat{\rho}\frac{-in'}{\rho} J_{-n'}(-k'_\rho \rho) - \hat{\phi}(-k'_\rho) J'_{-n'}(-k'_\rho \rho)] e^{-ik_z z + in'\phi} \\ =& [\hat{\rho}\frac{-in'}{\rho} J_{n'}(k'_\rho \rho) - \hat{\phi} k'_\rho J'_{n'}(k'_\rho \rho)] e^{-ik_z z + in'\phi}.\end{aligned} \tag{14}$$

In the above,

$$J'_n(kx) \triangleq \frac{dJ_n(kx)}{d(kx)}. \tag{15}$$

Hence,

$$\begin{aligned}&\mathbf{M}_n(k_\rho, k_z, \mathbf{r}) \cdot \mathbf{M}_{-n'}(-k'_\rho \rho, -k'_z, \mathbf{r}) \\ &= \left[\frac{nn'}{\rho^2} J_n(k_\rho \rho) J_{n'}(k'_\rho \rho) + k_\rho k'_\rho J'_n(k_\rho \rho) J'_{n'}(k'_\rho \rho)\right] \cdot e^{i(k_z - k'_z)z} e^{i(n-n')\phi}.\end{aligned} \tag{16}$$

Using the identity $J'_n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$ we can write the right hand side of (16) in the summation of the following terms multiplied by

$e^{i(k_z-k'_z)z}e^{i(n-n')\phi}$:

$$\frac{nn'}{\rho^2}J_n(k_\rho\rho)J_{n'}(k'_\rho\rho),$$
$$k_\rho k'_\rho J_{n-1}(k_\rho\rho)J'_{n-1}(k'_\rho\rho),$$
$$-k_\rho k'_\rho J_{n-1}(k_\rho\rho)\frac{n'}{k'_\rho\rho}J_{n'}(k'_\rho\rho),$$
$$-k_\rho k'_\rho J_{n-1}(k'_\rho\rho)\frac{n}{k_\rho\rho}J_n(k_\rho\rho),$$
$$-\frac{k_\rho k'_\rho nn'}{(k_\rho\rho)(k'_\rho\rho)}J_n(k_\rho\rho)J_{n'}(k'_\rho\rho).$$

The integration over ϕ results in $2\pi\delta_{nn'}$. Hence, when $n \neq n'$, the total integral will be zero. Therefore, we can get rid of the terms involving $J_n(k_\rho\rho)J_{n-1}(k'_\rho\rho)$ and $J_{n'}(k'_\rho\rho)J_{n-1}(k_\rho\rho)$. The first and the last terms are of the same form but of different sign, and they cancel each other. Finally, we get

$$k_\rho k_{\rho'}\int_0^\infty J_{n-1}(k_\rho\rho)J_{n'-1}(k'_\rho\rho)\rho d\rho = k_\rho\delta(k_\rho - k'_\rho). \tag{18}$$

The integration over z is once again $2\pi\delta(k_z - k'_z)$. Thus

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \int_0^\infty \mathbf{M}_n(k_\rho, k_z, \mathbf{r})\cdot\mathbf{M}_{-n'}(-k'_\rho, -k'_z, \rho)\rho\, d\rho = (2\pi)^2 k_\rho\delta_{nn'}\delta(k_z - k'_z)\delta(k_\rho - k'_\rho). \tag{19}$$

Before proving (7.2.39), we look at (7.2.40) first. By definition,

$$\begin{aligned}\mathbf{L}_n(k_\rho, k_z, \mathbf{r}) &= \nabla\psi_n(k_\rho, k_z, \mathbf{r})\\ &= [\hat{\rho}k_\rho J'_n(k_\rho\rho) + \hat{\phi}\frac{in}{\rho}J_n(k_\rho\rho) + \hat{z}(ik_z)J_n(k_\rho\rho)]e^{ik_z z + in\phi},\end{aligned}$$

$$\begin{aligned}&\mathbf{L}_{-n'}(-k'_\rho, -k'_z, \mathbf{r})\\ &= [\hat{\rho}(k'_\rho)J'_{n'}(k'_\rho\rho) + \hat{\phi}\frac{-in'}{\rho}J_{n'}(k'_\rho\rho) - \hat{z}(ik'_z)J_{n'}(k'_\rho\rho)]e^{-ik'_z z - in'\phi},\end{aligned}$$

$$\begin{aligned}\mathbf{L}_n(k_\rho, k_z, \mathbf{r})\cdot\mathbf{L}_{-n'}(-k'_\rho, -k'_z, \mathbf{r}) =& [k_\rho k'_\rho J'_n(k_\rho\rho) + \frac{nn'}{\rho^2}J_n(k_\rho\rho)J_{n'}(k'_\rho\rho)\\ &+ k_z k'_z J_n(k_\rho\rho)J_{n'}(k'_\rho\rho)]\cdot e^{i(k_z-k'_z)z+i(n-n')\phi}.\end{aligned} \tag{20}$$

Again, the integral over z and ϕ are $2\pi\delta(k_z - k'_z)$ and $2\pi\delta_{nn'}$, repectively, where we are only concerned with terms including Bessel functions. Comparing (20) with (16) we deduce that

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \int_0^{\infty} d\rho \mathbf{L}_n(k_\rho, k_z, \mathbf{r}) \cdot \mathbf{L}_{-n'}(-k'_\rho, -k'_z, \mathbf{r})$$
$$= (2\pi)^2 k_\rho \delta_{nn'} \delta(k_z - k'_z)\delta(k_\rho - k'_\rho) + (2\pi)^2 \frac{k_z^2}{k_\rho} \delta_{nn'} \delta(k_z - k'_z)\delta(k_\rho - k'_\rho)$$
$$= (2\pi)^2 k^2 \delta_{nn'} \delta(k_z - k'_z) \frac{\delta(k_\rho - k'_\rho)}{k_\rho}. \tag{21}$$

In the above, we have used $k^2 = k_\rho^2 + k_z^2$. Finally, we prove (7.2.39). Since

$$\mathbf{N}_n = \frac{1}{k} \nabla \times \nabla \times (\hat{z}\psi_n) = \frac{1}{k}[\nabla(\nabla \cdot \hat{z}\psi_n) - \nabla^2 \hat{z}\psi_n] = \frac{1}{k}[(ik_z)\nabla\psi_n + \hat{z}k^2\psi_n]$$
$$= \frac{ik_z}{k}\mathbf{L}_n + \hat{z}k\psi_n = \frac{ik_z}{k}[\hat{\rho}k_\rho J'_n(k_\rho\rho) + \hat{\phi}\frac{in}{\rho}J_n(k_\rho\rho)]e^{ik_z z + in\phi}$$
$$+ \frac{1}{k}(ik_z) \cdot ik_z \hat{z}\psi_n + \hat{z}k\psi_n,$$

we have

$$\mathbf{N}_n(k_\rho, k_z, \mathbf{r}) \cdot \mathbf{N}_{-n'}(-k'_\rho, -k'_z, \mathbf{r}) = \frac{k_z k'_z}{k}[k_\rho k'_\rho J'_{n'}(k'_\rho \rho) J'_n(k_\rho \rho)$$
$$+ \frac{nn'}{\rho^2} J_n J_{n'}] \cdot e^{i(k_z - k'_z)z + i(n-n')\phi} + \frac{k_\rho^2}{k}\psi_n \psi_{-n'}.$$

Using the previous result, we can easily show that

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \int_0^{\infty} \rho \quad d\rho \mathbf{N}_n(k_\rho, k_z, \mathbf{r}) \mathbf{N}_{-n'}(-k_\rho, -k'_z, \mathbf{r})$$
$$= (2\pi)^2 k_\rho \delta_{nn'} \delta(k_z - k'_z)\delta(k_\rho - k'_\rho). \tag{22}$$

§7.14

(a)

$$Y_{nm}(\theta, \phi) = \sqrt{\frac{(n-m)!(2n+1)}{(n+m)!4\pi}} P_n^m(\cos\theta) e^{im\phi}, \tag{1}$$

$$Y_{n,-m}(\theta, \phi) = \sqrt{\frac{(n+m)!(2n+1)}{(n-m)!4\pi}} P_n^{-m}(\cos\theta) e^{-im\phi}. \tag{2}$$

Since †

$$P_n^{-m}(\cos\theta) = \frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta), \tag{3}$$

$$\begin{aligned} Y_{n,-m}(\theta,\phi) &= \sqrt{\frac{(n+m)!(2n+1)}{(n-m)!4\pi}}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)e^{-im\phi} \\ &= \sqrt{\frac{(n-m)!(2n+1)!}{(n+m)!4\pi}}P_n^m(\cos\theta)e^{-im\phi} \\ &= Y_{nm}^*(\theta,\phi). \end{aligned} \tag{4}$$

The above formula is limited to m and n are integers and θ is real. For the orthoginality of $Y_{nm}(\theta,\phi)$ we first note that

$$\int_0^{2\pi} d\phi e^{i(m-m')\phi} = 2\pi\delta_{mm'}, \tag{5}$$

$$\int_0^{\pi} P_n^m(\cos\theta)P_{n'}^m(\cos\theta)\sin\theta d\theta = \delta_{nn'}\frac{2(n+m)!}{(2n+1)(n-m)!}. \tag{6}$$

Thus

$$\int_0^{2\pi}\int_0^{\pi} Y_{nm}(\theta,\phi)Y_{n',-m'}(\theta,\phi)\sin\theta d\theta d\phi$$

† A different formula is sometimes used here. See for example A. Messiah.

$$= \int_0^{2\pi} e^{i(m-m')\phi} d\phi \int_0^{\pi} P_n^m P_{n'}^{-m'}(\cos\theta)\sin\theta d\theta$$
$$\cdot \left[\frac{(n-m)!(n'+m')!}{(n+m)!(n'-m')!}\frac{(2n+1)(2n'+1)}{(4\pi)^2}\right]^{\frac{1}{2}}$$
$$=2\pi\delta_{mm'} \int_0^{\pi} P_n^m(\cos\theta)\cdot P_{n'}^{-m}(\cos\theta)\theta d\theta$$
$$\cdot \sqrt{\frac{(2n+1)(2n'+1)}{(4\pi)^2}}\sqrt{\frac{(n'+m)!(n-m)!}{(n'-m)!(n+m)!}}$$
$$=2\pi\delta_{mm'}\left[\frac{(n'+m)!(n-m)!}{(n'-m)!(n+m)!}\frac{(2n+1)(2n'+1)}{(4\pi)^2}\right]^{\frac{1}{2}}\left(\frac{(n'-m)!}{(n'+m)!}\right)$$
$$\cdot \int_0^{\pi} P_n^m(\cos\theta)P_{n'}^m(\cos\theta)\theta d\theta$$
$$=2\pi\delta_{mn'}\left[\frac{(n'+m)!(n-m)!}{(n'-m)!(n+m)!}\frac{(2n+1)(2n'+1)}{(4\pi)^2}\right]^{\frac{1}{2}}\left[\frac{(n'-m)!}{(n'+m)!}\right]$$
$$\cdot \delta_{nn'}\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$
$$=\delta_{mm'}\delta_{nn'}. \tag{7}$$

(b)

$$\psi_{nm}(k,\mathbf{r}) = j_n(kr)Y_{nm}(\theta,\phi),$$
$$\psi_{n',-m'}(k',\mathbf{r}) = j_{n'}(k'r)Y_{n',-m'}(\theta,\phi).$$

Therefore,

$$\int \psi_{nm}(k,\mathbf{r})\psi_{n',-m'}(k',\mathbf{r}) = \int_0^{\infty} drr^2 j_n(kr)j_{n'}(k'r)$$
$$\int_{\Omega} Y_{nm}(\theta,\phi)Y_{n',-m'}(\theta,\phi)d\Omega$$

$$= \delta_{mm'}\delta_{nn'}\int_0^{\infty} drr^2 j_n(kr)j_n(k'r) = \delta_{mm'}\delta_{nn'}\frac{\pi}{2k^2}\delta(k-k'). \tag{8}$$

In the above we have used the relation (7) and (7.2.46) on page 396. For (2.7.48), we have

$$\mathbf{M}_{nm}(k,\mathbf{r}) = \nabla\times[\mathbf{r}j_n(kr)Y_{nm}(\theta,\phi)]$$
$$= \nabla\times[\mathbf{r}\psi_{nm}(k,\mathbf{r})],$$
$$\mathbf{M}_{nm}(k,\mathbf{r})\cdot\mathbf{M}_{n',m'}(k',\mathbf{r}) = \nabla\times[\mathbf{r}\psi_a(\mathbf{r})]\cdot\nabla\times[\mathbf{r}\psi_b(\mathbf{r})],$$

where

$$\mathbf{M}_a \triangleq \mathbf{M}_{nm}(k, \mathbf{r}),$$
$$\mathbf{M}_b \triangleq \mathbf{M}_{-n',-m'}(k', \mathbf{r}).$$

Using the vector idenity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$ with $\mathbf{A} = \mathbf{r}\psi_a, \mathbf{B} = \nabla \times (\mathbf{r}\psi_b)$, we have

$$\begin{aligned}\nabla \times \overbrace{[\mathbf{r}\psi_a(\mathbf{r})]}^{A} \cdot \overbrace{\nabla \times [\mathbf{r}\psi_b(\mathbf{r})]}^{B} \\ = \nabla \cdot \underbrace{[\mathbf{r}\psi_a(\mathbf{r})]}_{A} \times \underbrace{\nabla \times [\mathbf{r}\psi_b(\mathbf{r})]}_{B} + \underbrace{\mathbf{r}\psi_a(\mathbf{r})}_{A} \cdot \nabla \times \underbrace{\nabla \times [\mathbf{r}\psi_b(\mathbf{r})]}_{B},\end{aligned}$$

and

$$\int_V d\mathbf{r} \mathbf{M}_a(\mathbf{r}) \cdot \mathbf{M}_b(\mathbf{r}) = \int_V d\mathbf{r} \nabla \cdot \{\mathbf{r}\psi_a(\mathbf{r}) \times \nabla \times [\mathbf{r}\psi_b(\mathbf{r})]\} + \int_V d\mathbf{r}\mathbf{r}\psi_a(\mathbf{r}) \cdot \nabla \times \nabla \times [\mathbf{r}\psi_b(\mathbf{r})]. \tag{9}$$

The first term on the right hand side, using divergence theorem, can be converted to

$$\int_V d\mathbf{r} \nabla \cdot \{\mathbf{r}\psi_a(\mathbf{r}) \times \nabla \times [\mathbf{r}\psi_b(\mathbf{r})]\} = \int_S ds\hat{n} \cdot \{\mathbf{r}\psi_a(\mathbf{r}) \times \nabla \times [\mathbf{r}\psi_b(\mathbf{r})]\}$$

where S is the surface enclosing V. If we chose V to be a sphere, then $\hat{n} = \hat{r}$. Hence, this integral vanishes since the quantity $\mathbf{r}\psi_a(\mathbf{r}) \times \mathbf{B}$ is a vector perpendicular to $\mathbf{r}$. As for the second term on the right hand side of (9), we note that

$$\begin{aligned}\mathbf{r} \cdot \nabla \times \nabla \times \mathbf{r}\psi &= \mathbf{r} \cdot \nabla \times \left\{ \frac{\hat{\theta}}{r \sin\theta} \frac{\partial}{\partial\phi}(r\psi) - \frac{\hat{\phi}}{r} \frac{\partial}{\partial\theta}(r\psi) \right\} \\ &= \mathbf{r} \cdot \nabla \times \left\{ \frac{\hat{\theta}}{\sin\theta} \frac{\partial}{\partial\phi}\psi - \hat{\phi} \frac{\partial}{\partial\theta}\psi \right\} \\ &= r \left\{ \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta}(-\sin\theta) \frac{\partial\psi}{\partial\theta}) - \frac{\partial}{\partial\phi}(\frac{1}{\sin\theta} \frac{\partial\psi}{\partial\phi}) \right] \right\} \\ &= -\left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin\theta} \frac{\partial^2}{\partial\phi^2} \right] \psi \\ &= n(n+1)\psi.\end{aligned} \tag{10}$$

In the last step of deriving the above relation, we made an assumption that $\psi = j_n(kr)Y_{nm}(\theta,\phi)$, i.e. P_n^m satisfy

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial P_n^m}{\partial\theta}) + [n(n+1) - \frac{m^2}{\sin^2\theta}]P_n^m(\cos\theta) = 0.$$

As a result,

$$\int_V d\mathbf{r}\mathbf{M}_{nm}(k,\mathbf{r})\mathbf{M}_{n,'-m'}(k',\mathbf{r}) = n(n+1)\int_V d\mathbf{r}\psi_{nm}(k\mathbf{r})\psi_{n',-m'}(k',\mathbf{r})$$
$$= n(n+1)\frac{\pi}{2k^2}\delta_{nn'}\delta_{mm'}\delta(k-k'). \tag{11}$$

Now, consider $\int_V d\mathbf{r}\mathbf{N}_{nm}(k,\mathbf{r})\cdot\mathbf{N}_{n',-m'}(k',\mathbf{r}') \triangleq I_N$. Let $\mathbf{N}_a(\mathbf{r}) \triangleq \mathbf{N}_{nm}(k,\mathbf{r})$, $\mathbf{N}_b(\mathbf{r}) \triangleq \mathbf{N}_{n',-m'}(k',\mathbf{r}')$. By definition,

$$\mathbf{N}_a(\mathbf{r}) = \frac{1}{k_a}\nabla\times\mathbf{M}_a(\mathbf{r}), \quad \mathbf{N}_b(\mathbf{r}) = \frac{1}{k_b}\nabla\times\mathbf{M}_b(\mathbf{r}),$$

and

$$\begin{aligned}\mathbf{N}_a\cdot\mathbf{N}_b &= \frac{1}{k_ak_b}\nabla\times\mathbf{M}_a\cdot\nabla\times\mathbf{M}_b\\ &= \frac{1}{k_ak_b}[\nabla\cdot(\mathbf{M}_a\times\nabla\times\mathbf{M}_b) + \mathbf{M}_a\cdot\nabla\times\nabla\times\mathbf{M}_b]\\ &= \frac{1}{k_ak_b}[\nabla\cdot(\mathbf{M}_a\times\nabla\times\mathbf{M}_b) + \mathbf{M}_a\cdot k_b^2\mathbf{M}_b].\end{aligned}$$

Hence,

$$\begin{aligned}I_N &= \int_V dr\frac{1}{k_ak_b}\nabla\cdot(\mathbf{M}_a(\mathbf{r})\times\nabla\times\mathbf{M}_b(\mathbf{r})) + \frac{k_b}{k_a}\int_V d\mathbf{r}\mathbf{M}_a\cdot\mathbf{M}_b\\ &= \int_V ds\hat{n}\cdot\frac{1}{k_ak_b}(\mathbf{M}_a\times\nabla\times\mathbf{M}_b) + \frac{k_b}{k_a}\int_V d\mathbf{r}\mathbf{M}_a\cdot\mathbf{M}_b.\end{aligned}$$

The first integrand vanishes since $\mathbf{M}_a$ or $\mathbf{M}_b$ satisfy either homogeneous Dirichlet or the homogeneous Neumann boundary condition on surface S. As for the second integral, it is just $\frac{k_b}{k_a}I_M$. Hence,

$$I_N = \int_V d\mathbf{r}\mathbf{N}_a\cdot\mathbf{N}_b = n(n+1)\frac{\pi}{2k^2}\delta_{nm'}\delta_{mm'}\delta(k-k'). \tag{12}$$

Finally, consider $I_L \triangleq \int_V dv\mathbf{L}_a\cdot\mathbf{L}_b$, where

$$\mathbf{L}_a(\mathbf{r}) \triangleq \mathbf{L}_{nm}(k,\mathbf{r}) = \frac{1}{k_a}\nabla\psi_a(\mathbf{r}),$$
$$\mathbf{L}_b(\mathbf{r}) \triangleq \mathbf{L}_{n',-m'}(k,\mathbf{r}) = \frac{1}{k_b}\nabla\psi_b(\mathbf{r}).$$

Again,

$$\mathbf{L}_a \cdot \mathbf{L}_b = \frac{1}{k_a}\nabla\psi_a \cdot \frac{1}{k_b}\nabla\psi_b$$
$$= \frac{1}{k_a k_b}\{\nabla \cdot [\psi_a \nabla\psi_b] - \psi_a \nabla \cdot \nabla\psi_b\}$$
$$= \frac{1}{k_a k_b}\nabla \cdot [\psi_a \nabla\psi_b] + \frac{k_b}{k_a}\psi_a\psi_b,$$

$$\int_V \mathbf{L}_a \cdot \mathbf{L}_b d\mathbf{r} = \frac{1}{k_a k_b}\int_V d\mathbf{r}\nabla \cdot [\psi_a \nabla\psi_b] + \frac{k_b}{k_a}\int_V d\mathbf{r}\psi_a\psi_b$$
$$= \frac{1}{k_a k_b}\int_S dS\hat{n} \cdot [\psi_a \nabla\psi_b] + \frac{\pi}{2k^2}\delta_{nm'}\delta_{mm'}\delta(k-k').$$

Neumann boundary conditions. Therefore,

$$\int_V \mathbf{L}_{nm}(k,\mathbf{r}) \cdot \mathbf{L}_{n',-m'}(k',\mathbf{r}) = \frac{\pi}{2k^2}\delta_{nm'}\delta_{mm'}\delta(k-k').$$

§7.15

(a) Dyadic Green's function $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ satisfy

$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') - k_0^2\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \mathbf{I}\delta(\mathbf{r}-\mathbf{r}'). \tag{1}$$

In cylindrical coordinates, we expand $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ by cylindrical vector wave function $\mathbf{M}, \mathbf{N}$, and $\mathbf{L}$ such that

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk_\rho k_\rho [\mathbf{M}_n(k_\rho,k_z,\mathbf{r})\mathbf{a}_n(k_\rho,k_z)+$$
$$\mathbf{N}_n(k_\rho,k_z,\mathbf{r})\mathbf{b}_n(k_\rho,k_z) + \mathbf{L}_n(k_\rho,k_z,\mathbf{r})\mathbf{c}_n(k_\rho,k_z)]. \tag{2}$$

Substituting (2) into (1), and noting that $\mathbf{M}, \mathbf{N}$, and $\mathbf{L}$ satisfies the vector wave equation $\nabla \times \nabla \times \mathbf{F} - k^2\mathbf{F} = 0$, we have

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk_\rho k_\rho \{(k^2-k_0^2)[\mathbf{M}_n(k_\rho,k_z,\mathbf{r})\mathbf{a}_n(k_\rho,k_z)$$
$$+ (k^2-k_0^2)\mathbf{N}_n(k_\rho,k_z,\mathbf{r})\mathbf{b}_n(k_\rho,k_z)$$
$$- k_0^2\mathbf{L}_n(k_\rho,k_z,\mathbf{r})\mathbf{c}_n(k_\rho,k_z)\} = \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'). \tag{3}$$

Left dot multiply the above equation by $\mathbf{M}_{-n'}(-k'_\rho,-k'_z,\mathbf{r})$ and integrate with respect to $\mathbf{r}$ over space, the right hand side becomes

$$\text{RHS} = \int_V \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}') \cdot \mathbf{M}_{-n'}(-k'_\rho,-k'_z,\mathbf{r}')d\mathbf{r}$$
$$= \mathbf{M}_{-n'}(-k'_\rho,-k'_z,\mathbf{r}').$$

As for the left hand side, we note that **M** and **N**, **M** and **L** are orthogonal. Also,

$$\int_V d\mathbf{r}\mathbf{M}_{-n'}(-k'_\rho, -k'_z, \mathbf{r}') \cdot \mathbf{M}_n(k_\rho, k_z, \mathbf{r}) = (2\pi)^2 k'_\rho \delta_{nn'} \delta(k_z - k'_z)\delta(k_\rho - k'_\rho).$$

Thus,

$$\begin{aligned}\text{LHS} = &\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_z \int_0^{\infty} k_\rho dk_\rho (2\pi)^2 k_\rho \delta_{nn'} \delta(k_z - k'_z) \\ &\cdot \delta(k_\rho - k'_\rho)\mathbf{a}_n(k_\rho, k_z)(k^2 - k_0^2) \\ =&(2\pi)^2 k'_\rho \mathbf{a}_{n'}(k'_\rho, k'_z).\end{aligned}$$

Hence,

$$\mathbf{a}_n(k_\rho, k_z) = \frac{1}{(2\pi)^2} \frac{\mathbf{M}_{-n}(-k_\rho, -k_z, \mathbf{r}')}{k_{\rho'}^2 (k^2 - k_0^2)}. \tag{4}$$

In the same way, we can find

$$\mathbf{b}_n = \frac{1}{(2\pi)^2} \frac{1}{k_\rho^2 (k^2 - k_0^2)} \mathbf{N}_{-n}(-k_\rho, -k_z, \mathbf{r}'), \tag{5}$$

$$\mathbf{c}_n = \frac{1}{(2\pi)^2} \frac{1}{k_\rho^2 k_0^2} \mathbf{L}_{-n}(-k_\rho, -k_z, \mathbf{r}'). \tag{6}$$

Using the coefficients in (4), (5), and (6) in equation (2), we obtain

$$\begin{aligned}\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = &\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk_\rho k_\rho \Big\{ \frac{\mathbf{M}_n(k_\rho, k_z, \mathbf{r})\mathbf{M}_{-n}(-k'_\rho, -k'_z, \mathbf{r}')}{(2\pi)^2 k_\rho^2 (k_2 - k_0^2)} \\ &+ \frac{\mathbf{N}_n(k_\rho, k_z, \mathbf{r})\mathbf{N}_{-n}(-k'_\rho, -k'_z, \mathbf{r}')}{(2\pi)^2 k_\rho^2 (k_2 - k_0^2)} - \frac{\mathbf{L}_n(k_\rho, k_z, \mathbf{r})\mathbf{L}_{-n}(-k'_\rho, -k'_z, \mathbf{r}')}{(2\pi)^2 k_\rho^2 (k_2 - k_0^2)} \Big\}.\end{aligned} \tag{7}$$

In spherical coordinates, we expand $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ in terms of spherical vector wave funtions:

$$\begin{aligned}\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = &\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \int_0^{\infty} dk [\mathbf{M}_{nm}(k, \mathbf{r})\mathbf{a}_{nm}(k) \\ &+ \mathbf{N}_{nm}(k, \mathbf{r})\mathbf{b}_{nm}(k) + \mathbf{L}_{nm}(k, \mathbf{r})\mathbf{c}_{nm}(k)].\end{aligned} \tag{8}$$

Since $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ must satisfy the vector wave equation, we can put this expression in

$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - k_0^2 \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}'),$$

to obtain

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\int_0^{\infty} dk\{(k^2-k_0^2)\mathbf{M}_{nm}(k,\mathbf{r})\mathbf{a}_{nm}(k)+(k^2-k_0^2)\mathbf{M}_{nm}(k,\mathbf{r})\mathbf{b}_n(k)$$
$$-k_0^2\mathbf{L}_{nm}(k,\mathbf{r})\mathbf{c}_{nm}(k)\}=\overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'). \tag{9}$$

Left dot multiplying the above equation by $\mathbf{M}_{n',-m'}$ and integrating over $\mathbf{r}$ in V, we have

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\int_0^{\infty} dk(k^2-k_0^2)\int_V \mathbf{M}_{n',-m'}(k,\mathbf{r})\cdot\mathbf{M}_{nm}(k,\mathbf{r})\mathbf{a}_{nm}(k)d\mathbf{r}$$
$$=\mathbf{M}_{n',-m'}(k',\mathbf{r}')$$

Using the orthogonality property of $\mathbf{M}_{nm}(k,\mathbf{r})$, the above becomes

$$n(n+1)(k^2-k_0^2)\frac{\pi}{(2k)}\mathbf{a}_{n',-m'}(k')=\mathbf{M}_{n',-m'}(k',\mathbf{r}'),$$

or

$$\mathbf{a}_{nm}(k)=\frac{2}{\pi}k^2\frac{\mathbf{M}_{n,-m}(k,\mathbf{r}')}{(k^2-k_0^2)n(n+1)}.$$

Similarly,

$$\mathbf{b}_{nm}(k)=\frac{2}{\pi}k^2\frac{\mathbf{N}_{n,-m}(k,\mathbf{r}')}{(k^2-k_0^2)n(n+1)},$$
$$\mathbf{c}_{nm}(k)=\frac{-2}{\pi}\frac{k^2}{k_0^2}\mathbf{L}_{n,-m}(k,\mathbf{r}').$$

Therefore, $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ is

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\frac{2}{\pi}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\int_0^{\infty} dkk^2\Big\{\frac{\mathbf{N}_{nm}(k,\mathbf{r})\mathbf{N}_{n,-m}(k,\mathbf{r}')}{(k^2-k_0^2)n(n+1)}$$
$$\frac{\mathbf{M}_{nm}(k,\mathbf{r})\mathbf{M}_{n,-m}(k,\mathbf{r}')}{(k^2-k_0^2)n(n+1)}-\frac{1}{k_0^2}\mathbf{L}_{nm}(k,\mathbf{r}')\mathbf{L}_{n,-m}(k,\mathbf{r})\Big\}. \tag{10}$$

(b) Inspecting (7.3.5), (7.3.7), and (7.3.8), we find that $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ is constructed by the superposition of the terms such as $\mathbf{M}_a(\mathbf{r})\mathbf{M}_b(\mathbf{r}')$, etc. Hence, we only need to show that

$$\mathbf{M}_a(\mathbf{r})\mathbf{M}_b(\mathbf{r}')=[\mathbf{M}_a(\mathbf{r}')\mathbf{M}_b(\mathbf{r})]^t,\quad b=-a. \tag{11}$$

First, we can easily show that

$$(\mathbf{AB})^t=(\mathbf{BA}). \tag{12}$$

Hence,

$$[\mathbf{M}_a(\mathbf{r}')\mathbf{M}_{-a}(\mathbf{r})] = \mathbf{M}_{-a}(\mathbf{r})\mathbf{M}_a(\mathbf{r}'). \tag{13}$$

However, in general, $\mathbf{M}_{-a}(\mathbf{r})\mathbf{M}_a(\mathbf{r}')$ may not be equal to $\mathbf{M}_a(\mathbf{r})\mathbf{M}_{-a}(\mathbf{r}')$. But what we need is that

$$\sum_a \mathbf{M}_{-a}(\mathbf{r})\mathbf{M}_a(\mathbf{r}') = \sum_a \mathbf{M}_a(\mathbf{r})\mathbf{M}_{-a}(\mathbf{r}'). \tag{14}$$

In the above, $\sum_a$ stands for the linear superposition of $\mathbf{M}_{-a}(\mathbf{r})\mathbf{M}_a(\mathbf{r}')$ over index a. It can be a summation or integration. In order to show (14), we consider different coordinates:

(i) Cartesian Coodinates

$$\iiint_{-\infty}^{\infty} d\mathbf{k}\frac{\mathbf{M}(-\mathbf{k},\mathbf{r})\mathbf{M}(\mathbf{k},\mathbf{r}')}{(k^2-k_0^2)k_z^2} = -\iiint_{\infty}^{-\infty} d\mathbf{u}\frac{\mathbf{M}(\mathbf{u},\mathbf{r})\mathbf{M}(-\mathbf{u},\mathbf{r}')}{(u^2-k_0^2)k_z^2}$$
$$= \iiint_{\infty}^{\infty} d\mathbf{k}\frac{\mathbf{M}(\mathbf{k},\mathbf{r})\mathbf{M}(-\mathbf{k},\mathbf{r}')}{(k^2-k_0^2)k_z^2}. \tag{15}$$

(ii) Cylindrical Coordinates:

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dk_z \int_0^{\infty} k_\rho dk_\rho \frac{\mathbf{M}_{-n}(-k_\rho,-k_z,\mathbf{r})\mathbf{M}_n(k_\rho,k_z,\mathbf{r}')}{k_\rho^2(k^2-k_0^2)}$$
$$= \sum_{n=\infty}^{-\infty}\int_{\infty}^{-\infty}(-dk_z)\int_0^{\infty} -k_\rho d{-k_\rho}\frac{\mathbf{M}_n(k_\rho,k_z,\mathbf{r})\mathbf{M}_{-n}(-k_\rho,-k_z,\mathbf{r}')}{k_\rho^2(k^2-k_0^2)}$$
$$= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk_\rho k_\rho \frac{\mathbf{M}_n(k_\rho,k_z,\mathbf{r})\mathbf{M}_{-n}(-k_\rho,-k_z,\mathbf{r}')}{k_\rho^2(k^2-k_0^2)}. \tag{16}$$

(iii) Spherical Coordinates:

$$\sum_{m=-n}^{m} \mathbf{M}_{n,-m}(\mathbf{r})\mathbf{M}_{nm}(\mathbf{r}') = \sum_{m=n}^{-n} \mathbf{M}_{nm}(k,\mathbf{r})\mathbf{M}_{n,-m}(k,\mathbf{r}')$$
$$= \sum_{m=-n}^{n} \mathbf{M}_{nm}(k,\mathbf{r})\mathbf{M}_{n,-m}(k,\mathbf{r}'). \tag{17}$$

Thus, we showed that (14) is true in three kinds of coordinate respectiviely. Combine (14) and (13), we see that

$$\Big[\sum_a \mathbf{M}_a(\mathbf{r}')\mathbf{M}_{-a}(\mathbf{r})\Big]^t = \sum_a \mathbf{M}_a(\mathbf{r})\mathbf{M}_{-a}(\mathbf{r}').$$

As stated previousely, $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ is constructed by the superposition of the terms $\mathbf{M}_a(\mathbf{r})$, $\mathbf{M}_{-a}(\mathbf{r}')$. Hence, we conclude that

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = [\overline{\mathbf{G}}(\mathbf{r}',\mathbf{r})]^t,$$

provided $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ is given by (7.3.5), (7.3.7), and (7.3.8).

§7.16

By definition,

$$\begin{aligned}\mathbf{N}(k,\mathbf{r}) = \frac{1}{k}\nabla\times\mathbf{M}(\mathbf{k},\mathbf{r}) &= \frac{1}{k}\nabla\times\nabla\times\hat{z}\psi(\mathbf{k},\mathbf{r})\\ &= \frac{1}{k}[ik_z\nabla\psi(\mathbf{k},\mathbf{r}) - \hat{z}\nabla^2\psi(\mathbf{k},\mathbf{r})]\\ &= \frac{ik_z}{k}\mathbf{L}(\mathbf{k},\mathbf{r}) + k\hat{z}\psi(\mathbf{k},\mathbf{r}).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{F}(\mathbf{k},\mathbf{r}) \triangleq \frac{\mathbf{N}(\mathbf{k},\mathbf{r})\mathbf{N}(-\mathbf{k},\mathbf{r}')}{(k^2-k_0^2)k_s^2} &= \frac{k_z^2}{k_2(k^2-k_0^2)k_s^2}\mathbf{L}(\mathbf{k},\mathbf{r})\mathbf{L}(-\mathbf{k},\mathbf{r}')\\ &+ \frac{-k^2}{(k^2-k_0^2)k_s^2}\hat{z}\hat{z}\psi(\mathbf{k},\mathbf{r})\psi(-\mathbf{k},\mathbf{r}').\end{aligned}$$

As a function of k_z, $\mathbf{F}(\mathbf{k},\mathbf{r})$ has poles at $k=\pm k_0$ and at $k=0$. We are going to consider the contribution of the pole at $k=0$:

$$I_1 = \oint_{(k=0)} \mathbf{F}(\mathbf{k},\mathbf{r})dk_z = \frac{\pi}{k_0^2k_s}\mathbf{L}(iks,\mathbf{r})\mathbf{L}(-iks,\mathbf{r}'). \tag{1}$$

On the other hand,

$$\begin{aligned}I_2 = \oint_{(k=0)} &\left[\frac{\mathbf{L}(\mathbf{k},\mathbf{r})\mathbf{L}(-\mathbf{k},\mathbf{r}')}{k_0^2k^2} - \hat{z}\hat{z}\frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}}{k^2}\right]\\ &= \frac{\pi}{k_0^2k_s}\mathbf{L}(iks,\mathbf{r})\mathbf{L}(-iks,\mathbf{r}').\end{aligned} \tag{2}$$

Thus, $I_1 - I_2 = 0$.

§7.17

(a) First, assume $z > z'$. Then,

$$\psi(\mathbf{k},\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z},$$

$$\begin{aligned}\mathbf{M}(\mathbf{k},\mathbf{r}) &= \nabla\times[\hat{z}\psi(\mathbf{k},\mathbf{r})] = (i\mathbf{k}\times\hat{z})\psi(\mathbf{k},\mathbf{r}),\\ \mathbf{M}(-\mathbf{k},\mathbf{r}') &= (-i\mathbf{k}\times\hat{z})\psi(-\mathbf{k},\mathbf{r}').\end{aligned}$$

Consequently, we have

$$\begin{aligned}\mathbf{M}(\mathbf{k},\mathbf{r})\mathbf{M}(-\mathbf{k},\mathbf{r}') =&(i\mathbf{k}\times\hat{z})(-i\mathbf{k}\times\hat{z})\psi(\mathbf{k},\mathbf{r})\psi(-\mathbf{k},\mathbf{r}')\\ =&(k_y^2\hat{x}\hat{x}+k_x^2\hat{y}\hat{y}-k_xk_y\hat{x}\hat{y}-k_yk_x\hat{y}\hat{x})\\ &\psi(\mathbf{k},\mathbf{r})\psi(-\mathbf{k},\mathbf{r}'),\end{aligned} \tag{1}$$

$$\begin{aligned}\mathbf{N}(\mathbf{k},\mathbf{r}) = \frac{1}{k}\nabla\times\mathbf{M}(\mathbf{k},\mathbf{r}) &= \frac{1}{k}i\mathbf{k}\times(i\mathbf{k}\times\hat{z})\psi(\mathbf{k},\mathbf{r})\\ &= -\frac{1}{k}(k_{0z}\mathbf{k}-k^2\hat{z})\psi(\mathbf{k},\mathbf{r}),\end{aligned}$$

$$\begin{aligned}\mathbf{N}(\mathbf{k},\mathbf{r})\mathbf{N}(-\mathbf{k},\mathbf{r}') &= \frac{1}{k^2}(k_{0z}\mathbf{k}-k^2\hat{z})(k_{0z}\mathbf{k}-k^2\hat{z})\psi(\mathbf{k},\mathbf{r})\psi(-\mathbf{k},\mathbf{r}')\\ &= \left[\frac{k_{0z}^2}{k^2}\mathbf{k}\mathbf{k}-k_{0z}\mathbf{k}\hat{z}-k_{0z}\hat{z}\mathbf{k}+k^2\hat{z}\hat{z}\right]\psi(\mathbf{k},\mathbf{r})\psi(-\mathbf{k},\mathbf{r}'),\end{aligned} \tag{2}$$

and

$$\begin{aligned}&\mathbf{M}(\mathbf{k},\mathbf{r})\mathbf{M}(-\mathbf{k},\mathbf{r}')+\mathbf{N}(\mathbf{k},\mathbf{r})\mathbf{N}(-\mathbf{k},\mathbf{r}')\\ &\quad=[k_y^2\hat{x}\hat{x}+k_x^2\hat{y}\hat{y}-k_xk_y\hat{x}\hat{y}-k_yk_x\hat{y}\hat{x}+\frac{k_{0z}^2}{k^2}\mathbf{k}\mathbf{k}\\ &\qquad-k_{0z}\mathbf{k}\hat{z}-k_{0z}\hat{z}\mathbf{k}+k^2\hat{z}\hat{z}]\psi(\mathbf{k},\mathbf{r})\psi(-\mathbf{k},\mathbf{r}')\\ &\quad=\{\hat{x}\hat{x}(-k_y^2-\frac{k_{0z}^2}{k^2}k_x^2)+\hat{x}\hat{y}(-k_xk_y+k_xk_y\frac{k_{0z}^2}{k^2})+\hat{x}\hat{z}(-k_xk_z+k_xk_z\frac{k_{0z}^2}{k^2})\\ &\qquad+\hat{y}\hat{x}(-k_yk_x-k_yk_x\frac{k_{0z}^2}{k^2})+\hat{y}\hat{y}(k^2-\frac{k_{0z}^2}{k^2}k_y^2)+\hat{y}\hat{z}(-k_yk_z+k_yk_z\frac{k_{0z}^2}{k^2})\\ &\qquad+\hat{z}\hat{x}(-k_{0z}k_x+k_{0z}k_x\frac{k_{0z}^2}{k^2})+\hat{z}\hat{y}(-k_{0z}k_y)(1-\frac{k_{0z}^2}{k^2})\\ &\qquad+\hat{z}\hat{z}+(\frac{k_{0z}^4}{k^2}-2k_{0z}^2+k^2)\}\psi(\mathbf{k},\mathbf{r})\psi(-\mathbf{k},\mathbf{r}')\\ &\quad=(k_s^2\overline{\mathbf{I}}-\frac{k_s^2}{k^2}\mathbf{k}\mathbf{k})\psi(\mathbf{k},\mathbf{r})\psi(-\mathbf{k},\mathbf{r}')\\ &\quad=\frac{k_s^2(k^2\overline{\mathbf{I}}-\mathbf{k}\mathbf{k})}{k^2}e^{i\mathbf{k}_s\cdot(\mathbf{r}-\mathbf{r}')+ik_{0z}(z-z')},\quad z>z',\end{aligned} \tag{3}$$

where $\mathbf{k}_s=\hat{x}k_x+k_y\hat{y}$, $k_{0z}=\sqrt{k^2-k_s^2}$. For $z<z'$, we can show that

$$\mathbf{M}(\mathbf{k},\mathbf{r})\mathbf{M}(-\mathbf{k},\mathbf{r}')=\frac{k_s^2}{k^2}(k^2\overline{\mathbf{I}}-\mathbf{k}\mathbf{k})e^{i\mathbf{k}_s\cdot(\mathbf{r}-\mathbf{r}')-ik_{0z}(z-z')},\quad z<z'. \tag{4}$$

Combine (3) and (4), we obtain

$$\frac{1}{k_{0z}k_s^2}\{\mathbf{M}(\mathbf{k},\mathbf{r})\mathbf{M}(-\mathbf{k},\mathbf{r})+\mathbf{N}(\mathbf{k},\mathbf{r})\mathbf{N}(-\mathbf{k},\mathbf{r})\}$$
$$=\frac{k^2\overline{\mathbf{I}}-\mathbf{k}\mathbf{k}}{k_{0z}k^2}e^{i\mathbf{k}_s\cdot(\mathbf{r}-\mathbf{r}')+ik_{0z}(z-z')}. \tag{5}$$

This is the integrand of (7.1.54) where

$$\mathbf{k}=\hat{x}k_x+\hat{y}k_y+\hat{z}k_{0z}\quad \text{sign}\quad (z-z'),$$
$$k_{0z}^2=k^2-k_s^2.$$

(b) First, we have

$$\nabla\cdot\mathbf{E}(\mathbf{r})=i\omega\mu\int\nabla\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')dr.$$

But

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\frac{i}{8\pi^2}\int\int_{-\infty}^{\infty}d\mathbf{k}_s e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}_s')+ik_{0z}|z-z'|}\left[\frac{\overline{\mathbf{I}}k_0^2-\mathbf{k}_0\mathbf{k}_0}{k_0^2k_{0z}}\right]-\frac{\hat{z}\hat{z}}{k_0^2}\delta(\mathbf{r}-\mathbf{r}'),$$

$$\mathbf{k}_s=\hat{x}k_x+\hat{y}k_y, d\mathbf{k}_s=dk_xdk_y, k_{0z}=\sqrt{k_0^2-k_x^2-k_y^2}, \mathbf{r}_s=\hat{x}x+\hat{y}y,$$

and

$$\mathbf{k}_0=\begin{cases}\mathbf{k}_s+\hat{z}k_{0z}, z-z'>0\\ \mathbf{k}_y-\hat{z}k_0z, z-z'<0,\end{cases}$$

is a function of z. Therefore,

$$\nabla\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\frac{i}{8\pi}\int\int_{-\infty}^{\infty}dk_s e^{i\mathbf{k}_0\cdot\mathbf{r}}\frac{-\frac{\partial}{\partial z}\hat{z}\cdot\mathbf{k}_0\mathbf{k}_0}{k_0^2k_{0z}}-\frac{\hat{z}}{k_0^2}\frac{\partial}{\partial z}\delta(\mathbf{r}-\mathbf{r}').$$

But

$$\hat{z}\cdot\frac{\partial}{\partial z}\mathbf{k}_0\mathbf{k}_0=2\hat{z}\cdot[\mathbf{k}_0\hat{z}\delta(z-z')+\hat{z}\mathbf{k}_0\delta(z-z')]k_{0z}$$
$$=2k_{0z}[\pm k_{0z}\hat{z}+\mathbf{k}_0]\delta(z-z')$$

or

$$\nabla\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\frac{i}{4\pi^2}\int\int_{-\infty}^{\infty}d\mathbf{k}_s e^{i\mathbf{k}_s\cdot\mathbf{r}_s}\frac{(\pm k_{0z}\hat{z}+\mathbf{k}_0)}{k_0^2}\delta(z-z')-\frac{\hat{z}}{k_0^2}\frac{\partial}{\partial z}\delta(\mathbf{r}-\mathbf{r}')$$
$$=\frac{-i}{4\pi k_0^2}\int\int_{-\infty}^{\infty}dk_s e^{i\mathbf{k}_s\cdot\mathbf{r}_s}\mathbf{k}_s\delta(z-z')-\frac{\hat{z}}{k_0^2}\frac{\partial}{\partial z}\delta(\mathbf{r}-\mathbf{r}')$$
$$=-\frac{\nabla_s}{k_0^2}\delta(\mathbf{r}-\mathbf{r}')-\frac{\hat{z}}{k_0^2}\frac{\partial}{\partial z}\delta(\mathbf{r}-\mathbf{r}')=-\frac{1}{k_0^2}\nabla\delta(\mathbf{r}-\mathbf{r}').$$

Therefore,

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{i\omega\epsilon} \int d\mathbf{r} \nabla\delta(\mathbf{r}-\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')$$

$$= \frac{\nabla \cdot \mathbf{J}(\mathbf{r})}{i\omega\epsilon} = \frac{\rho(\mathbf{r})}{\epsilon}.$$

§7.18

(a) We have

$$I_1 = P.V.\frac{1}{2}\int_{-\infty}^{\infty} dk_\rho \frac{J_n(k_\rho\rho)H_n^{(1)}(k_\rho\rho)}{k_\rho(k^2-k_0^2)}. \tag{1}$$

In the integral, there are poles at $k_\rho = \pm k_{0\rho}$ and $k_\rho = 0$. In this problem, we will find the residue at $k_\rho = 0$:

$$I_{10} = \frac{1}{2}\oint_{k_\rho=0} \frac{J_n(k_\rho\rho)H_n^{(1)}(k_\rho\rho')}{k_\rho(k^2-k_0^2)} dk_\rho = \frac{1}{2}2\pi i \lim_{k_\rho\to 0} \frac{J_n(k_\rho\rho)H_n^{(1)}(k_\rho\rho')}{k_\rho^2+k_z^2-k_0^2)}.$$

At $k_\rho = 0$,

$$k_\rho\rho \to 0, k_\rho\rho' \to 0,$$
$$J_n(k_\rho\rho) \to \frac{1}{\Gamma(n+1)}\left(\frac{k_\rho\rho}{2}\right)^n, H_n^{(1)}(k_\rho\rho') \to -i\frac{\Gamma(n)}{\pi}\left(\frac{k_\rho\rho'}{2}\right)^{-n}.$$

Hence,

$$J_n(k_\rho\rho)H_n^{(1)}(k_\rho\rho') \to -\frac{i}{\pi}\frac{\Gamma(n)}{\Gamma(n+1)}\left(\frac{\rho}{\rho'}\right)^n, \quad (n \neq 0).$$

Therefore,

$$I_{10} = \pi i \lim_{k_\rho\to 0} \frac{1}{2}\frac{J_n(k_\rho\rho)H_n^{(1)}(k_\rho\rho')}{k_\rho(k^2-k_0^2)} dk_\rho = \frac{1}{2}2\pi i \lim_{k_\rho\to 0} \frac{J_n(k_\rho\rho)H_n^{(1)}(k_\rho\rho')}{k_\rho^2+k_z^2-k_0^2)}$$
$$= \pi i \frac{1}{k_z^2-k_0^2}\frac{i}{n\pi}\left(\frac{\rho}{\rho'}\right)^n = -\frac{1}{nk_{0\rho}^2}\left(\frac{\rho}{\rho'}\right)^n.$$

Note that I_{10} is the residue of the integrand at $k_\rho = 0$, but we are only integrating on the half circle. Hence the contribution of the pole at $k_\rho = 0$ is only half of I_{10} or

$$\frac{1}{2}I_{10} = -\frac{1}{2nk_{0\rho}^2}\left(\frac{\rho}{\rho'}\right)^n. \tag{2}$$

This is the last term on the right hand side of (7.3.19).

(b) We consider I_1 first. In part (a) of this problem, we have derived the contribution from the pole at $k_\rho = 0$. Now we consider the pole at $k_\rho = \pm k_{\rho 0}$. For simplicity, we express I as

$$I_1 = I_{11} + \frac{1}{2} I_{10} \tag{3}$$

where $\frac{1}{2}I_{10}$ is given by (2), and

$$I_{11} = \frac{1}{2} \int_c dk_\rho \frac{J_n(k_\rho \rho) H_n^{(1)}(k_\rho \rho')}{k_\rho (k_\rho^2 - k_{0\rho}^2)}. \tag{4}$$

In (4), we have replaced $k^2 - k_0^2$ by $k_\rho^2 - k_{0\rho}^2$. In practical cases, k_0 has an imaginary part due to loss in the media. Hence $\pm k_{0\rho}$ will be off the real k_ρ axis a little, as shown in Figure 7.3.2 in the book. As a result, we expect to close the integral path. In doing so, we have to consider the behaviour of the integrand at $k_\rho \to \infty$. When $k_\rho \to \infty$,

$$J_n(k_\rho \rho) \to \frac{1}{2} \sqrt{\frac{2}{\pi k_\rho \rho}} \left[e^{i(k_\rho \rho - \frac{2n+1}{4}\pi)} + e^{-i(k_\rho \rho - \frac{2n+1}{4}\pi)} \right],$$

$$H_n^{(1)}(k_\rho \rho) \to \sqrt{\frac{2}{\pi k_\rho \rho}} e^{i(k_\rho \rho' - \frac{2n+1}{4}\pi)}.$$

Hence,

$$J_n(k_\rho \rho) \cdot H_n^{(1)}(k_\rho \rho') \to \frac{1}{\pi k_\rho \sqrt{\rho \rho'}} \left[a_n e^{ik_\rho(\rho + \rho')} + b_n e^{ik_\rho(\rho' - \rho)} \right]$$

where $a_n = e^{i\frac{2n+1}{2}\pi}, b = 1$. Since $\rho > 0$, $\rho' > 0$, the term $e^{ik_\rho(\rho+\rho')} \to 0$ when $k_\rho \to \infty$. Another term, $e^{ik_\rho(\rho'-\rho)}$ will be dependent on the sign of $(\rho' - \rho)$. It is easy to see that when $\rho' - \rho > 0$ or $\rho' > \rho$, we can close the path by adding an integral along the upper infinite half circle without changing the value of I_{11}. Now the pole $k_\rho = k_{0\rho}$ lies inside the closed path, and residue theorem gives

$$\begin{aligned} I_{11} &= \frac{1}{2} \cdot 2\pi i \lim_{k_\rho \to k_{0\rho}} (k_\rho - k_{0\rho}) \frac{J_n(k_\rho \rho) H_n^{(1)}}{k_\rho^2 - k_{0\rho}^2} \\ &= \frac{\pi i}{2k_{0\rho}} J_n(k_{0\rho\rho}) H_n^{(1)}(k_{0\rho}\rho), \quad \rho' > \rho. \end{aligned} \tag{5}$$

Thus

$$\begin{aligned} I_1 &= I_{11} + \frac{1}{2} I_{10} \\ &= \frac{\pi i}{2k_{0\rho}} J_n(k_{0\rho}\rho) H_n^{(1)}(k_{0\rho}\rho') - \frac{1}{2nk_{0\rho}^2} \left(\frac{\rho}{\rho'}\right), \quad \rho' > \rho. \end{aligned} \tag{6}$$

For the case that $\rho' < \rho$, since the integrand of I_1 is symetric for ρ and ρ' [see Equation (7.3.16a)], we can do the same analysis by change the position of ρ and ρ' in each equation to get

$$I_1 = \frac{\pi i}{2k_0\rho} J_n(k_{0\rho}\rho')H_n^{(1)}(k_{0\rho}\rho) - \frac{1}{2nk_{0\rho}^2}\left(\frac{\rho'}{\rho}\right)^n, \quad \rho' < \rho. \tag{7}$$

If we denote $\rho_> = \max(\rho, \rho')$ and $\rho_< = \min(\rho, \rho')$, then we can combine (6) and (7) into one expression

$$I_1 = \frac{\pi i}{2k_0\rho} J_n(k_{0\rho}\rho_<)H_n^{(1)}(k_{0\rho}^{\rho_>}) - \frac{1}{2nk_{0\rho}^2}\left(\frac{\rho_<}{\rho_>}\right)^n, \quad n \neq 0. \tag{8}$$

The reason for the requirment of $n \neq 0$ is that, when $n = 0$, the Hankel function has another form of small argument approximation. Now, we consider I_2:

$$I_2 = \int_0^\infty dk_\rho \frac{J_n(k_\rho\rho)J_n(k_\rho\rho')}{k_\rho k^2(k^2 - k_0^2)}. \tag{9}$$

Comparing with I_1 (see 7.3.16a), we find that we can almost do the same analysis to I_1, except that we have one more pole at $k_\rho = \pm ik_z$. The contribution from pole at $k_\rho = ikz$ is

$$\begin{aligned} I_{22} &\triangleq \frac{1}{2} \oint_{k_\rho = ik_z} \frac{J_n(k_\rho\rho_<)H_n^{(1)}(k_\rho\rho_>)}{k_\rho k^2(k^2 - k_0^2)} dk_\rho \\ &= \frac{1}{2}(2\pi i) \lim_{k_\rho \to ik_z} (k_\rho - ikz) \frac{J_n(k_\rho\rho_<)H_n^{(1)}(k_\rho\rho_>)}{k_\rho(k_\rho + ik_z)(k_\rho - ik_z)(k_\rho^2 - k_{0\rho}^2)} \\ &= \frac{\pi i}{2k_z^2 k_0^2} J_n(ik_z\rho_<)H_n^{(1)}(ik_z\rho_>). \end{aligned} \tag{10}$$

Thus

$$\begin{aligned} I_2 = \frac{\pi i}{2k_o^2\rho} J_n(k_{0\rho}\rho_<)H_n^{(1)}(k_{0\rho}\rho_>) &+ \frac{\pi i}{2k_z^2 k_\rho^2} J_n(ik_z\rho_<)H_n^{(1)}(ik_z\rho_>) \\ &- \frac{1}{2nk_z^2 k_\rho^2}\left(\frac{\rho_<}{\rho_>}\right)^n, \quad \rho \neq 0. \end{aligned} \tag{11}$$

Finally, as for I_3, we only have pole at $k_\rho = \pm ik_z$. In similar manner as deriving Equation (10), we can find

$$I_3 = \frac{\pi i}{2k_0^2} J_n(ik_z\rho_<)H_n^{(1)}(ik_z p_>). \tag{12}$$

§7.19

(a) From(7.3.7)

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dk_z \int_0^{\infty} k_\rho dk_\rho \Big[\frac{\mathbf{M}_n(k_\rho,k_z,\mathbf{r})\mathbf{M}_{-n}(-k_\rho,-k_z,\mathbf{r}')}{(k^2-k_0^2)k_\rho^2}$$
$$+\frac{\mathbf{N}_n(k_\rho,k_z,\mathbf{r})\mathbf{N}_{-n}(-k_\rho,-k_z,\mathbf{r}')}{(k^2-k_0^2)k_\rho^2} - \frac{\mathbf{L}_n(k_\rho,k_z,\mathbf{r})\mathbf{L}_{-n}(-k_\rho,-k_z,\mathbf{r}')}{(k^2-k_0^2)k_\rho^2}\Big]. \quad (1)$$

Since

$$\mathbf{L}_n(k_\rho,k_z,\mathbf{r}) \to \hat{\rho} i k_\rho \psi_n(k_\rho,k_z,\mathbf{r}), \quad \text{when} \quad k_\rho \to \infty, \quad (2)$$

the term involving $\mathbf{L}_n$ in the acove integrand will tend to constant when $k_\rho \to \infty$ (in this case, $k^2 \to k_\rho^2$ for fixed k_z). In order to perform the integral over k_ρ, we first extract the singularity in the integrand. The resulting expression for $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ becomes

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dk_z \int_0^{\infty} k_\rho dk_\rho \Big[\frac{\mathbf{M}_n(k_\rho,k_z,\mathbf{r})\mathbf{M}_{-n}(-k_\rho,-k_z,\mathbf{r}')}{(k^2-k_0^2)k_\rho^2}$$
$$+\frac{\mathbf{N}_n(k_\rho,k_z,\mathbf{r})\mathbf{N}_{-n}(-k_\rho,-k_z,\mathbf{r}')}{(k^2-k_0^2)k_\rho^2} - \Big(\frac{\mathbf{L}_n(k_\rho,k_z,\mathbf{r})\mathbf{L}_{-n}(-k_\rho,-k_z,\mathbf{r}')}{(k^2-k_0^2)k_\rho^2}$$
$$-\frac{\hat{\rho}\hat{\rho}}{k_0^2}\psi_n(k_\rho,k_z,\mathbf{r})\psi_{-n}(-k_\rho,-k_z,\mathbf{r}')\Big)\Big]$$
$$-\frac{1}{(2\pi)^2}\sum_{n=-m}^{\infty}\int_{-\infty}^{\infty} dk_z \int_0^{\infty} dk_\rho k_\rho \frac{\hat{\rho}\hat{\rho}}{k_0^2}\psi_n(k_\rho,k_z,\mathbf{r})\psi_{-n}(-k_\rho,-k_z,\mathbf{r}'). \quad (3)$$

The last term in (3) is easily identified to be

$$\frac{\hat{\rho}\hat{\rho}}{-k_0^2}\delta(\mathbf{r}-\mathbf{r}'). \quad (4)$$

By definition,

$$\mathbf{M}_n(k_\rho,k_z,\mathbf{r}) = \nabla\times[\hat{z}\psi_n(k_\rho,k_z,\mathbf{r})] = (\nabla\times\hat{z})\psi_n(k_\rho,k_z,\mathbf{r}) \quad (5)$$
$$\mathbf{N}_n(k_\rho,k_z,\mathbf{r}) = \frac{1}{k}\nabla\times\nabla\times[\hat{z}\psi_n] = (\nabla\times\nabla\times\hat{z})\psi_n(k_\rho,k_z,\mathbf{r}) \quad (6)$$
$$\mathbf{L}_n(k_\rho,k_z,\mathbf{r}) = \nabla\psi_n(k_\rho,k_z,\mathbf{r}).$$

The differential operator is with respect to the space coordinate $\mathbf{r}$ and $\mathbf{r}'$, while the integration is over the spectral variables. Hence, we can change the order of differentiation and integration. At this point, we are only interested in the integration of k_ρ. We can see that the integrateion of k_ρ

can be classified into the following three types:

$$I_1 = \int_0^\infty dk_\rho \frac{J_n(k_\rho\rho')J_n(k_\rho\rho)}{k_\rho(k^2 - k_0^2)}, \tag{7a}$$

$$I_2 = \int_0^\infty dk_\rho \frac{J_n(k_\rho\rho')J_n(k_\rho\rho)}{k_\rho k^2(k^2 - k_0^2)}, \tag{7b}$$

$$I_3 = \int_0^\infty dk_\rho k_\rho \frac{J_n(k_\rho\rho')J_n(k_\rho\rho)}{k^2 k_0^2}. \tag{7c}$$

In Exercise (7.18), we have derived the closed form expression for I_1, I_2, and I_3 [see (7.3.21), (7.3.22) and (7.3.23)]. Now, we can substitute the closed form of I_1, I_2, and I_3 into (3). Before doing further derivation, we note that the intagrand involving $\mathbf{N}_n$ has two terms. One of them is

$$(ik_z\nabla\psi_n)(-ik_z\nabla'\psi(k_\rho, k_z, \mathbf{r}') = k_z^2\mathbf{L}_n\mathbf{L}_{-n}. \tag{8}$$

The integration of this term will be cancelled by the third term in (3). To see this, we note that the integration of (8) corresponds to the multiplication of k_z^2 to the second term in (7.3.22), i.e.,

$$k_z^2 \cdot \left[\frac{\pi i}{k_z^2 k^2} J_n(ik_z\rho_<)H_n^{(1)}(ik_z\rho_>)\right]$$
$$\frac{\pi i}{2k^2} J_n(ik_z\rho_<)H_n^{(1)}(ik_z\rho_>) = I_3. \tag{9}$$

Hence, after this cancellation, it will not appear in $\overline{\mathbf{G}}$. Now I_1 and I_2 are expressed in $J_n(k_{0\rho}\rho_<)H_n^{(1)}(k_{0\rho}\rho_>)$. In order to write $\overline{\mathbf{G}}$ in a compact form, we will define a new form $\mathbf{M}_n$ and $\mathbf{N}_n$. That is to say, we replace Bessel function in $\mathbf{M}_n$ and $\mathbf{N}_n$ by Hankel function if $\rho > \rho'$. Similarly, we replace Bessel functions in $\mathbf{M}_{-n}$ and $\mathbf{N}_{-n}$ by Hankel function if $\rho < \rho'$. After this replacement, we can see

$$\psi_n(k_{0\rho}, k_z, \mathbf{r})\psi_n(k_{0\rho}, k_z, \mathbf{r}) = J_n(k_{0\rho}\rho_<)H_n^{(1)}(k_{0\rho}\rho_>)e^{ik_z(z-z')+in(\phi-\phi')}.$$

This is to say that we can replace

$$J_n(k_\rho\rho)J_n(k_\rho\rho')$$

in (3) by

$$J_n(k_{0\rho}\rho_<)H_n^{(1)}(k_{0\rho}\rho_>),$$

to get the same form (accordingly, some coefficients will change). The coefficients for $\mathbf{M}_n\mathbf{M}_{-n}$ is

$$\left(\frac{1}{2\pi}\right)^2 \cdot \frac{1}{2} \cdot \frac{\pi i}{k_{0\rho}^2} = \frac{i}{8\pi}\frac{1}{k_{0\rho}^2}.$$

The coefficients for $\mathbf{N}_n\mathbf{N}_{-n}$ is

$$\left(\frac{1}{2\pi}\right)^2 k_0^2 \left(\frac{1}{2}\cdot\frac{\pi i}{k_0^2 k_{0\rho}^2}\right) = \frac{1}{8\pi}\frac{1}{k_{0\rho}^2}.$$

Thus,

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{i}{8\pi}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dk_z \frac{1}{k_{0\rho}^2}[\mathbf{M}_n(k_{0\rho},k_z,\mathbf{r})\mathbf{M}_{-n}(-k_{0\rho},-k_z,\mathbf{r}')$$
$$+ \mathbf{N}_n(k_{0\rho},k_z,\mathbf{r})\mathbf{N}_{-n}(k_{0\rho},-k_z,\mathbf{r}')] - \frac{\hat{\rho}\hat{\rho}}{k_0^2}\delta(\mathbf{r},\mathbf{r}').$$

(b) For $n=0$, the expressions for I_1 and I_2 [Equations (7.3.21) and (7.3.22)] is not valid. The problem is caused by the nonintegrable pole at $k_\rho = 0$. In fact, this is caused by interchanging the order of differentiation and integration. If we use the integrand of I_1 and I_2 by $\frac{\partial}{\partial\rho'}$ or $\frac{\partial}{\partial\rho}$, then $J_n(k_\rho\rho)$ will be changed to $k_\rho J_n'(k_\rho\rho)$. The factor k_ρ will cancel the $k_\rho = 0$ pole. This is to say that $k_\rho = 0$ is not a pole when $n=0$. In substituting I_1 and I_2 in the expression for $\overline{\mathbf{G}}$, just ignore the last term in (7.3.21) and (7.3.22).

§7.20

(a) For spherical coordinates, the first term in (7.3.8) has the form (7.3.32)

$$I_1 = \int_0^\infty dk\; k^2 \frac{j_n(kr)j_n(kr')}{k^2 - k_0^2}.$$

Use the identity $j_n(x) = \frac{1}{2}[h_n^{(1)}(x) + h_n^{(2)}(x)]$ and two terms arise. In the second term we change variables such that $k' = -k$. The identities $h_n^{(2)}(-x) = (-1)^n h_n^{(1)}(x)$ and $j_n(-x) = (-1)^n j_n(x)$ are used to unfold the integral

$$I_1 = \frac{1}{2}\int_{-\infty}^{\infty} dk\; k^2 \frac{j_n(kr)h_n^{(1)}(kr')}{k^2 - k_0^2}.$$

Notice that poles exist at $\pm k_0$ which are on the real axis for lossless media. But with the introduction of a small loss, these poles are displaced from the real axis as shown in Figure 7.3.1 (replace k_z with k) and the integral is well defined. When $k \to \infty$ we can use the large argument approximations to show

$$j_n(kr)h_n^{(1)}(kr') \sim \frac{\cos(kr - \frac{\pi}{2}(n+1))}{kr}\cdot\frac{e^{ikr' - i\frac{\pi}{2}(n+1)}}{kr'}, \quad k\to\infty$$
$$\sim a\frac{e^{ik(r'+r)}}{k^2 rr'} + b\frac{e^{ik(r'-r)}}{k^2 rr'}, \qquad k\to\infty.$$

Thus, for $r' > r$ the integrand becomes exponentially small in the upper half-plane of complex k-space. Consequently, by virtue of Jordan's lemma and Cauchy's theorem, the integral along the real k-axis can be deformed as in Figure 7.3.1 and then vanishes, leaving a residue contribution at $k = k_0$. Hence

$$I_1 = 2\pi i \ \mathrm{Res}[\mathcal{I}_1(k_0)],$$

where "Res" stands for "residue of" and $\mathcal{I}_1$ is the integrand of I_1. We can simplify

$$\begin{aligned} I_1 &= 2\pi i \frac{1}{2} k_0^2 \lim_{k \to k_0} (k - k_0) \frac{j_n(kr) h_n^{(1)}(kr')}{k^2 - k_0^2}, \\ &= \frac{\pi i}{2} k_0 \ j_n(k_0 r) h_n^{(1)}(k_0 r'), \quad r' > r. \end{aligned}$$

When $r > r'$ we can use a similar technique to show

$$I_1 = \frac{\pi i}{2} k_0 \ j_n(k_0 r_<) h_n^{(1)}(k_0 r_>),$$

where $r_>$ is the larger of r and r' and $r_<$ is the smaller of r and r'. The second term in (7.3.8) has the form (7.3.33)

$$\begin{aligned} I_2 &= \int_0^\infty dk \frac{j_n(kr) j_n(kr')}{k^2 - k_0^2}, \\ &= \frac{I_1}{k_0^2} + \int_0^\infty dk \left(1 - \frac{k^2}{k_0^2}\right) \frac{j_n(kr) j_n(kr')}{k^2 - k_0^2}, \\ &= \frac{I_1}{k_0^2} - \frac{1}{k_0^2} \int_0^\infty dk \ j_n(kr) j_n(kr'), \\ &= \frac{I_1}{k_0^2} - \frac{\pi}{2k_0^2} \int_0^\infty dk \frac{J_{n+\frac{1}{2}}(kr) J_{n+\frac{1}{2}}(kr')}{k r^{\frac{1}{2}} r'^{\frac{1}{2}}}. \end{aligned}$$

The second term in I_2 above can be found in closed form [Abramowitz and Stegun, 1965]

$$\int_0^\infty dk \frac{J_v(kr) J_v(kr')}{k} = \frac{1}{2v} \left(\frac{r_<}{r_>}\right)^v.$$

So, we arrive at

$$I_2 = \frac{I_1}{k_0^2} - \frac{\pi}{2(2n+1)k_0^2} \frac{r_<^n}{r_>^{n+1}}.$$

The third term in (7.3.8) is (7.3.34)

$$I_3 = \int_0^\infty dk \frac{j_n(kr)j_n(kr')}{k_0^2},$$

and we have already shown this above to be

$$I_3 = \frac{\pi}{2(2n+1)k_0^2} \frac{r_<^n}{r_>^{n+1}}.$$

(b) The spherical dyadic Green's function (7.3.8) has a term due to the irrotational vector wave function $\mathbf{L}(\mathbf{r})$ which can be written

$$\overline{\mathbf{G}}_L(\mathbf{r},\mathbf{r}') \triangleq \frac{-2}{\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \int_0^\infty dk\ k^2 \frac{\mathbf{L}_{nm}(k,\mathbf{r})\mathbf{L}_{n,-m}(k,\mathbf{r}')}{k_0^2},$$

$$= \frac{-2}{\pi} \frac{1}{k_0^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \int_0^\infty dk \nabla j_n(kr) Y_{nm}(\theta,\phi) \nabla' j_n(kr') Y_{n,-m}(\theta',\phi').$$

After exchanging the order of integration and differentiation,

$$\overline{\mathbf{G}}_L(\mathbf{r},\mathbf{r}') = \frac{-2}{\pi} \frac{1}{k_0^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \nabla\nabla' Y_{nm}(\theta,\phi) Y_{n,-m}(\theta',\phi') \int_0^\infty dk\ I(k),$$

where the integrand is

$$I(k) = \left[j_n(kr)j_n(kr') - \frac{\cos(kr-\theta_n)}{kr} \frac{\cos(kr'-\theta_n)}{kr'} \right] + \frac{\cos(kr-\theta_n)}{kr} \frac{\cos(kr'-\theta_n)}{kr'},$$

where $\theta_n = \frac{\pi}{2}(n+1)$. The third term above arises from the large argument asymtote of the spherical Bessel function. Notice that the terms between the brackets of $I(k)$ cancel each other as $k \to \infty$ and is analytic everywhere in k-space. By Cauchy's theorem and Jordan's Lemma, this term is equal to zero. The remaining term in $I(k)$ can be expanded such that

$$I(k) \sim \frac{e^{ik(r+r')}}{4k^2rr'} + \frac{e^{-ik(r+r')}}{4k^2rr'} + \frac{e^{ik(r-r')}}{4k^2rr'} + \frac{e^{ik(r'-r)}}{4k^2rr'}, \qquad k \to \infty$$

However, the stationary phase point occurs at

$$I(k) \sim \frac{e^{ik(r-r')}}{4k^2rr'} + \frac{e^{ik(r'-r)}}{4k^2rr'}, \qquad k \to \infty$$

$$\sim \frac{\cos(k(r-r'))}{2k^2rr'}, \qquad k \to \infty$$

Next, we unfold (2.2.6) to show

$$\delta(r-r') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ik(r-r')},$$
$$= \frac{1}{\pi}\int_{0}^{\infty} dk\;\cos(k(r-r')).$$

We can use the above Fourier expansion of the Dirac delta function to show

$$\nabla\nabla'\int_0^\infty dk\frac{\cos(k(r-r'))}{2k^2rr'} \sim \int_0^\infty dk\nabla_r\nabla'_r\frac{\cos(k(r-r'))}{2k^2rr'},$$
$$\sim \hat{r}\hat{r}\int_0^\infty dk\frac{\cos(k(r-r'))}{2rr'},$$
$$\sim \frac{\pi}{2}\frac{\hat{r}\hat{r}}{rr'}\delta(r-r').$$

Notice that the above derivation has replaced $\nabla\nabla'$ by $\nabla_r\nabla'_r$. This is justified by the $\frac{1}{k^2}$ dependence in the integrand. Each factor of ∇_r was shown to create a k factor, canceling the $\frac{1}{k^2}$ dependence. This term dominates as $k \to \infty$ because ∇_θ and ∇_ϕ are both $O(k^0)$. After collecting together the above information we show

$$\overline{\mathbf{G}}_L(\mathbf{r},\mathbf{r}') \sim -\frac{1}{k_0^2}\delta(r-r')\frac{\hat{r}\hat{r}}{rr'}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y_{nm}(\theta,\phi)Y_{n,-m}(\theta',\phi').$$

Then we use the completeness relation (Wyld, Mathematical Methods for Physics, p. 99)

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y_{nm}(\theta,\phi)Y^*_{n,m}(\theta',\phi') = \frac{\delta(\theta-\theta')\delta(\phi-\phi')}{\sin\theta'},$$

and $Y_{n,-m}(\theta,\phi) = Y^*_{nm}(\theta,\phi)$ (see Exercise 7.14) to arrive at

$$\overline{\mathbf{G}}_L(\mathbf{r},\mathbf{r}') \sim -\frac{\hat{r}\hat{r}}{k_0^2}\frac{\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')}{rr'\sin\theta'},$$
$$\sim -\frac{\hat{r}\hat{r}}{k_0^2}\delta(\mathbf{r}-\mathbf{r}').$$

Finally, we use (7.3.35) and (7.3.36) to write the dyadic Green's function as

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = ik_0\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{n(n+1)}[\mathbf{M}_{nm}(k_0,\mathbf{r})\mathbf{M}_{n,-m}(k_0,\mathbf{r}')$$
$$+\mathbf{N}_{nm}(k_0,\mathbf{r})\mathbf{N}_{n,-m}(k_0,\mathbf{r}')] - \frac{\hat{r}\hat{r}}{k_0^2}\delta(\mathbf{r}-\mathbf{r}'),$$

where we define

$$\mathbf{M}_{nm}(k_0,\mathbf{r}) = \nabla \times \hat{r}\; j_n(kr)Y_{nm}(\theta,\phi),$$
$$\mathbf{N}_{nm}(k_0,\mathbf{r}) = \frac{1}{k}\nabla \times \nabla \times \hat{r}\; j_n(kr)Y_{nm}(\theta,\phi),$$

and the spherical Bessel function $j_n(kr)$ should be replaced by a spherical Hankel function $h_n^{(1)}(kr)$ for the greater of r and r'. Notice that the static term in (7.3.36) does not contribute because it arises as a residue near $k = 0$. Similarly, I_3 has a residue at $k = 0$ given by (7.3.37) that will cancel the second term in (7.3.36). On the other hand, the Dirac delta function arises from I_3 due to the "high frequency" stationary phase point as $k \to \infty$. One can see that I_2 does not have a stationary phase point as $k \to \infty$ because the integrand varies as $\frac{k^2}{k^2-k_0^2}\frac{\cos(k(r-r'))}{rr'}$, altering magnitude of the spectral content.

§7.21

(a) First, we use the vector identity (A.7)

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B},$$

to show

$$\nabla\cdot[(\mu^{-1}(\mathbf{r})\nabla\times\mathbf{E}(\mathbf{r}))\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') + \mathbf{E}(\mathbf{r})\times(\mu^{-1}(\mathbf{r})\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}'))]$$
$$= \nabla\times\mu^{-1}(\mathbf{r})\nabla\times\mathbf{E}(\mathbf{r})\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') - \mathbf{E}(\mathbf{r})\cdot\nabla\times\mu^{-1}(\mathbf{r})\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}').$$

Then we integrate (7.4.3) over all space $\mathbf{r}$

$$\int_V d\mathbf{r}\nabla\cdot\Big[(\mu^{-1}(\mathbf{r})\nabla\times\mathbf{E}(\mathbf{r}))\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a}$$
$$+\mathbf{E}(\mathbf{r})\times(\mu^{-1}(\mathbf{r})\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}'))\cdot\mathbf{a}\Big]$$
$$= i\omega\int_V d\mathbf{r}\;\overline{\mathbf{J}}(\mathbf{r})\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a} - \mu^{-1}(\mathbf{r}')\mathbf{E}(\mathbf{r}')\cdot\mathbf{a}.$$

Next we invoke the divergence theorem $\int_V d\mathbf{r}\nabla\cdot\mathbf{A} = \oint_S dS\hat{n}\cdot\mathbf{A}$

$$\oint_{S_\infty} dS\;\hat{n}\cdot\Big[(\mu^{-1}(\mathbf{r})\nabla\times\mathbf{E}(\mathbf{r}))\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a}$$
$$+\mathbf{E}(\mathbf{r})\times(\mu^{-1}(\mathbf{r})\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}'))\cdot\mathbf{a}\Big]$$
$$= i\omega\int_V d\mathbf{r}\;\overline{\mathbf{J}}(\mathbf{r})\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a} - \mu^{-1}(\mathbf{r}')\mathbf{E}(\mathbf{r}')\cdot\mathbf{a}.$$

At $S_\infty, \nabla \to \hat{r}ik$ since all fields look like plane waves. Thus the surface integral vanishes by virtue of the radiation condition. So we have

$$\mathbf{E}(\mathbf{r}') = i\omega\mu(\mathbf{r}') \int_V d\mathbf{r}\; \mathbf{J}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}'), \quad \text{for } \mathbf{r} \neq \mathbf{r}'.$$

(b) The reciprocity theorem states

$$\langle \mathbf{E}_1, \mathbf{J}_2 \rangle = \langle \mathbf{E}_2, \mathbf{J}_1 \rangle,$$

where $\mathbf{J}_1$ and $\mathbf{J}_2$ are the sources at $\mathbf{r}$ and $\mathbf{r}'$ and $\mathbf{E}_1$ and $\mathbf{E}_2$ are the measured fields at $\mathbf{r}$ and $\mathbf{r}'$, respectively. Using (7.4.4) and the definition of the inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \int d\mathbf{r} \mathbf{A} \cdot \mathbf{B}$ we find

$$\int d\mathbf{r}' \Big(i\omega\mu(\mathbf{r}') \int d\mathbf{r} \mathbf{J}_1(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \Big) \cdot \mathbf{J}_2(\mathbf{r}')$$
$$= \int d\mathbf{r} \Big(i\omega\mu(\mathbf{r}) \int d\mathbf{r}' \mathbf{J}_2(\mathbf{r}') \cdot \overline{\mathbf{G}}(\mathbf{r}', \mathbf{r}) \Big) \cdot \mathbf{J}_1(\mathbf{r}),$$

$$\int d\mathbf{r}' \int d\mathbf{r}\; i\omega\mu(\mathbf{r}') \mathbf{J}_1(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_2(\mathbf{r}')$$
$$= \int d\mathbf{r} \int d\mathbf{r}'\; i\omega\mu(\mathbf{r}) \mathbf{J}_1(\mathbf{r}) \cdot \overline{\mathbf{G}}^t(\mathbf{r}', \mathbf{r}) \cdot \mathbf{J}_2(\mathbf{r}'),$$

where t denotes transpose. By uniqueness, we find that the following identity must be satisfied

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')\mu(\mathbf{r}') = \overline{\mathbf{G}}^t(\mathbf{r}', \mathbf{r})\mu(\mathbf{r})$$

§7.22

Maxwell's curl equations in source-free regions are

$$\nabla \times \mathbf{E} = i\omega\mathbf{B},$$
$$\nabla \times \mathbf{H} = -i\omega\mathbf{D}.$$

By writing

$$\mathbf{D} = \mathbf{D}_s + \hat{z}\mathbf{D}_z,$$
$$\mathbf{B} = \mathbf{B}_s + \hat{z}\mathbf{B}_z.$$

and letting $\nabla = \nabla_s + \hat{z}\frac{\partial}{\partial z}$, we find that the $\hat{z}$ component of Maxwell's curl equations are

$$\hat{z} \cdot \nabla_s \times \mathbf{E} = i\omega B_z,$$
$$\hat{z} \cdot \nabla_s \times \mathbf{H} = -i\omega D_z.$$

We can use the following identity

$$\nabla_s \times \mathbf{E} = \nabla_s \times \hat{z}E_z + \hat{z}\hat{z}\cdot\nabla_s \times \mathbf{E},$$

and the constitutive relation to show

$$\begin{aligned}\nabla_s \times \mathbf{E} &= \nabla \times \hat{z}\epsilon^{-1}D_z + i\omega\hat{z}B_z,\\ &= i\omega\Big[\frac{1}{i\omega\epsilon}\nabla \times \hat{z}D_z + \hat{z}B_z\Big].\end{aligned}$$

Then, we operate $(\nabla\times)$ on the above equation to show

$$\nabla \times \nabla_s \times \mathbf{E} = i\omega\Big[\frac{1}{i\omega\epsilon}\nabla \times \nabla \times \hat{z}D_z + \nabla \times \hat{z}B_z\Big].$$

At this point we note $\nabla \times \nabla_s \times \mathbf{E}_s = \nabla_s \times \nabla_s \times \mathbf{E}_s$. Also, for plane waves $\nabla_s \to i\mathbf{k}_s$ and $i\mathbf{k}_s \times i\mathbf{k}_s \times \mathbf{E} = k_s^2\mathbf{E}$. Finally,

$$\mathbf{E} = \frac{i\omega}{k_s^2}\left[\frac{1}{i\omega\epsilon}\nabla \times \nabla \times \hat{z}D_z + \nabla \times \hat{z}B_z\right].$$

§7.23

By comparing (7.4.8a) and (7.4.8b) to (7.4.9) one notices (8a) is due to a field generated by B_z, while (8b) is due to a field generated by D_z. Hence $\overline{\mathbf{M}}(\mathbf{k}_s,\mathbf{r},\mathbf{r}')$ and $\overline{\mathbf{N}}(\mathbf{k}_s,\mathbf{r},\mathbf{r}')$ are the TE and TM fields in cartesian coordinates. For this reason, the $\hat{z}$ propagation in layered media is given by $F_\pm(z,z')$ in Section 2.4. The TE and TM forms differ only by reflection coefficients (e.g. R_{ij}^{TM} versus R_{ij}^{TE}).

However, in the transverse directions, the medium appears homogeneous so the propagation is described merely by $e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)}$. Furthermore, orthogonal dyads are necessary in each layer of the stratified medium.

For these reasons we can write the vector wave functions as

$$\begin{aligned}\overline{\mathbf{M}}(\mathbf{k}_s,\mathbf{r},\mathbf{r}') &= (\nabla \times \hat{z})(\nabla' \times \hat{z})e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)}F_\pm^{TE}(z,z'),\\ \overline{\mathbf{N}}(\mathbf{k}_s,\mathbf{r},\mathbf{r}') &= \left(\frac{\nabla \times \nabla \times \hat{z}}{i\omega\epsilon_n}\right)\left(\frac{\nabla' \times \nabla' \times \hat{z}}{-i\omega\mu_m}\right)e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)}F_\pm^{TM}(z,z').\end{aligned}$$

Note that ϵ_n and μ_m are in the denominators of the first two factors of $\mathbf{N}(\mathbf{k}_s,\mathbf{r},\mathbf{r}')$ because the source at $\mathbf{r}'$ is in region m and the field at $\mathbf{r}$ is in region n.

§7.24

Given the integrand of the homogeneous dyadic Green's function in cylindrical coordinates

$$\overline{\mathbf{C}}_n(k_z,\mathbf{r},\mathbf{r}') = \frac{1}{k_\rho^2}[\mathbf{N}_n(k_\rho,k_z,\mathbf{r}),\mathbf{M}_n(k_\rho,k_z,\mathbf{r})]\cdot\begin{bmatrix}\mathbf{N}_{-n}(-k_\rho,-k_z,\mathbf{r}')\\ \mathbf{M}_{-n}(-k_\rho,-k_z,\mathbf{r}')\end{bmatrix}.$$

Using the vector wave functions (7.2.33), we can write

$$\overline{\mathbf{C}}_n(k_z,\mathbf{r},\mathbf{r}') = \frac{1}{k_\rho^2}\left[\frac{1}{k}\nabla\times\nabla\times\hat{z}, \nabla\times\hat{z}\right]\cdot\begin{bmatrix}\frac{1}{k}\nabla'\times\nabla'\times\hat{z}\\ \nabla'\times\hat{z}\end{bmatrix}$$
$$\cdot\,\psi_n(k_\rho,k_z,\mathbf{r})\psi_{-n}(-k_\rho,-k_z,\mathbf{r}').$$

Note that the Bessel functions above are to be replaced by Hankel functions for $\rho_>$ (the larger of ρ and ρ'). This is because the "static" term has been cancelled in (7.3.26). With the aid of the identity $J_{-n}(-x) = J_n(x)$, we can show

$$\psi_n(k_\rho,k_z,\mathbf{r})\psi_{-n}(-k_\rho,-k_z,\mathbf{r}') = J_n(k_\rho\rho_<)H_n^{(1)}(k_\rho\rho_>)e^{in(\phi-\phi')+ik_z(z-z')}.$$

Consequently, we can write

$$\overline{\mathbf{C}}_n(k_z,\mathbf{r},\mathbf{r}') = \frac{1}{(kk_\rho)^2}\overline{\mathbf{D}}_\mu J_n(k_\rho\rho_<)H_n^{(1)}(k_\rho\rho_>)e^{in(\phi-\phi')+ik_z(z-z')}\overleftarrow{\overline{\mathbf{D}}}'^{\,t}_\epsilon,$$

where the operators are defined by

$$\overline{\mathbf{D}}_\mu = [\nabla\times\nabla\times\hat{z}, i\omega\mu\nabla\times\hat{z}],$$
$$\overleftarrow{\overline{\mathbf{D}}}'_\epsilon = [\nabla'\times\nabla'\times\hat{z}, -i\omega\epsilon\nabla'\times\hat{z}].$$

§7.25

(a) For planarly layered media, the dyadic Green's function is given by (7.4.14)

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{i}{8\pi^2}\iint_{-\infty}^{\infty}\frac{d\mathbf{k}_s}{k_{mz}k_s^2}\left[\overline{\mathbf{M}}(\mathbf{k}_s,\mathbf{r},\mathbf{r}') + \overline{\mathbf{N}}(\mathbf{k}_s,\mathbf{r},\mathbf{r}')\right] - \frac{\hat{z}\hat{z}}{k_m^2}\delta(\mathbf{r}-\mathbf{r}'),$$

for $\mathbf{r}' \in$ region m and $\mathbf{r}' \in$ region n. According to (7.4.5), reciprocity implies $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\mu(\mathbf{r}') = \overline{\mathbf{G}}^t(\mathbf{r}',\mathbf{r})\mu(\mathbf{r})$ (Exercise 7.21). So for this case

$$\mu_m\iint_{-\infty}^{\infty}\frac{dks}{k_{mz}k_s^2}[\overline{\mathbf{M}}(\mathbf{k}_s,\mathbf{r},\mathbf{r}') + \overline{\mathbf{N}}(\mathbf{k}_s,\mathbf{r},\mathbf{r}')] - \frac{\hat{z}\hat{z}}{k_m^2}\mu_m\delta(\mathbf{r}-\mathbf{r}')$$
$$= \mu_n\iint_{-\infty}^{\infty}\frac{d\mathbf{k}_s}{k_{nz}k_s^2}\left[\overline{\mathbf{M}}^t(\mathbf{k}_s,\mathbf{r}',\mathbf{r}) + \overline{\mathbf{N}}^t(\mathbf{k}_s,\mathbf{r}',\mathbf{r})\right] - \frac{\hat{z}\hat{z}}{k_n^2}\mu_n\delta(\mathbf{r}-\mathbf{r}'),$$

where t denotes transpose. Notice that the singularity terms are non-zero only at $\mathbf{r} = \mathbf{r}'$ where $n = m$ and these terms cancel each other. Due to the orthogonality of $\overline{\mathbf{M}}$ and $\overline{\mathbf{N}}$, the remaining equation is true when

$$\overline{\mathbf{M}}(\mathbf{k}_s,\mathbf{r},\mathbf{r}')\frac{\mu_m}{k_{mz}} = \overline{\mathbf{M}}^t(\mathbf{k}_s,\mathbf{r}',\mathbf{r})\frac{\mu_n}{k_{nz}},$$
$$\overline{\mathbf{N}}(\mathbf{k}_s,\mathbf{r},\mathbf{r}')\frac{\mu_m}{k_{mz}} = \overline{\mathbf{N}}^t(\mathbf{k}_s,\mathbf{r}',\mathbf{r})\frac{\mu_n}{k_{nz}}.$$

From the definitions in (7.4.12) and (7.4.13) we find

$$\frac{\mu_m}{k_{mz}}(\nabla\times\hat{z})(\nabla'\times\hat{z})e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)}F^{TE}_{\pm}(z,z')$$
$$=\frac{\mu_n}{k_{nz}}[(\nabla'\times\hat{z})(\nabla\times\hat{z})]^t e^{-i\mathbf{k}_s\cdot(\mathbf{r}'_s-\mathbf{r}_s)}(F^{TE}_{\pm}(z',z))^*,$$

$$\frac{\mu_m}{k_{mz}}\left(\frac{\nabla\times\nabla\times\hat{z}}{i\omega\epsilon_n}\right)\left(\frac{\nabla'\times\nabla'\times\hat{z}}{-i\omega\mu_m}\right)e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)}F^{TM}_{\pm}(z,z')$$
$$=\frac{\mu_n}{k_{nz}}\left[\left(\frac{\nabla'\times\nabla'\times\hat{z}}{i\omega\epsilon_m}\right)\left(\frac{\nabla\times\nabla\times\hat{z}}{-i\omega\mu_n}\right)\right]^t e^{-i\mathbf{k}_s\cdot(\mathbf{r}'_s-\mathbf{r}_s)}(F^{TM}_{\pm}(z',z))^*.$$

Now we observe $(F_{\pm}(z',z))^*=F_{\mp}(z',z)$ because $F(z,z')\sim e^{ik_z|z-z'|}$. So we find

$$\frac{\mu_m}{k_{mz}}F^{TE}_{\pm}(z,z')=\frac{\mu_n}{k_{nz}}F^{TE}_{\mp}(z',z),$$
$$\frac{\epsilon_m}{k_{mz}}F^{TM}_{\pm}(z,z')=\frac{\epsilon_n}{k_{nz}}F^{TM}_{\mp}(z',z),$$

where the $+$ case is used when $z>z'$ and the $-$ case is used when $z<z'$. When n and m are two different regions, we can use (2.4.15) and (2.4.16) to show that the above equations reduce to

$$\frac{\mu_m}{k_{mz}}\tilde{T}^{TE}_{mn}=\frac{\mu_n}{k_{nz}}\tilde{T}^{TE}_{nm},$$
$$\frac{\epsilon_m}{k_{mz}}\tilde{T}^{TM}_{mn}=\frac{\epsilon_n}{k_{nz}}\tilde{T}^{TM}_{nm}.$$

(b) For cylindrically layered media the dyadic Green's function is given by (7.4.24)

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\frac{i}{8\pi}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dk_z\frac{1}{(k_jk_{i\rho})^2}\overline{\mathbf{D}}_{i\mu}\cdot\overline{\mathbf{F}}_n(\rho,\rho')f_n(z,\phi;z',\phi')$$
$$\cdot\overleftarrow{\overline{\mathbf{D}}}'^{\,t}_{j\epsilon}-\frac{\hat{\rho}\hat{\rho}}{k_j^2}\delta(\mathbf{r}-\mathbf{r}'),$$

for $\mathbf{r}'\in$ region j and $\mathbf{r}\in$ region i. According to the reciprocity relation (7.4.5), we know

$$\mu_j\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dk_z\frac{1}{(k_jk_{i\rho})^2}\overline{\mathbf{D}}_{i\mu}\cdot\overline{\mathbf{F}}_n(\rho,\rho')f_n(z,\phi;z',\phi')\cdot\overleftarrow{\overline{\mathbf{D}}}'^{\,t}_{j\epsilon}-\frac{\hat{\rho}\hat{\rho}}{k_j^2}\mu_j\delta(\mathbf{r}-\mathbf{r}')$$
$$=\mu_i\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dk_z\frac{1}{(k_ik_{j\rho})^2}\left[\overline{\mathbf{D}}'_{j\mu}\cdot\overline{\mathbf{F}}_n(\rho',\rho)f_n(z',\phi';z,\phi)\cdot\overleftarrow{\overline{\mathbf{D}}}^{\,t}_{i\epsilon}\right]^t$$
$$-\frac{\hat{\rho}\hat{\rho}}{k_i^2}\mu_i\delta(\mathbf{r}-\mathbf{r}').$$

Once again the singularities cancel each other at $\mathbf{r} = \mathbf{r}'$ (where $i = j$). By othogonality of $e^{in\phi}$ and $e^{im\phi}$, the remaining terms are equal when

$$\frac{\mu_j}{(k_j k_{i\rho})^2}\overline{\mathbf{D}}_{i\mu} \cdot \overline{\mathbf{F}}_n(\rho,\rho') f_n(z,\phi;z',\phi') \cdot \overleftarrow{\overline{\mathbf{D}}}'^{t}_{j\epsilon}$$
$$= \frac{\mu_i}{(k_i k_{j\rho})^2}\overline{\mathbf{D}}_{i\epsilon} \cdot \overline{\mathbf{F}}^t_n(\rho',\rho) f^*_n(z',\phi';z,\phi) \cdot \overleftarrow{\overline{\mathbf{D}}}'^{t}_{j\mu}.$$

Since $f_n(z,\phi;z',\phi') = e^{in(\phi-\phi')+ik_z(z-z')}$, we know $f_n(z,\phi;z',\phi') = f^*_n(z',\phi';z,\phi)$. So

$$\frac{\mu_j}{(k_j k_{i\rho})^2}\overline{\mathbf{D}}_{i\mu} \cdot \overline{\mathbf{F}}_n(\rho,\rho') \cdot \overleftarrow{\overline{\mathbf{D}}}'^{t}_{j\epsilon} = \frac{\mu_i}{(k_i k_{j\rho})^2}\overline{\mathbf{D}}_{i\epsilon} \cdot \overline{\mathbf{F}}^t_n(\rho',\rho) \cdot \overleftarrow{\overline{\mathbf{D}}}'^{t}_{j\mu}.$$

The operator $\overline{\mathbf{D}}_\mu = [\nabla\times\nabla\times\hat{z}, i\omega\mu\nabla\times\hat{z}]$ acts on $\overline{\mathbf{F}}_n(\rho,\rho')$, which involves Bessel and Hankel functions $B_n(k_{i\rho}\rho)$. In general,

$$\begin{aligned}\nabla\times\nabla\times\hat{z}B_n(k_{i\rho}\rho) &= -\hat{z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\rho^2}\right)B_n(k_{i\rho}\rho),\\ &= \hat{z}\left[k^2_{i\rho} - \frac{n(n-1)}{\rho^2}\right]B_n(k_{i\rho}\rho),\\ \nabla\times\hat{z}B_n(k_{i\rho}\rho) &= -\hat{\phi}\frac{\partial}{\partial\rho}B_n(k_{i\rho}\rho).\end{aligned}$$

Since the TM to z waves and the TE to z waves are orthogonal when acted upon by $\overline{\mathbf{D}}_{i\mu}$, the two waves decouple. The two waves can be written

$$\frac{\mu_j}{k_j^2 k_{i\rho}^2}\hat{z}\hat{z}\left(k^2_{i\rho} - \frac{n(n-1)}{\rho^2}\right)\left(k^2_{j\rho} - \frac{n(n-1)}{\rho'^2}\right)\overline{\mathbf{F}}_n(\rho,\rho')$$
$$= \frac{\mu_i}{k_i^2 k_{j\rho}^2}\hat{z}\hat{z}\left(k^2_{i\rho} - \frac{n(n-1)}{\rho^2}\right)\left(k^2_{j\rho} - \frac{n(n-1)}{\rho'^2}\right)\overline{\mathbf{F}}^t_n(\rho',\rho),$$

$$-\frac{\mu_j}{k_j^2 k_{i\rho}^2}\hat{\phi}\hat{\phi}\omega^2\mu_i\epsilon_j\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho'}\overline{\mathbf{F}}_n(\rho,\rho') = -\frac{\mu_i}{k_i^2 k_{j\rho}^2}\hat{\phi}\hat{\phi}\omega^2\mu_j\epsilon_i\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho'}\overline{\mathbf{F}}^t_n(\rho',\rho).$$

Note that $\overline{\mathbf{F}}_n(\rho,\rho')$ is smooth with respect to ρ and ρ', so if $\frac{\partial}{\partial\rho}\overline{\mathbf{F}}_n(\rho,\rho')$ is continuous, then $\overline{\mathbf{F}}_n(\rho,\rho')$ is continuous. We can condense these two equations with new notation

$$\frac{\mu_j}{k_j^2 k_{i\rho}^2}\overline{\boldsymbol{\mu}}_i \cdot \overline{\mathbf{F}}_n(\rho,\rho') \cdot \overline{\epsilon}_j = \frac{\mu_i}{k_i^2 k_{j\rho}^2}\overline{\epsilon}_i \cdot \overline{\mathbf{F}}^t_n(\rho',\rho) \cdot \overline{\boldsymbol{\mu}}_j,$$

where

$$\overline{\boldsymbol{\mu}}_i = \begin{bmatrix}1 & 0\\ 0 & \mu_i\end{bmatrix}, \overline{\epsilon}_j = \begin{bmatrix}1 & 0\\ 0 & -\epsilon_j\end{bmatrix}.$$

(c) For spherically layered media, the dyadic Green's function is given by (7.4.32)

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = ik_j \sum_{n=0}^{\infty} \frac{1}{n(n+1)} [\overline{\mathbf{m}}_n(\mathbf{r},\mathbf{r}') + \overline{\mathbf{n}}_n(\mathbf{r},\mathbf{r}')] - \frac{\hat{r}\hat{r}}{k_j^2}\delta(\mathbf{r}-\mathbf{r}'),$$

where $\mathbf{r} \in$ region i and $\mathbf{r}' \in$ region j. According to the reciprocity relation (7.4.5), we know

$$\mu_j k_j \sum_{n=0}^{\infty} \frac{1}{n(n+1)} [\mathbf{m}_n(\mathbf{r},\mathbf{r}') + \mathbf{n}_n(\mathbf{r},\mathbf{r}')] - \mu_j \frac{\hat{r}\hat{r}}{k_j^2}\delta(\mathbf{r}-\mathbf{r}')$$
$$= \mu_i k_i \sum_{n=0}^{\infty} \frac{1}{n(n+1)} [\mathbf{m}_n^t(\mathbf{r}',\mathbf{r}) + \mathbf{n}_n^t(\mathbf{r}',\mathbf{r})] - \mu_i \frac{\hat{r}\hat{r}}{k_i^2}\delta(\mathbf{r}-\mathbf{r}').$$

The singularities cancel each other at $\mathbf{r} = \mathbf{r}'$ (where $i = j$). By orthogonality of $\mathbf{m}_n$ and $\mathbf{n}_n$, the remaining terms are equal when

$$\mathbf{m}_n(\mathbf{r},\mathbf{r}')k_j\mu_j = \mathbf{m}_n^t(\mathbf{r}',\mathbf{r})k_i\mu_i,$$
$$\mathbf{n}_n(\mathbf{r},\mathbf{r}')k_j\mu_j = \mathbf{n}_n^t(\mathbf{r}',\mathbf{r})k_i\mu_i.$$

From the definitions (7.4.30) and (7.4.31), we find

$$(\nabla \times \mathbf{r})(\nabla' \times \mathbf{r}')F_n^{TE}(r,r')A_n(\theta,\phi;\theta',\phi')k_j\mu_j$$
$$= [(\nabla' \times \mathbf{r}')(\nabla \times \mathbf{r})]^t F_n^{TE}(r',r)A_n^*(\theta',\phi';\theta,\phi)k_i\mu_i,$$

$$\left(\frac{\nabla \times \nabla \times \mathbf{r}}{-i\omega\epsilon_i}\right)\left(\frac{\nabla' \times \nabla' \times \mathbf{r}'}{i\omega\mu_j}\right)F_n^{TM}(r,r')A_n(\theta,\phi;\theta',\phi')k_j\mu_j$$
$$= \left[\left(\frac{\nabla' \times \nabla' \times \mathbf{r}'}{-i\omega\epsilon_j}\right)\left(\frac{\nabla \times \nabla \times \mathbf{r}}{i\omega\mu_i}\right)\right]^t F_n^{TM}(r',r)A_n^*(\theta',\phi';\theta,\phi)k_i\mu_i.$$

Clearly $A_n(\theta,\phi;\theta',\phi') = A_n^*(\theta',\phi';\theta,\phi)$ from (7.4.25c) and Exercise 7.14. Now we can simplify

$$k_j\mu_j F_n^{TE}(r,r') = k_i\mu_i F_n^{TE}(r',r),$$
$$k_j\epsilon_j F_n^{TM}(r,r') = k_i\epsilon_i F_n^{TM}(r',r).$$

When i and j are two different regions, we use (3.7.27) and (3.7.28) to show that the above equations reduce to

$$k_j\mu_j \tilde{T}_{ji}^{TE} = k_i\mu_i \tilde{T}_{ij}^{TE},$$
$$k_j\epsilon_j \tilde{T}_{ji}^{TM} = k_i\epsilon_i \tilde{T}_{ij}^{TM}.$$

§7.26

(a) Given the cylindrical reflection matrix (7.4.38a)

$$\overline{\mathbf{r}}_{j,j\pm1} = \overline{\boldsymbol{\mu}}_j \cdot \overline{\mathbf{R}}_{j,j\pm1} \cdot \overline{\boldsymbol{\epsilon}}_j.$$

By examining $\overline{\mathbf{R}}_{j,j+1}$ in (3.1.11a) and $\overline{\mathbf{R}}_{j,j-1}$ in (3.1.17a), they both are the product of two symmetric matrices so they can be written in the form

$$\begin{aligned}\overline{\mathbf{R}}_{j,j\pm1} &= \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \cdot \begin{bmatrix} e & f \\ f & g \end{bmatrix}, \\ &= \frac{1}{ac-b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \cdot \begin{bmatrix} e & f \\ f & g \end{bmatrix}, \\ &= \frac{1}{ac-b^2}\begin{bmatrix} ce-bf & cf-bg \\ af-be & ag-bf \end{bmatrix}.\end{aligned}$$

In general, $\overline{\mathbf{R}}_{j,j\pm1}$ is not symmetric. In fact, for the case $\overline{\mathbf{R}}_{j,j+1}$, we can use (3.1.11) to specify

$$\begin{aligned}a &= i\omega\left[\frac{\epsilon_j}{k_{j\rho}}J_n'(k_{j\rho}a_j)H_n^{(1)}(k_{j+1,\rho}a_j) - \frac{\epsilon_{j+1}}{k_{j+1,\rho}}H_n^{(1)\prime}(k_{j+1,\rho}a_j)J_n(k_{j\rho}a_j)\right], \\ c &= -i\omega\left[\frac{\mu_j}{k_{j\rho}}J_n'(k_{j\rho}a_j)H_n^{(1)}(k_{j+1,\rho}a_j) - \frac{\mu_{j+1}}{k_{j+1,\rho}}H_n^{(1)\prime}(k_{j+1,\rho}a_j)J_n(k_{j\rho}a_j)\right], \\ b &= \frac{-n}{\rho}k_z\left[\frac{1}{k_{j\rho}^2}J_n(k_{j\rho}a_j)H_n^{(1)}(k_{j+1,\rho}a_j) - \frac{1}{k_{j+1,\rho}^2}H_n^{(1)}(k_{j+1,\rho}a_j)J_n(k_{j\rho}a_j)\right], \\ e &= i\omega\left[\frac{\epsilon_{j+1}}{k_{j+1,\rho}}H_n^{(1)\prime}(k_{j+1,\rho}a_j)H_n^{(1)}(k_{j\rho}a_j) - \frac{\epsilon_j}{k_{j\rho}}H_n^{(1)\prime}(k_{j\rho}a_j)H_n^{(1)}(k_{j+1,\rho}a_j)\right], \\ g &= -i\omega\left[\frac{\mu_{j+1}}{k_{j+1,\rho}}H_n^{(1)\prime}(k_{j+1,\rho}a_j)H_n^{(1)}(k_{j\rho}a_j) - \frac{\mu_j}{k_{j\rho}}H_n^{(1)\prime}(k_{j\rho}a_j)H_n^{(1)}(k_{j+1,\rho}a_j)\right], \\ f &= \frac{-n}{\rho}k_z\left[\frac{1}{k_{j+1,\rho}^2}H_n^{(1)}(k_{j+1,\rho}a_j)H_n^{(1)}(k_{j\rho}a_j) - \frac{1}{k_{j\rho}^2}H_n^{(1)}(k_{j\rho}a_j)H_n^{(1)}(k_{j+1,\rho}a_j)\right].\end{aligned}$$

However, for $\overline{\mathbf{r}}_{j,j\pm1}$, we can write

$$\begin{aligned}\overline{\mathbf{r}}_{j,j\pm1} &= \overline{\boldsymbol{\mu}}_j \cdot \overline{\mathbf{R}}_{j\cdot j\pm1} \cdot \overline{\boldsymbol{\epsilon}}_j, \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \mu_j \end{bmatrix} \cdot \frac{1}{ac-b^2}\begin{bmatrix} ce-bf & cf-bg \\ af-be & ag-bf \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -\epsilon_j \end{bmatrix}, \\ &= \frac{1}{ac-b^2}\begin{bmatrix} ce-bf & -\epsilon_j(cf-bg) \\ \mu_j(af-be) & -\mu_j\epsilon_j(ag-bf) \end{bmatrix}.\end{aligned}$$

In order to show that this reflection matrix is symmetric, we must show

$$\mu_j(af-be) = -\epsilon_j(cf-bg),$$

or equivalently

$$\mu_j(a) + \epsilon_j(c) = \frac{b}{f}[\mu_j(e) + \epsilon_j(g)].$$

For the case $\overline{\mathbf{r}}_{j,j+1}$, we can use the matrix elements defined above and we can show

$$\frac{b}{f} = -\frac{J_n(k_{j\rho}a_j)}{H_n^{(1)}(k_{j\rho}a_j)}$$

$$\mu_j(a) + \epsilon_j(c) = i\omega[-\mu_j\epsilon_{j+1} + \epsilon_j\mu_{j+1}]\frac{1}{k_{j+1,\rho}}H_n^{(1)\prime}(k_{j+1,\rho}a_j)J_n(k_{j\rho}a_j)$$

$$\mu_j(e) + \epsilon_j(g) = i\omega[\mu_j\epsilon_{j+1} - \epsilon_j\mu_{j+1}]\frac{1}{k_{j+1,\rho}}H_n^{(1)\prime}(k_{j+1,\rho}a_j)H_n^{(1)}(k_{j\rho}a_j)$$

Hence $\overline{\mathbf{r}}_{j,j+1}$ is symmetric. Similarly the case $\overline{\mathbf{r}}_{j,j-1}$ is symmetric, but to show this, one must redefine elements e, f, g from (3.1.17) and show that the off-diagonal terms are equal using the same steps above.

(b) Now examine $\overline{\mathbf{t}}_{j,j+1}$

$$\begin{aligned}
\overline{\mathbf{t}}_{j,j+1} &\triangleq \overline{\boldsymbol{\mu}}_{j+1}\cdot\overline{\mathbf{T}}_{j,j+1}\cdot\overline{\boldsymbol{\epsilon}}_j,\\
&= \begin{bmatrix}1 & 0\\ 0 & \mu_{j+1}\end{bmatrix}\cdot\frac{2\omega}{\pi k_{j\rho}^2 a_j}\overline{\mathbf{D}}^{-1}\cdot\begin{bmatrix}\epsilon_j & 0\\ 0 & -\mu_j\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 0 & -\epsilon_j\end{bmatrix},\\
&= \frac{2\omega}{\pi k_{j\rho}^2 a_j}\begin{bmatrix}1 & 0\\ 0 & \mu_{j+1}\end{bmatrix}\cdot\begin{bmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{bmatrix}\cdot\begin{bmatrix}\epsilon_j & 0\\ 0 & \mu_j\epsilon_j\end{bmatrix},\\
&= \frac{2\omega\epsilon_j}{\pi k_{j\rho} a_j}\begin{bmatrix}D_{11} & \mu_j D_{12}\\ \mu_{j+1}D_{21} & \mu_{j+1}\mu_j D_{22}\end{bmatrix}.
\end{aligned}$$

In the above derivation, we have used the definitions (7.4.38b) and (3.1.12). Next examine $\overline{\mathbf{t}}_{j+1,j}$

$$\begin{aligned}
\overline{\mathbf{t}}_{j+1,j} &\triangleq \overline{\boldsymbol{\mu}}_{j}\cdot\overline{\mathbf{T}}_{j+1,j}\cdot\overline{\boldsymbol{\epsilon}}_{j+1},\\
&= \begin{bmatrix}1 & 0\\ 0 & \mu_{j}\end{bmatrix}\cdot\frac{2\omega}{\pi k_{j+1,\rho}^2 a_j}\overline{\mathbf{D}}^{-1}\cdot\begin{bmatrix}\epsilon_{j+1} & 0\\ 0 & -\mu_{j+1}\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 0 & -\epsilon_{j+1}\end{bmatrix},\\
&= \frac{2\omega}{\pi k_{j+1,\rho}^2 a_j}\begin{bmatrix}1 & 0\\ 0 & \mu_{j}\end{bmatrix}\cdot\begin{bmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{bmatrix}\cdot\begin{bmatrix}\epsilon_{j+1} & 0\\ 0 & \mu_{j+1}\epsilon_{j+1}\end{bmatrix},\\
&= \frac{2\omega\epsilon_{j+1}}{\pi k_{j+1,\rho}^2 a_j}\begin{bmatrix}D_{11} & \mu_{j+1} D_{12}\\ \mu_{j}D_{21} & \mu_{j}\mu_{j+1} D_{22}\end{bmatrix}.
\end{aligned}$$

In the above derivation we used definitions (7.4.38c) and (3.1.17b). Also, if one examines the elements of $\overline{\mathbf{D}}$ given by (3.1.11c), one can see that $\overline{\mathbf{D}}$ is symmetric. So $\overline{\mathbf{D}}^{-1}$ is symmetric and hence $D_{12} = D_{21}$ as defined above. Then the following symmetry is elucidated

$$\frac{\epsilon_{j+1}}{k_{j+1,\rho}^2}\overline{\mathbf{t}}_{j,j+1} = \frac{\epsilon_j}{k_{j\rho}^2}\overline{\mathbf{t}}^t_{j+1,j}.$$

§7.27

From Chapter 3,

$$\tilde{\mathbf{R}}_{j,j+1} = \mathbf{R}_{j,j+1} + \mathbf{T}_{j+1,j}\cdot\tilde{\mathbf{R}}_{j+1,j+2}\cdot(\mathbf{I} - \mathbf{R}_{j+1,j}\cdot\tilde{\mathbf{R}}_{j+1,j+2})^{-1}\cdot\mathbf{T}_{j,j+1}. \quad (3.2.8)$$

Thus,

$$\tilde{\mathbf{r}}_{j,j+1} = \boldsymbol{\mu}_j \tilde{\mathbf{R}}_{j,j+1} \boldsymbol{\epsilon}_j = \mathbf{r}_{j,j+1} + (\mathbf{t}_{j+1,j} \boldsymbol{\epsilon}_{j+1}^{-1}) \cdot (\boldsymbol{\mu}_{j1}^{-1} \cdot \tilde{\mathbf{r}}_{j+1,j+2} \cdot \boldsymbol{\epsilon}_{j+1}^{-1}) \cdot [\mathbf{I} - (\boldsymbol{\mu}_{j+1}^{-1} \cdot \mathbf{r}_{j+1,j} \cdot \boldsymbol{\epsilon}_{j+1}^{-1}) \cdot (\boldsymbol{\mu}_{j+1}^{-1} \cdot \tilde{\mathbf{r}}_{j+1,j+2} \cdot \boldsymbol{\epsilon}_{j+1}^{-1})]^{-1} \cdot (\boldsymbol{\mu}_{j+1}^{-1} \cdot \mathbf{t}_{j,j+1}).$$

In the above, use has been made of the relationships in (7.4.38). The proof of the symmetry of $\tilde{\mathbf{r}}_{j,j+1}$ is by induction. Using the result of Exercise 7.26, $\tilde{\mathbf{r}}_{N-1,N} = \mathbf{r}_{N-1,N}$ is symmetric. To prove the general case, assume the symmetry of $\tilde{\mathbf{r}}_{j+1,j+2}$, and show that $\tilde{\mathbf{r}}_{j,j+1}$ must also be symmetric. Transposing the above expression for $\tilde{\mathbf{r}}_{j,j+1}$,

$$\tilde{\mathbf{r}}_{j,j+1}^t = \mathbf{r}_{j,j+1}^t + (\mathbf{t}_{j,j+1}^t \cdot \boldsymbol{\mu}_{j+1}^{-1}) \cdot [\mathbf{I} - (\boldsymbol{\epsilon}_{j+1}^{-1} \cdot \tilde{\mathbf{r}}_{j+1,j+2}^t \cdot \boldsymbol{\mu}_{j+1}^{-1}) \cdot (\boldsymbol{\epsilon}_{j+1}^{-1} \cdot \mathbf{r}_{j+1,j}^t \cdot \boldsymbol{\mu}_{j+1}^{-1})]^{-1} \cdot (\boldsymbol{\epsilon}_{j+1}^{-1} \cdot \tilde{\mathbf{r}}_{j+1,j+2}^t \cdot \boldsymbol{\mu}_{j+1}^{-1}) \cdot (\boldsymbol{\epsilon}_{j+1}^{-1} \mathbf{t}_{j+1,j}^{-t}).$$

Note that $\boldsymbol{\epsilon}^{-1}$ and $\boldsymbol{\mu}^{-1}$ are diagonal, and hence symmetric.

Now, using (7.4.39), the result of Exercise 7.2(b) and the induction hypothesis ($\tilde{\mathbf{r}}_{j+1,j+2}^t = \tilde{\mathbf{r}}_{j+1,j+2}$),

$$\begin{aligned}\tilde{\mathbf{r}}_{j,j+1}^t &= \mathbf{r}_{j,j+1} + (\mathbf{t}_{j+1,j} \cdot \boldsymbol{\mu}_{j+1}^{-1}) \cdot [\mathbf{I} - (\boldsymbol{\epsilon}_{j+1}^{-1} \cdot \tilde{\mathbf{r}}_{j+1,j+2}^t \cdot \boldsymbol{\mu}_{j+1}^{-1}) \\ &\quad \cdot (\boldsymbol{\epsilon}_{j+1}^{-1} \cdot \mathbf{r}_{j+1,j} \cdot \boldsymbol{\mu}_{j+1}^{-1})]^{-1} \cdot (\boldsymbol{\epsilon}_{j+1}^{-1} \cdot \tilde{\mathbf{r}}_{j+1,j+2}^t \cdot \boldsymbol{\mu}_{j+1}^{-1}) \cdot (\boldsymbol{\epsilon}_{j+1}^{-1} \mathbf{t}_{j,j+1}) \\ &= \boldsymbol{\mu}_j \cdot \mathbf{R}_{j,j+1} \boldsymbol{\epsilon}_j + (\boldsymbol{\mu}_j \cdot \mathbf{T}_{j+1,j} \cdot \boldsymbol{\mu}_{j+1}^{-1}) \cdot [\mathbf{I} - (\boldsymbol{\epsilon}_{j+1}^{-1} \cdot \boldsymbol{\mu}_{j+1} \cdot \tilde{\mathbf{R}}_{j+1,j+2} \cdot \boldsymbol{\epsilon}_{j+1} \cdot \\ &\quad \boldsymbol{\mu}_{j+1}^{-1}) \cdot (\boldsymbol{\epsilon}_{j+1}^{-1} \cdot \boldsymbol{\mu}_{j+1} \mathbf{R}_{j+1,j} \cdot \boldsymbol{\epsilon}_{j+1} \cdot \boldsymbol{\mu}_{j+1}^{-1})]^{-1} \\ &\quad \cdot (\boldsymbol{\epsilon}_{j+1}^{-1} \cdot \boldsymbol{\mu}_{j+1} \mathbf{R}_{j+1,j+2} \cdot \boldsymbol{\epsilon}_{j+1} \cdot \boldsymbol{\mu}_{j+1}^{-1}) \cdot (\boldsymbol{\epsilon}_{j+1}^{-1} \cdot \boldsymbol{\mu}_{j+1} \mathbf{T}_{j,j+1} \cdot \boldsymbol{\epsilon}_j).\end{aligned}$$

Defining $\mathbf{x} = \boldsymbol{\epsilon}_{j+1} \cdot \boldsymbol{\mu}_{j+1}^{-1}$,

$$\tilde{\mathbf{r}}_{j,j+1}^t = \boldsymbol{\mu}_j \mathbf{R}_{j,j+1} \cdot \boldsymbol{\epsilon}_j + \boldsymbol{\mu}_j \cdot \mathbf{T}_{j+1,j} \cdot \mathbf{x} \cdot [\mathbf{I} - (\mathbf{x}^{-1} \cdot \tilde{\mathbf{R}}_{j+1,j+2}) \cdot (\mathbf{R}_{j+1,j} \cdot \mathbf{x})]^{-1} \cdot (\mathbf{x}^{-1} \cdot \tilde{\mathbf{R}}_{j+1,j+2}) \cdot \mathbf{T}_{j,j+1} \cdot \boldsymbol{\epsilon}_j.$$

The fact that $\boldsymbol{\mu}_j, \boldsymbol{\epsilon}_j$, and their inverses are all diagonal and hence commute with one another has been used above.

At this point it is helpful to consider the following identity,

$$\begin{aligned}[I - \mathbf{AB}]^{-1} &= [A^{-1} - \mathbf{A}^{-1}\mathbf{B}^{-1}] = [\mathbf{A}^{-1} - \mathbf{B}]^{-1} = [\mathbf{A}^{-1} - \mathbf{BAA}^{-1}]^{-1} \\ &= \mathbf{A}[\mathbf{I} - \mathbf{BA}]^{-1}.\end{aligned}$$

Setting $\mathbf{A} = \mathbf{x}^{-1} \cdot \tilde{\mathbf{R}}_{j+1,j+2}, \mathbf{B} = \mathbf{R}_{j+1,j} \cdot \mathbf{x}$, yields

$$\begin{aligned}\tilde{\mathbf{r}}_{j,j+1} &= \boldsymbol{\mu}_j \cdot \mathbf{R}_{j,j+1} \cdot \boldsymbol{\epsilon}_j + \boldsymbol{\mu}_j \cdot \mathbf{T}_{j+1,j} \cdot \mathbf{x} \cdot \mathbf{x}^{-1} \tilde{\mathbf{R}}_{j+1,j+2} \cdot \\ &\quad [\mathbf{I} - \mathbf{R}_{j+1,j} \cdot \mathbf{x}\mathbf{x}^{-1} \tilde{\mathbf{R}}_{j+1,j+2}] \cdot \mathbf{T}_{j,j+1} \cdot \boldsymbol{\epsilon}_j \\ &= \boldsymbol{\mu}_j \cdot \{\mathbf{R}_{j,j+1} + \mathbf{T}_{j+1,j} \cdot \tilde{\mathbf{R}}_{j+1,j+2} \cdot [\mathbf{I} - \mathbf{R}_{j+1,j} \cdot \tilde{\mathbf{R}}_{j+1,j+2}]^{-1} \mathbf{T}_{j,j+1}\} \cdot \mathbf{t}_j \\ &= \tilde{\mathbf{r}}_{j,j+1}.\end{aligned}$$

A similar result can be proved for $\tilde{\mathbf{r}}_{j,j+1}$ using Equation (3.2.19) and the induction hypothesis $\tilde{\mathbf{r}}_{j-1,j-2} = \tilde{\mathbf{r}}_{j-1,j-2}$.

§7.28

By definition,

$$\tilde{\overline{\overline{\mathbf{M}}}}_{j\pm} = (\overline{\mathbf{I}} - \tilde{\overline{\mathbf{R}}}_{j,j\mp1} \cdot \tilde{\overline{\mathbf{R}}}_{j,j\pm1})^{-1}.$$

From the defining equation (7.4.42) we have

$$\begin{aligned}\tilde{\overline{\mathbf{m}}}_{j}\pm &= \overline{\mu}_j \cdot (\overline{\mathbf{I}} - \tilde{\overline{\mathbf{R}}}_{j,j\mp1} \cdot \tilde{\overline{\mathbf{R}}}_{j,j\pm1})^{-1} \cdot \overline{\epsilon}_j \\ &= [\overline{\epsilon}_j^{-1}\overline{\mu}_j^{-1} - \overline{\epsilon}_j^{-1}(\overline{\mu}_j^{-1} \cdot \tilde{\overline{\mathbf{r}}}_{j,j\mp1} \cdot \overline{\epsilon}_j^{-1}) \cdot (\overline{\mu}_j^{-1}\tilde{\overline{\mathbf{r}}}_{j,j\pm1} \cdot \overline{\epsilon}_j^{-1}) \cdot \overline{\mu}_j^{-1}]^{-1} \\ &= (\overline{p}_j^{-1} - \overline{p}_j^{-1} \cdot \tilde{\overline{\mathbf{r}}}_{j,j\mp1} \cdot \overline{p}_j^{-1} \cdot \tilde{\overline{\mathbf{r}}}_{j,j\pm1} \cdot \overline{p}_j^{-1})^{-1}.\end{aligned}$$

Thus, using (7.4.40) and the fact that $\overline{p}_j^{-1}$ is diagonal and hence symmetric,

$$\begin{aligned}\tilde{\overline{\mathbf{m}}}_{j+}^{t} &= [(\overline{p}_j^{-1} - \overline{p}_j^{-1} \cdot \tilde{\overline{\mathbf{r}}}_{j,j-1} \cdot \overline{p}_j^{-1} \cdot \tilde{\overline{\mathbf{r}}}_{j,j+1} \cdot \overline{p}_j^{-1})^{-1}]^{t} \\ &= (\overline{p}_j^{-1} - \overline{p}_j^{-1} \cdot \tilde{\overline{\mathbf{r}}}_{j,j+1} \cdot \overline{p}_j^{-1} \cdot \tilde{\overline{\mathbf{r}}}_{j,j-1} \cdot \overline{p}_j^{-1})^{-1} \\ &= \tilde{\overline{\mathbf{m}}}_{j-}.\end{aligned}$$

§7.29

(a) A simple way to verify the equality is to invert both sides, then expand $\tilde{\overline{\mathbf{R}}}_{23}$ and $\tilde{\overline{\mathbf{R}}}_{32}$ using

$$\begin{aligned}\tilde{\overline{\mathbf{R}}}_{23} &= \overline{\mathbf{R}}_{23} + \overline{\mathbf{T}}_{32} \cdot \tilde{\overline{\mathbf{R}}}_{34} \cdot (\overline{\mathbf{I}} - \overline{\mathbf{R}}_{32}\tilde{\overline{\mathbf{R}}}_{34})^{-1} \cdot \overline{\mathbf{T}}_{23}, \\ \tilde{\overline{\mathbf{R}}}_{32} &= \overline{\mathbf{R}}_{32} + \overline{\mathbf{T}}_{23} \cdot \overline{\mathbf{R}}_{21} \cdot (\overline{\mathbf{I}} - \overline{\mathbf{R}}_{23}\overline{\mathbf{R}}_{21})^{-1} \cdot \overline{\mathbf{T}}_{32}.\end{aligned}$$

The above expressions are from Equations (3.2.8) and (3.2.19). After multiplying out all the terms, the inverses of the left- and right-hand-sides are found to be equal, with

$$\begin{aligned}(\mathrm{LHS})^{-1} =& (\mathrm{RHS})^{-1} \\ =& \overline{\mathbf{T}}_{12}^{-1}\{\overline{\mathbf{T}}_{23}^{-1} - \overline{\mathbf{T}}_{23}^{-1}\overline{\mathbf{R}}_{32}\tilde{\overline{\mathbf{R}}}_{34} - \overline{\mathbf{R}}_{21}\overline{\mathbf{R}}_{23}\overline{\mathbf{T}}_{23}^{-1} \\ &+ \overline{\mathbf{R}}_{21}\overline{\mathbf{R}}_{23}\overline{\mathbf{T}}_{23}^{-1}\overline{\mathbf{R}}_{32}\tilde{\overline{\mathbf{R}}}_{34} - \overline{\mathbf{R}}_{21}\overline{\mathbf{T}}_{32}\tilde{\overline{\mathbf{R}}}_{34}\}.\end{aligned}$$

(b) The generalized transmission coefficient is given by (3.2.21) and (3.2.22),

$$\begin{aligned}\tilde{\overline{\mathbf{T}}}_{1N} = \overline{\mathbf{T}}_{N-1,N} \cdot (\overline{\mathbf{I}} - \overline{\mathbf{R}}_{N-1,N-2} \cdot \overline{\mathbf{R}}_{N-1,N})^{-1} \cdot \overline{\mathbf{T}}_{N-2,N-1}(\overline{\mathbf{I}} - \overline{\mathbf{R}}_{N-2,N-3} \\ \cdot \tilde{\overline{\mathbf{R}}}_{N-2,N-1})^{-1} \cdot \overline{\mathbf{T}}_{N-3,N-2} \cdot \cdots \cdot (\overline{\mathbf{I}} - \overline{\mathbf{R}}_{21} \cdot \tilde{\overline{\mathbf{R}}}_{23})^{-1} \cdot \overline{\mathbf{T}}_{12}.\end{aligned}$$

Applying the idea of part (a), the tilde signs can be shifted to give

$$\begin{aligned}\tilde{\overline{\mathbf{T}}}_{1\mathrm{N}} = \overline{\mathbf{T}}_{N-1,N} \cdot (\overline{\mathbf{I}} - \tilde{\overline{\mathbf{R}}}_{N-1,N-2} \cdot \overline{\mathbf{R}}_{N-1,N})^{-1} \cdot \overline{\mathbf{T}}_{N-2,N-1}(\overline{\mathbf{I}} - \tilde{\overline{\mathbf{R}}}_{N-2,N-3} \\ \cdot \overline{\mathbf{R}}_{N-2,N-1})^{-1} \cdot \overline{\mathbf{T}}_{N-3,N-2} \cdots (\overline{\mathbf{I}} - \overline{\mathbf{R}}_{21} \cdot \overline{\mathbf{R}}_{23})^{-1} \cdot \overline{\mathbf{T}}_{12}.\end{aligned}$$

Using the definition $\overline{\mathbf{p}}_j = \overline{\boldsymbol{\mu}}_j \cdot \overline{\boldsymbol{\epsilon}}_j$, we can write

$$\begin{aligned}\tilde{\overline{\mathbf{t}}}_{1N} =& \overline{\boldsymbol{\mu}}_N \cdot \tilde{\overline{\mathbf{T}}}_{1N} \cdot \overline{\boldsymbol{\epsilon}}_1 \\ =& \overline{\mathbf{t}}_{N-1,N}(\overline{\mathbf{p}}_{N-1} - \tilde{\overline{\mathbf{r}}}_{N-1,N-2} \cdot \overline{\mathbf{p}}_{N-1}^{-1} \cdot \overline{\mathbf{r}}_{N-1,N})^{-1} \cdot \overline{\mathbf{t}}_{N-2,N-1} \cdot \\ & \cdots \cdot \overline{\mathbf{t}}_{23} \cdot (\overline{\mathbf{p}}_2 - \overline{\mathbf{r}}_{21} \cdot \overline{\mathbf{p}}_2^{-1} \cdot \overline{\mathbf{r}}_{23})^{-1} \cdot \overline{\mathbf{t}}_{12}.\end{aligned}$$

Now, applying (7.4.39) and (7.4.40), and since $\overline{\mathbf{p}}_j$ is symmetric,

$$\begin{aligned}\tilde{\overline{\mathbf{t}}}^t_{1N} =& \left(\frac{\epsilon_1 k_{2\rho}^2}{\epsilon_2 k_{1\rho}^2}\right)\overline{\mathbf{t}}_{21} \cdot (\overline{\mathbf{p}}_2 - \overline{\mathbf{r}}_{23} \cdot \overline{\mathbf{p}}_2^{-1} \cdot \overline{\mathbf{r}}_{21})^{-1} \cdot \left(\frac{\epsilon_2 k_{3\rho}^2}{\epsilon_3 k_{2\rho}^2}\right)\overline{\mathbf{t}}_{32} \cdot \\ & \cdots \cdot \left(\frac{\epsilon_{N-2} k_{N-1,\rho}^2}{\epsilon_{N-1} k_{N-2,\rho}^2}\right)\overline{\mathbf{t}}_{N-1,N-2} \cdot (\overline{\mathbf{p}}_{N-1} - \overline{\mathbf{r}}_{N-1,N} \cdot \overline{\mathbf{p}}_{N-1}^{-1} \cdot \tilde{\overline{\mathbf{r}}}_{N-1,N-2})^{-1} \\ & \cdot \left(\frac{\epsilon_{N-1} k_{N\rho}^2}{\epsilon_N k_{N-1,\rho}^2}\right)\overline{\mathbf{t}}_{N,N-1} \\ =& \left(\frac{\epsilon_1 k_{N\rho}^2}{k_{1\rho}^2 \epsilon_N}\right) \cdot \overline{\boldsymbol{\mu}}_1 \cdot [\overline{\mathbf{T}}_{21} \cdot (\overline{\mathbf{I}} - \overline{\mathbf{R}}_{23} \cdot \overline{\mathbf{R}}_{21})^{-1} \cdot \overline{\mathbf{T}}_{32} \cdot \\ & \cdots \cdot \overline{\mathbf{T}}_{N-1,N-2} \cdot (\overline{\mathbf{I}} - \overline{\mathbf{R}}_{N-1,N} \cdot \tilde{\overline{\mathbf{R}}}_{N-1,N-2})^{-1} \cdot \overline{\mathbf{T}}_{N,N-1}] \cdot \overline{\boldsymbol{\epsilon}}_N.\end{aligned}$$

Comparing with Equations (3.2.21) and (3.2.22), the term in brackets is $\tilde{\overline{\mathbf{T}}}_{N1}$, so we have

$$\tilde{\overline{\mathbf{t}}}^t_{1N} = \left(\frac{\epsilon_1 k_{N\rho}^2}{k_{1\rho}^2 \epsilon_N}\right)\tilde{\overline{\mathbf{t}}}_{N1},$$

which proves (7.4.46).

§7.30

From Equations (3.5.21), (3.5.22), (3.5.27) and (3.5.28), we have

$$\frac{T_{12}^{TM}}{T_{21}^{TM}} = \frac{\epsilon_2 k_2}{\epsilon_1 k_1}$$

and

$$\frac{T_{12}^{TE}}{T_{21}^{TE}} = \frac{\mu_2 k_2}{\mu_1 k_1}.$$

From (3.6.7) and (3.6.9),

$$\tilde{T}_{1N} = \left(\prod_{i=1}^{N-2} \frac{T_{i,i+1}}{1 - R_{i+1,i}\tilde{R}_{i+1,i+2}}\right) T_{N-1,N},$$

while

$$\tilde{T}_{N1} = \left(\prod_{i=2}^{N-1} \frac{T_{i+1,i}}{1 - R_{i,i+1}\tilde{R}_{i,i-1}}\right) T_{21}.$$

Applying the idea of Exercise 7.27 we are free to shift the tildes in the expression for $\widetilde{T}_{1N}$,

$$\widetilde{T}_{1N} = \left(\prod_{i=1}^{N-2} \frac{T_{i,i+1}}{1 - \widetilde{R}_{i+1,i} R_{i+1,i+2}} \right) T_{N-1,N}.$$

The proof is exactly the same as for Exercise 7.29a, except the matrices of the cylindrical case are replaced by scalars for the spherical case. Next, changing the order of the subscripts on the $T_{i,i+1}$ factors,

$$\widetilde{T}_{1N} = \left(\prod_{i=1}^{N-2} \frac{T_{i+1,i}\alpha_{i+1,i}}{1 - R_{i+1,i+2}\widetilde{R}_{i+1,i}} \right) T_{N,N-1}\alpha_{N,N-1},$$

where

$$\begin{aligned} \alpha_{i+1,i} &= \frac{\epsilon_{i+1}k_{i+1}}{\epsilon_i k_i}, \quad TM \quad \text{case} \\ &= \frac{\mu_{i+1}k_{i+1}}{\mu_i k_i}, \quad TE \quad \text{case.} \end{aligned}$$

Thus,

$$\begin{aligned} \widetilde{T}_{1N} &= \left(\prod_{i=1}^{N-1} \alpha_{i+1,i} \right) \left(\prod_{i=2}^{N-1} \frac{T_{i,i-1}}{1 - R_{i,i+1}\widetilde{R}_{i,i-1}} \right) T_{N,N-1} \\ &= \left(\prod_{i=1}^{N-1} \alpha_{i+1,i} \right) \left(\prod_{i=2}^{N-1} \frac{T_{i+1,i}}{1 - R_{i,i+1}\widetilde{R}_{i,i-1}} \right) T_{21} \\ &= \left(\prod_{i=1}^{N-1} \alpha_{i+1,i} \right) \widetilde{T}_{N1}. \end{aligned}$$

The first factor is $\frac{\epsilon_N k_N}{\epsilon_1 k_1}$ for the TM case and $\frac{\mu_N k_N}{\mu_1 k_1}$ for the TE case. So, we have the result

$$k_1\mu_1\widetilde{T}_{1N}^{TE} = k_N\mu_N\widetilde{T}_{N1}^{TE}, \quad (7.4.49a)$$
$$k_1\epsilon_1\widetilde{T}_{1N}^{TM} = k_N\epsilon_N\widetilde{T}_{N1}^{TM}. \quad (7.4.49b)$$

CHAPTER 8

EXERCISE SOLUTIONS

by
J. H. Lin[1], G. P. Otto[2], R. L. Wagner[3],
Y. M. Wang[4] and W. H. Weedon[5]

§8.1

(a) Equation (8.1.3) is

$$(\nabla^2 + k_1^2) g_1(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$$

where $\mathbf{r}, \mathbf{r}' \in V_1$. One of the solutions which satisfies the radiation condition is given by (8.1.8). It is

$$g_1(\mathbf{r}, \mathbf{r}') = \frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \,. \tag{1}$$

This solution satisfies both the singularity property at $\mathbf{r} = \mathbf{r}'$ and the radiation condition as $\mathbf{r} \to \infty$. If we add another term such that it satisfies Equation (8.1.3) and the radiation condition at infinity but does not have any singularity for $\forall \mathbf{r} \in V_1$, then this new solution is also a solution of (8.1.3) in V_1. In the light of (8.1.8) with any vector $\mathbf{A} \in V_2$, we have

$$g_1'(\mathbf{r}, \mathbf{r}') = \frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik_1|\mathbf{r}-\mathbf{A}|}}{|\mathbf{r}-\mathbf{A}|}. \tag{2}$$

(2) satisfies Equation (8.1.3) in V_1 and the radiation condition at infinity.

(b) Suppose that S_{inf} is a sphere surface with radius $r \to \infty$, then,

$$I = \int_{S_{inf}} dS\hat{n} \cdot [g_1(\mathbf{r}, \mathbf{r}')\nabla\phi_1(\mathbf{r}) - \phi_1(\mathbf{r})\nabla g_1(\mathbf{r}, \mathbf{r}')] \tag{3a}$$

$$= \int_{S_{inf}} r^2 d\theta d\psi \hat{r} \cdot [g_1(\mathbf{r}, \mathbf{r}')\nabla\phi_1(\mathbf{r}) - \phi_1(\mathbf{r})\nabla g_1(\mathbf{r}, \mathbf{r}')] \,. \tag{3b}$$

[1] Exercises 8.29–8.35

[2] Reviewed Exercises 8.36–8.43

[3] Exercises 8.19–8.21

[4] Exercises 8.1–8.18 and 8.36–8.43

[5] Exercises 8.22–8.28

¿From the radiation condition at $r \to \infty$, we have

$$g_1(\mathbf{r},\mathbf{r}')|_{r\to\infty} \longrightarrow A\frac{e^{ik_1|\mathbf{r}|}}{|\mathbf{r}|}, \tag{4a}$$

$$\phi_1(\mathbf{r})|_{r\to\infty} \longrightarrow B\frac{ik_1|\mathbf{r}|}{|\mathbf{r}|}, \tag{4b}$$

A and B are constant. From the above two relations,

$$\nabla g_1(\mathbf{r},\mathbf{r}')|_{r\to\infty} \longrightarrow ik_1 A\frac{e^{ik_1|\mathbf{r}|}}{|\mathbf{r}|}\hat{r}, \tag{5a}$$

$$\nabla \phi_1(\mathbf{r})|_{r\to\infty} \longrightarrow ik_1 B\frac{e^{ik_1|\mathbf{r}|}}{|\mathbf{r}|}\hat{r}. \tag{5b}$$

In the above, the higher order terms $O(|r|^{-2})$ have been neglected. Substituting Equations (4) and (5) into (3b), we have,

$$I = \int_{S_{inf}} r^2 d\theta d\psi \hat{r}\cdot\hat{r}\left[ik_1 AB\frac{e^{2ik_1|\mathbf{r}|}}{|\mathbf{r}|^2} - ik_1 AB\frac{e^{2ik_1|\mathbf{r}|}}{|\mathbf{r}|^2}\right] = 0. \tag{6}$$

§8.2

The geometry of the problem is shown on the right. Equations (8.1.2) and (8.1.4) are

$$(\nabla^2 + k_2^2)\phi_2(\mathbf{r}) = 0, \tag{1}$$

$$(\nabla^2 + k_2^2)g_2(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'), \tag{2}$$

where $\mathbf{r} \in V_2$ in Equation (1).

On multiplying Equation (1) by $g_2(\mathbf{r},\mathbf{r}')$ and Equation (2) by $\phi_2(\mathbf{r})$, subtracting the two resultant equations, and integrating over region V_2, we have for $\mathbf{r} \in V_2$,

$$\int_{V_2} dV\,[g_2(\mathbf{r},\mathbf{r}')\nabla^2\phi_2(\mathbf{r}) - \phi_2(\mathbf{r})\nabla^2 g_2(\mathbf{r},\mathbf{r}')] = \phi_2(\mathbf{r}'), \quad \mathbf{r}' \in V_2. \tag{3}$$

Applying the relation of

$$g\nabla^2\phi - \phi\nabla^2 g = \nabla\cdot(g\nabla\phi - \phi\nabla g),$$

and Gauss' theorem, the volume integral on the left-hand side of (3) becomes a surface integral over S. Thus,

$$\int_S dS\hat{n}\cdot[g_2(\mathbf{r},\mathbf{r}')\nabla\phi_2(\mathbf{r}) - \phi_2(\mathbf{r})\nabla g_2(\mathbf{r},\mathbf{r}')] = \phi_2(\mathbf{r}'), \quad \mathbf{r}' \in V_2. \tag{4}$$

If $\mathbf{r}' \in V_1$, the right-hand side of (4) would be zero. Finally, we have Equation (8.1.11) which is

$$\left.\begin{array}{ll}\mathbf{r} \in V_2, & \phi_2(\mathbf{r}) \\ \mathbf{r} \in V_1, & 0\end{array}\right\} = \int_S dS'\hat{n} \cdot [g_2(\mathbf{r},\mathbf{r}')\nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}')\nabla' g_2(\mathbf{r},\mathbf{r}')] . \quad (5)$$

In the above, $g_2(\mathbf{r},\mathbf{r}')$ does not need to satisfy the radiation condition.

(b) First, suppose that a homogeneous Dirichlet boundary condition on S is imposed on $g_2(\mathbf{r},\mathbf{r}')$, that is

$$g_2(\mathbf{r},\mathbf{r}') = 0, \quad \mathbf{r} \in S. \quad (6)$$

Then, Equation (8.1.11) becomes

$$\mathbf{r} \in V_2, \quad \phi_2(\mathbf{r}) = -\int_S dS'\hat{n} \cdot [\phi_2(\mathbf{r}')\nabla' g_2(\mathbf{r},\mathbf{r}')] . \quad (7)$$

For a homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \hat{n}} g_2(\mathbf{r},\mathbf{r}') = 0 \qquad \mathbf{r} \in S, \quad (8)$$

Equation (5) becomes

$$\mathbf{r} \in V_2, \quad \phi_2(\mathbf{r}) = \int_S dS'\hat{n} \cdot [g_2(\mathbf{r},\mathbf{r}')\nabla'\phi_2(\mathbf{r})] . \quad (9)$$

In these cases, $g_2(\mathbf{r},\mathbf{r}')$ is only defined in V_2. It is completely undefined beyond S, the boundary of V_2. Thus, the lower part of (5) does not hold anymore.

§8.3

(a) An unbounded homogeneous-medium dyadic Green's function satisfies

$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') - k^2\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'). \quad (1)$$

We can write $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ in terms of a scalar Green's function $g(\mathbf{r},\mathbf{r}')$

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) g(\mathbf{r},\mathbf{r}'), \quad (2)$$

where, $g(\mathbf{r},\mathbf{r}')$ satisfies

$$\left(\nabla^2 + k^2\right) g(\mathbf{r}-\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'). \quad (3)$$

¿From (2), we have

$$\nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \nabla \times \overline{\mathbf{I}} g(\mathbf{r},\mathbf{r}') = -\nabla' \times \overline{\mathbf{I}} g(\mathbf{r},\mathbf{r}'). \quad (4)$$

Thus, we have

$$\left[\nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]^t = -(\nabla' \times \overline{\mathbf{I}})^t g(\mathbf{r},\mathbf{r}') \tag{5}$$

$$\begin{aligned}\left\{(\nabla' \times \overline{\mathbf{I}})^t g(\mathbf{r},\mathbf{r}')\right\}_{ij} &= \epsilon_{jlm}\frac{\partial}{\partial x_l'}\delta_{mi}g(\mathbf{r},\mathbf{r}') \\ &= \epsilon_{jli}\frac{\partial}{\partial x_l'}g(\mathbf{r},\mathbf{r}') \\ &= -\epsilon_{ilj}\frac{\partial}{\partial x_l'}g(\mathbf{r},\mathbf{r}') \\ &= -\epsilon_{ilk}\frac{\partial}{\partial x_l}\delta_{kj}g(\mathbf{r},\mathbf{r}') \\ &= -\left\{(\nabla' \times \overline{\mathbf{I}})\, g(\mathbf{r},\mathbf{r}')\right\}_{ij}.\end{aligned} \tag{6}$$

Thus, we have

$$-\left(\nabla' \times \overline{\mathbf{I}}\right)^t g(\mathbf{r},\mathbf{r}') = \nabla' \times \overline{\mathbf{I}}g(\mathbf{r},\mathbf{r}'). \tag{7}$$

Using (5) in (4), we have

$$\left[\nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]^t = \nabla' \times \overline{\mathbf{I}}g(\mathbf{r},\mathbf{r}') = \nabla' \times \overline{\mathbf{G}}(\mathbf{r}',\mathbf{r}). \tag{8}$$

(b) Equations (8.1.15) and (8.1.17) are

$$\nabla \times \nabla \times \overline{\mathbf{E}}_2(\mathbf{r}) - \omega^2\mu_2\epsilon_2\overline{\mathbf{E}}_2(\mathbf{r}) = 0 \tag{9}$$

$$\nabla \times \nabla \times \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') - \omega^2\mu_2\epsilon_2\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') = \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}') \tag{10}$$

On post-multiplying Equation (8) with $\overline{\mathbf{G}}_2$ and pre-multiplying Equation (9) with $\overline{\mathbf{E}}_2(\mathbf{r})$, subtracting the two equations, and integrating the result over V_2, we have

$$\int_{V_2} dV\left[\nabla \times \nabla \times \mathbf{E}_2(\mathbf{r}) \cdot \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') - \mathbf{E}_2(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\right] = -\mathbf{E}_2(\mathbf{r}'), \qquad \mathbf{r}' \in V_2. \tag{11}$$

Using the fact that

$$\begin{aligned}&\nabla \cdot \left\{[\nabla \times \mathbf{E}_2(\mathbf{r})] \times \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') + \mathbf{E}_2(\mathbf{r}) \times \left[\nabla \times \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\right]\right\} \\ =&\nabla \times \nabla \times \mathbf{E}_2(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') - \mathbf{E}_2(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}'),\end{aligned} \tag{12}$$

we have

$$\mathbf{E}_2(\mathbf{r}') = -\int_S dS\hat{n} \cdot \left\{[\nabla \times \mathbf{E}_2(\mathbf{r})] \times \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') + \mathbf{E}_2(\mathbf{r}) \times \left[\nabla \times \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\right]\right\}. \tag{13}$$

Using (8.1.22) and (8), we have

$$\hat{n}\cdot[\nabla\times\mathbf{E}_2(\mathbf{r})]\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') = \hat{n}\times[\nabla\times\mathbf{E}_2(\mathbf{r})]\cdot\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}) = i\omega\mu_2\overline{\mathbf{G}}_2(\mathbf{r}',\mathbf{r})\cdot\hat{n}\times\mathbf{H}_2(\mathbf{r}), \tag{14}$$

and

$$\hat{n}\cdot\mathbf{E}_2(\mathbf{r})\times\left[\nabla\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\right] = \hat{n}\times\mathbf{E}_2(\mathbf{r})\cdot\nabla\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') = -\left[\nabla\times\overline{\mathbf{G}}_2(\mathbf{r}',\mathbf{r})\right]\cdot\hat{n}\times\mathbf{E}_2(\mathbf{r}). \tag{15}$$

Using (14) and (15) in (3), finally we have

$$\mathbf{r}\in V_2,\quad \mathbf{E}_2(\mathbf{r}) = -\int_S dS'\left\{i\omega\mu_2\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\hat{n}\times\mathbf{H}_2(\mathbf{r}') - \left[\nabla\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\right]\cdot\hat{n}\times\mathbf{E}_2(\mathbf{r}')\right\}. \tag{16}$$

when $\mathbf{r}\in V_1$, we have the left-hand side of (11) is zero, therefore,

$$\left.\begin{array}{ll}\mathbf{r}\in V_2, & \mathbf{E}_2(\mathbf{r})\\ \mathbf{r}\in V_1, & 0\end{array}\right\} = -\int_S dS'\left\{i\omega\mu_2\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\hat{n}\times\mathbf{H}_2(\mathbf{r}') - \left[\nabla\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\right]\cdot\hat{n}\times\mathbf{E}_2(\mathbf{r}')\right\} \tag{17}$$

(c) In deriving (13), the radiation condition is not applied. Thus, $\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')$ need not satisfy the radiation condition at infinity in this case.

(d) In deriving (8.1.28), the volume integration is over V_1. Thus, we can choose $\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r})$ satisfy Equation (8.1.16) but with inhomogeneous medium in V_2.

§8.4

(a) The identity in Equation (8.1.37) is

$$\begin{aligned}&\nabla\cdot\left\{\left[\overline{\boldsymbol{\mu}}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\right]\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b} + \mathbf{E}_1(\mathbf{r})\times\left[(\overline{\boldsymbol{\mu}}_1^t)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}\right]\right\}\\ &=\nabla\times\left[\overline{\boldsymbol{\mu}}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\right]\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\mathbf{b} - \mathbf{E}_1(\mathbf{r})\cdot\nabla\times\left[(\overline{\boldsymbol{\mu}}_1^t)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}\right].\end{aligned} \tag{1}$$

Using the identity of

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \tag{2}$$

for the first term on the right-hand side of (1), we have

$$\begin{aligned}&\nabla\left\{\left[\overline{\boldsymbol{\mu}}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\right]\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\mathbf{b}\right\}\\ &=\nabla\times\left[\overline{\boldsymbol{\mu}}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\right]\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b} - \overline{\boldsymbol{\mu}}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b},\end{aligned} \tag{3}$$

for the second term on the right-hand side of (1), we have

$$\begin{aligned}&\nabla\cdot\left\{\mathbf{E}_1(\mathbf{r})\times\left[\left(\overline{\boldsymbol{\mu}}_1^t\right)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}\right]\right\}\\ =&\nabla\times\mathbf{E}_1(\mathbf{r})\cdot(\overline{\boldsymbol{\mu}}_1^t)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}\\ &-\mathbf{E}_1(\mathbf{r})\cdot\nabla\times\left[\left(\overline{\boldsymbol{\mu}}_1^t\right)^{-1}\cdot\nabla\times G_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}\right]\end{aligned}\tag{4}$$

Notice that

$$\nabla\times\mathbf{E}_1(\mathbf{r})\cdot\left(\overline{\boldsymbol{\mu}}_1^t\right)^{-1}=(\overline{\boldsymbol{\mu}}_1)^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\tag{5}$$

Using (3), (4), and (5), Equation (1) follows.

(b) $\mathbf{E}_2(\mathbf{r})$ and $\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')$ satisfy

$$\nabla\times\overline{\boldsymbol{\mu}}_2^{-1}\cdot\nabla\times\mathbf{E}_2(\mathbf{r})-\omega^2\overline{\epsilon}_2\cdot\mathbf{E}_2=0,\quad \mathbf{r}\in V_2,\tag{6}$$

$$\nabla\times\left(\overline{\boldsymbol{\mu}}_2^t\right)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')-\omega^2\overline{\epsilon}_2^t\cdot\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')=\left(\boldsymbol{\mu}_2^t\right)^{-1}\overline{\mathbf{I}}\delta(\mathbf{r},\mathbf{r}').\tag{7}$$

On post-multiplying (6) by $\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}$, we have

$$\nabla\times\overline{\boldsymbol{\mu}}_2^{-1}\cdot\nabla\times\mathbf{E}_2(\mathbf{r})\cdot\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}-\omega^2\overline{\epsilon}_2\cdot\mathbf{E}_2\cdot\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}=0,\tag{8}$$

where $\mathbf{b}$ is an arbitrary constant vector. Then, after pre-multiplying (7) by $\mathbf{E}_2(\mathbf{r})$, and post-multiplying it by $\mathbf{b}$, we have,

$$\begin{aligned}\mathbf{E}_2(\mathbf{r})\cdot\nabla\times\left(\overline{\boldsymbol{\mu}}_2^t\right)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}-\omega^2\mathbf{E}_2(\mathbf{r})\cdot\overline{\mathbf{G}}^t\cdot G_2(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}&\\ =\mathbf{E}_2(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\cdot\left(\overline{\boldsymbol{\mu}}_2^t\right)^{-1}\cdot\mathbf{b}.&\end{aligned}\tag{9}$$

Next, integrating the difference of (8) and (9) over V_2, for $\mathbf{r}'\in V_2$, yields

$$\begin{aligned}\int_{V_1}dV\Big[\nabla\times\overline{\boldsymbol{\mu}}_2^{-1}\cdot\nabla\times\mathbf{E}_2(\mathbf{r})\cdot\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\mathbf{b}-\mathbf{E}_2(\mathbf{r})\cdot\nabla\times\left(\overline{\boldsymbol{\mu}}_2^t\right)^{-1}\cdot\nabla&\\ \times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}\Big]=-\mathbf{E}_2(\mathbf{r}')\cdot\left(\overline{\boldsymbol{\mu}}_2^t\right)^{-1}\cdot\mathbf{b}.&\end{aligned}\tag{10}$$

Using the identity of Equation (8.1.37) and Gauss' theorem, we have

$$\begin{aligned}\mathbf{E}_2(\mathbf{r}')=-\int_S dS\hat{n}\cdot\Big[(\overline{\boldsymbol{\mu}}_2)^{-1}\cdot\nabla\times\mathbf{E}_2(\mathbf{r})\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\overline{\boldsymbol{\mu}}_2^t&\\ +\mathbf{E}_2(\mathbf{r})\times\left(\overline{\boldsymbol{\mu}}_2^t\right)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\overline{\boldsymbol{\mu}}_2^t\Big],\quad \mathbf{r}\in V_2.&\end{aligned}\tag{11}$$

For $\mathbf{r}'\in V_1$, the right-hand side of (11) is zero. Thus, we have

$$\begin{aligned}\left.\begin{array}{l}\mathbf{r}\in V_2,\ \mathbf{E}_2(\mathbf{r})\\ \mathbf{r}\in V_1,\ 0\end{array}\right\}=-\int_S dS'\,\Big\{i\omega\hat{n}\times\mathbf{H}_2(\mathbf{r}')\cdot\overline{\mathbf{G}}_2(\mathbf{r}',\mathbf{r})\cdot\overline{\boldsymbol{\mu}}_2^t(\mathbf{r})&\\ +\hat{n}\times\mathbf{E}_2(\mathbf{r}')\cdot[\overline{\boldsymbol{\mu}}_2^t(\mathbf{r}')]^{-1}\cdot\nabla'\times\overline{\mathbf{G}}_2(\mathbf{r}',\mathbf{r})\cdot\overline{\boldsymbol{\mu}}_2^t(\mathbf{r})\Big\}.&\end{aligned}\tag{12}$$

Using lower part of Equation (8.1.40) and (8.1.41), we have the integral equations (8.1.42a) and (8.1.42b).

§8.5

The wave equation for E-field is

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = i\omega\mu\mathbf{J}. \tag{1}$$

Applying the identity of

$$\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A}, \tag{2}$$

(1) becomes

$$\nabla^2\mathbf{E}(\mathbf{r}) + k^2\mathbf{E}(\mathbf{r}) = -i\omega\mu\mathbf{J} + \nabla\nabla \cdot \mathbf{E} \tag{3}$$

Since,

$$\nabla \cdot \mathbf{D} = \rho \tag{4}$$

and

$$\mathbf{D} = \epsilon\mathbf{E} \tag{5}$$

we have

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}. \tag{6}$$

Substituting (3) into (2), we have

$$\nabla^2\mathbf{E}(\mathbf{r}) + k^2\mathbf{E}(\mathbf{r}) = -i\omega\mu\mathbf{J} + \nabla\frac{\rho}{\epsilon} \tag{7}$$

using (8.1.43) and (8.1.44), and noticing that k^2 is only a function of x and y, we have for the z component of (4),

$$\frac{1}{2\pi}\int dk_z\left[\nabla_s^2 + k^2 - k_z^2\right]E_z(\rho, k_z)\,e^{ik_z z} = \frac{1}{2\pi}\int dk_z\left[-i\omega\mu J_z(\rho, k_z) + ik_z\frac{\rho}{\epsilon}\right]e^{ik_z z} \tag{8}$$

Then, we have†

$$\left[\nabla_s^2 + k^2 - k_z^2\right]E_z(\boldsymbol{\rho}, k_z) = -i\omega\mu J_z(\boldsymbol{\rho}, k_z) + \frac{ik_z}{\epsilon}\rho(\boldsymbol{\rho}, k_z) \tag{9}$$

Similarly, the wave equation for H-field is

$$\nabla \times \nabla \times \mathbf{H} - k^2\mathbf{H} = \nabla \times \mathbf{J}. \tag{10}$$

† The last term on the right-hand side of (8.1.46a) should be $\frac{ik_z}{\epsilon}\rho(\boldsymbol{\rho}, k_z)$ instead of $\frac{\partial}{\partial z}\frac{\rho}{\epsilon}$.

Using the relation of (2) and $\nabla \cdot \mathbf{H} = 0$, we have

$$(\nabla^2 + k^2)\mathbf{H} = -\nabla \times \mathbf{J} \tag{11}$$

The z-component of (11) is

$$(\nabla^2 + k^2)H_z = -(\nabla_s \times J_s)_z. \tag{12}$$

Using (8.1.43) and (8.1.44), we finally have

$$(\nabla^2 + k^2 - k_z^2)H_z = -(\nabla_s \times J_s)_z. \tag{13}$$

§8.6

In region 2, the wave equation is

$$(\nabla_s^2 + k_{2s}^2)E_{2z}(\rho) = 0 \tag{1}$$

$$(\nabla_s^2 + k_{2s}^2)H_{2z}(\rho) = 0 \tag{2}$$

since region 2 is source free. Assume that $G_2(\rho, \rho)$ satisfies

$$(\nabla_s^2 + k_{2s}^2)G_2(\rho, \rho') = -\delta(\rho - \rho'). \tag{3}$$

After multiplying Equation (1) with G_2 and Equation (3) with E_{2z}, subtracting the two equations and integrating over region 2, we have for $\rho' \in V_2$

$$\int_{V_2} d\rho[G_2(\rho, \rho')\nabla_s^2 E_{2z}(\rho) - E_{2z}(\rho)\nabla_s^2 G_2(\rho, \rho')]$$
$$= E_{2z}(\rho'). \tag{4}$$

Using the identity

$$\phi_1 \nabla_s^2 \phi_2 - \phi_2 \nabla_s^2 \phi_1 = \nabla \cdot (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \tag{5}$$

and Gauss' theorem, we have

$$\rho \in V_2 \quad E_{2z}(\rho) = \int_S dS' \hat{n} \cdot [G_2(\rho, \rho')\nabla'_s E_{2z}(\rho') - E_{2z}(\rho')\nabla' G_2(\rho, \rho')]. \tag{6}$$

For $\rho \notin V_2$, we have $E_{2z}(\rho) = 0$. Thus

$$\left.\begin{matrix} \rho \in V_2, & E_{2z}(\rho) \\ \rho \in V_1, & 0 \end{matrix}\right\} = \int_S dS' \, \hat{n} \cdot [G_2(\rho, \rho')\nabla'_s E_{2z}(\rho') - E_{2z}(\rho')\nabla'_s G_2(\rho, \rho')]. \tag{7}$$

Similarity, for H_{2z}, we have

$$\left.\begin{array}{ll}\rho \in V_2, & H_{2z}(\rho) \\ \rho \in V_1, & 0\end{array}\right\} = \int_S dS'\,\hat{n}\cdot[G_2(\rho,\rho')\nabla'_s H_{2z}(\rho') - H_{2z}(\rho')\nabla'_s G_2(\rho,\rho')]. \tag{8}$$

In deriving (7) and (8), the radiation condition of G_2 is not imposed. Therefore, another choice of G_2 is

$$G_2(\rho,\rho') = \frac{i}{4}H_0^{(1)}(k_{2s}|\rho-\rho'|) + AJ_0(k_{2s}|\rho-\rho'|) \tag{9}$$

which satisfies Equation (3) and the singularity property at $\rho = \rho'$.

§8.7

The transverse components of the electromagnetic fields are related to the z-component fields by

$$\mathbf{E}_s = \frac{1}{k_s^2}\left[k_z\nabla_s E_z - \omega\mu\hat{z}\times\nabla_s H_z\right], \tag{1}$$

$$\mathbf{H}_s = \frac{i}{k_s^2}\left[k_z\nabla_s H_z + \omega\epsilon\hat{z}\times\nabla_s E_z\right]. \tag{2}$$

Thus, we have

$$\mathbf{E}_{1s} = \frac{i}{k_{1s}^2}[k_z\nabla_s E_{1z} - \omega\mu_1\hat{z}\times\nabla_s H_{1z}], \tag{3}$$

$$\mathbf{E}_{2s} = \frac{i}{k_s^2}[k_z\nabla_s E_{2z} - \omega\mu_2\hat{z}\times\nabla_s H_{2z}], \tag{4}$$

$$\mathbf{H}_{1s} = \frac{i}{k_{1s}^2}[k_z^2\nabla_s H_{1z} + \omega\epsilon_2\hat{z}\times\nabla_s E_{1z}], \tag{5}$$

$$\mathbf{H}_{2s} = \frac{i}{k_{2s}^2}[k_z\nabla_s H_{2z} + \omega\epsilon_2\hat{z}\times\nabla_s E_{2z}]. \tag{6}$$

The boundary conditions are

$$\hat{n}\times\mathbf{E}_{1s} = \hat{n}\times\mathbf{E}_{2s}, \quad \hat{n}\times\mathbf{H}_{1s} = \hat{n}\times\mathbf{H}_{2s} \quad \text{on } S. \tag{7}$$

¿From (7), we have

$$\hat{n}\times\frac{i}{k_{1s}^2}[k_z\nabla_s E_{1z} - \omega\mu_1\hat{z}\times\nabla_s H_{1z}] = \hat{n}\times\frac{i}{k_{2s}^2}[k_z\nabla_s E_{2z} - \omega\mu_2\hat{z}\times\nabla_s H_{2z}], \tag{8}$$

$$\hat{n}\times\frac{i}{k_{1s}^2}[k_z\nabla_s H_{1z} + \omega\epsilon_1\hat{z}\times\nabla_s E_{1z}] = \hat{n}\times\frac{i}{k_{2s}^2}[k_z\nabla_s H_{2z} + \omega\epsilon_2\hat{z}\times\nabla_s E_{2z}]. \tag{9}$$

Regrouping (8) and (9), and using $\hat{n}\times\hat{z}\times\nabla_s E = \hat{z}(\hat{n}\cdot\nabla_s F)$, we have

$$\frac{k_{2s}^2}{k_{1s}^2}[k_z\hat{n}\times\nabla_s E_{1z} - \omega\mu_1\hat{z}(\hat{n}\cdot\nabla_s H_{1z})] = k_z\hat{n}\times\nabla_s E_{2z} - \omega\mu_2\hat{z}(\hat{n}\cdot\nabla_s H_{2z}), \tag{10}$$

$$\frac{k_{2s}^2}{k_{1s}^2}[k_z\hat{n}\times\nabla_s H_{1z} + \omega\epsilon_1\hat{z}(\hat{n}\cdot\nabla_s E_{1z})] = k_z\hat{n}\times\nabla_s H_{2z} + \omega\epsilon_2\hat{z}(\hat{n}\cdot\nabla_s E_{2z}). \tag{11}$$

Taking the dot products of Equations (10) and (11) with $\hat{z}$, we have

$$\frac{k_{2s}^2}{k_{1s}^2}[k_z\hat{z}\cdot\hat{n}\times\nabla_s E_{1z} - \omega\mu_1\hat{n}\cdot\nabla_s H_{1z}] = k_z\hat{z}\cdot\hat{n}\times\nabla_s E_{2z} - \omega\mu_2\hat{n}\cdot\nabla_s H_{2z}, \tag{12}$$

$$\frac{k_{2s}^2}{k_{1s}^2}[k_z\hat{z}\cdot\hat{n}\times\nabla_s H_{1z} + \omega\epsilon_1\hat{n}\cdot\nabla_s E_{1z}] = k_z\hat{z}\cdot\hat{n}\times\nabla_s H_{2z} + \omega\epsilon_2\hat{n}\cdot\nabla_s H_{2z}. \tag{13}$$

Since $\hat{n}\times\nabla_s E_z$ and $\hat{n}\times\nabla_s H_z$ are on S, and

$$E_{1z} = E_{2z}, \quad H_{1z} = H_{2z} \qquad \text{on} \quad S, \tag{14}$$

we have

$$\hat{n}\times\nabla_s H_{1z} = \hat{n}\times\nabla_s H_{2z}, \quad \hat{n}\times\nabla_s E_{1z} = \hat{n}\times\nabla_s E_{2z}. \tag{15}$$

Using (15), (12) and (13) become

$$\hat{n}\cdot\nabla_s H_{2z} = -\frac{k_z}{\omega\mu_2}\left(\frac{k_{2s}^2}{k_{1s}^2}-1\right)(\hat{z}\cdot\hat{n}\times\nabla_s)E_{1z} + \frac{\mu_1 k_{2s}^2}{\mu_2 k_{1s}^2}\hat{n}\cdot\nabla_s H_{1z}. \tag{16}$$

$$\hat{n}\cdot\nabla_s E_{2z} = \frac{k_z}{\omega\epsilon_2}\left(\frac{k_{2s}^2}{k_{1s}^2}-1\right)(\hat{z}\cdot\hat{n}\times\nabla_s)H_{1z} + \frac{\epsilon_1 k_{2s}^2}{\epsilon_2 k_{1s}^2}\hat{n}\cdot\nabla_s E_{1z}. \tag{17}$$

The above two equations can be combined as

$$\begin{bmatrix}\hat{n}\cdot\nabla_s E_{2z}\\ \hat{n}\cdot\nabla_s H_{2z}\end{bmatrix} = \overline{\mathbf{M}}\cdot\begin{bmatrix}E_{1z}\\ H_{1z}\end{bmatrix} + \overline{\mathbf{N}}\cdot\begin{bmatrix}\hat{n}\cdot\nabla_s E_{1s}\\ \hat{n}\cdot\nabla_s H_{1s}\end{bmatrix}$$

where $M_{11} = M_{22} = N_{12} = N_{21} = 0$

$$\begin{aligned} M_{12} &= \tfrac{k_z}{\omega\epsilon_2}\left(\tfrac{k_{2s}^2}{k_{1s}^2}-1\right)\hat{z}\cdot\hat{n}\times\nabla_s, \qquad N_{11} = \tfrac{\epsilon_1 k_{2s}^2}{\epsilon_2 k_{1s}^2},\\ M_{21} &= -\tfrac{k_z}{\omega\mu_2}\left(\tfrac{k_{2s}^2}{k_{1s}^2}-1\right)\hat{z}\cdot\hat{n}\times\nabla_s, \quad N_{22} = \tfrac{\mu_1 k_{2s}^2}{\mu_2 k_{1s}^2}. \end{aligned} \tag{18}$$

§8.8

The second equations of (8.1.10) and (8.1.11) are

$$\phi_{inc}(\mathbf{r}) = \int_S dS'\hat{n}\cdot[g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')], \quad \mathbf{r}\in V_2, \tag{1}$$

$$0 = \int_S dS'\hat{n}\cdot[g_2(\mathbf{r},\mathbf{r}')\nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r})\nabla' g_2(\mathbf{r},\mathbf{r}')], \quad \mathbf{r}\in V_1. \tag{2}$$

Figure for Exercise Solution 8.8

Due do the singularity in $\hat{n} \cdot g(\mathbf{r}, \mathbf{r}')$, the above equations are not defined when $\mathbf{r} \in S$. But we can let $\mathbf{r} \to S$ from the two regions. For the first equation, we can deform the S as shown in the Figure.

The semi-sphere surface is denoted by C_δ, the other part is the principle integral in which there is no singularity. Thus, we have

$$\begin{aligned}\phi_{inc}(\mathbf{r}) = & \fint_S dS'\hat{n} \cdot [g_1(\mathbf{r}, \mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla' g_1(\mathbf{r}, \mathbf{r}') \\ & + \int_{C_\delta} dS'\hat{n} \cdot [g_1(\mathbf{r}, \mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla' g_1(\mathbf{r}, \mathbf{r}')]. \end{aligned} \tag{3}$$

The first term in C_δ approaches zero when $\delta \to 0$, since it only contains $\frac{1}{\delta}$ singularity and $dS' = \delta^2 \sin\theta d\phi d\theta$. For the second term in C_δ, we look at the equation for $g_1(\mathbf{r}, \mathbf{r}')$

$$\nabla^2 g_1(\mathbf{r}, \mathbf{r}') + k_1^2 g_1(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'). \tag{4}$$

Taking integration over $\{V : |\mathbf{r} - \mathbf{r}'| < \delta\}$, and let $\delta \to 0$, we have

$$\lim_{\delta \to 0} \int_{|\mathbf{r}-\mathbf{r}'| \leq \delta} dV \nabla^2 g_1(\mathbf{r}, \mathbf{r}') = -1,$$

$$\lim_{\delta \to 0} \int dV \nabla_s \nabla g_1(\mathbf{r}, \mathbf{r}') = \lim_{\delta \to 0} \int_S dS \hat{n} \cdot \nabla g_1(\mathbf{r}, \mathbf{r}') = -1. \quad (5)$$

Therefore, in the second term in C_δ, we can take $\phi_1(\mathbf{r}')$ out of the integration, since $\phi_1(\mathbf{r})$ is a continuous function on S. Meanwhile, C_δ is a half-sphere surface. So we have

$$\int_{C_\delta} dS' \hat{n} \cdot [g_1(\mathbf{r}_1, \mathbf{r}') \nabla' \phi_1(\mathbf{r}') - \phi_1(\mathbf{r}') \nabla' g_1(\mathbf{r}, \mathbf{r}')] = \frac{1}{2} \phi_1(\mathbf{r}) \quad \mathbf{r} \in S. \quad (6)$$

Consequently,

$$\phi_{inc}(\mathbf{r}) = \frac{1}{2} \phi_1(\mathbf{r}) + \int_S dS' \hat{n} \cdot [g_1(\mathbf{r}, \mathbf{r}') \nabla' \phi_1(\mathbf{r}') - \phi_1(\mathbf{r}') \nabla' g_1(\mathbf{r}, \mathbf{r}')] \quad r \in S. \quad (7)$$

For Equation (2), S is deformed as shown in the right. Since $\hat{n}$ is in the negative $\hat{\delta}$ direction. We have

$$\int_{C_\delta} dS \hat{n} \cdot [g_2(\mathbf{r}, \mathbf{r}') \nabla' \phi_2(\mathbf{r}') - \phi_2(\mathbf{r}') \nabla' g_2(\mathbf{r}, \mathbf{r}')] = -\frac{1}{2} \phi_2(\mathbf{r}). \quad (8)$$

Thus, we get

$$0 = \frac{1}{2} \phi_2(\mathbf{r}) - \int_S d\Delta' \hat{n} \cdot [g_2(\mathbf{r}, \mathbf{r}') \nabla' \phi_2(\mathbf{r}') - \phi_2(\mathbf{r}') \nabla' g_2(\mathbf{r}, \mathbf{r}')], \quad \mathbf{r} \in S. \quad (9)$$

§8.9

First, the Green function $g_i(\mathbf{r}, \mathbf{r}')$ has the following properties.

$$g_i(\mathbf{r}, \mathbf{r}') = g_i(\mathbf{r}, \mathbf{r}'),$$
$$\nabla' g_i(\mathbf{r}, \mathbf{r}') = -\nabla g_i(\mathbf{r}, \mathbf{r}') = -\nabla g_i(\mathbf{r}', \mathbf{r}) = -\nabla' g_i(\mathbf{r}', \mathbf{r}). \quad (1)$$

To prove L_{ij}'s symmetric properties, we need to operate L_{ij} on same function $f(\mathbf{r}')$. Thus, for L_{11}, we have

$$L_{11}(\mathbf{r}, \mathbf{r}') \cdot f(\mathbf{r}') = \int_S dS' g(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') \quad (2)$$

¿From (1), we have

$$L_{11}(\mathbf{r}, \mathbf{r}') \cdot f(\mathbf{r}') = \int_S dS' g(\mathbf{r}', \mathbf{r}) f(\mathbf{r}') = L_{11}(\mathbf{r}', \mathbf{r}) \cdot f(\mathbf{r}') \quad (3)$$

So, $L_{11}(\mathbf{r}, \mathbf{r}')$ is symmetric with respect to $\mathbf{r}$ and $\mathbf{r}'$.

Similarly, we have

$$L_{21}(\mathbf{r},\mathbf{r}') = L_{21}(\mathbf{r}',\mathbf{r}'). \tag{4}$$

For L_{12}, we have

$$L_{12}(\mathbf{r},\mathbf{r}')\cdot f(\mathbf{r}') = \int_S dS'\hat{n}\cdot\nabla' g_1(\mathbf{r},\mathbf{r}')f(\mathbf{r}'). \tag{5}$$

Using (2), we have

$$\begin{aligned} L_{12}(\mathbf{r},\mathbf{r}')f(\mathbf{r}') &= -\int_S dS'\hat{n}\cdot\nabla' g_1(\mathbf{r}',\mathbf{r})f(\mathbf{r}') \\ &= -L_{12}(\mathbf{r}',\mathbf{r})\cdot f(\mathbf{r}'). \end{aligned} \tag{6}$$

So, $L_{11}(\mathbf{r},\mathbf{r}')$ is skew-symmetric with respect to $\mathbf{r}$ and $\mathbf{r}'$. Following the same argument, L_{22} is also skew-symmetric respect to $\mathbf{r}$ and $\mathbf{r}'$.

The matrix element of L_{ij} in Galerkin's method is

$$\{L_{ij}\}_{nm} = \langle f_n(\mathbf{r}), L_{ij}(\mathbf{r},\mathbf{r}')\cdot f_m(\mathbf{r}')\rangle. \tag{7}$$

For L_{11}, we have

$$\begin{aligned} \{L_{11}\}_{nm} &= \langle f_n(\mathbf{r}), L_{11}(\mathbf{r},\mathbf{r}')\cdot f_m(\mathbf{r}')\rangle \\ &= \int_S dS' f_m(\mathbf{r}')\int_S dS' g_1(\mathbf{r},\mathbf{r}')f_m(\mathbf{r}') \\ &= \int_S dS' f_m(\mathbf{r}')\int_S dS g_1(\mathbf{r},\mathbf{r}')f_n(\mathbf{r}) \\ &= \langle f_m(\mathbf{r}), L_{11}(\mathbf{r},\mathbf{r}')\cdot f_n(\mathbf{r})\rangle = \{L_{11}\}_{mn}. \end{aligned} \tag{8}$$

So, $\{L_{11}\}_{nm}$ is symmetric with respect to m and n. Similarly, $\{L_{21}\}_{nm}$ is also symmetric.

For $\{L_{12}\}_{nm}$, we have

$$\begin{aligned} \{L_{12}\}_{nm} &= \langle f_n(\mathbf{r}), L_{12}(\mathbf{r},\mathbf{r}')f_m(\mathbf{r}')\rangle \\ &= \int_S dS f_n(\mathbf{r})\int_S dS'\hat{n}\cdot\nabla' g_1(\mathbf{r},\mathbf{r}')f_m(\mathbf{r}') \\ &= -\int_S dS' f_m(\mathbf{r}')\int_S dS\hat{n}\cdot\nabla g_1(\mathbf{r}',\mathbf{r})f_n(\mathbf{r}') \\ &= -\langle f_m(\mathbf{r}'), L_{12}(\mathbf{r}',\mathbf{r})f_n(\mathbf{r})\rangle = -\{L_{12}\}_{mn}. \end{aligned} \tag{9}$$

So, $\{L_{12}\}_{nm}$ is a skew-symmetric matrix. So is $\{L_{22}\}_{nm}$.

§8.10

(a) Equation (8.2.14) is

$$I_1 = \int_S dS' \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot \hat{n} \times \mathbf{H}(\mathbf{r}') \tag{1}$$

To prove its continuity, we only need to prove that the contribution from the singularity is zero. From the Figure,

Figure for Exercise Solution 8.10

$$I_1 = \mathop{\int\!\!\!\!\!\!-}_S dS' \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot \hat{n} \times \mathbf{H}(\mathbf{r}') + \text{Res} \tag{2}$$

where

$$\mathop{\int\!\!\!\!\!\!-}_S dS'$$

denotes a principal value integral, which is the integration on S excluding S_δ, while,

$$\begin{aligned}
\text{Res} &= \lim_{\delta\to 0} \int_{S_\delta} dS' \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot \hat{n} \times \mathbf{H}(\mathbf{r}') \\
&= \hat{n} \times \mathbf{H}(\mathbf{r}) \lim_{\delta\to 0} \int_{S_\delta} dS' \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}').
\end{aligned} \tag{3}$$

Recall that

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) g(\mathbf{r},\mathbf{r}'), \tag{4}$$

we have

$$\begin{aligned}
\text{Res} &= \hat{n} \times \mathbf{H}(\mathbf{r}) \cdot \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) \lim_{\delta\to 0} \int_0^{\pi} \int_0^{2\pi} \delta^2 \sin\theta d\theta d\psi \frac{e^{ik\delta}}{4\pi\delta} \\
&= \hat{n} \times \mathbf{H}(\mathbf{r}) \cdot \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) \lim_{\delta\to 0} \frac{1}{2}\delta = 0.
\end{aligned} \tag{5}$$

Since "Res" is zero and the principal value integral is continuous, I_1 is continuous.

(b) Using Figure 1, we can write eq. (8.2.18) as

$$\hat{n} \times \mathbf{I}_2 = -\fint_S dS'\hat{n} \times \mathbf{E}(\mathbf{r}')\hat{n} \cdot \nabla' g(\mathbf{r}, \mathbf{r}') - \text{Res} \tag{6}$$

where

$$\begin{aligned} \text{Res} &= \lim_{\delta \to 0} \int_{S_\delta} dS'\hat{n} \times \mathbf{E}(\mathbf{r}')\hat{n} \cdot \nabla' g(\mathbf{r}, \mathbf{r}') \\ &= \hat{n} \times \mathbf{E}(\mathbf{r}) \lim_{\delta \to 0} \int_{S_\delta} dS'\hat{n} \cdot \nabla g(\mathbf{r}, \mathbf{r}'). \end{aligned} \tag{7}$$

Applying Equation (8.2.11), we have

$$\text{Res} = -\hat{n} \times \mathbf{E}(\mathbf{r})\frac{1}{2}. \tag{8}$$

Substituting the above to (5), we have

$$\hat{n} \times \mathbf{I}_2 = \frac{1}{2}\hat{n} \times \mathbf{E}(\mathbf{r}) - \fint_S dS'\hat{n} \times \mathbf{E}(\mathbf{r}')\hat{n} \cdot \nabla'(\mathbf{r}, \mathbf{r}'). \tag{9}$$

§8.11

(a) From (8.2.23), $\hat{n} \times \mathbf{H}_n$ on the n-th patch can be expanded as

$$\hat{n} \times \mathbf{H}_n = \sum_{i=1}^{3} [h_{i1n}\mathbf{N}_{i1n} + h_{i2n}\mathbf{N}_{i2n}] \tag{1}$$

$$= \sum_{i=1}^{3} [\mathbf{N}_{i1n}, \mathbf{N}_{i2n}] \begin{bmatrix} h_{i1n} \\ h_{i2n} \end{bmatrix} \tag{2}$$

$$= \sum_{i=1}^{3} \mathbf{N}_{in} \cdot \mathbf{h}_{in} = \mathbf{N}_n^t \cdot \mathbf{h}_n, \tag{3}$$

where

$$\mathbf{N}_n^t = [\mathbf{N}_{1n}, \mathbf{N}_{2n}, \mathbf{N}_{3n}], \quad h_n^t = [\mathbf{h}_{1n}^t, \mathbf{h}_{2n}^t, \mathbf{h}_{3n}^t], \tag{4}$$

and

$$\mathbf{N}_{in} = [\mathbf{N}_{i1n}, \mathbf{N}_{i2n}], \quad \mathbf{h}_{in}^t = [h_{i1n}, h_{i2n}]. \tag{5}$$

It is clear that the dimension of $\mathbf{h}_n$ is six.

(b) From (a), we know the expression of $\hat{n} \times \mathbf{H}$ on the n-th patch. Thus, for a surface S which is approximated by a union of N triangular patches, we have

$$\hat{n} \times \mathbf{H} = \sum_{i=1}^{N} \mathbf{N}_n^t \cdot \mathbf{h}_n = \mathbf{N}^t \cdot \mathbf{h}. \tag{6}$$

$$\text{where} \quad \mathbf{N}^t = [\mathbf{N}_1^t, \mathbf{N}_2^t, \cdots, \mathbf{N}_N^t], \quad \text{and} \quad \mathbf{h}^t = [\mathbf{h}_1^t, \mathbf{h}_2^t, \cdots \mathbf{h}_N^t]$$

Since $\mathbf{h}_i$ is a column vector of length 6, $\mathbf{h}$ is a column vector of length $6N$.

(c)

Figure for Exercise Solution 8.11

From the Figure, for patch P_1, there are two component on l_{n_1,n_2} which are $h_{p_1n_1}$, $h_{p_1n_2}$. Similarly, for patch P_2 there are two component on $l_{n_1n_2}$, which are $h_{p_2n_1}, h_{p_2n_2}$. In order to maintain the normal component of $\hat{n} \times \mathbf{H}$ across $l_{n_1n_2}$ continuous. We need $h_{p_1n_1} = h_{p_2n_1}$, $h_{p_1n_2} = h_{p_2n_2}$. Thus, for edge $l_{n_1n_2}$, there are two independent component left. So, the total independent unknowns will be $2M$, where M is the total number of edges on S.

For each patch, there are three edges. If all N patches are disconnected, there will be $3N$ edges. However, S is a continuous surface, for each patch, it has to share one edge with another patch. Consequently, the total number of edges $M < 3N$. Thus we have $2M < 6N$.

(d) From (c), we know that there are only $2M$ fundamental unknowns in $\mathbf{h}$. Thus there are $6N - 2M$ components which are given by continuity of normal components of $\hat{n} \times \mathbf{H}$, given which is a column vector of length $2M$ containing the fundamental unknowns, by a matrix $\mathbf{M}$,

$$\mathbf{h} = \mathbf{M}^t \cdot \boldsymbol{\eta}. \tag{7}$$

where $\mathbf{M}$ is a $2M \times 6N$ matrix. So, we have

$$\hat{n} \times \mathbf{H} = \mathbf{N}^t \cdot \mathbf{h} = \mathbf{N}^t \cdot \mathbf{M}^t \cdot \boldsymbol{\eta}. \tag{8}$$

(e) Equation (8) can be interpreted as that $\hat{n} \times \mathbf{H}$ is expanded on the basis $\mathbf{N}^t \cdot \mathbf{M}^t$ with unknown vector $\boldsymbol{\eta}$. Thus for $\int_S dS' \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \hat{n} \times \mathbf{H}(\mathbf{r}')$, the matrix representation is

$$\begin{aligned} &\langle \mathbf{M} \cdot \mathbf{N}, \int_S dS' \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{N}^t \cdot \mathbf{M}^t \rangle \\ =& \mathbf{M} \cdot \langle \mathbf{N}, \int_S dS' \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{N}^t \rangle \cdot \mathbf{M}^t. \end{aligned} \tag{9}$$

The dimension of the above matrix is $2M \times 2M$.

§8.12

$\mathbf{g}_n$ can be also defined as

$$\begin{bmatrix} 0 & g_{1n}(\rho') \\ g_{2n}(\rho') & 0 \end{bmatrix} \tag{1}$$

which forms a complete set. In this set, (8.2.27) becomes

$$\begin{bmatrix} E_{1z} \\ H_{1z} \end{bmatrix} = \sum_n \begin{bmatrix} 0 & g_{1n}(\rho') \\ g_{2n}(\rho) & 0 \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} = \sum_{n=1}^{N} \mathbf{g}_n \cdot \mathbf{r}_n \tag{2}$$

$$E_{1z} = \sum_n \beta_n g_{1n}(\rho') \tag{3}$$

$$H_{1z} = \sum_n \alpha_n g_{2n}(\rho'). \tag{4}$$

§8.13

(a) Since $e^{-ik\rho\cos\phi}$ is regular at the origin, we can expand it in terms of $\{J_n(k\rho)e^{in\phi}\}$. This is

$$e^{-ik\rho\cos\phi} = \sum_{n=-\infty}^{\infty} a_n J_n(k\rho) e^{in\phi}. \tag{1}$$

Equation (2.2.17) is

$$J_n(k_\rho \rho) e^{in\phi} = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \, e^{ik\rho\cos(\alpha-\phi)+in\alpha-in\frac{\pi}{2}}. \tag{2}$$

We can rewrite Equation (1) as

$$e^{-ik\rho\cos\phi} = \sum_{n=-\infty}^{\infty} a_n (-1)^n J_n(-k\rho) e^{in\phi}. \tag{3}$$

Applying (2.2.17), we have

$$e^{-ik\rho\cos\phi} = \sum_{n=-\infty}^{\infty} a_n(-1)^n \frac{1}{2\pi}\int_0^{2\pi} d\alpha e^{-ik\rho\cos(\alpha-\phi)+in\alpha-in\frac{\pi}{2}}$$
$$= \int_0^{2\pi} d\alpha e^{-ik\rho\cos(\alpha-\phi)} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} a_n e^{in\frac{\pi}{2}} e^{in\alpha}. \tag{4}$$

In order to make the right-hand side of the above equation equal to the left-hand side, we need

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} a_n e^{in\frac{\pi}{2}} e^{in\alpha} = \delta(\alpha). \tag{5}$$

¿From the complements of $\{e^{in\alpha}\}$, we get

$$a_n = e^{-in\frac{\pi}{2}}. \tag{6}$$

Finally, we have

$$e^{-ik\rho\cos\phi} = \sum_{n=-\infty}^{\infty} J_n(k\rho)e^{in\phi-in\frac{\pi}{2}}. \tag{7}$$

(b) Using the relation of

$$g(\boldsymbol{\rho}-\boldsymbol{\rho}') = \frac{1}{4}H_0^{(1)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) = \frac{i}{4}\sum_{n=-\infty}^{\infty} J_n(k\rho_<)H_n^{(1)}(k\rho_>)e^{in(\phi-\phi')}, \tag{8}$$

and Equation (7) in (8.2.31), we have

$$\sum_{n=-\infty}^{\infty} J_n(k\rho)e^{in\phi-in\frac{\pi}{2}}$$
$$= \frac{i}{4}\sum_{n=-\infty}^{\infty} J_n(k\rho)e^{in\phi}H_n^{(1)}(ka)\int_0^{2\pi} d\phi e^{-in\phi'}\hat{n}\cdot\nabla'\phi_1(\mathbf{r}'), \quad \rho < a. \tag{9}$$

(c) Substituting $\hat{n}\cdot\nabla'\phi_1(\mathbf{r}') = \sum_{m=-M}^{M} a_m e^{im\phi'}$ into the above equation, we have

$$\sum_{n=-\infty}^{\infty} J_n(k\rho)e^{in\phi-in\frac{\pi}{2}} = \frac{i}{4}\cdot\sum_{m=-M}^{M} J_m(k\rho)e^{in\phi}H_m^{(1)}(ka)2\pi a_m. \tag{10}$$

We test (1) with $\rho = a$ by multiplying the side of (1) by $e^{-ip\phi}$ and integrating over ϕ. Then

$$J_p(ka)e^{-ip\frac{\pi}{2}} = \frac{i}{4} J_p(ka)H_p^{(1)}(ka)2\pi a_p. \tag{11}$$

(d) At the internal resonance, $J_p(ka) = 0$. Thus, the both sides of the equation in (c) are zero. Therefore, a_p is undefined.

(e) Applying (2) and $\hat{n} \cdot \nabla' \phi_1(\mathbf{r}') = \sum_{m=-M}^{M} a_m e^{im\phi'}$ to Equation (8.2.32), we have

$$\begin{aligned}
&\int_S dS' g_1(\mathbf{r}, \mathbf{r}')\hat{n} \cdot \nabla' \phi_1(\mathbf{r}') \\
&= \sum_{p=-\infty}^{\infty} H_n(k\rho)e^{in\phi} J_n(ka) \int_0^{2\pi} d\phi' e^{-in\phi'} \sum_{p=-M}^{M} e^{ip\phi'} a_p \\
&= \sum_p H_p(k\rho)e^{ip\phi} J_p(ka) 2\pi a_p.
\end{aligned} \tag{12}$$

Here, a_p is the resonant sources. From (c) we know that

$$J_p(ka) = 0. \tag{13}$$

So, we have

$$\int_S dS' g_1(\mathbf{r}, \mathbf{r}')\hat{n} \cdot \nabla' \phi_1(\mathbf{r}') = 0, \tag{14}$$

for the resonant sources.

Using the relation in (a), we have

$$\begin{aligned}
&\int_S dS' \phi_{inc}(\mathbf{r}')\hat{n} \cdot \nabla \phi_1(\mathbf{r}') \\
&= \sum_{n=-\infty}^{\infty} J_n(k\rho)e^{-in\frac{\pi}{2}} \int_0^{2\pi} d\phi' e^{in\phi'} \sum_p e^{ip\phi'} a_p. \\
&= \sum_p J_p(k\rho)e^{ip\frac{\pi}{2}} 2\pi a_p = \sum_p (-1)^p J_p(k\rho)e^{ip\frac{\pi}{2}} 2\pi a_p \\
&= 0.
\end{aligned} \tag{15}$$

In the last step, $J_p(k\rho) = 0$ for the internal resonant case has been used.

§8.14

From Exercise 8.13. the combined field integral equation can be written as

$$\sum_{n=-\infty}^{\infty}[J_n(ka)+\lambda kJ_n'(ka)]e^{in\phi-in\frac{\pi}{2}}$$

$$=\frac{i}{4}\sum_{n}[J_n(ka)+\lambda kJ_n'(ka)]\,e^{in\phi}H_n^{(1)}(ka)\int_0^{2\pi}d\phi' e^{-in\phi'}\hat{n}\cdot\nabla\phi_1(\mathbf{r}'), \qquad (1)$$

where λ is complex or pure imaginary. From the above, we can see that the right-hand side of the equation will not be zero even at internal resonances, $J_n(ka)$ and $J_n'(ka)$ cannot be zero at same point. Therefore, the indeterminacy in Exercise 8.13(d) due to internal resonances does not exist for this integral equation.

§8.15

Using

$$\hat{n}\cdot\nabla'\phi_1(\mathbf{r}')=\sum_{m=-M}^{M}a_m e^{im\phi'} \qquad (1)$$

in the integral equation of Exercise 8.13(b), we have

$$\sum_{n=-\infty}^{\infty}J_n(k\rho)e^{-in\phi-in\frac{\pi}{2}}$$

$$=\frac{i}{4}\sum_{n=-\infty}^{\infty}J_n(k\rho)e^{in\phi}H_n^{(1)}(ka)\int_0^{2\pi}d\phi' e^{in\phi'}\sum_{m=-M}^{M}a_m e^{im\phi'}. \qquad (2)$$

¿From the above, we obtain

$$\sum_{n=-\infty}^{\infty}J_n(k\rho)e^{in\phi-in\frac{\pi}{2}}=\frac{i}{4}\sum_{n=-M}^{M}J_n(k\rho)e^{in\phi}H_n^{(1)}(ka)2\pi a_n. \qquad (3)$$

Testing Equation (8.15.3) with $\delta(\rho-a)e^{-ip\phi}$ and $\delta(\rho-a+\Delta)e^{-ip\phi}$, where $p=-M,\ \cdots,\ M$ and $\Delta\ll a$, we have

$$J_p(ka)e^{-ip\frac{\pi}{2}}=\frac{i}{4}J_p(ka)H_p^{(1)}(ka)2\pi a_p, \quad p=-M,\ \cdots,\ M, \qquad (4a)$$

$$J_p\left(k(a-\Delta)\right)e^{-ip\frac{\pi}{2}}=\frac{i}{4}J_p\left(k(a-\Delta)\right)H_p^{(1)}\left(k(a-\Delta)\right)2\pi a_p,$$

$$p=-M,\ \cdots,\ M. \qquad (4b)$$

The above is a set overdetermined equations with $2(2M+1)$ equations but only $2M+1$ unknowns. A least-square solution can be obtained from the above equations. At the internal resonant frequency, where $J_q(ka)=0$, Equation (4a) is zero on both sides when $p=q$, but Equation (4b) is still well defined for $p=q$. Thus, a_q can be solved for even at the internal resonant frequency. However, if both $J_q(ka)$ and $J_q(k(a-\Delta))$ are equal to zero, then a_q will be undefined. The above conditions are exactly the resonant conditions of the annular region bounded by a and $a-\Delta$. Since $\Delta \ll a$, the resonant frequency of the annular region will be much higher then the internal resonant frequency.

§8.16

In two dimensions, the scalar Green's function for a homogeneous medium is

$$g(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{i}{4} H_0^{(1)}\left(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|\right). \tag{1}$$

Using the addition theorem of Equation (3.3.2), we have

$$g(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{i}{4}\sum_{n=-\infty}^{\infty} J_n(k\rho_<)H_n^{(1)}(k\rho_>)e^{in(\phi-\phi')}. \tag{2}$$

Define

$$\psi_n(k,\rho_>) = \begin{cases} \dfrac{1}{2\sqrt{k}}H_n^{(1)}(k\rho)e^{in\phi}, & \rho>\rho' \\ \dfrac{1}{2\sqrt{k}}H_n^{(1)}(k\rho')e^{-in\phi'}, & \rho'>\rho \end{cases}$$

$$\Re e\psi_n(k,\rho_<) = \begin{cases} \dfrac{1}{2\sqrt{k}}J_n(k\rho')e^{-in\phi'}, & \rho>\rho' \\ \dfrac{1}{2\sqrt{k}}J_n(k\rho)e^{in\phi}, & \rho'>\rho. \end{cases}$$

Then Equation (2) becomes

$$g(\boldsymbol{\rho},\boldsymbol{\rho}') = ik\sum_{n=-\infty}^{\infty} \psi_n(k,\rho_>)\Re e\psi_n(k,\rho_<). \tag{5}$$

In three dimensions, the scalar Green's function is

$$g(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{6}$$

¿From Equation (3.7.4), we have

$$g(\mathbf{r},\mathbf{r}') = ik\sum_{n=0}^{\infty}\sum_{m=-n}^{n} j_n(kr_<)h_n^{(1)}(kr_>)Y_{nm}(\theta,\phi)Y_{n,-m}(\theta',\phi'). \tag{7}$$

Equation (7) may be written as

$$g(\mathbf{r},\mathbf{r}') = ik \sum_{n=0}^{\infty} \psi_n^t(k,\mathbf{r}_>) \cdot \Re e\psi_n(k,\mathbf{r}_<) \tag{8}$$

if we define ψ and $\Re e\psi$ properly. Hence, we let

$$[\psi_n(k,\mathbf{r}_>)]_m = \begin{cases} h_n^{(1)}(kr)Y_{nm}(\theta,\phi), & r > r' \\ h_n^{(1)}(kr')Y_{n,-m}(\theta',\phi'), & r' > r, \end{cases} \tag{9}$$

$$[\Re e\psi_n(k,\mathbf{r}_<)]_m = \begin{cases} j_n(kr')Y_{n,-m}(\theta',\phi') & r > r' \\ j_n(kr)Y_{nm}(\theta,\phi) & r' > r \end{cases} \tag{10}$$

where ψ_n and $\Re e\psi$ are now vectors of length $2n+1$.

§8.17

(a) For $\mathbf{r} \in V_1$, using the extinction theorem of Equation (8.1.11), we have

$$0 = \int_S dS'\hat{n} \cdot [g_2(\mathbf{r},\mathbf{r}')\nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}')\nabla' g_2(\mathbf{r},\mathbf{r}')] \,. \tag{1}$$

Applying the boundary condition $\phi_2(\mathbf{r}') = 0$ on S, we obtain

$$0 = \int_S dS' g_2(\mathbf{r},\mathbf{r}')\hat{n}\nabla'\phi_2(\mathbf{r}'), \quad \mathbf{r} \in V_1. \tag{2}$$

Now let

$$g_2(\mathbf{r},\mathbf{r}') = ik_2 \sum_n \psi_n(k_2,\mathbf{r})\Re e\psi_n(k_2,\mathbf{r}). \tag{3}$$

Here, $\mathbf{r} \in V_1$, $\mathbf{r}' \in S$, and $|\mathbf{r}| > |\mathbf{r}'|$. Substituting (8.17.3) into (8.17.2), we have

$$0 = ik_2 \sum_n \psi_n(k_2,\mathbf{r}') \int_S dS'\Re e\psi_n(k_2,\mathbf{r}')\hat{n}\nabla'\phi_2(\mathbf{r}') \tag{4}$$

Since $\{\psi_n, n = -\infty, \cdots, \infty\}$ is an orthogonal set, we have

$$0 = \int_S dS'\Re e\psi_n(k_2,\mathbf{r}')\hat{n} \cdot \nabla'\phi_2(\mathbf{r}'), \quad \mathbf{r} \in V_1, \quad \text{for all } n. \tag{5}$$

At the nonresonant frequencies, the above has only a trivial solution for $\hat{n} \cdot \nabla'\phi_2(\mathbf{r}')$. This means that $\{\Re e\psi_n(k_2\mathbf{r}'),\ n = -\infty, \cdots, \infty\}$ is complete on S, except at the resonant frequencies of the cavity.*

(b) Similarly, if we impose the boundary condition

$$\hat{n} \cdot \phi_2(\mathbf{r}) = 0, \quad \text{on } S. \tag{6}$$

* This proof, due to Waterman, may be flawed.

then, Equation (1) becomes

$$0 = \int dS' \phi_2(\mathbf{r}')\hat{n} \cdot \nabla' g_2(\mathbf{r}, \mathbf{r}'), \quad \mathbf{r} \in V_1. \tag{7}$$

Using (3) in (7), we have

$$0 = ik_2 \underset{n}{\;} \psi_n(k_2, \mathbf{r}) \int_S dS' \phi_2(\mathbf{r}')\hat{n} \cdot \nabla' \Re e\psi_n(k_2, \mathbf{r}') \tag{8}$$

Finally, we obtain

$$0 = \int_S dS' \phi_2(\mathbf{r}')\hat{n} \cdot \nabla' \Re e\psi_n(k_2, \mathbf{r}'), \quad \mathbf{r} \in V_1, \quad \text{for all n} \tag{9}$$

At the nonresonant frequencies, (8.17.9) has only a trivial solution for $\phi_2(\mathbf{r}')$. This means that at the nonresonant frequencies, the only function that is orthogonal to $\{\hat{n} \cdot \nabla \Re e\psi_n(k_2, \mathbf{r}), \quad n = -\infty, \cdots, \infty\}$ is zero. Consequently, $\{\hat{n} \cdot \nabla \Re e\psi_n(k_2, \mathbf{r}'), \quad n = -\infty, \cdots, \infty\}$ is complete on S, except at the resonant frequencies of the cavity.* The boundary condition in this case is $\hat{n} \cdot \nabla' \phi(\mathbf{r}') = 0$, for $\mathbf{r}' \in S$.

(c) For particular geometries, the resonant frequencies for the Dirichlet and the Neumann problems may coincide. For example, in a rectangular cavity, the fields for the Dirichlet and Neumann problem are given as

$$\phi_D(\mathbf{r}) = \phi_{OD} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{c}\right) \tag{10}$$

$$\phi_N(\mathbf{r}) = \phi_{ON} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{c}\right) \tag{11}$$

The resonant frequencies in both cases are given by

$$k_2 = \frac{\omega}{c_2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2}\,. \tag{12}$$

§8.18

$$F(\mathbf{r}') = \nabla' \cdot [\Re e\psi_n(k_2, \mathbf{r}')\nabla' \Re e\psi_m(k_2, \mathbf{r}') \\ - \Re e\psi_m(k_2, \mathbf{r}')\nabla' \Re e\psi_n(k_2, \mathbf{r}')]\,. \tag{1}$$

Since $\Re e\psi_n(k_2, \mathbf{r}')$ satisfies the wave equation,

$$\left(\nabla^2 + k_2^2\right) \Re e\psi_n(k_2, \mathbf{r}') = 0, \quad \text{for all } n, \tag{2}$$

we have $F(\mathbf{r}) = 0$.

Now, we integrate (1) over a volume V bounded by S and S_2, we obtain

$$\int_V dV \nabla' \cdot [\Re e\psi_n(k_2,\mathbf{r}')\nabla'\Re e\psi_m(k_2,\mathbf{r}') \cdot$$
$$\Re e\psi_m(k_2,\mathbf{r}')\nabla'\Re e\psi_n(k_2,\mathbf{r}')] = 0. \quad (3)$$

Applying Gauss' theorem, we get

$$\int_{S+S_2} dS' [\Re e\psi_n(k_2,\mathbf{r}')\hat{n}\cdot\nabla'\Re e\psi_m(k_2,\mathbf{r}')$$
$$-\Re e\psi_m(k_2,\mathbf{r}')\hat{n}\cdot\nabla'\Re e\psi_n(k_2,\mathbf{r}')] = 0. \quad (4)$$

On S_2, the above the integral is zero due to the orthogonality of spherical harmonics on a spherical surface. Finally, we have

$$\int_S dS' [\Re e\psi_n(k_2,\mathbf{r}')\hat{n}\cdot\nabla'\Re e\psi_m(k_2,\mathbf{r}')]$$
$$= \int_S dS [\Re e\psi_m(k_2,\mathbf{r}')\hat{n}\cdot\nabla\Re e\psi_n(k_2,\mathbf{r}')] . \quad (5)$$

§8.19

(a) From Equation (8.3.7), with the boundary condition $\phi_1(\mathbf{r}') = 0$, $\mathbf{r}' \in S$,

$$a_n = ik_1 \int_S dS'\psi_n(k_1,\mathbf{r}')\hat{n}\cdot\nabla'\phi_1(\mathbf{r}'), \quad \forall n.$$

Expanding $\hat{n}\cdot\nabla'\phi_1(\mathbf{r}')$ as in (8.3.10),

$$\hat{n}\cdot\nabla'\phi_1(\mathbf{r}') = \sum_m \alpha_m \hat{n}\cdot\nabla'\Re e\psi_m(k_1,\mathbf{r}'),$$

yields the equation

$$a_n = ik_1 \sum_m \alpha_m \int_S dS'\psi_n(k_1,\mathbf{r}')\hat{n}\cdot\nabla'\Re e\psi_m(k_1,\mathbf{r}').$$

This has the form

$$a_n = i\sum_m Q_{nm}\alpha_m,$$

where

$$Q_{nm} = k_1 \int_S dS'\psi_n(k_1,\mathbf{r}')\hat{n}\cdot\nabla'\Re e\psi_m(k_1,\mathbf{r}').$$

(b) As discussed in Exercise 8.17, the set $\{\hat{n} \cdot \nabla' \Re e \psi_m(k_1, \mathbf{r}')\}$ is incomplete on S at the resonant frequencies of the cavity formed by S, with interior wavenumber k_1 and boundary conditon $\hat{n} \cdot \nabla' \phi(\mathbf{r}') = 0$, $\mathbf{r}' \in S$. That is, there exists a non-zero function $\phi(\mathbf{r}')$ such that

$$\int_S dS' \phi(\mathbf{r}') \hat{n} \cdot \nabla' \Re e \psi_m(k_1, \mathbf{r}') = 0, \qquad \forall m.$$

If the set $\{\psi_n(k_1, \mathbf{r}')\}$ is complete at such a resonant frequency, then $\phi(\mathbf{r}')$ can be expanded as

$$\phi(\mathbf{r}') = \sum_n c_n \psi_n(k_1, \mathbf{r}').$$

Then,

$$\sum_n c_n \int_S dS' \psi_n(k_1, \mathbf{r}') \hat{n} \cdot \nabla' \Re e \psi_m(k_1, \mathbf{r}') = 0.$$

¿From part (a), this is

$$\frac{1}{k_1} \sum_n c_n Q_{nm} = 0.$$

Thus, $\overline{\mathbf{Q}}$ has a nullspace, and is ill-conditioned.

(c) If $\hat{n} \cdot \nabla' \phi(\mathbf{r}')$ is expanded using a compete set $\{u_m(\mathbf{r}')\}$,

$$\hat{n} \cdot \nabla' \phi(\mathbf{r}') = \sum_m \alpha_m u_m(\mathbf{r}'),$$

then the elements of $\overline{\mathbf{Q}}$ are given by

$$Q_{nm} = k_1 \int_S dS' \psi_n(k_1, \mathbf{r}') u_m(\mathbf{r}').$$

The matrix $\overline{\mathbf{Q}}$ is singular if and only if one of the following is true. Either

$$\sum_m Q_{nm} \beta_m = \int_S dS' \psi_n(k_1, \mathbf{r}') \left[\sum_m \beta_m u_m(\mathbf{r}') \right] = 0, \qquad \forall n,$$

with β_m not all zero, or

$$\sum_n \gamma_n Q_{nm} = \int_S dS' \left[\sum_n \gamma_n \psi_n(k_1, \mathbf{r}') \right] u_m(\mathbf{r}') = 0, \qquad \forall m,$$

with γ_n not all zero. The second case above cannot be true if $\{u_m\}$ is a complete set on S. The first case corresponds to the incompleteness of the set $\{\psi_n(k_1, \mathbf{r}')\}$. So, as long as both $\{u_m(\mathbf{r}')\}$ and $\{\psi_n(k_1, \mathbf{r}')\}$ are complete on S, $\overline{\mathbf{Q}}$ will be nonsingular.

(d) For a penetrable scatterer,

$$Q_{nm} = k_1 \int_S dS' \Big[\psi_n(k_1, \mathbf{r}') \hat{n} \cdot \nabla' \Re e \psi_m(k_2, \mathbf{r}') \frac{p_2}{p_1} - \Re e \psi_m(k_2, \mathbf{r}') \hat{n} \cdot \nabla' \psi_n(k_1, \mathbf{r}') \Big]. \tag{8.3.16}$$

For most object shapes, the two types of resonance (with homogeneous Dirichlet or Neumann boundary conditions) do not occur at the same frequency. Therefore, from the results of Exercise (8.17), the sets $\{\hat{n} \cdot \nabla' \Re e \psi_m(k_2, \mathbf{r}')\}$ and $\{\Re e \psi_m(k_2, \mathbf{r}')\}$ do not ususally become incomplete simultaneously. Based on parts (b) and (c), we expect $\overline{\mathbf{Q}}$ to be well-conditioned in this case. However, for some shapes, such as a square scatterer, it is possible for the two types of resonance to occur together. In this case, $\overline{\mathbf{Q}}$ may become ill-conditioned.

§8.20

For a metallic cylinder $\phi(\mathbf{r}') = 0$, $\mathbf{r}' \in S$, so the integral equation (8.3.1) becomes

$$\phi_{inc}(\mathbf{r}) = \int_S dS' g_1(\mathbf{r}, \mathbf{r}') \hat{n} \cdot \nabla' \phi_1(\mathbf{r}'), \qquad \mathbf{r} \in V_2.$$

In two dimensions, the Green's function is

$$g(\mathbf{r}, \mathbf{r}') = \frac{i}{4} H_0^{(1)}(k_1 |\boldsymbol{\rho} - \boldsymbol{\rho}'|) = \frac{i}{4} \sum_{n=-\infty}^{\infty} J_n(k_1 \rho_<) H_n^{(1)}(k_1 \rho_>) e^{in(\phi - \phi')}.$$

Expanding $\phi_{inc}(\mathbf{r})$ as in (8.3.4), with $\psi_n = H_n^{(1)}(k_1 \rho) e^{in\phi}$, the integral equation becomes

$$\sum_n a_n J_n(k_1 \rho) e^{in\phi} = \frac{i}{4} \sum_n J_n(k_1 \rho) e^{in\phi} \int_S dS' H_n^{(1)}(k_1 \rho') e^{-in\phi'} \hat{n} \cdot \nabla' \phi_1(\mathbf{r}'), \qquad \mathbf{r} \in S_2,$$

where S_2 is chosen to be a cylindrical surface inside of S. For a circular cylinder of radius a, the integral on the right is

$$a H_n^{(1)}(k_1 a) \int_0^{2\pi} d\phi' e^{-in\phi'} \hat{n} \cdot \nabla' \phi_1(\mathbf{r}').$$

Then, from the orthogonality of the cylindrical harmonics on S_2, we have the equivalent of (8.3.7),

$$a_n = \frac{i}{4} a H_n^{(1)}(k_1 a) \int_0^{2\pi} d\phi' e^{-in\phi'} \hat{n} \cdot \nabla' \phi_1(\mathbf{r}').$$

Note that in the above step we were free to choose the radius "b" of the testing circle S_2 such that $J_n(k_1 b) \neq 0$, thus allowing the cancellation of $J_n(k_1 \rho)$ from both sides of the equation. This equation is easily solved by expanding $\hat{n} \cdot \nabla' \phi(\mathbf{r}')$ in a Fourier series, $\hat{n} \cdot \nabla' \phi(\mathbf{r}') = \sum_m \alpha_m e^{im\phi'}$. Then we have

$$\int_0^{2\pi} d\phi' e^{-in\phi'} \hat{n} \cdot \nabla' \phi(\mathbf{r}') = \sum_m \alpha_m \int_0^{2\pi} d\phi' e^{i(m-n)\phi'} = 2\pi\alpha_n,$$

so that

$$a_n = \frac{i}{4} a H_n^{(1)}(k_1 a) \cdot 2\pi\alpha_n.$$

Clearly, the internal resonance problem of Exercise 8.13, which occurs when $J_n(k_1 a) = 0$, is not a problem here.

§8.21

The dyadic Green's function can be written as

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = ik_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{n(n+1)} \Big[\mathbf{M}_{nm}(k_0, \mathbf{r}) \mathbf{M}_{n,-m}(k_0, \mathbf{r}') + \mathbf{N}_{nm}(k_0, \mathbf{r}) \mathbf{N}_{n,-m}(k_0, \mathbf{r}') \Big] - \frac{\hat{r}\hat{r}}{k_0^2} \delta(\mathbf{r} - \mathbf{r}'). \tag{7.3.40}$$

The vector wave functions $\mathbf{M}_{nm}$ and $\mathbf{N}_{nm}$ are defined in Equations (7.2.42-43),

$$\mathbf{M}_{nm}(k, \mathbf{r}) = \nabla \times \mathbf{r} j_n(kr) Y_{nm}(\theta, \phi), \tag{7.2.42}$$

$$\mathbf{N}_{nm}(k, \mathbf{r}) = \frac{1}{k} \nabla \times \nabla \times \mathbf{r} j_n(kr) Y_{nm}(\theta, \phi). \tag{7.2.43}$$

In (7.3.40) it is understood that the spherical Bessel functions in $\mathbf{M}_{nm}(k_0, \mathbf{r})$ and $\mathbf{N}_{nm}(k_0, \mathbf{r})$ should be replaced by the spherical Hankel functions when $r > r'$, and similarly for $\mathbf{M}_{n,-m}(k_0, \mathbf{r}')$ and $\mathbf{N}_{n,-m}(k_0, \mathbf{r}')$ when $r < r'$.

To write $\overline{\mathbf{G}}$ as in (8.3.19), we must reorder the double summation over n, m as a single summation. For example, we can construct a vector of vectors, $\overline{\psi}(k_0, \mathbf{r})$,

$$\overline{\psi}(k_0, \mathbf{r}) \equiv \begin{bmatrix} \psi_1(k_0, \mathbf{r}) \\ \psi_2(k_0, \mathbf{r}) \\ \vdots \\ \psi_l(k_0, \mathbf{r}) \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{00}(k_0, \mathbf{r}) \\ \mathbf{N}_{00}(k_0, \mathbf{r}) \\ \mathbf{M}_{1,-1}(k_0, \mathbf{r}) \\ \mathbf{M}_{10}(k_0, \mathbf{r}) \\ \mathbf{M}_{11}(k_0, \mathbf{r}) \\ \mathbf{N}_{1,-1}(k_0, \mathbf{r}) \\ \mathbf{N}_{10}(k_0, \mathbf{r}) \\ \mathbf{N}_{11}(k_0, \mathbf{r}) \\ \vdots \end{bmatrix},$$

where the $\mathbf{M}_{nm}(k_0, \mathbf{r})$ and $\mathbf{N}_{nm}(k_0, \mathbf{r})$ are defined as in (7.2.42-43) but with spherical Hankel functions instead of spherical Bessel functions, and a small change in normalization. Then a typical element of $\overline{\psi}$ is

$$\psi_l(k_0, \mathbf{r}) = \frac{1}{\sqrt{n(n+1)}} \nabla \times \mathbf{r} h_n(k_0 r) Y_{nm}(\theta, \phi)$$

or

$$\psi_l(k_0, \mathbf{r}) = \frac{1}{\sqrt{n(n+1)}} \frac{1}{k_0} \nabla \times \nabla \times \mathbf{r} h_n(k_0 r) Y_{nm}(\theta, \phi).$$

Furthermore, if the operator $\Re e$ is defined so that

$$\Re e \psi_l(k_0, \mathbf{r}) = \frac{1}{\sqrt{n(n+1)}} \nabla \times \mathbf{r} j_n(k_0 r) Y_{n,-m}(\theta, \phi)$$

or

$$\Re e \psi_l(k_0, \mathbf{r}) = \frac{1}{\sqrt{n(n+1)}} \frac{1}{k_0} \nabla \times \nabla \times \mathbf{r} j_n(k_0 r) Y_{n,-m}(\theta, \phi),$$

then $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ may be written as

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = ik_0 \sum_l \psi_l(k_0, \mathbf{r}_>) \Re e \psi_l(k_0, \mathbf{r}_<), \qquad \mathbf{r}_> \neq \mathbf{r}_<,$$

where $\mathbf{r}_>$ is the larger of $\mathbf{r}$ and $\mathbf{r}'$, and $\mathbf{r}_<$ is the smaller of $\mathbf{r}$ and $\mathbf{r}'$ in magnitude. In writing the above, use has been made of the fact that, by a simple reversal of summation order,

$$\sum_{m=-n}^{n} \mathbf{M}_{nm}(k_0, \mathbf{r}) \mathbf{M}_{n,-m}(k_0, \mathbf{r}') = \sum_{m=-n}^{n} \mathbf{M}_{n,-m}(k_0, \mathbf{r}) \mathbf{M}_{nm}(k_0, \mathbf{r}'),$$

and similarly for $\mathbf{N}_{nm}$. This expansion for $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ matches (8.3.19), and the vector wave functions $\psi_l(k_0, \mathbf{r})$ and operator $\Re e$ are defined above.

§8.22

Substitution of Equations (8.3.19) and (8.3.20) into Equation (8.3.17) gives

$$\begin{aligned} &\sum_n a_n \Re e(\psi) n(k, \mathbf{r}) \\ &= ik_1 \sum_n \int_S dS' \left[i\omega\mu_1 \psi_n(k_1, \mathbf{r}_>) \Re e \psi_n(k_1, \mathbf{r}_<) \cdot \hat{n} \times \mathbf{H}_1(\mathbf{r}') \right. \\ &\left. - \nabla' \times \psi_n(k_1, \mathbf{r}_>) \Re e \psi_n(k_1, \mathbf{r}_<) \cdot \hat{n} \times \mathbf{E}_1(\mathbf{r}') \right], \\ &\qquad \mathbf{r} \in V_2, \quad \mathbf{r}_> \neq \mathbf{r}_<. \end{aligned} \tag{1}$$

However, the above applied for $\mathbf{r} < \mathbf{r}'$ which means that $\mathbf{r}_> = \mathbf{r}'$, $\mathbf{r}_< = \mathbf{r}$. Equation (1) becomes

$$\sum_n a_n \Re e\psi_n(k_1, \mathbf{r}) = ik_1 \sum_n \Re e\psi_n(k_1, \mathbf{r}) \int_S dS' \left[i\omega\mu_1 \psi_n(k_1, \mathbf{r}') \cdot \hat{n} \times \mathbf{H}_1(\mathbf{r}') - \nabla' \times \psi_n(k_1, \mathbf{r}') \cdot \hat{n} \times \mathbf{E}_1(\mathbf{r}') \right], \qquad \mathbf{r} \in V_2. \tag{2}$$

Then applying the orthogonality property of the vector harmonics, we have

$$a_n = ik_1 \int_S dS' \left[i\omega\mu_1 \psi_n(k_1, \mathbf{r}') \cdot \hat{n} \times \mathbf{H}_1(\mathbf{r}') - \nabla' \times \psi_n(k_1, \mathbf{r}') \cdot \hat{n} \times \mathbf{E}_1(\mathbf{r}') \right], \quad \mathbf{r} \in V_2. \tag{3}$$

Substitution of Equation (8.3.19) into Equation (8.3.18) gives

$$0 = ik_2 \int_S dS' \left[i\omega\mu_2 \psi_n(k_2, \mathbf{r}_>) \Re e\psi_n(k_2, \mathbf{r}_<) \cdot \hat{n} \times \mathbf{H}_2(\mathbf{r}') - \nabla' \times \psi_n(k_2, \mathbf{r}_>) \Re e\psi_n(k_2, \mathbf{r}_<) \cdot \hat{n} \times \mathbf{E}_2(\mathbf{r}') \right], \quad \mathbf{r} \in V_1, \mathbf{r}_> \neq \mathbf{r}_<. \tag{4}$$

But Equation (4) applies for $\mathbf{r} > \mathbf{r}'$ and hence $\mathbf{r}_> = \mathbf{r}$, $\mathbf{r}_< = \mathbf{r}'$. Equation (4) becomes

$$0 = ik_2 \sum_n \psi_n(k_e, \mathbf{r}) \int_S dS' [i\omega\mu_3 \Re e\psi_n(k_2, \mathbf{r}') \cdot \hat{n} \times \mathbf{E}_2(\mathbf{r}')], \quad \mathbf{r} \in V_1, \tag{5}$$

and application of the orthogonality property gives

$$0 = ik_2 \int_d S' [i\omega\mu_2 \Re e\psi_n(k_2, \mathbf{r}') \cdot \hat{n} \times \mathbf{H}_2(\mathbf{r}') - \nabla' \times \Re e\psi_n(k_2, \mathbf{r}') \cdot \hat{n} \times \mathbf{E}_2(\mathbf{r}')], \quad \mathbf{r} \in V_2. \tag{6}$$

§8.23

For a cavity filled with a material of wavenumber k_2, $\hat{n} \times \mathbf{E}_2(\mathbf{r}') = 0$, $\mathbf{r}' \in S$ for a perfect electric conductor (PEC) and $\hat{n} \times \mathbf{H}_2(\mathbf{r}) = 0$, $\mathbf{r}' \in S$ for a perfect magnetic conductor (PMC) cavity. For the PEC case, Equation (8.3.22) becomes

$$\omega \int_S dS' \Re e\psi_n(k_2, \mathbf{r}') \cdot \hat{n} \times \mathbf{H}_2(\mathbf{r}') = 0, \quad \forall n, \tag{1}$$

which may be rewritten as

$$\omega \int_S dS'[\hat{n} \times \Re e\psi_n(k_2, \mathbf{r}')] \cdot \mathbf{H}_2(\mathbf{r}') = 0, \quad \forall n, \tag{2}$$

Except at the resonance frequencies of the PEC cavity where $\mathbf{H}_2(\mathbf{r}') = 0$, Equation (2) is an orthogonality relationship and tells us that the only vector $\mathbf{H}_2(\mathbf{r}')$ that is orthogonal to $\{\hat{n} \times \Re e\psi_n(k_2, \mathbf{r}')\}$, $n = 0, 1, \ldots, \infty$ is the zero vector. Therefore the vectors $\{\hat{n} \times \Re e\psi_m(k_2, \mathbf{r}')\}$ form a complete basis set inside the PEC cavity.

Similarly for the PMC cavity, the relevant equation is

$$\omega \int_S dS'[\nabla' \times \Re e\psi_n(k_0, \mathbf{r}')] \cdot \hat{n} \times \mathbf{E}_2(\mathbf{r}') = 0, \quad \forall n, \tag{3}$$

which becomes

$$\omega \int_S dS'[\hat{n} \times \nabla' \times \Re e(k_2, \mathbf{r}')] \cdot \mathbf{E}_2(\mathbf{r}') = 0, \quad \forall n. \tag{4}$$

By a similar argument, the only vector $E_2(\mathbf{r}')$ that is orthogonal to the set $\{\hat{n} \times \nabla' \times \Re e\psi_n(k_2, \mathbf{r}')\}$, $n = 0, 1, \ldots, \infty$ is the zero vector. Hence the set $\{\hat{n} \times \nabla' \times \Re e\psi_n(k_2, \mathbf{r}')\}$ forms a complete basis set inside the PMC cavity.*

§8.24

$$I = \int_V dV'\nabla' \cdot \{[\nabla' \times \Re e\psi_m(k_2, \mathbf{r}')] \times \Re e\psi_n(k_2, \mathbf{r}') + \Re e\psi_m(k_2, \mathbf{r}') \times [\nabla' \times \Re e\psi_n(k_2, \mathbf{r}')]\} \tag{1}$$

$$= \int_V dV'\{\Re e\psi_n \cdot \nabla' \times \nabla' \times \Re e\psi_m - \Re e\psi_m \cdot \nabla' \times \nabla' \times \Re e\psi_n\} \tag{2}$$

$$= k_2^2 \int_V dV'[\Re e\psi_n \cdot \Re e\psi_m - \Re e\psi_m \cdot \Re e\psi_n] = 0, \tag{3}$$

since the vector harmonics satisfy the source-free wave equations

$$\nabla \times \nabla \times \psi(k_2, \mathbf{r}') - k_2^2\psi_n(k_2, \mathbf{r}) = 0. \tag{4}$$

Application of Gauss' divergence to Equation (1) gives

$$I = \int_S dS'\hat{n} \cdot \{[\nabla' \times \Re e\psi_n(k_2, \mathbf{r}')] \times \Re e\psi_n(k_2, \mathbf{r}') + \Re e\psi_m(k_2, \mathbf{r}') \times [\nabla' \times \Re e\psi_n(k_2, \mathbf{r}')] = 0, \tag{5}$$

* These proofs may be flawed.

which is the same as Equation (1). This results in $\alpha_m = -\beta_m$.

§8.25

If

$$(\nabla^2 + k^2)\phi_l(\mathbf{r}) = -\delta(\mathbf{r} - \mathbf{r}_1'), \quad (1)$$
$$(\nabla^2 + k^2)\phi_2(\mathbf{r}) = -\delta(\mathbf{r} - \mathbf{r}_2'), \quad (2)$$

then

$$\phi_1(\mathbf{r}) = \frac{i}{4}H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_1'|) + \psi^t(\mathbf{r}_0') \cdot \overline{\mathbf{T}} \cdot \alpha_{01}\frac{i}{4}, \quad (3)$$
$$\phi_2(\mathbf{r}) = \frac{i}{4}H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_2'|) + \psi^t(\mathbf{r}_0) \cdot \overline{\mathbf{T}} \cdot \alpha_{02}\frac{i}{4}. \quad (4)$$

where $\mathbf{r}_0 = \mathbf{r} - \mathbf{r}_0'$,

$$[\alpha_{0i}]_n = H_n^{(1)}(k|\mathbf{r}_0' - \mathbf{r}_i'|)e^{-in\phi_{0i}}, \quad (5)$$
$$[\psi(\mathbf{r}_0)]_n = H_n^{(1)}(k|\mathbf{r} - \mathbf{r}_0'|)e^{in\phi_0}, \quad (6)$$

and ϕ_{0i} is the angle the vector $\mathbf{r}_i' - \mathbf{r}_0'$ makes with the x axis and ϕ_0 is the angle the vector $\mathbf{r} - \mathbf{r}_0'$ makes with the x axis.
Reciprocity requires that $\phi_1(\mathbf{r}_2') = \phi_2(\mathbf{r}_1')$. The first terms in (3) and (4) clearly satisfy this. In order for the second terms to satisfy this, we need

$$\psi^t(\mathbf{r}_2' - \mathbf{r}_0') \cdot \overline{\mathbf{T}} \cdot \alpha_{01} = \psi^t(\mathbf{r}_1' - \mathbf{r}_0') \cdot \overline{\mathbf{T}} \cdot \alpha_{02}. \quad (7)$$

The above is the same as

$$\alpha_{20}^t \cdot \overline{\mathbf{T}} \cdot \alpha_{01} = \alpha_{10}^t \cdot \overline{\mathbf{T}} \cdot \alpha_{02}, \quad (8)$$

where

$$[\alpha_{i0}]_m = H_m^{(1)}(k|\mathbf{r}_i' - \mathbf{r}_0'|)e^{im\phi_{0i}}. \quad (9)$$

Notice that $[\alpha_{i0}]_{-m} = (-1)^m\,[\alpha_{0i}]_m$. Taking the transpose of (8), we have

$$\alpha_{01}^t \cdot \overline{\mathbf{T}}^t \cdot \alpha_{20} = \alpha_{10}^t \cdot \overline{\mathbf{T}} \cdot \alpha_{02}. \quad (10)$$

The above is the same as

$$\sum_{nm}[\alpha_{01}]_m T_{nm}[\alpha_{20}]_n = \sum_{nm}[\alpha_{10}]_m T_{mn}[\alpha_{02}]_n,$$
$$= \sum_{nm}[\alpha_{01}]_m(-1)^m T_{-m,-n}[\alpha_{20}]_n(-1)^n. \quad (11)$$

Therefore

$$T_{nm} = (-1)^{m+n}T_{-m,-n}. \quad (12)$$

The above is not exactly symmetrical because we have used $e^{im\phi}$ expansion in the cylindrical harmonics. If $\cos n\phi$ and $\sin n\phi$ are used, we will get a symmetric $\overline{\mathbf{T}}$. In this case, $\overline{\mathbf{S}}$ will be symmetric too.

§8.26

(a) When k is real, we have the two wave equations

$$\phi^*(\nabla^2 + k^2)\phi = 0, \tag{1}$$
$$\phi(\nabla^2 + k^2)\phi^* = 0. \tag{2}$$

Subtracting the above,

$$(\phi^*\nabla^2\phi - \phi\nabla^2\phi^*) = 0 \tag{3}$$

from which we have

$$\nabla \cdot [\phi^*\nabla\phi - \phi\nabla\phi^*] = 0. \tag{4}$$

We now take the volume integral of Equation (4) and apply Gauss' divergence theorem to obtain

$$\int_S dS\hat{n} \cdot (\phi^*\nabla\phi - \phi\nabla\phi^*) = 0. \tag{5}$$

(b) To solve this problem we need the Bessel function identities

$$H_n^{(1)}(-z) = -e^{in\pi}H_n^{(2)}(z), \tag{6}$$
$$h_n^{(1)}(-z) = e^{in\pi}h_n^{(2)}(z). \tag{7}$$

Also, we break ψ_n into cosinusoidal and sinusoidal components as

$$\psi_n(k, \rho_>) = \psi_n^c(k, \rho_>) + i\psi_n^s(k, \rho_>). \tag{8}$$

For the 2-D case, we have

$$\psi_n^c(k, \rho) = \frac{1}{2\sqrt{k}}H_n^{(1)}(k\rho)\cos(n\phi), \tag{9}$$
$$\psi_n^s(k, \rho) = \frac{1}{2\sqrt{k}}H_n^{(1)}(k\rho)\sin(n\phi), \tag{10}$$

and for the 3-D harmonics,

$$\psi_{nm}^c(k, \mathbf{r}) = h_n^{(1)}(kr)Y_{nm}^0(\theta, \phi)\cos(m\phi), \tag{11}$$
$$\psi_{nm}^s(k, \mathbf{r}) = h_n^{(1)}(kr)Y_{nm}^0(\theta, \phi)\sin(m\phi). \tag{12}$$

We now redefine the harmonics so as to take care of the phase factors appearing in Equations (6) and (7). For the 2-D case, we let

$$\widetilde{\psi}_n^c(k, \rho) = \psi_n^c(k, \rho)e^{-i\pi/2(n+1/2)} \tag{13}$$
$$\widetilde{\psi}_n^s(k, \rho) = \psi_n^s(k, \rho)e^{-i\pi/2(n+1/2)} \tag{14}$$

and for the 3-D case we let

$$\widetilde{\psi}_n^c(k,\mathbf{r}) = e^{-in\pi/2}\psi_n^c(k,\mathbf{r}) \tag{15}$$
$$\widetilde{\psi}_n^s(k,\mathbf{r}) = e^{-in\pi/2}\psi_n^s(k,\mathbf{r}). \tag{16}$$

It then follows that

$$\widetilde{\psi}_n^c(-k,\mathbf{r}) = [\widetilde{\psi}_n^c(k,\mathbf{r})]^* \tag{17}$$
$$\widetilde{\psi}_n^c(-k,\mathbf{r}) = [\widetilde{\psi}_n^s(k,\mathbf{r})]^* \tag{18}$$

when the medium is lossless for both the cylindrical and spherical harmonics.

(c)

$$\begin{aligned} A_{nm} &= \left\{ \int_S dS\hat{n}\cdot\left[\psi\nabla\psi^t - (\nabla\psi)\psi^t\right]\right\}_{nm} \\ &= \int_S dS\hat{n}\cdot[\psi_n\nabla\psi_m - \nabla\psi_n\psi_m] \\ &= \int_{S+S_{inf}} dS\hat{n}\cdot[\psi_n\nabla\psi_m - \nabla\psi_n\psi_m] \end{aligned} \tag{19}$$

because the surface integral at infinity vanishes by the radiation condition. Using Gauss' theorem, we have

$$\begin{aligned} A_{nm} &= \int_V dV\nabla\cdot[\psi_n\nabla\psi_m - \nabla\psi_n\psi_m] \\ &= \int_V dV[\psi_n\nabla^2\psi_m - \nabla^2\psi_n\psi_m] = 0 \end{aligned} \tag{20}$$

where V is the surface bounded by $S + S_{inf}$.

(d) From Equation (8.4.12),

$$\phi(\mathbf{r}) = \frac{1}{2}[\psi^t(-k_1,\mathbf{r}) + \psi^t(k_1,\mathbf{r})\cdot\mathbf{S}]\cdot\mathbf{a}. \tag{21}$$

Then,

$$\begin{aligned} \phi\nabla\phi^* = \frac{1}{4}\mathbf{a}^t\cdot[\psi(-k_1,\mathbf{r}) + \mathbf{S}^t\cdot\psi(k_1,\mathbf{r})] \\ [\nabla\psi^\dagger(-k_1,\mathbf{r}) + \nabla\psi^\dagger(k_1,\mathbf{r})\cdot\mathbf{S}^*]\cdot\mathbf{a}^* \end{aligned} \tag{22a}$$

$$\begin{aligned} (\nabla\phi)\phi^* = \frac{1}{4}\mathbf{a}^t\cdot[\nabla\psi(-k_1,\mathbf{r}) + \mathbf{S}^t\cdot\nabla\psi(k_1,\mathbf{r})] \\ [\psi^\dagger(-k_1,\mathbf{r}) + \psi^\dagger(k_1,\mathbf{r})\cdot\mathbf{S}^*]\cdot\mathbf{a}^* \end{aligned} \tag{22b}$$

and hence

$$\phi\nabla\phi^* - \phi^*\nabla\phi = \frac{1}{4}\mathbf{a}^t \cdot \{[\boldsymbol{\psi}(-k_1,\mathbf{r})\nabla\boldsymbol{\psi}^\dagger(-k_1,\mathbf{r}) - \nabla\boldsymbol{\psi}(-k_1,\mathbf{r})\boldsymbol{\psi}^\dagger(-k_1,\mathbf{r})]$$
$$+ \mathbf{S}^t \cdot [\boldsymbol{\psi}(k_1,\mathbf{r})\nabla\boldsymbol{\psi}^\dagger(-k_1,\mathbf{r}) - \nabla\boldsymbol{\psi}(k_1,\mathbf{r})\boldsymbol{\psi}^\dagger(-k_1,\mathbf{r})]$$
$$+ [\boldsymbol{\psi}(-k_1,\mathbf{r})\nabla\boldsymbol{\psi}^\dagger(k_1,\mathbf{r}) - \nabla\boldsymbol{\psi}(-k_1,\mathbf{r})\boldsymbol{\psi}^\dagger(k_1,\mathbf{r})] \cdot \mathbf{S}^*$$
$$+ \mathbf{S}^t \cdot [\boldsymbol{\psi}(k_1,\mathbf{r})\nabla\boldsymbol{\psi}^\dagger(k_1,\mathbf{r}) - \nabla\boldsymbol{\psi}(k_1,\mathbf{r})\boldsymbol{\psi}^\dagger(k_1,\mathbf{r})] \cdot \mathbf{S}^*\} \cdot \mathbf{a}^*. \tag{23}$$

If we let $\boldsymbol{\psi}$ denote either $\boldsymbol{\psi}^c$ or $\boldsymbol{\psi}^s$ and perform the integration

$$\int_S dS\hat{n} \cdot [\phi\nabla\phi^* - \phi^*\nabla\phi], \tag{24}$$

the second and third terms in Equation (23) integrate to zero by virtue of part (c) of this problem. We can use the result of part (b) to rewrite Equation (23) as

$$\phi\nabla\phi^* - \phi^*\nabla\phi = \frac{1}{4}\mathbf{a}^t\{[\boldsymbol{\psi}^*(k_1,\mathbf{r})\nabla\boldsymbol{\psi}^t(k_1,\mathbf{r}) - (\nabla\boldsymbol{\psi}^*(k_1,\mathbf{r}))\boldsymbol{\psi}^t(k_1,\mathbf{r})]$$
$$+ \mathbf{S}^t \cdot [\boldsymbol{\psi}(k_1,\mathbf{r})\nabla\boldsymbol{\psi}^t(k_1,\mathbf{r}) - (\nabla\boldsymbol{\psi}(k_1,\mathbf{r}))\boldsymbol{\psi}^t(k_1,\mathbf{r})] \cdot \mathbf{S}^*\} \cdot \mathbf{a}^* \tag{25}$$

which gives the desired result upon integration.

(e)

$$A_{nm} = \left\{ \int_S dS\hat{n} \cdot [\boldsymbol{\psi}^*\nabla\boldsymbol{\psi}^t - (\nabla\boldsymbol{\psi}^*)\boldsymbol{\psi}^t] \right\}_{nm}$$
$$= \int_S dS\hat{n} \cdot [\psi_n^*\nabla\psi_m - \nabla\psi_n^*\psi_m]. \tag{26}$$

When $n \neq m$,

$$A_{nm} = \int_{S+S_0} dS\hat{n} \cdot [\psi_n^*\nabla\psi_m - \nabla\psi_m^*\psi_m] \tag{27}$$

because the integral over S_0, which is a spherical or circular surface, is zero because of the orthogonality of ψ_n on such a surface. Using Gauss' theorem to convert (27) into a volume integral over the volume bounded by $S + S_0$, we can show that it is zero as in (20).

When $n = m$,

$$A_{nn} = \int_S dS\hat{n} \cdot [\psi_n^*\nabla\psi_n - \nabla\psi_n^*\psi_n], \tag{28}$$

which is a pure imaginary number. Therefore

$$A_{nm} = \delta_{nm} ic. \tag{29}$$

(f) The second bracketed term in Equation (30) is just the conjugate of the first bracketed term. The integral becomes

$$\int_S dS\ \hat{n}\cdot(\phi\nabla\phi^* - \phi^*\nabla\phi) = \frac{1}{4}\mathbf{a}^t\cdot\left[i\left(\frac{n}{k\rho}\right)\overline{\mathbf{I}} - i\left(\frac{n}{k\rho}\right)\overline{\mathbf{S}}^t\cdot\overline{\mathbf{S}}^*\right]\cdot\mathbf{a}^*. \tag{30}$$

In order for the integral above to vanish, we must have

$$\overline{\mathbf{S}}^t\cdot\overline{\mathbf{S}}^* = \overline{\mathbf{I}}. \tag{31}$$

§8.27

(a) For the impenetrable scatterer with a homogeneous Dirichlet boundary condition, Equation (8.5.3) becomes

$$\sum_n [a_n \Re e\psi_n(k_1,\mathbf{r}) + f_n\psi_n(k_1,\mathbf{r})] = 0, \quad r\in S. \tag{1}$$

Testing Equation (1) with $\hat{n}\cdot\nabla\Re e\psi_m(k_1,\mathbf{r})$ and integrating over S, we have

$$\sum_n \left[a_n\int_S dS\ \Re e\psi_n(k_1,\mathbf{r})\ \hat{n}\cdot\nabla\Re e\psi_m(k_1,\mathbf{r}) + f_n\int_S dS\ \psi_n(k_1,\mathbf{r})\ \hat{n}\cdot\nabla\Re e\psi_m(k_1,\mathbf{r})\right] = 0 \tag{2}$$

or

$$\sum_n a_n\Re e Q_{nm} = -\sum_n f_n Q_{nm} \tag{3}$$

where

$$Q_{nm} = \int_S dS\ \psi_n(k_1,\mathbf{r})\ \hat{n}\cdot\nabla\Re e\psi_m(k_2,\mathbf{r}). \tag{4}$$

Equation (3) is the same as Equation (8.5.8) and so from Equation (8.5.11),

$$\overline{\mathbf{T}} = -(\overline{\mathbf{Q}}^t)^{-1}\cdot\Re e\overline{\mathbf{Q}}^t \tag{5}$$

When we apply the EBC method to the impenetrable scatterer, Equation (8.3.7) becomes

$$a_n = ik_1\int_S dS'\ \psi_n(k_1,\mathbf{r}')\ \hat{n}\cdot\nabla'\phi_1(\mathbf{r}'). \tag{6}$$

Then, similar to Equation (8.3.10), we expand the normal derivative of the field as

$$\hat{n} \cdot \nabla' \phi_1(\mathbf{r}') = \sum_m \alpha_m \hat{n} \cdot \nabla' \Re e \psi_m(k_1, \mathbf{r}')$$

from which we have

$$a_n = ik_1 \sum_m \alpha_m \int_S dS' \psi_n(k_1, \mathbf{r}') \hat{n} \cdot \nabla' \Re e \psi_m(k_1, \mathbf{r}'). \tag{7}$$

The above may be written as

$$a_n = ik_1 \sum_m \alpha_m Q_{nm} \tag{8}$$

where Q_{nm} is defined in Equation (4) above.

From Equation (8.1.9),

$$\phi_{sca}(\mathbf{r}) = -\int_S dS' g_i(\mathbf{r}, \mathbf{r}') \hat{n} \cdot \nabla' \phi_1(\mathbf{r}'), \quad \mathbf{r} \in V, \tag{9a}$$

$$= -ik_1 \int_S dS' \sum_n \psi_n(k_1, \mathbf{r}) \Re e \psi_n(k_1, \mathbf{r}') \sum_m \alpha_m \hat{n} \cdot \nabla' \Re e \psi_m(k_1, \mathbf{r}') \tag{9b}$$

$$= \sum_n \left(-ik_1 \sum_m \alpha_m \Re e Q_{nm} \right) \psi_n(k_1, \mathbf{r}) \tag{9c}$$

$$\equiv \sum_n f_n \psi_n(k_1, \mathbf{r}). \tag{9d}$$

Rewriting Equations (8) and (9c) in vector notation,

$$\mathbf{a} = ik_1 \overline{\mathbf{Q}} \cdot \boldsymbol{\alpha} \tag{10}$$

$$\mathbf{f} = -ik_1 (\Re e \overline{\mathbf{Q}}) \cdot \boldsymbol{\alpha} \tag{11}$$

or

$$\mathbf{f} = -(\Re e \overline{\mathbf{Q}}) \cdot \overline{\mathbf{Q}}^{-1} \cdot \mathbf{a}. \tag{17}$$

Since from Equation (8.4.5)

$$\mathbf{f} = \overline{\mathbf{T}} \cdot \mathbf{a}, \tag{13}$$

$$\overline{\mathbf{T}} = -(\Re e \overline{\mathbf{Q}}) \cdot \overline{\mathbf{Q}}^{-1} \tag{14}$$

which is the transpose of Equation (5).

(b) The method of Rayleigh's hypothesis and the EBC method are approximately equivalent. The same basis functions are used in both methods and the only difference is that the $\overline{\mathbf{Q}}$ matrix used in the method of Rayleigh's hypothesis is the transpose of that used in the EBC method.

Therefore, if the same number of terms are used in both methods, we should expect that the errors in the two methods are of the same order.

§8.28

(a) We define the cylindrical harmonics as

$$\psi_m(k,\rho_0) = H_m^{(1)}(k|\rho - \rho_0|)e^{im\phi_0} \tag{1}$$

where ϕ_0 is the angle from ρ_0 to ρ defined with respect to an arbitrary reference direction. The translation matrices may then be defined as

$$[\overline{\alpha}_{10}]_{mn} = H_{n-m}^{(1)}(k|\rho_1 - \rho_0|)e^{i(n-m)\phi_{10}}, \tag{2}$$

$$\begin{aligned}[\overline{\beta}_{10}]_{mn} &= \Re e\left\{[\overline{\alpha}_{10}]_{mn}\right\} \\ &= J_{n-m}^{(1)}(k|\rho_1 - \rho_0|)e^{i(n-m)\phi_{10}},\end{aligned} \tag{3}$$

where ϕ_{10} is the angle from ρ_0 to ρ_1. Using the above definition, we may write the addition theorem as

$$\psi^{\dagger}(k,\rho_0) = \begin{cases} \Re e\psi^{\dagger}(k,\rho_1)\cdot\overline{\alpha}_{10}, & |\rho - \rho_1| < |\rho_1 - \rho_0|, \\ \psi^{\dagger}(k,\rho_1)\cdot\overline{\beta}_{10}, & |\rho - \rho_1| > |\rho_1 - \rho_0|. \end{cases} \tag{4}$$

(b) Similarly, we may define the spherical harmonics as

$$\psi_{lm}(k,\mathbf{r}_0) = h_l^{(1)}(k|\mathbf{r} - \mathbf{r}_0|)Y_{lm}(\theta_0,\phi_0). \tag{5}$$

The addition theorem for spherical harmonics may now be written as

$$\psi_{lm}^{\dagger}(k,\mathbf{r}_0) = \begin{cases} \Re e\psi^{\dagger}(k,\mathbf{r}_1)\cdot\overline{\alpha}_{10}, & |\mathbf{r} - \mathbf{r}_1| < |\mathbf{r}_1 - \mathbf{r}_0| \\ \psi^{\dagger}(k,\mathbf{r}_1)\cdot\overline{\beta}_{10}, & |\mathbf{r} - \mathbf{r}_1| > |\mathbf{r}_1 - \mathbf{r}_0| \end{cases} \tag{6}$$

where the dot product indicates a double summation over $l' \in [0,\infty)$ and $m' \in [-l', l']$. The translation matrices are then given as

$$\alpha_{l'm',lm} = \sum_{l''=0}^{\infty} 4\pi i^{(l'+l''-l)}Y_{l'',m-m'}(\theta_{10},\phi_{10})h_{l''}^{(1)}(k|\mathbf{r}_1 - \mathbf{r}_0|)A(m,l,-m',l',l''), \tag{7}$$

$$\begin{aligned}\beta_{l'm',lm} &= \Re e\{\alpha_{l'm',lm}\} \\ &= \sum_{l''=0}^{\infty} 4\pi i^{(l'+l''-l)}Y_{l'',m-m'}(\theta_{10},\phi_{10})j_{l''}^{(1)}(k|\mathbf{r}_1 - \mathbf{r}_0|)A(m,l,-m',l',l'')\end{aligned} \tag{8}$$

where

$$\begin{aligned}A(m,l,-m',l',l'') &= (-1)^m[(2l+1)(2l'+1)(2l''+1)/4\pi]^{1/2} \\ &\cdot\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ -m & m' & m-m' \end{pmatrix}.\end{aligned} \tag{9}$$

The angles θ_{10} and ϕ_{10} in Equations (7) and (8) above are the angles from $\mathbf{r}_0$ to $\mathbf{r}_1$ in spherical coordinates.

§8.29

The dimensions of matrices $\overline{\mathbf{T}}_{i(n)}$, $\overline{\boldsymbol{\beta}}_{i0}$, and $\overline{\boldsymbol{\alpha}}_{i,n+1}$ are $M \times M$, $M \times P$, and $M \times M$ respectively. Recognizing that the number of operations required to multiply an $M \times P$ matrix with a $P \times M$ matrix, or an $M \times M$ matrix with an $M \times P$ matrix is equal to M^2P and that an $M \times M$ matrix with an $M \times M$ matrix requires M^3 operations, the number of multiplications of Equation (8.6.17) is $(2n+2)M^3 + (2n+3)M^2P$ for n-th iteration, and that of Equation (8.6.18) is $4M^2P$. Note that for the inversion of an $M \times M$ matrix, it requires M^3 operations approximately. Hence, in total, for n-th iteration, we need $(2n+2)M^3 + (2n+7)M^2P$ multiplications. When there exist N scatterers, the number of multiplications amounts to

$$\sum_{n=1}^{N} (2n+2)M^3 + (2n+7)M^2P = N(N+3)M^3 + N(N+8)M^2P.$$

When $N \to \infty$, it is easy to see from the above equation that the number of multiplications is proportional to N^2.

§8.30

Since the field is incident at the interface form the inside, we can have two integral equations similar to (8.3.5) and (8.3.6).

$$\begin{aligned}
&\sum_n a_n\psi_n(k_1,\mathbf{r}) \\
&\quad = ik_1 \sum_n \psi_n(k_1,\mathbf{r}) \int_S ds'[\Re e\psi_n(k_1,\mathbf{r}')\hat{n}\cdot\nabla'\phi_1(\mathbf{r}') \\
&\qquad - \phi_1(\mathbf{r}')\hat{n}\cdot\nabla'\Re e\psi_n(k_1,\mathbf{r}')], \quad \mathbf{r}\in S_2, \qquad (1) \\
&0 = ik_o \sum_n \Re e\psi_n(k_0,\mathbf{r}) \int_S ds'[\psi_n(k_0,\mathbf{r}')\hat{n}\cdot\nabla'\phi_2(\mathbf{r}') \\
&\qquad - \phi_2(\mathbf{r}')\hat{n}\cdot\nabla'\psi_n(k_0,\mathbf{r})], \quad \mathbf{r}\in S_1. \qquad (2)
\end{aligned}$$

By the same argument given in Section 8.3.1, the solutions can be found as follows:

$$\phi_2(\mathbf{r}') = \sum_m \alpha_m\psi_m(k_0,\mathbf{r}') = \phi_1(\mathbf{r}')$$

$$\hat{n}\cdot\nabla'\phi_2(\mathbf{r}') = \sum_m \alpha_m\hat{n}\cdot\nabla'\psi_m(k_0,\mathbf{r}') = \frac{p_1}{p_2}\hat{n}\cdot\nabla'\phi_1(\mathbf{r}')$$

where α_m can be solved for by the matrix equations

$$a_n = i \sum_m \alpha_m \Re_{nm}$$

$$\Re_{nm} = k_1 \int_S ds' [\Re e\psi_n(k_1, \mathbf{r}') \frac{P_2}{P_1} \hat{n} \cdot \nabla' \psi_m(k_0, \mathbf{r}') - \hat{n} \cdot \nabla' \Re e\psi_n(k_1, \mathbf{r}') \psi_m(k_0, \mathbf{r}')].$$

Hence $\alpha = -i\overline{\mathbf{R}} \cdot \mathbf{a}$. Since the field external to S can be expressed as

$$\phi_2(\mathbf{r}) = \boldsymbol{\psi}^t(k_0, \mathbf{r}) \cdot \boldsymbol{\alpha} = \boldsymbol{\psi}^t(k_o, \mathbf{r}) \cdot (-i\overline{\mathbf{R}}) \cdot \mathbf{a},$$

we can define a transmission matrix

$$\overline{\mathbf{T}}_{10} = -i\overline{\mathbf{Q}}.$$

For the scattered field inside S.

$$\begin{aligned}\psi_{sca}(\mathbf{r}) &= -ik_1 \sum_{nm} \alpha_m \Re e\psi_n(k_1, \mathbf{r}) \int_S ds' [\psi_n(k_1, \mathbf{r}') \hat{n} \cdot \nabla' \phi_m(k_0, \mathbf{r}') \frac{p_2}{p_1} \\ &\quad - \psi_m(k_0, \mathbf{r}') \hat{n} \cdot \nabla' \psi_n(k_1, \mathbf{r}')] \\ &= -i\Re e\boldsymbol{\psi}^t(k_1, \mathbf{r}) \cdot \overline{\mathbf{P}} \cdot \boldsymbol{\alpha}\end{aligned}$$

where

$$\overline{\mathbf{P}} = k_1 \int_S dS' [\psi_n(k_1, \mathbf{r}') \hat{n} \cdot \nabla' \psi_m(k_0, \mathbf{r}') \frac{p_2}{p_1} - \psi_m(k_0, \mathbf{r}') \hat{n} \cdot \nabla' \psi_n(k_1, \mathbf{r}')].$$

Therefore

$$\begin{aligned}\psi_{sca} &= -i\Re e\boldsymbol{\psi}^t(k_1, \mathbf{r}) \cdot \overline{\mathbf{P}} \cdot (-i\overline{\mathbf{Q}}) \cdot \mathbf{a}, \\ &= \Re e\boldsymbol{\psi}^t(k_1, \mathbf{r}) \cdot (-\overline{\mathbf{P}} \cdot \overline{\mathbf{R}}) \cdot \mathbf{a}.\end{aligned}$$

We can define $\overline{\mathbf{R}}_{10} = -\overline{\mathbf{P}} \cdot \overline{\mathbf{R}}$.

§8.31

(a) (8.7.23) can be expanded into a geometric series.

$$\begin{aligned}\overline{\mathbf{M}}_{i+1,-} &= \overline{\mathbf{I}} + \overline{\mathbf{R}}_{i+1,i} \cdot \overline{\boldsymbol{\beta}}_{i+1,i+2} \cdot \tilde{\overline{\mathbf{R}}}_{i+1,i+2} \cdot \overline{\boldsymbol{\beta}}_{i+2,i+1} \\ &+ \overline{\mathbf{R}}_{i+1,i} \cdot \overline{\boldsymbol{\beta}}_{i+1,i+2} \cdot \tilde{\overline{\mathbf{R}}}_{i+1,i+2} \cdot \overline{\boldsymbol{\beta}}_{i+2,i+1} \cdot \overline{\mathbf{R}}_{i+1,i} \cdot \overline{\boldsymbol{\beta}}_{i+1,i+2} \cdot \tilde{\overline{\mathbf{R}}}_{i+1,i+2} \cdot \overline{\boldsymbol{\beta}}_{i+2,i+1} \\ &+ \dots\end{aligned}$$

The first term $\overline{\mathbf{I}}$ represents pure transmission from the field outside. The second term means that the wave bounces back and forth one time between surface $i+1$ and $i+2$. The third term just represents that the wave bounces two times between the two surfaces. So $\overline{\mathbf{M}}_{i+1,-}$ accounts for the multiple reflections between $(i+1)$-th surface and $(i+2)$-th surface. The same explanation applies to the expanded terms of $\overline{\mathbf{M}}_{i,+}$.

(b) $\mathbf{a}_{i+1}$ is the amplitude of inward field at $(i+1)$-th region and it can be related to $\mathbf{a}_i$. First, we express $\mathbf{a}_i$ in the coordinate of $(i+1)$-th surface by $\overline{\boldsymbol{\beta}}_{i+1,i}\cdot\mathbf{a}_i$ and then multiply it by transmission matrix $\overline{\mathbf{T}}_{i,i+1}$. Hence the resultant vector $\overline{\mathbf{T}}_{i,i+1}\cdot\overline{\boldsymbol{\beta}}_{i+1,i}\cdot\mathbf{a}_i$ represents the initial inward wave and in order to find out the final inward wave, we have to take multiple reflections in $(i+1)$-th layer into account. To achieve this simply multiply the vector by $\overline{\mathbf{M}}_{i+1,-}$ we discussed in (a). So,

$$\mathbf{a}_{i+1}=\overline{\mathbf{M}}_{i,-}\cdot\overline{\mathbf{T}}_{i,i+1}\cdot\boldsymbol{\beta}_{i+1,i}\cdot\mathbf{a}_i. \tag{8.7.25}$$

Since $\phi_i(\mathbf{r})=\Re e\psi^t(k_i,\mathbf{r}_i)\cdot\mathbf{a}_i+\psi^t(k_i,\mathbf{r}_{i+1})\cdot\mathbf{b}_i$, by addition theorem, it can be expressed as

$$\phi_i(\mathbf{r})=\Re e\psi^t(k_i,\mathbf{r}_{i+1})\cdot\overline{\boldsymbol{\beta}}_{i+1,i}\cdot\mathbf{a}_i+\psi^t(k_i,\mathbf{r}_{i+1})\cdot\mathbf{b}_i$$

Hence,

$$\mathbf{b}_i=\tilde{\overline{\mathbf{R}}}_{i,i+1}\cdot(\overline{\boldsymbol{\beta}}_{i+1,i}\cdot\mathbf{a}_i). \tag{8.7.26}$$

where $\tilde{\overline{\mathbf{R}}}_{i,i+1}$ is the generalized reflection matrix at the $(i+1)$-th surface.

§8.32

(8.8.3) reads

$$[\nabla\cdot p(\mathbf{r})\nabla+k_1^2(\mathbf{r})]g_1(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}') \tag{1}$$

Let $g_1(\mathbf{r},\mathbf{r}')=\sum_{n=1}^{N}a_nf_n(\mathbf{r})$, $\mathbf{r},\mathbf{r}'\in V_1$ Substituting the above in (1) yields

$$\sum_{n=1}^{N}a_n[\nabla\cdot p(\mathbf{r})\nabla f_n(\mathbf{r})+k_1^2(\mathbf{r})f_n(\mathbf{r})]=-\delta(\mathbf{r}-\mathbf{r}')$$

Multiplying the above equation by $f_m(\mathbf{r})$ and integrating over the region V_1, we have

$$\sum_{n=1}^{N}a_n[\int_{V_1}d\mathbf{r}f_m(\mathbf{r})\nabla\cdot p(\mathbf{r})\nabla f_n(\mathbf{r})+\int_{V_1}d\mathbf{r}f_m(\mathbf{r})k_1^2(\mathbf{r})f_n(\mathbf{r})]=-\int_{V_1}d\mathbf{r}f_m(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'). \tag{2}$$

The first term on the left-hand side can be written as

$$\int_{V_1}d\mathbf{r}\nabla\cdot[f_m(\mathbf{r})p(\mathbf{r})\nabla f_n(\mathbf{r})]-\int_{V_1}d\mathbf{r}\nabla f_m(\mathbf{r})\cdot p(\mathbf{r})\nabla f_n(\mathbf{r})$$

$$=\oint_S ds f_m(\mathbf{r})p(\mathbf{r})\hat{n}\cdot\nabla f_n(\mathbf{r})-\int_{V_1}d\mathbf{r}\nabla f_m(\mathbf{r})\cdot p(\mathbf{r})\nabla f_n(\mathbf{r}).$$

Since we impose the boundary condition $\hat{n}\cdot\nabla g_1(\mathbf{r},\mathbf{r}')=0$ on S which implies that the basis functions can be chosen to satisfy $\hat{n}\cdot\nabla f_n(\mathbf{r})=0$, (2) becomes

$$\sum_{n=1}^{N} a_n - \int_{V_1} d\mathbf{r}\nabla f_m(\mathbf{r})\cdot p(\mathbf{r})\nabla f_n(\mathbf{r}) + \int_{V_1} d\mathbf{r} f_m(\mathbf{r})k_1^2(\mathbf{r})f_n(\mathbf{r})] = -f_m(\mathbf{r}').$$

So, the matrix equation appears as

$$\sum_{n=1}^{N} L_{mn}a_n = b_m, \qquad m=1,\ldots,N$$

where

$$L_{mn} = -\langle\nabla f_m(\mathbf{r}), p(\mathbf{r})\nabla f_n(\mathbf{r})\rangle + \langle f_m(\mathbf{r})\cdot k_1^2(\mathbf{r})f_n(\mathbf{r})\rangle.$$

and

$$b_m = -f_m(\mathbf{r}')$$

§8.33

(a) At the resonant frequencies of the cavity, there exist non-trivial solutions satisfying

$$[\nabla\cdot p(\mathbf{r})\nabla + k_1^2(\mathbf{r})]h(\mathbf{r}) = 0 \tag{1}$$

and the boundary conditions

$$\hat{n}\cdot\nabla h(\mathbf{r}) = 0, \qquad \text{on } S$$

As in the Exercise 8.32, the above differential equation can be converted to a matrix equation

$$\sum_{n=1}^{N} L_{mn}a_n = 0, \qquad m=1,\ldots,N.$$

In order for non-trivial solutions existing in the matrix equation. $\overline{\mathbf{L}}$ must be a singular matrix.

Since $\overline{\mathbf{M}} = \overline{\mathbf{A}}\cdot\overline{\mathbf{L}}^{-1}\cdot\overline{\mathbf{A}}^t$,

$$\overline{\mathbf{M}}^{-1} = (\overline{\mathbf{A}}^t)^{-1}\cdot\overline{\mathbf{L}}\cdot\overline{\mathbf{A}}^{-1} = ((\overline{\mathbf{A}}^{-1})^t\cdot\overline{\mathbf{L}}\cdot\overline{\mathbf{A}}^{-1}.$$

If $\overline{\mathbf{L}}\cdot\mathbf{a}=0$, for some $\mathbf{a}\neq 0$, then

$$\overline{\mathbf{M}}^{-1}\cdot\overline{\mathbf{A}}\cdot\mathbf{a} = (\overline{\mathbf{A}}^{-1})^t\cdot\overline{\mathbf{L}}\cdot\mathbf{a} = (\overline{\mathbf{A}}^{-1})^t\cdot 0 = 0.$$

In general $\overline{\mathbf{A}}\cdot\mathbf{a}\neq 0$; therefore, $\overline{\mathbf{M}}^{-1}$ is a singular matrix as well.

The fact that there exist non-trivial solutions to Equation (1) is equivalent to that there are no unique solutions to

$$[\nabla\cdot p(\mathbf{r})\nabla + k_1^2(\mathbf{r})]g_1(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}').$$

Suppose $g_2(\mathbf{r},\mathbf{r}')$ is another solution to the above equation, then

$$[\nabla\cdot p(\mathbf{r})\nabla + k_1^2(\mathbf{r})]\delta g(\mathbf{r},\mathbf{r}') = 0, \quad \text{where} \quad \delta g(\mathbf{r},\mathbf{r}') = g_1(\mathbf{r},\mathbf{r}') - g_2(\mathbf{r},\mathbf{r}').$$

Multiplying the above by $\delta g^*(\mathbf{r},\mathbf{r}')$ and integrating over V_1, it becomes

$$\int_{V_1} d\mathbf{r}\delta g^*(\mathbf{r},\mathbf{r}')\nabla\cdot p(\mathbf{r})\nabla\delta g(\mathbf{r},\mathbf{r}') + \int_{V_1} d\mathbf{r}k_1^2|\delta g(\mathbf{r},\mathbf{r}')|^2 = 0$$

The first term of the above equations can be expressed as

$$\int_{V_1} d\mathbf{r}\nabla\cdot[\delta g^*(\mathbf{r},\mathbf{r}')p(\mathbf{r})\nabla\delta g(\mathbf{r},\mathbf{r}')] - \int_{V_1} d\mathbf{r}p(\mathbf{r})\nabla\delta g^*(\mathbf{r},\mathbf{r}')\cdot\nabla\delta g(\mathbf{r},\mathbf{r}')$$

$$= \oint_S dS\delta g^*(\mathbf{r},\mathbf{r})p(\mathbf{r})\hat{n}\cdot\nabla\delta g(\mathbf{r},\mathbf{r}') - \int_{V_1} d\mathbf{r}p(\mathbf{r})|\nabla\delta g(\mathbf{r},\mathbf{r}')|^2.$$

Since $g_1(\mathbf{r},\mathbf{r}')$ and $g_2(\mathbf{r},\mathbf{r}')$ satisfy the Neumann boundary condition, the above surface integral is zero.

Hence,

$$-\int_{V_1} d\mathbf{r}p(\mathbf{r})|\nabla\delta g(\mathbf{r},\mathbf{r}')|^2 + \int_{V_1} d\mathbf{r}k_1^2(\mathbf{r})|\delta g(\mathbf{r},\mathbf{r}')|^2 = 0.$$

Taking the imaginary part of the above, we obtain

$$-\int_{V_1} d\mathbf{r}\Im m[p(\mathbf{r})]|\nabla\delta g(\mathbf{r},\mathbf{r}')|^2 + \int_{V_1} d\mathbf{r}\Im m[k_1^2(\mathbf{r})]|\delta(g(\mathbf{r},\mathbf{r}')|^2 = 0.$$

For a lossy medium, the imaginary parts of $-p(\mathbf{r})$ and $k_1^2(\mathbf{r})$ have the same positive sign (for example, in a two dimensional problem, $\nabla\cdot(\epsilon^{-1}\nabla H_z) + \omega^2\mu H_z = 0$), so the above equation is only possible if $\delta g(\mathbf{r},\mathbf{r}') = 0$ everywhere in V_1.

So the unique solution only exists in a lossy medium and for lossless medium, it always imposes a singular problem on $\overline{\mathbf{L}}$.

(b) Since $\mathbf{d} = -\overline{\mathbf{M}}^{-1}\cdot\overline{\mathbf{F}}\cdot\mathbf{c}$, when at the resonant frequencies, $\overline{\mathbf{M}}^{-1}$ is singular, in order for a finite $\mathbf{d}$, $\mathbf{c}$ must be infinite.

(c) Assume $\mathbf{a}$ is an eigenvector of $\overline{\mathbf{L}}$ and λ_i is the corresponding eigenvalue. In the singular value decomposition method $\overline{\mathbf{S}}^{-1} = \overline{\mathbf{S}}^t$, so $\overline{\mathbf{L}} = \overline{\mathbf{S}}^t\cdot\overline{\boldsymbol{\lambda}}\cdot\overline{\mathbf{S}}$ can be written as $\overline{\mathbf{S}}\cdot\overline{\mathbf{L}} = \overline{\boldsymbol{\lambda}}\cdot\overline{\mathbf{S}}$. Then, $\overline{\mathbf{S}}\cdot\overline{\mathbf{L}}\cdot\mathbf{a} = \overline{\mathbf{S}}\cdot\lambda_i\mathbf{a} = \lambda_i\overline{\mathbf{S}}\cdot\mathbf{a}$, which is equal to $\overline{\boldsymbol{\lambda}}\cdot\overline{\mathbf{S}}\cdot\mathbf{a}$. Hence, λ_i is an eigenvalue of $\overline{\boldsymbol{\lambda}}$ and since $\overline{\boldsymbol{\lambda}}$ is a diagonal matrix, its diagonal elements are just λ_i, the eigenvalues of $\overline{\mathbf{L}}$.

(d) Since $\overline{\mathbf{M}} = \overline{\mathbf{A}} \cdot \overline{\mathbf{L}}^{-1} \cdot \overline{\mathbf{A}}^t$ and $\overline{\mathbf{A}}$ is an invertable matrix, $\overline{\mathbf{M}}^{-1} = (\overline{\mathbf{A}}^{-1})^t \cdot \overline{\mathbf{L}} \cdot (\overline{\mathbf{A}}^{-1})$. At the resonant frequency, assume that $\overline{\mathbf{L}}$ has an eigenvector $\mathbf{a}$ with zero eigenvalue. Then,

$$\overline{\mathbf{M}}^{-1} \cdot \overline{\mathbf{A}} \cdot \mathbf{a} = (\overline{\mathbf{A}}^{-1})^t \cdot \overline{\mathbf{L}} \cdot \mathbf{a} = 0 \cdot \mathbf{a},$$

which implies that $\overline{\mathbf{M}}^{-1}$ has zero eigenvalues. In other words, $\overline{\mathbf{M}}$ has infinite eigenvalues.

(e) By setting the zero eigenvalues of $\overline{\mathbf{L}}$ to a small nonzero number,

$$\overline{\mathbf{M}}^{-1} = (\overline{\mathbf{A}}^{-1})^t \cdot (\overline{\mathbf{S}}^t \cdot \overline{\boldsymbol{\lambda}}^{-1} \cdot \overline{\mathbf{S}}) \cdot (\overline{\mathbf{A}}^{-1}),$$

where the singular value decomposition is invoked. The matrices on the right hand side of the above equation are all invertable, so $\overline{\mathbf{M}}$ is computable. Since $\text{cond}(\overline{\mathbf{M}}^{-1}) \leq \text{cond}(\overline{\mathbf{A}}^{-1})^2 \text{cond}(\overline{\boldsymbol{\lambda}})$, and $\overline{\boldsymbol{\lambda}}$ is almost a singular matrix, we can expect that $\text{cond}(\overline{\mathbf{M}}^{-1})$ is quite large.

§8.34

(a) When the internal resonance exists in a cavity, there is no unique solution to this problem. But we know that if the medium is lossless, the uniqueness of the solution is not guaranteed at any frequency. Therefore, the internal resonance poses a problem only for lossless media in S. Since for lossless media, $p(\mathbf{r})$ and $k_1^2(\mathbf{r})$ are real values in Equation (8.8.9a), $\overline{\mathbf{L}}$ is a real symmetric matrix (assume that the basis functions are real). Therefore, for a real symmetric matrix, $\overline{\mathbf{L}}$, its eigenvalues are real.

(b) A new $\overline{\overline{\mathbf{L}}}$ is defined such that $\overline{\overline{\mathbf{L}}} = \overline{\mathbf{L}} + i\delta\overline{\mathbf{I}}$, where $i\delta$ is a pure imaginary number, Assume that λ is one of eigenvalues of $\overline{\mathbf{L}}$ and its corresponding eigenvector is $\mathbf{a}$. Then,

$$(\overline{\mathbf{L}} + i\delta\overline{\mathbf{I}}) \cdot \mathbf{a} = \lambda\mathbf{a} + i\delta\mathbf{a} = (\lambda + i\delta)\mathbf{a}.$$

Therefore, $\mathbf{a}$ is an eigenvector of $\overline{\overline{\mathbf{L}}}$ and the eigenvalue is $\lambda + i\delta$, which can not be zero since λ is real and $i\delta$ is pure imaginary.

§8.35

(a) Let us impose the impedance boundary condition $\hat{n} \cdot \nabla g_1 = Z g_1$ on S. Then, from Exercise 8.32, the elements of $\overline{\mathbf{L}}$ can be expressed as

$$L_{mn} = \oint_S dS f_m(\mathbf{r}) p(\mathbf{r}) Z(\mathbf{r}) f_n(\mathbf{r}) - \int_{V_1} d\mathbf{r} \nabla f_m(\mathbf{r}) \cdot p(\mathbf{r}) \nabla f_n(\mathbf{r}) + \int_{V_1} d\mathbf{r} f_m(\mathbf{r}) k_1^2(\mathbf{r}) f_n(\mathbf{r}).$$

(b) Since Z is complex, $\overline{\mathbf{L}}$ is a complex matrix, which requires its eigenvalues to be complex. Therefore, the problem can be treated as a lossy cavity problem in which the internal resonance does not exist.

§8.36

For the field in V_1, we write $\phi_1(\mathbf{r}') = \phi_{inc}(\mathbf{r}') + \phi_{sca}(\mathbf{r}')$, where ϕ_{sca} is the scattered field due to the inhomogeneity in V_1.

(a) The Green's function $g_1(\mathbf{r},\mathbf{r}')$ satisfies (8.8.3)

$$[\nabla \cdot p(\mathbf{r})\nabla + k_1^2(\mathbf{r})]g_1(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{1}$$

and the radiation condition at infinity. Then, for $\mathbf{r} \in V_1$ we define

$$F(\mathbf{r}) = \int_S dS' \, [g_1(\mathbf{r},\mathbf{r}')\hat{n} \cdot \nabla'\phi_{sca}(\mathbf{r}') - \phi_{sca}(\mathbf{r}')\hat{n} \cdot \nabla' g_1(\mathbf{r},\mathbf{r}')]. \tag{2}$$

Using Gauss' theorem, we show

$$F(\mathbf{r}) = \int_{V_1} dV' \, \nabla' \cdot [g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_{sca}(\mathbf{r}') - \phi_{sca}(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')], \tag{3}$$

$$= \int_{V_1+V_0} dV' \, \nabla' \cdot [g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_{sca}(\mathbf{r}') - \phi_{sca}(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')]. \tag{4}$$

In the above, we have used the fact that in V_0 both $\phi_{sca}(\mathbf{r}')$ and $g_1(\mathbf{r},\mathbf{r}')$ are nonsingular, and so are their gradients in Figure 8.8.1.

Using Gauss' theorem again for (4) we have

$$F(\mathbf{r}) = \int_{S_\infty} dS' \, [g_1(\mathbf{r},\mathbf{r}')\hat{n} \cdot \nabla'\phi_{sca}(\mathbf{r}') - \phi_{sca}(\mathbf{r}')\hat{n} \cdot \nabla' g_1(\mathbf{r},\mathbf{r}')]. \tag{5}$$

The above integral is equal to zero due to the radiation condition at infinity for both $\phi_{sca}(\mathbf{r})$ and $g_1(\mathbf{r},\mathbf{r}')$. Thus, finally we have

$$F(\mathbf{r}) = 0 = \int_S dS'[g_1(\mathbf{r},\mathbf{r}')\hat{n} \cdot \nabla'\phi_{sca}(\mathbf{r}') - \phi_{sca}(\mathbf{r}')\hat{n} \cdot \nabla' g_1(\mathbf{r},\mathbf{r}')], \quad \mathbf{r} \in V_1 \tag{6}$$

Using the above equation, Equation (8.8.6) can be rewritten as

$$\phi_1(\mathbf{r}) = \int_S dS'[g_1(\mathbf{r},\mathbf{r}')\hat{n} \cdot \nabla'\phi_{inc}(\mathbf{r}') - \phi_{inc}(\mathbf{r}')\hat{n} \cdot \nabla' g_1(\mathbf{r},\mathbf{r}')], \quad \mathbf{r} \in V_1. \tag{7}$$

(b) If $\phi(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \phi_{sca}(\mathbf{r})$ in V_0, then we define

$$F(\mathbf{r}) = \int_S dS' \, [g_0(\mathbf{r},\mathbf{r}')\hat{n} \cdot \nabla'\phi_{inc}(\mathbf{r}') - \phi_{inc}(\mathbf{r}')\hat{n} \cdot \nabla' g_0(\mathbf{r},\mathbf{r}')]. \tag{8}$$

For $\mathbf{r} \in V_0$, $F(\mathbf{r})$ must be zero, since this situation is equivalent to the entire space being homogeneous. Thus, we have

$$\int_S dS' \, [g_0(\mathbf{r},\mathbf{r}')\hat{n} \cdot \nabla'\phi_{inc}(\mathbf{r}') - \phi_{inc}(\mathbf{r}')\hat{n} \cdot \nabla' g_0(\mathbf{r},\mathbf{r}')] = 0, \quad \mathbf{r} \in V_0 \quad (9)$$

¿From the above, (8.8.5) reduces to

$$\phi_{sca}(\mathbf{r}) = -\int_S dS' \, [g_0(\mathbf{r},\mathbf{r}')\hat{n} \cdot \nabla'\phi_{sca}(\mathbf{r}') - \phi_{sca}(\mathbf{r}')\hat{n} \cdot \nabla' g_0(\mathbf{r},\mathbf{r}')], \quad \mathbf{r} \in V_0. \quad (10)$$

§8.37

¿From Maxwell's equations in the frequency domain

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu\mathbf{H}(\mathbf{r}) \quad (1)$$
$$\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\epsilon\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \quad (2)$$

we can interprete the $-i\omega\epsilon\mathbf{E}(\mathbf{r})$ term as the electric polarization current (displacement current) and $-i\omega\mu\mathbf{H}(\mathbf{r})$ as the magnetic polarization current. We can add a fictitious magnetic current $\mathbf{M}(\mathbf{r})$ into (1), giving

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu\mathbf{H}(\mathbf{r}) - \mathbf{M}(\mathbf{r}). \quad (3)$$

¿From (3), we have

$$\begin{aligned} \mu\nabla \times \mu^{-1}\nabla \times \mathbf{E} &= i\omega\mu\nabla \times \mathbf{H} - \mu\nabla \times \mu^{-1}\mathbf{M}, \\ &= \omega^2\mu\epsilon\mathbf{E} + i\omega\mu\mathbf{J} - \mu\nabla \times \mu^{-1}\mathbf{M}. \end{aligned} \quad (4)$$

Thus, the wave equation becomes

$$\mu\nabla \times \mu^{-1}\nabla \times \mathbf{E} - k^2(\mathbf{r})\mathbf{E}(\mathbf{r}) = i\omega\mu\mathbf{J} - \mu\nabla \times \mu^{-1}\mathbf{M}. \quad (5)$$

¿From (1) and (2), we can see that the induced electric current is in the $\mathbf{E}$ direction and the induced magnetic current is in the $\mathbf{H}$ direction. The second term in (8.9.13) is

$$\mathbf{F}(\mathbf{r}) = -\int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot (\mu\nabla'\mu^{-1}) \times [\nabla' \times \mathbf{E}(\mathbf{r}')]. \quad (6)$$

Using (1), we have

$$\begin{aligned} \mathbf{F}(\mathbf{r}) &= -\int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot (\mu\nabla'\mu^{-1}) \times [i\omega\mu\mathbf{H}(\mathbf{r}')], \\ &= \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot \mu\nabla' \times \mu^{-1}\mathbf{M}^i(\mathbf{r}'). \end{aligned} \quad (7)$$

Here $\mathbf{M}^i$ is the induced magnetic current which is defined as

$$\mathbf{M}^i(\mathbf{r}) = -i\omega\mu\mathbf{H}(\mathbf{r}). \tag{8}$$

Comparing (1) and (8), we can see that this term is related to the induced magnetic current.

§8.38

The matrix in (8.9.23) is

$$\begin{aligned} N_{mn} &= \int_V d\mathbf{r}\; O(\mathbf{r})\; \mathbf{E}_m(\mathbf{r}) \cdot \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k_b^2}\right) \cdot \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}') O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}'), \\ &= \int_V d\mathbf{r}\; O(\mathbf{r})\; \mathbf{E}_m(\mathbf{r}) \cdot \int d\mathbf{r}' g(\mathbf{r},\mathbf{r}') O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}') \\ &\quad + \int_V d\mathbf{r}\; O(\mathbf{r})\; \mathbf{E}_m(\mathbf{r}) \frac{\nabla\nabla}{k_b^2} \cdot \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}') O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}'). \end{aligned} \tag{1}$$

The last term of the above is

$$F_{nm} = \frac{1}{k_b^2} \int_V d\mathbf{r}\; O(\mathbf{r})\; \mathbf{E}_m(\mathbf{r}) \nabla \int_V d\mathbf{r}'\; [\nabla g(\mathbf{r},\mathbf{r}')] \cdot O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}'). \tag{2}$$

Using the identity of $\nabla g(\mathbf{r},\mathbf{r}') = -\nabla' g(\mathbf{r},\mathbf{r}')$, we can show

$$\begin{aligned} F_{nm} &= \frac{1}{k_b^2} \int_V d\mathbf{r}\; O(\mathbf{r})\; \mathbf{E}_m(\mathbf{r}) \nabla \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}') \nabla' \cdot [O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}')], \\ &= -\frac{1}{k_b^2} \int_V d\mathbf{r}\; \nabla \cdot [O(\mathbf{r})\mathbf{E}_m(\mathbf{r})] \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}') \nabla' \cdot [O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}')]. \end{aligned} \tag{3}$$

In the above, integration by parts has been used twice, and $O(\mathbf{r})$ equals zero for $r \to \infty$ has also been applied.

¿From (1) and (3), we have

$$\begin{aligned} N_{nm} &= \int_V d\mathbf{r}\; O(\mathbf{r})\; \mathbf{E}_m(\mathbf{r}) \cdot \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}') O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}') \\ &\quad - \frac{1}{k_b^2} \int_V d\mathbf{r} \nabla \cdot [O(\mathbf{r})\mathbf{E}_m(\mathbf{r})] \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}') \nabla' \cdot [O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}')]. \end{aligned} \tag{4}$$

Since $g(\mathbf{r},\mathbf{r}') = g(\mathbf{r}',\mathbf{r})$, (4) is symmetric with respect to n and m. Therefore, $[N_{nm}]$ is a symmetric matrix. It is clear that the multiplication of $O(\mathbf{r})$ has forced both integrands with respect to $\mathbf{r}$ and $\mathbf{r}'$ contain the same function of $O(\mathbf{r})\mathbf{E}(\mathbf{r})$. Meanwhile, $g(\mathbf{r},\mathbf{r}')$ is symmetric with respect to $\mathbf{r}$

and $\mathbf{r}'$. Therefore, the final matrix is symmetric. Without the multiplication of $O(\mathbf{r})$ as in (8.9.19) the matrix $[N_{nm}]$ will not be symmetric.

§8.39

The volume integral equation for the field in materials with homogeneous μ is (8.9.14)

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot O(\mathbf{r}')\mathbf{E}(\mathbf{r}'). \tag{1}$$

If the scattering is very weak we can consider that the scattered field is a small perturbation to the incident field. In this case, the second term on the right hand side of (1) will be much smaller than the first term on the right hand side of (1), and $O(\mathbf{r}) \ll 1$. We can have a Newmann series expansion of (1)

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) &+ \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot O(\mathbf{r}')\mathbf{E}_{inc}(\mathbf{r}') \\
&+ \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot O(\mathbf{r}') \int_V d\mathbf{r}'' \, \overline{\mathbf{G}}(\mathbf{r}', \mathbf{r}'') \cdot O(\mathbf{r}'')\mathbf{E}_{inc}(\mathbf{r}'') \\
&+ \ldots \\
= \mathbf{E}_{inc}(\mathbf{r}) &+ \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot O(\mathbf{r}')\mathbf{E}_{inc}(\mathbf{r}') + \text{Order}(O(\mathbf{r})^2).
\end{aligned} \tag{2}$$

The first two terms are the Born approximation. The error is higher order in $(k^2 - k_b^2)$.

§8.40

(a) Given an integral of the form

$$I = \int_V d\mathbf{r}' g(\mathbf{r}, \mathbf{r}')q(\mathbf{r}'), \tag{1}$$

where $g(\mathbf{r}, \mathbf{r}') = e^{ik_b|\mathbf{r}-\mathbf{r}'|}/4\pi|\mathbf{r} - \mathbf{r}'|$. By normalizing $\boldsymbol{\eta} = \mathbf{r}/L$ and $\boldsymbol{\eta}' = \mathbf{r}'/L$, we have

$$\begin{aligned}
I &= L^3 \int_{V/L^3} d\boldsymbol{\eta}' \frac{e^{ik_bL|\boldsymbol{\eta}-\boldsymbol{\eta}'|}}{4\pi L|\boldsymbol{\eta} - \boldsymbol{\eta}'|} q(\boldsymbol{\eta}'L), \\
&= L^2 \int_{V/L^3} d\boldsymbol{\eta}' \frac{e^{ik_bL|\boldsymbol{\eta}-\boldsymbol{\eta}'|}}{4\pi|\boldsymbol{\eta} - \boldsymbol{\eta}'|} q(\boldsymbol{\eta}'L).
\end{aligned} \tag{2}$$

If $k_b L \ll 1$, then $\exp(ik_b L|\boldsymbol{\eta} - \boldsymbol{\eta}'|) \sim 1$, we approximate

$$I \simeq L^2 \int\limits_{V/L^3} d\boldsymbol{\eta}' \frac{q(\boldsymbol{\eta}'L)}{4\pi|\boldsymbol{\eta} - \boldsymbol{\eta}'|} \sim O(L^2)\tilde{q}(\boldsymbol{\eta}'L). \tag{3}$$

If $|q(\mathbf{r})| \propto C$, where C is a constant, then $\tilde{q}$ is of the same order as q. In this case

$$I \sim O(L^2)\tilde{q}(\boldsymbol{\eta}'L) \sim O(L^2 q). \tag{4}$$

(b) Using dimensional analysis $g(\mathbf{r}, \mathbf{r}') \sim 1/L$, $\int d\mathbf{r}' \sim L^3$, we have

$$I \sim O\left(L^3 \frac{1}{L} q(\mathbf{r}')\right) \sim O(L^2 q). \tag{5}$$

§8.41

(a) Consider a dielectric slab as in Figure 2.1.3 with $d_1 = 0$ and $d_2 = L$. For a normal incident plane wave

$$\mathbf{E}_{in} = \hat{x} e^{-ik_b z}, \tag{1}$$

then in region 1, we have an incident and reflected wave

$$e_{1x} = e^{-ik_b z} + \tilde{R}_{12} e^{ik_b z}. \tag{2}$$

In region 2 of a dielectric slab, we have

$$e_{2x} = A_2 \left[e^{-ikz} + R_{23} e^{2ikL + ikz}\right]. \tag{3}$$

In region 3, the downgoing wave can be written as

$$e_{3x} = A_3 e^{-ik_b z}. \tag{4}$$

Now we enforce the constraint conditions. In region 2 the downgoing wave is a consequence of the transmission of the downgoing wave in region 1 plus a reflection of the upgoing wave in region 2. Thus at the first interface $z = 0$, we have

$$A_2 = T_{12} + A_2 R_{23} R_{21} e^{+2ikL}. \tag{5}$$

In region 1 the upgoing wave is caused by the reflection of the downgoing wave in region 1 plus a transmission of the upgoing wave in region 2. Thus we have the condition of

$$\tilde{R}_{12} = R_{12} + T_{21} R_{23} A_2 e^{+2ikL}. \tag{6}$$

¿From (5), we have

$$A_2 = \frac{T_{12}}{1 - R_{21} R_{23} e^{2ikL}}. \tag{7}$$

Substituting (7) into (6), we have

$$\tilde{R}_{12} = R_{12} + \frac{T_{21}T_{12}R_{23}e^{2ikL}}{1 - R_{21}R_{23}e^{2ikL}}, \tag{8}$$

where the TE reflection and transmission coefficients are

$$T_{12} = \frac{2k_b}{k_b + k}, \qquad T_{21} = \frac{2k}{k_b + k}, \tag{9a}$$

$$R_{21} = \frac{k - k_b}{k_b + k}, \qquad R_{23} = \frac{k - k_b}{k_b + k}. \tag{9b}$$

(b) In order to simplify we can rewrite (8) as

$$\tilde{R}_{12} = \frac{R_{12} + R_{23}e^{2ikL}}{1 - R_{21}R_{23}e^{2ikL}}. \tag{10}$$

Using (9b) we have

$$\begin{aligned}\tilde{R}_{12} &= \frac{\frac{\sqrt{\epsilon_b}-\sqrt{\epsilon}}{\sqrt{\epsilon_b}+\sqrt{\epsilon}} + \frac{\sqrt{\epsilon}-\sqrt{\epsilon_b}}{\sqrt{\epsilon_b}+\sqrt{\epsilon}}e^{2ikL}}{1 - \left(\frac{\sqrt{\epsilon}-\sqrt{\epsilon_b}}{\sqrt{\epsilon}+\sqrt{\epsilon_b}}\right)^2 e^{2ikL}}, \\ &\simeq \frac{\frac{\epsilon_b-\epsilon}{(\sqrt{\epsilon_b}+\sqrt{\epsilon})^2}(1 - e^{2ikL})}{1 - O(\delta\epsilon^2)}, \qquad \epsilon \to \epsilon_b \\ &\simeq -\frac{\delta\epsilon}{4}(1 - e^{2ikL}), \quad \epsilon \to \epsilon_b\end{aligned} \tag{11}$$

where $\delta\epsilon = (\epsilon - \epsilon_b)/\epsilon$.

(c) In one dimension, the Green's function is

$$g(z, z') = \frac{i}{2k_b}e^{ik_b|z-z'|}. \tag{12}$$

The reflected field from a slab is exactly

$$E_R(z) = \int_{-L}^{0} dz' g(z, z')E(z')k_b^2\delta\epsilon, \qquad z > 0 \tag{13}$$

In the Born approximation, $E(z) \simeq E_{inc}(z)$ so the above becomes

$$\begin{aligned}E_R(z) &\simeq \int_{-L}^{0} dz' \frac{i}{2k_b}e^{ik_b|z-z'|}e^{-ik_bz'}k_b^2\delta\epsilon \\ &\simeq \int_{-L}^{0} dz' \frac{ik_b}{2}\delta\epsilon\, e^{ik_bz}e^{-2ik_bz'} \\ &\simeq e^{ik_bz}\frac{\delta\epsilon}{4}[e^{2ik_bL} - 1].\end{aligned} \tag{14}$$

Thus, we have

$$\tilde{R}_{21} \simeq -\frac{\delta\epsilon}{4}\left[1 - e^{2ik_bL}\right]. \tag{15}$$

The Born field reduces to the low contrast slab field only if $k_b\delta\epsilon L \ll 1$.

§8.42

(a) The volume integral equation for the scalar wave case is

$$\phi(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \int_V dV' g(\mathbf{r},\mathbf{r}') \cdot O(\mathbf{r}')\phi(\mathbf{r}'). \tag{1}$$

The Born approximation is

$$\phi(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \int_V dV' g(\mathbf{r},\mathbf{r}') \cdot O(\mathbf{r}')\phi_{inc}(\mathbf{r}'). \tag{2}$$

We want to find a condition in which the second term on the right-hand side of (2) will be much smaller than the first term on the right-hand side of (2). Using dimensional analysis in 3-D, we have

$$\begin{aligned} g(\mathbf{r},\mathbf{r}') &\sim \frac{1}{L}, \\ O(\mathbf{r}') &= (k^2 - k_b^2) \sim k_b^2\Delta\epsilon_r, \\ \int_V dV' &\sim L^3. \end{aligned} \tag{3}$$

Therefore, the second term on the right-hand side of (2) is of the order

$$L^2k_b^2\Delta\epsilon_r\phi_{inc}. \tag{4}$$

Thus, the constraint of the Born approximation for the scalar wave is

$$k_b^2L^2\Delta\epsilon_r \ll 1. \tag{5}$$

(b) In two dimensions, the Green's function is

$$g(\mathbf{r},\mathbf{r}') = \frac{i}{4}H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|). \tag{6}$$

At low frequencies, $H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|) \sim i\frac{2}{\pi}\ln(k|\mathbf{r}-\mathbf{r}'|)$. Applying dimensional analysis in 2-D, we have

$$\begin{aligned} g(\mathbf{r},\mathbf{r}') &\sim \ln(kL), \\ \int_V dV' &\sim L^2, \\ O(\mathbf{r}') &= k^2 - k_b^2 \sim k_b^2\Delta\epsilon_r. \end{aligned} \tag{7}$$

Substituting the above into (2), the order of the second term on the right hand side of (2) is

$$L^2 k_b^2 \ln(kL) \Delta\epsilon_r \phi_{inc}. \tag{8}$$

Thus, the constraint for two dimensions is

$$k_b^2 L^2 \ln(kL) \Delta\epsilon_r \ll 1. \tag{9}$$

In one dimension, the Green's function is

$$g(z, z') = \frac{i}{2k_b} e^{ik_b|z-z'|} \sim \frac{1}{k_b}. \tag{10}$$

Similarly, the constraint in one dimension is

$$k_b L \Delta\epsilon_r \ll 1. \tag{11}$$

§8.43

(a) From Exercise 8.41, we have that the reflected field is

$$\phi_R(z) = \tilde{R}_{12} e^{ik_b z}, \qquad z > 0, \tag{1}$$

and the field inside the slab is

$$\phi(z) = A_2 \left(e^{-ikz} + R_{23} e^{2ikL + ikz} \right), \tag{2}$$

where

$$\tilde{R}_{12} = \frac{R_{12} + R_{23} e^{2ikL}}{1 - R_{21} R_{23} e^{2ikL}}, \tag{3}$$

$$A_2 = \frac{T_{12}}{1 - R_{21} R_{23} e^{2ikL}}, \tag{4}$$

and for a dielectric slab

$$T_{12} = \frac{2\sqrt{\epsilon_b}}{\sqrt{\epsilon_b} + \sqrt{\epsilon}},$$

$$R_{21} = R_{23} = \frac{\sqrt{\epsilon} - \sqrt{\epsilon_b}}{\sqrt{\epsilon} + \sqrt{\epsilon_b}},$$

$$R_{12} = \frac{\sqrt{\epsilon_b} - \sqrt{\epsilon}}{\sqrt{\epsilon_b} + \sqrt{\epsilon}}. \tag{5}$$

(b) The form of the Rytov solution is

$$\phi(z) \simeq \phi_0(z) e^{i\psi_1(z)}, \qquad O < z < L, \tag{6}$$

where

$$\phi_0(z) = e^{-ik_b z}, \tag{7}$$

and the phase perturbation is given by (8.10.26)

$$\begin{aligned}
\psi_1(z) &= -\frac{i}{\phi_0(z)}\int_{-L}^{0} dz'\frac{i}{2k_b}e^{ik_b|z-z'|}\phi_0(z')O(z') \\
&= -ie^{ik_bz}\int_{-L}^{0} dz'\frac{i}{2k_b}e^{ik_b|z-z'|}e^{-ik_bz'}k_b^2\Delta\epsilon \\
&= -ie^{ik_bz}\int_{-L}^{z} dz'\frac{i}{2}e^{ik_b(z-z')}e^{-ik_bz'}k_b\Delta\epsilon \\
&\quad - ie^{ik_bz}\int_{z}^{0} dz'\frac{i}{2}e^{-ik_b(z-z')}e^{-ik_bz'}k_b\Delta\epsilon \\
&= -ie^{2ik_bz}\int_{-L}^{z} dz'\frac{ik_b}{2}\Delta\epsilon e^{-2ik_bz'} + \frac{k_b}{2}\Delta\epsilon\int_{z}^{0} dz' \\
&= \frac{ie^{2ik_bz}}{4}\Delta\epsilon e^{-2ik_bz'}\Big|_{-L}^{z} - z\frac{k_b}{2}\Delta\epsilon \\
&= \frac{i\Delta\epsilon}{4} - \frac{i\Delta\epsilon}{4}e^{2ik_bz+2ik_bL} - z\frac{k_b}{2}\Delta\epsilon.
\end{aligned} \tag{8}$$

Substituting (8) into (6), we have

$$\begin{aligned}
\phi(z) &= e^{-ik_bz}e^{i[-z\frac{k_b}{2}\Delta\epsilon+\frac{i\Delta\epsilon}{4}-\frac{i\Delta\epsilon}{4}e^{2ik_bL+2ik_bz}]} \\
&= e^{-ik_b(1+\frac{\Delta\epsilon}{2})z}e^{[-\frac{\Delta\epsilon}{4}+\frac{\Delta\epsilon}{4}e^{2ik_bL+2ik_bz}]}
\end{aligned} \tag{9}$$

Notice that

$$\begin{aligned}
\frac{\epsilon}{\epsilon_b} - 1 &= \Delta\epsilon, \\
\epsilon &= \epsilon_b(1+\Delta\epsilon),
\end{aligned} \tag{10}$$

and

$$k = \omega\sqrt{\mu\epsilon} = k_b\sqrt{1+\Delta\epsilon} \simeq k_b\left(1+\frac{\Delta\epsilon}{2}\right).$$

Using the above in (9) becomes

$$\begin{aligned}
\phi(z) &= e^{-ikz}e^{-\frac{\Delta\epsilon}{4}}e^{[\frac{\Delta\epsilon}{4}e^{2ik_bL+2ik_bz}]} \\
&= e^{-ikz}\left(1-\frac{\Delta\epsilon}{4}\right)\left(1+\frac{\Delta\epsilon}{4}e^{2ik_bL+2ik_bz}\right) + O(\Delta\epsilon^2) \\
&= \left(1-\frac{\Delta\epsilon}{4}\right)e^{-ikz} + \frac{\Delta\epsilon}{4}e^{2ik_bL+ikz} + O(\Delta\epsilon^2).
\end{aligned} \tag{11}$$

In part (a), if $\Delta\epsilon \ll 1$, then

$$\begin{aligned} A_2 &= \frac{T_{12}}{1 - R_{21}R_{23}e^{2ikL}} \\ &\simeq T_{12} + O(\Delta\epsilon^2) = \frac{2\sqrt{\epsilon_b}}{\sqrt{\epsilon_b} + \sqrt{\epsilon}} + O(\Delta\epsilon^2) \\ &= 1 - \frac{\Delta\epsilon}{4} + O(\Delta\epsilon^2). \end{aligned} \tag{12}$$

Thus, the first term on the right-hand size of (11) is equal to the first term of (2) when $\Delta\epsilon \ll 1$.

Using (12), we can further obtain that

$$\begin{aligned} A_2R_{23} &= \left(1 - \frac{\Delta\epsilon}{4}\right)\frac{\sqrt{\epsilon} - \sqrt{\epsilon_b}}{\sqrt{\epsilon} + \sqrt{\epsilon_b}} + O(\Delta\epsilon^2) \\ &= \frac{\Delta\epsilon}{4} + O(\Delta\epsilon^2). \end{aligned}$$

And when $\Delta\epsilon \ll 1$, we can approximate e^{2ik_bL} with e^{2ikL}. Therefore, we have that the second term on the right-hand side of (11) is the same as the second term on the right-hand side of (2) when $\Delta\epsilon \ll 1$. In conclusion, the Rytov solution (11) reduces to that in part (a) when (8.10.33) is satisfied.

CHAPTER 9

EXERCISE SOLUTIONS

by W. H. Weedon

§9.1

From Equation (9.1.1), we have

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \int_V d\mathbf{r}'\, \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot O(\mathbf{r}')\mathbf{E}(\mathbf{r}') \tag{1}$$

where $O(\mathbf{r}) = k^2(\mathbf{r}) - k_b^2$. In operator form,

$$\mathcal{E}\rangle = \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \mathcal{E}\rangle. \tag{2}$$

The above is a Fredholm integral equation of the second kind, since $\mathbf{E}(\mathbf{r})$ appears both inside and outside the integral. A Neumann series may be obtained by several methods. The most direct way is to recursively substitute the expression for $\mathcal{E}\rangle$ given by Equation (2) into the second term on the right of Equation (2):

$$\begin{aligned}\mathcal{E}\rangle &= \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \left[\mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \mathcal{E}\rangle\right] \\ &= \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \left[\mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \mathcal{E}\rangle\right] \\ &= \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \mathcal{E}\rangle \\ &= \ldots \end{aligned} \tag{3}$$

Bringing the incident field $\mathcal{E}_{inc}\rangle$ to the left side of Equation (2), the scattered field is given by

$$\mathcal{E}_s\rangle = \mathcal{E}\rangle - \mathcal{E}_{inc}\rangle = \overline{\mathcal{G}}\cdot\overline{\mathcal{O}}\cdot\mathcal{E}_{inc}\rangle + \overline{\mathcal{G}}\cdot\overline{\mathcal{O}}\cdot\overline{\mathcal{G}}\cdot\overline{\mathcal{O}}\cdot\mathcal{E}_{inc}\rangle + \overline{\mathcal{G}}\cdot\overline{\mathcal{O}}\cdot\overline{\mathcal{G}}\cdot\overline{\mathcal{O}}\cdot\overline{\mathcal{G}}\cdot\overline{\mathcal{O}}\cdot\mathcal{E}_{inc}\rangle + \ldots \tag{4}$$

The above Neumann series is more commonly known as the Born series and is valid provided that this series converges uniformly (See Courant and Hilbert, Vol. 1, 1953). It is analogous to expanding Equation (2) in a Taylor series about the object function, since successive terms depend (loosely) on $\overline{\mathcal{O}}$,$\overline{\mathcal{O}}^2$,$\overline{\mathcal{O}}^3$, etc. Hence, the scattered field is a nonlinear function of the scattering object $\overline{\mathcal{O}}$.

The first terms in Equation (4) is linear in $\overline{\mathcal{O}}$ and corresponds to single scattering within the object. Higher order terms correspond to multiple scattering. If the object contrast is small (recall $\overline{\mathcal{O}}(\mathbf{r}) = k^2(\mathbf{r}) - k_b$), then

Equation (4) may be approximated with the first term only, neglecting higher order terms. This is known as the first Born approximation and is valid in the weak scattering limit only.

§9.2

Using the formulation given in Section 8.6 of the text, we may express an arbitrary incident field as

$$\phi_{inc}(\mathbf{r}) = \Re e \psi^t(k_0, \mathbf{r}_0) \cdot \mathbf{a}, \tag{1}$$

where $\mathbf{a}$ is the vector of expansion coefficients of the standing wave harmonic basis functions $\Re e \psi^t(k_0, \mathbf{r}_0)$. In the single scatterer case, we express the scattered field in outgoing wave harmonics,

$$\phi_{sca}(\mathbf{r}) = \psi^t(k_0, \mathbf{r}) \cdot \mathbf{f} \tag{2}$$

where the scattering amplitude coefficients $\mathbf{f}$ may be written in terms of the incident field amplitude vector $\mathbf{a}$ as

$$\mathbf{f} = \overline{\mathbf{T}}_{(1)} \cdot \mathbf{a}, \tag{3}$$

and $\overline{\mathbf{T}}_{(1)}$ is the single-scatterer transition matrix. When two scatterers are present, we write the scattered field as

$$\phi_{sca}(\mathbf{r}) = \psi^t(k_0, \mathbf{r}_1) \cdot \mathbf{f}_1 + \psi^t(k_0, \mathbf{r}_2) \cdot \mathbf{f}_2. \tag{4}$$

The scattering coefficients $\mathbf{f}_1$ and $\mathbf{f}_2$ may again be expressed in terms of the incident field amplitude as

$$\mathbf{f}_1 = \overline{\mathbf{T}}_{1(2)} \cdot \overline{\beta}_{10} \cdot \mathbf{a} \tag{5}$$

$$\mathbf{f}_2 = \overline{\mathbf{T}}_{2(2)} \cdot \overline{\beta}_{20} \cdot \mathbf{a}. \tag{6}$$

Now, $\overline{\mathbf{T}}_{1(2)}$ and $\overline{\mathbf{T}}_{2(2)}$ are the transition matrices in the presence of two scatterers, and $\overline{\beta}_{10}$ and $\overline{\beta}_{20}$ are appropriate translation matrices. From Equations (8.6.7) and (8.6.8) of the text, the two-scatterer $\overline{\mathbf{T}}$ matrices may be written in terms of the one-scatterer $\overline{\mathbf{T}}$ matrices as

$$T_{1(2)} = \left[\overline{\mathbf{I}} - \overline{\mathbf{T}}_{1(1)} \cdot \overline{\alpha}_{12} \cdot \overline{\mathbf{T}}_{2(1)} \cdot \overline{\alpha}_{21}\right]^{-1} \cdot \overline{\mathbf{T}}_{1(1)} \cdot \left[\overline{\mathbf{I}} + \overline{\alpha}_{12} \cdot \overline{\mathbf{T}}_{2(1)} \cdot \overline{\beta}_{20} \cdot \overline{\beta}_{10}^{-1}\right] \tag{7}$$

$$\overline{\mathbf{T}}_{2(2)} = \left[\overline{\mathbf{I}} - \overline{\mathbf{T}}_{2(1)} \cdot \overline{\alpha}_{21} \cdot \overline{\mathbf{T}}_{1(1)} \cdot \overline{\alpha}_{12}\right]^{-1} \cdot \overline{\mathbf{T}}_{2(1)} \cdot \left[\overline{\mathbf{I}} + \overline{\alpha}_{21} \cdot \overline{\mathbf{T}}_{1(1)} \cdot \overline{\beta}_{10} \cdot \overline{\beta}_{20}^{-1}\right]. \tag{8}$$

Hence, the single-scatterer $\overline{\mathbf{T}}$ matrices are not the same as the double-scatterer $\overline{\mathbf{T}}$ matrices and the two-scatterer solution is not a simple linear superposition of the one-scatterer solutions. As is clear from the form of Equations (7) and (8), the two-scatterer solution takes into account multiple scattering between the two scatterers.

§9.3

The phase delay may be obtained from the WKB approximation as

$$\phi(b,\omega) = \phi(a,\omega)e^{i\omega\tau}, \tag{1}$$

where

$$\tau = \int_a^b s(z')dz', \tag{2}$$

and $s(z) = k_z/\omega$ is the slowness of the medium. From elementary Fourier theory, a phase shift in the frequency domain corresponds to a time delay in the time domain. That is, if

$$f(t) \to F(\omega), \tag{3}$$

then

$$f(t-\tau) \to e^{i\omega\tau}F(\omega). \tag{4}$$

Taking the inverse Fourier transform of (1) and using property (4), we have

$$\phi(b,t) = \phi(a,t-\tau). \tag{5}$$

That is, the field at b is just a time delayed version of the field at point a.

§9.4

(a)

$$s(x,y) = \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} dk_x dk_y\, e^{ik_x x+ik_y y}S(k_x,k_y). \tag{1}$$

Let, $x = \rho\cos\phi, y = \rho\sin\phi$, and $k_x = k_\rho\cos\alpha, k_y = k_\rho\sin\alpha$. Hence, $dk_x dk_y = k_\rho dk_\rho d\alpha$ and we have

$$s(\rho,\phi) = \frac{1}{(2\pi)^2}\int\limits_0^{2\pi} d\alpha \int\limits_0^{\infty} dk_\rho\, k_\rho e^{ik_\rho\cos(\alpha-\phi)}S(k_\rho,\alpha). \tag{2}$$

where we have used the identity $\cos(\alpha-\phi) = \cos\alpha\cos\phi + \sin\alpha\sin\phi$. Note that in the above integral both the exponential $e^{ik_\rho\cos(\alpha-\phi)}$ and the Fourier transform slice $S(k_\rho,\alpha)$ remain invariant if α is replaced by $\alpha+\pi$ and k_ρ is replaces by $-k_\rho$. The invariance of the exponential is obvious. To see that $S(k_\rho,\alpha)$ remains invariant, we write

$$S(k_\rho,\alpha) = \int\limits_{-\infty}^{\infty} d\xi e^{-ik_\rho\xi}P(\xi,\alpha). \tag{3}$$

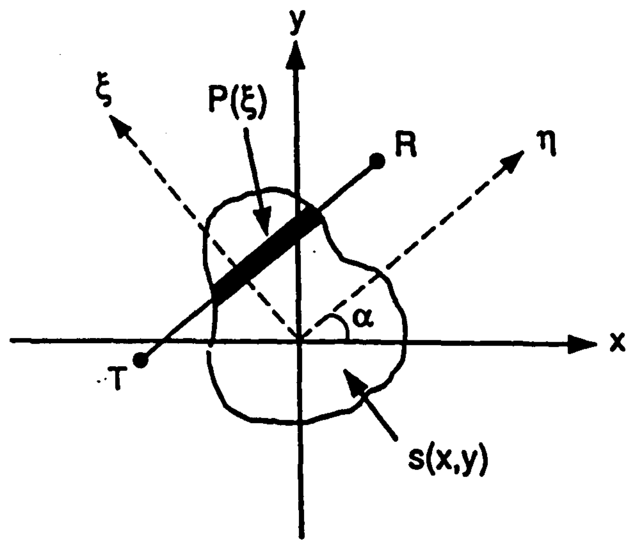

Figure 1 for Exercise Solution 9.4

By observing Figure 1, it should be clear that if we were to rotate the $\hat{\xi}, \hat{n}$ axis by π radians, we would find that

$$P(-\xi, \alpha + \pi) = P(\xi, \alpha). \tag{4}$$

Substituting this expression into Equation (3) above and replacing ξ by $-\xi$, we have

$$S(k_\rho, \alpha) = S(-k_\rho, \alpha + \pi). \tag{5}$$

Thus, Equation (2) can be rewritten as

$$s(\rho, \phi) = \frac{1}{(2\pi)^2} \int_0^\pi d\alpha \int_{-\infty}^{\infty} dk_\rho \, |k_\rho| e^{ik_\rho \rho \cos(\alpha - \phi)} S(k_\rho, \alpha) \tag{6}$$

where we have replaced the factor k_ρ by $|k_\rho|$ to account for its asymmetry.

(b) We are asked to prove

$$P.V. \int_{-\infty}^{\infty} dy \, \frac{e^{-ixy}}{y} = -\pi i \, \text{sgn}(x). \tag{7}$$

Without the "*P.V.*", the above is an improper integral due to the singularity at the origin. The principal value is obtained by integrating along the real axis from $y = -\infty$ to $-\epsilon$ and $y = +\epsilon$ to $+\infty$ and then letting $\epsilon \to 0$. The evaluation is simplified by using analytic continuation to

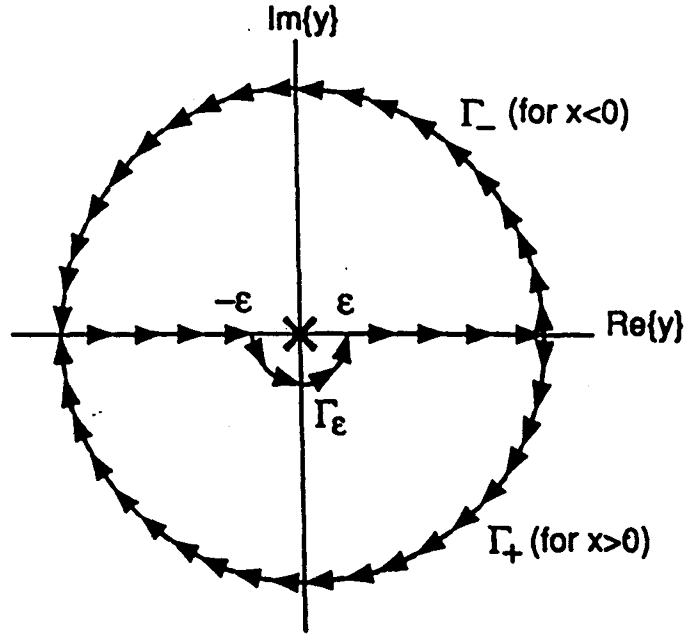

Figure 2 for Exercise Solution 9.4

extend the integral into the complex plane and using Cauchy's integral theorem along with Jordan's lemma.

In Figure 2, we show a closed contour for evaluating this integral. We have chosen a semicircle below the singularity at the origin, extending from $-\epsilon$ to $+\epsilon$. The contour is closed at ∞ by either Γ_+ or Γ_-, depending on the sign of x. For $x > 0$, we want $\Re e\{y\} < 0$, so that the exponential factor e^{-ixy} decays as $|y| \to \infty$, and so we choose Γ_+. Similarly, for $x < 0$ we want to choose Γ_-.

Then

$$P.V. \int_{-\infty}^{\infty} dy \frac{e^{-ixy}}{y} = \oint_C dy \frac{e^{-ixy}}{y} - \int_{\Gamma_\epsilon} dy \frac{e^{-ixy}}{y} \tag{8}$$

where C is the contour along the real axis around Γ_ϵ and closing at either Γ_+ or Γ_-. The integral around Γ_ϵ is subtracted off because we only want the integral along the real axis. We have

$$P.V. \int_{-\infty}^{\infty} dy \frac{e^{-ixy}}{y} = \begin{cases} 0 - [\pi i\, Res(0)], & x > 0 \\ [2\pi i\, Res(0)] - [\pi i\, Res(0)], & x < 0 \end{cases}$$

$$= \begin{cases} -\pi i, & x > 0 \\ +\pi i, & x < 0 \end{cases}$$
$$= -\pi i \,\mathrm{sgn}(x). \tag{9}$$

Hence,

$$P.V. \int_{-\infty}^{\infty} dy \, \frac{e^{-ixy}}{y} = -\pi i \, \text{sgn}(x). \tag{10}$$

Taking the inverse Fourier transform of the above,

$$\begin{aligned} P.V. \frac{1}{y} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx (-\pi i) \, \text{sgn}(x) e^{ixy}, \\ &= -\frac{i}{2} \int_{-\infty}^{\infty} dx \, \text{sgn}(x) e^{ixy}. \end{aligned} \tag{11}$$

Taking the partial derivative with respect to y,

$$\frac{\partial}{\partial y} P.V. \frac{1}{y} = \frac{1}{2} \int_{-\infty}^{\infty} dx \, |x| e^{ixy} \tag{12}$$

where we have used $|x| = x \, \text{sgn}(x)$.From Equations (9.1.10) and (9.1.11) in the text,

$$\begin{aligned} s(\rho, \phi) &= \frac{1}{(2\pi)^2} \int_0^{\pi} d\alpha \int_{-\infty}^{\infty} dk_\rho \, |k_\rho| e^{ik_\rho \rho \cos(\alpha - \phi)} \int_{-\infty}^{\infty} d\xi \, e^{-ik_\rho \xi} P(\xi, \alpha) \\ &= \frac{1}{(2\pi)^2} \int_0^{\pi} d\alpha \int_{-\infty}^{\infty} d\xi \, P(\xi, \alpha) \int_{-\infty}^{\infty} dk_\rho \, |k_\rho| e^{ik_\rho \rho \cos(\alpha - \phi) - ik_\rho \xi} \\ &= \frac{1}{2\pi^2} \int_0^{\pi} d\alpha \int_{-\infty}^{\infty} d\xi \, P(\xi, \alpha) \frac{\partial}{\partial[\rho \cos(\alpha - \phi) - \xi]} \\ &\quad \times P.V. \frac{1}{\rho \cos(\alpha - \phi) - \xi}. \end{aligned} \tag{13}$$

The above expression may be further reduced by integrating the ξ integral by parts. However, a generalized integration by parts formula must be used since partial derivatives are involved. From the product rule for partial derivatives, we may obtain

$$\int_{-\infty}^{\infty} d\xi \, u \frac{\partial v}{\partial \xi} = uv \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} d\xi \, v \frac{\partial u}{\partial \xi}. \tag{14}$$

Let $u = P(\xi, \alpha)$ and $\frac{\partial v}{\partial \xi} = \frac{\partial}{\partial y} P.V. \frac{1}{y}$ where $y = \rho \cos(\alpha - \phi) - \xi$. Invoking the Chain rule,

$$\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \xi} = -\frac{\partial v}{\partial y} \tag{15}$$

and thus $v = -P.V.(\frac{1}{y})$. Then

$$\int_{-\infty}^{\infty} d\xi\, P(\xi,\alpha)\frac{\partial}{\partial y}\, P.V.\left(\frac{1}{y}\right) = -P(\xi,\alpha)P.V.\left(\frac{1}{y}\right)\Big|_{-\infty}^{\infty} + P.V.\int_{-\infty}^{\infty} d\xi\, \frac{\partial P(\xi,\alpha)}{\partial \xi}\cdot\left(\frac{1}{y}\right). \tag{16}$$

If we assume that the scattering object has compact support, then $P(\infty,\alpha) = P(-\infty,\alpha) = 0$. It follows that

$$s(\rho,\phi) = \frac{1}{2\pi^2}\int_0^{\pi} d\alpha\, P.V.\int_{-\infty}^{\infty} \frac{\frac{\partial P(\xi,\alpha)}{\partial \xi}}{\rho\cos(\alpha-\phi)-\xi}\, d\xi. \tag{17}$$

(c) For the fixed measurement configuration shown in Figure 9.1.3 of the text, we obtain the slowness profile as a function of the y' coordinate only as

$$P(y') = \int_{-\infty}^{\infty} s(x',y')\, dx'. \tag{18}$$

If we were to rotate the entire measurement apparatus by an angle of α, but keep the object fixed, we would obtain

$$P(\xi,\alpha) = \int_{-\infty}^{\infty} d\eta\, s(\xi,\eta) \tag{19}$$

where ξ,η represent the the rotated coordinates. It is desirable to express the slowness profile in terms of the fixed coordinates (x',y'). To do this we must integrate $s(x',y')$ over both x' and y' at a fixed value of ξ given as the projection of (x',y') in the $\hat{\xi}$ direction, $\xi = \boldsymbol{\rho}\cdot\hat{\xi}$ where $\boldsymbol{\rho} = \hat{x}x' + \hat{y}y'$. Hence

$$P(\xi,\alpha) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy'\, \delta(\boldsymbol{\rho}\cdot\hat{\xi}-\xi)s(x',y'). \tag{20}$$

We may also write the above in terms of the fixed cylindrical coordinates ρ,ϕ where $x' = \rho\cos\phi$, $y' = \rho\sin\phi$. We have $dx'dy' = \rho d\rho d\phi$, $\hat{\xi} = -\hat{x}\sin\alpha + \hat{y}\cos\alpha$, and $\boldsymbol{\rho}\cdot\hat{\xi} = -\rho\cos\phi\sin\alpha + \rho\sin\phi\cos\alpha = \rho\sin(\phi-\alpha)$. Thus,

$$P(\xi,\alpha) = \int_0^{2\pi} d\phi \int_0^{\infty} \rho d\rho\, \delta[\rho\sin(\phi-\alpha)-\xi]s(\rho,\phi). \tag{21}$$

§9.5

(a) Starting with Equation (9.1.22) in the text,

$$s(r,\theta,\phi) = \frac{1}{(2\pi)^3} \int_0^\pi d\beta \int_0^\pi d\alpha \sin\alpha \int_{-\infty}^{\infty} d\xi\, P(\xi,\beta,\alpha) \int_{-\infty}^{\infty} dk\, k^2 e^{ik(\hat{k}\cdot\mathbf{r}-\xi)}. \tag{1}$$

The last integral may be simplified using

$$\int_{-\infty}^{\infty} dk\, k^2 e^{ik(\hat{k}\cdot\mathbf{r}-\xi)} = -2\pi\delta''(\hat{k}\cdot\mathbf{r}-\xi). \tag{2}$$

Hence the last two integrals in Equation (1) may be rewritten as

$$-2\pi \int_{-\infty}^{\infty} d\xi\, P(\xi,\beta,\alpha)\, \delta''(\hat{k}\cdot\mathbf{r}-\xi) = -2\pi \frac{\partial^2 P(\hat{k}\cdot\mathbf{r},\beta,\alpha)}{\partial^2(\hat{k}\cdot\mathbf{r})} \tag{3}$$

and thus

$$s(r,\theta,\phi) = \frac{-1}{(2\pi)^2} \int_0^\pi d\beta \int_0^\pi d\alpha\, \sin\alpha \frac{\partial^2}{\partial^2(\hat{k}\cdot\mathbf{r})} P\left(\hat{k}\cdot\mathbf{r},\beta,\alpha\right). \tag{4}$$

The forward transform is derived in a manner exactly analogous to the 2-D forward Radon transform.

(b) In any number of dimensions, the slowness projection along a direction $\hat{\xi}$ is given as

$$P(\xi,\hat{\xi}) = \int d\eta_1 \cdots \int d\eta_{n-1} s(\xi,\boldsymbol{\eta}) \tag{5}$$

where $\hat{\eta}_i \perp \hat{\xi}$ for $i = 1, n-1$ and n is the number of dimensions. But this is just the Fourier transform of $s(\xi,\boldsymbol{\eta})$ with respect to the $(n-1)$ dimensional vector $\boldsymbol{\eta}$ evaluated at $\mathbf{k}_\eta = 0$.

Writing

$$S(k_\xi,\mathbf{k}_\eta) = \int d\mathbf{x}\, s(\xi,\boldsymbol{\eta}) e^{i\mathbf{k}\cdot\mathbf{x}} = \int d\xi \int d\eta_1 \ldots \int d\eta_{n-1}\, s(\xi,\boldsymbol{\eta}) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{6}$$

with $\mathbf{x} = (\xi,\boldsymbol{\eta})$ and $\mathbf{k} = (k_\xi,\mathbf{k}_\eta)$, we have

$$S(k_\xi,\hat{\xi}) = S(k_\xi,\boldsymbol{\eta}=0) = \int_{-\infty}^{\infty} d\xi\, P(\xi,\hat{\xi}) e^{-ik_\xi\xi} \tag{7}$$

where we have discarded $\eta = 0$ and included the parameter $\hat{\xi}$ as the second argument to $S(\cdot,\cdot)$. The inverse Fourier transform of $S(k,\hat{k})$ may be written as

$$s(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_B d\hat{k} \int_0^\infty dk\, k^{n-1} e^{ik\cdot\mathbf{x}} S(k,\hat{k}). \tag{8}$$

By geometrical arguments (see solution for Exercise 9.4(a)) we can show that

$$P(-\xi,-\hat{\xi}) = P(\xi,\hat{\xi}) \tag{9}$$

Similarly,

$$S(-k_\xi,-\hat{\xi}) = S(k_\xi,\hat{\xi}) \tag{10}$$

since $\hat{k}_\xi = \hat{\xi}$ (this may also be derived from Equations (7) and (9)). This means that we may rewrite Equation (8) as

$$s(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{B/2} d\hat{k} \int_{-\infty}^\infty dk\, |k|^{n-1} e^{ik\cdot\mathbf{x}} S(k,\hat{k}), \tag{11}$$

where $B/2$ is an arbitrary half of a unit ball. Substituting (7) into (11),

$$s(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{B/2} d\hat{k} \int_{-\infty}^\infty d\xi P(\xi,\hat{k}) \int_{-\infty}^\infty dk\, |k|^{n-1} e^{ik\cdot\mathbf{x}-ik\xi}. \tag{12}$$

From Equation (9.1.14) of the text it follows that

$$\frac{\partial^{n-1}}{\partial y^{n-1}} P.V.\frac{1}{y} = -\frac{i^n}{2}\int_{-\infty}^\infty dk\, |k|^{n-1} e^{iky}, \tag{13}$$

and so

$$s(\mathbf{x}) = \frac{-2}{(2\pi i)^n}\int_{B/2} d\hat{k} \int_{-\infty}^\infty d\xi P(\xi,\hat{k}) \frac{\partial^{n-1}}{\partial y^{n-1}} P.V.\frac{1}{y} \tag{14}$$

where $y = \hat{k}\cdot\mathbf{x} - \xi$. Integrating by parts $(n-1)$ times gives

$$s(\mathbf{x}) = \frac{-2}{(2\pi i)^n}\int_{B/2} d\hat{k} \int_{-\infty}^\infty d\xi \frac{1}{\hat{k}\cdot\mathbf{x}-\xi}\frac{\partial^{n-1}P(\xi,\hat{k})}{\partial \xi^{n-1}}. \tag{15}$$

(c) In odd dimensions, Equation (12) can be written as

$$s(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{B/2} d\hat{k} \int_{-\infty}^\infty d\xi P(\xi,\hat{k}) \int_{-\infty}^\infty dk\, k^{n-1} e^{ik\cdot\mathbf{x}-ik\xi}. \tag{16}$$

Using the identity

$$\frac{i^{n-1}}{2\pi}\int_{-\infty}^{\infty} dk\; k^{n-1}e^{ikx} = \frac{\partial^{n-1}}{\partial x^{n-1}}\delta(x) \triangleq \delta^{n-1}(x), \tag{17}$$

the last two integrals in (16) above may be rewritten as

$$\frac{2\pi}{i^{n-1}}\int_{-\infty}^{\infty} d\xi P(\xi,\hat{k})\delta^{n-1}(\hat{k}\cdot\mathbf{x}-\xi) = \frac{2\pi}{i^{n-1}}\frac{\partial^{n-1}}{\partial(\hat{k}\cdot\mathbf{x})}P(\hat{k}\cdot\mathbf{x},\hat{k}). \tag{18}$$

Then

$$s(\mathbf{x}) = \frac{1}{(2\pi i)^{n-1}}\int_{B/2} d\hat{k}\frac{\partial^{n-1}}{\partial^{n-1}(\hat{k}\cdot\mathbf{x})}P(\hat{k}\cdot\mathbf{x},\hat{k}). \tag{19}$$

§9.6

Since we only have a single transducer, we are only allowed backscatter data where $\mathbf{k}_R = -\mathbf{k}_T$. But from (9.1.31) of the text, the scattered field is proportional to

$$\widetilde{O}(\mathbf{k}_R - \mathbf{k}_T) = \widetilde{O}(2\mathbf{k}_R) = \widetilde{O}(2k_0\hat{k}_R). \tag{1}$$

Since $k_0 = \omega/c_0$ where c_0 is the free-space velocity, it is clear that the Fourier space of the object may be filled out by varying both the frequency ω and the incidence direction of the single transducer $\hat{k}_R$. Since the transducer is bandlimited by $0 < \omega < \Omega$, we obtain a windowed version of the object Fourier transform $\widetilde{O}(k,\hat{k})$ where $0 < k < \frac{2\Omega}{c_0}$. Assuming that the transducer has unity gain over the entire bandwidth, the inverse Fourier transform of $\widetilde{O}(k,\hat{k})$ gives the true object $O(\mathbf{r})$ convolved with a sinc function (in rectangular coordinates). Hence, the resolution is determined by the width of the main lobe of the sinc function, given as $\Delta x = \frac{2\pi c_0}{\Omega}$.

§9.7

(a) We are asked to consider the locus of $\mathbf{k}' - \mathbf{k}$ with both $\mathbf{k}'$ and $\mathbf{k}$ sweeping over the angles 0 to 180°. It should be clear upon examining Figure 9.1.9 of the text that the locus is the two disks shown. Otherwise, the locus may be determined by letting $\mathbf{k}' = k(\hat{x}\cos\phi' + \hat{y}\sin\phi')$ and $\mathbf{k} = k(\hat{x}\cos\phi + \hat{y}\sin\phi)$ and plotting $\mathbf{k}' - \mathbf{k}$ for $0 \le \phi', \phi \le 180°$.

(b) For the backscatter experiment, we have $\mathbf{k}' = -\mathbf{k}$ and so $\mathbf{k}' - \mathbf{k} = -2\mathbf{k}$. Since $\mathbf{k}$ varies from 0 to 180°, it is clear that the locus of $-2\mathbf{k}$ is a semicircle of radius $2k_o$ in the lower-half $\mathbf{k}$ plane. If the backscatter experiment were performed varying the transmitter and receiver angles independently, we would fill in the semicircle in the lower-half $\mathbf{k}$ plane.

§9.8

(a) From Equation (9.2.1) of the text,

$$A\phi_{zz} + 2B\phi_{zt} + C\phi_{tt} = \Phi(\phi, \phi_t, \phi_z, z, t). \tag{1}$$

Assuming ϕ to be discontinuous in the vicinity of the characteristic, we have

$$\phi(z,t) \approx u(z - vt - a). \tag{2}$$

Hence

$$\phi_z \approx u'(z - vt - a), \quad \phi_t \approx -vu'(z - vt - a) \tag{3}$$

and

$$\phi_{zz} \approx u''(z - vt - a), \phi_{zt} \approx -vu''(z - vt - a), \phi_{tt} \approx v^2u''(z - vt - a). \tag{4}$$

However, the most singular terms of Equation (1) above cancel each other and since the right hand side of Equation (1) is of order $u'(\cdot)$ (due to ϕ_z, ϕ_t) and $u''(\cdot)$ is more singular than $u'(\cdot)$, we must have

$$A - 2Bv + Cv^2 = 0. \tag{5}$$

(b) For the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(z,t) = 0 \tag{6}$$

the characteristic curve is given by

$$z = \pm c(z)t + a \tag{7}$$

where we assume that $c(z)$ is non-dispersive (does not depend on time). If we assume that ϕ has a discontinuity given by

$$\phi(z,t) = u(z \mp ct - a), \tag{8}$$

it follows that

$$\phi_{zz} = u''(z \mp ct - a) \quad \text{and} \quad \phi_{tt} = c^2u''(z \mp ct - a) \tag{9}$$

and hence $\phi(z,t)$ satisfies Equation (6). If c were frequency dispersive, then we could express the characteristic curve in the vicinity of time t_0 as

$$z = \pm c(z, t_o)t \tag{10}$$

(c) Using the chain rule, we may write

$$\frac{d\phi_z}{dt} = v\phi_{zz} + \phi_{zt} \text{ and } \frac{d\phi_t}{dz} = \phi_{zt} + \phi_{tt}v^{-1} \tag{11}$$

then

$$A\phi_{zz} + 2B\phi_{zt} + C\phi_{tt}$$
$$= Av^{-1}\frac{d\phi_z}{dt} + Cv\frac{d\phi_t}{dz} - \phi_{zt}(Av^{-1} - 2B + Cv) \qquad (12)$$

The last term in Equation (12) alone is zero by virtue of Equation (5) above. Then using Equation (1),

$$Av^{-1}\frac{d\phi_z}{dt} + Cv\frac{d\phi_t}{dz} = \Phi. \qquad (13)$$

But

$$v^{-1}\frac{d\phi_z}{dt} = \frac{dt}{dz}\frac{d\phi_z}{dt} = \frac{d\phi_z}{dz}. \qquad (14)$$

Substituting Equation (14) into Equation (13) and multiplying through by dz gives

$$Ad\phi_z + Cvd\phi_t = \Phi dz. \qquad (15)$$

§9.9

(a) For the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(z,t) = 0, \qquad (1)$$

we have $A = 1$, $B = 0$, $C = -\frac{1}{c^2}$, $\Phi = 0$ and hence $v = \pm c$. Then Equations (9.2.9a) and (9.2.9b) become

$$d\phi_z - \frac{1}{c}d\phi_t = 0 \text{ on } C_+ \qquad (2)$$

and

$$d\phi_z + \frac{1}{c}d\phi_t = 0 \text{ on } C_- \qquad (3)$$

(b) With knowledge of ϕ and ϕ_t for all z at $t = 0$, we may solve for ϕ, ϕ_z, and ϕ_t for all time using a finite difference scheme. First, we discretize the time and space variables as shown in the Figure below. We are given values of ϕ and ϕ_t at points N, Q, M and we wish to derive the fields ϕ, ϕ_z, ϕ_t at point P. Once we have a formula for deriving the fields at P from those at N, Q, M the algorithm may be used recursively to derive the fields everywhere. Of course, an absorbing boundary condition must be used to take care of the endpoints of the grid.

Using the notation of the Figure, Equations (2) and (3) may be rewritten as

$$\phi_z(P) = \phi_z(N) - \frac{1}{c(Q)}[\phi_t(P) - \phi_t(N)] \qquad (4)$$

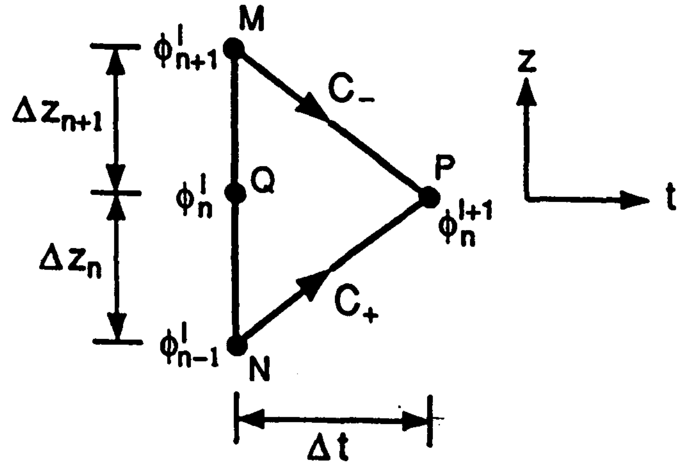

Figure 1 for Exercise Solution 9.9

$$\phi_z(P) = \phi_z(M) + \frac{1}{c(Q)}[\phi_t(P) - \phi_t(M)]. \tag{5}$$

Equations (4) and (5) may be combined to give

$$\phi_z(P) = \frac{1}{2}[\phi_z(N) + \phi_z(M)] + \frac{1}{2c(Q)}[\phi_t(N) - \phi_t(M)] \tag{6}$$

$$\phi_t(P) = \frac{c(Q)}{2}[\phi_z(N) - \phi_z(M)] + \frac{1}{2}[\phi_t(N) + \phi_t(M)]. \tag{7}$$

Next, we let $\Delta z_n = c_n \Delta t$ and so

$$\phi(z,t) = \phi(n\Delta z_n, l\Delta t) \equiv \phi^l_n. \tag{8}$$

Then, Equations (6) and (7) become

$$\phi^{l+1}_{z,n} = \frac{1}{2}[\phi^l_{z,n-1} + \phi^l_{z,n+1}] + \frac{1}{2c_n}[\phi^l_{t,n-1} - \phi^l_{t,n+1}], \tag{9}$$

$$\phi^{l+1}_{t,n} = \frac{c_n}{2}[\phi^l_{z,n-1} - \phi^l_{z,n+1}] + \frac{1}{2}[\phi^l_{t,n-1} + \phi^l_{t,n+1}]. \tag{10}$$

The initial value of the derivative $\phi^l_{z,n}$ may be obtained using

$$\phi^0_{z,n} = \frac{\phi^0_n - \phi^0_{n-1}}{\Delta z_n}. \tag{11}$$

and finally the field ϕ^l_n may be obtained everywhere from

$$\phi^l_{n+1} = \phi^l_n + \phi^l_{z,n}\Delta z_n \tag{12}$$

Equations (9), (10), and (12) may be used recursively to obtain ϕ, ϕ_z, and ϕ_t for all time and for all z.

§9.10

(a)

$$\mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\psi + \frac{\partial^2}{\partial x^2}\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \mu\sigma\frac{\partial}{\partial t}\psi = 0 \tag{1}$$

Let

$$\psi(z,x,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x e^{ik_x x}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\phi(z,k_x,\omega) \tag{2}$$

$$\equiv \mathcal{F}^{-1}_{k_x,\omega}\{\phi(z,k_x,\omega)\} \tag{3}$$

Assume $\mu = \mu(z)$, $c = c(z)$, $\sigma = \sigma(z)$. Then by linearity,

$$\mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\mathcal{F}^{-1}_{k_x,\omega}\{\phi\} = \mathcal{F}^{-1}_{k_x,\omega}\left\{\mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\phi\right\}, \tag{4}$$

$$\frac{\partial^2}{\partial x^2}\mathcal{F}^{-1}_{k_x,\omega}\{\phi\} = \mathcal{F}^{-1}_{k_x,\omega}\{-k_x^2\phi\}, \tag{5}$$

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathcal{F}^{-1}_{k_x,\omega}\{\phi\} = \mathcal{F}^{-1}_{k_x,\omega}\left\{\frac{\omega^2}{c^2}\phi\right\}, \tag{6}$$

$$-\mu\sigma\frac{\partial}{\partial t}\mathcal{F}^{-1}_{k_x,\omega}\{\phi\} = \mathcal{F}^{-1}_{k_x,\omega}\{i\omega\mu\sigma\phi\}. \tag{7}$$

Substituting these quantities in Equation (9.10.1) above,

$$\mathcal{F}^{-1}_{k_x,\omega}\left\{\left(\mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z} + \frac{\omega^2}{c^2} - k_x^2 + i\omega\mu\sigma\right)\phi\right\} = 0. \tag{8}$$

Since the above holds for all x, t, we may perform the forward transform corresponding to Equation (2) on both sides of Equation (8), or equivalently remove the $\mathcal{F}^{-1}_{k_x,\omega}\{\cdot\}$ operator from the left hand side of Equation (8) to obtain

$$\left(\mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z} + \frac{\omega^2}{c^2} - k_x^2 + i\omega\mu\sigma\right)\phi(z,k_x,\omega) = 0 \tag{9}$$

which is the same as Equation (9.2.11).

(b) If we let $k_x = \frac{\omega}{c}\cos\theta$ in Equation (9), we have

$$\left(\mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z} + \frac{\omega^2}{c^2}\sin^2\theta + i\omega\mu\sigma\right)\phi\left(z,\frac{\omega}{c}\cos\theta,\omega\right) = 0. \tag{10}$$

Now let

$$\phi\left(z,\frac{\omega}{c}\cos\theta,\omega\right) = \mathcal{F}_t\{\phi(z,\theta,t)\} \tag{11}$$

Substituting Equation (11) into Equation (10) and moving the $\mathcal{F}_t$ operator to the outside, we have

$$\mathcal{F}_t\left\{\left(\mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}-\frac{1}{c_e^2}\frac{\partial^2}{\partial t^2}-\mu\sigma\frac{\partial}{\partial t}\right)\phi(z,\theta,t)\right\}=0 \tag{12}$$

where we have defined $c_e^2=\frac{c^2}{\sin^2\theta}$. Since Equation (12) must hold for all ω, we may remove the $\mathcal{F}_t$ operator to obtain

$$\left(\mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}-\frac{1}{c_e^2}\frac{\partial^2}{\partial t^2}-\mu\sigma\frac{\partial}{\partial t}\right)\phi(z,\theta,t)=0 \tag{13}$$

which is equivalent to Equation (9.2.13) in the text. The equation

$$\psi(z,x,t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_x e^{ik_x x}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega t}\phi(z,k_x,\omega) \tag{14}$$

may be rewritten as

$$\phi(z,k_x,\omega)=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dt\,\psi(z,x,t)e^{-ik_x x+i\omega t} \tag{15}$$

or

$$\phi\left(z,\frac{\omega}{c}\cos\theta,\omega\right)=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dt\,\psi(z,x,t)e^{-i\frac{\omega}{c}x\cos\theta+i\omega t} \tag{16}$$

then

$$\phi(z,\theta,t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega t'}\phi\left(z,\frac{\omega}{c}\cos\theta,\omega\right) \tag{17}$$

$$=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dt\,\psi(z,x,t)\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega[t'-t+\frac{\omega x}{c}\cos\theta]} \tag{18}$$

$$\phi(z,\theta,t')=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dt\,\psi(z,x,t)\,\delta\left(t-t'-\frac{\omega x}{c}\cos\theta\right) \tag{19}$$

$$=\int_{-\infty}^{\infty}dx\,\psi\left(z,x,t'+\frac{\omega x}{c}\cos\theta\right). \tag{20}$$

§9.11

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\phi(\rho,t)-\frac{1}{c^2(\rho)}\frac{\partial^2}{\partial t^2}\phi(\rho,t)=0 \tag{1}$$

The above may be rewritten as

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\rho, t) = 0 \tag{2}$$

From (9.2.1) in the text, $A = 1$, $B = 0$, $C = -\frac{1}{c^2}$, $\Phi(\rho, \phi) = -\frac{1}{\rho}\frac{\partial \phi}{\partial \rho}$ and hence $v_\pm = \pm c$. Equations (9.2.9) in the text now become

$$d\phi_\rho - \frac{1}{c}d\phi_t + \frac{1}{\rho}\phi_\rho = 0 \quad \text{on} \quad C_+ \tag{3}$$

$$d\phi_\rho + \frac{1}{c}d\phi_t + \frac{1}{\rho}\phi_\rho = 0 \quad \text{on} \quad C_- \tag{4}$$

The layer-stripping algorithm follows from Equation (9.2.17) in the text which we repeat as

$$\phi^l_{\rho,m+1} - \phi^l_{\rho,m} - \frac{1}{c_m}(\phi^l_{t,m+1} - \phi^l_{t,m}) - \Phi^l_m \Delta\rho_m = 0 \text{ on } C_+ \tag{5}$$

$$\phi^l_{\rho,m+1} - \phi^{l+1}_{\rho,m} + \frac{1}{c_m}(\phi^l_{t,m+1} - \phi^{l+1}_{t,m}) - \Phi^{l+1}_m \Delta\rho_m = 0 \text{ on } C_- \tag{6}$$

where

$$\phi^l_{\rho,m} = \frac{\partial}{\partial \rho}\phi[m\Delta\rho_m, (m+l)\Delta t], c_m = c(\rho_m), \Phi^l_m = -\frac{1}{\rho_m}\frac{\partial \phi^l_m}{\partial \rho}. \tag{7}$$

Then from Equation (9.2.19) in the text,

$$-\phi^1_{\rho,m} - \frac{1}{c_m}\phi^1_{t,m} - \Phi^1_m c_m \Delta t = 0 \ \forall \text{ m}. \tag{8}$$

or

$$-\phi^1_{\rho,m} - \frac{1}{c_m}\phi^1_{t,m} + \frac{1}{\rho_m}\phi^1_{\rho,m} c_m \Delta t = 0 \ \forall \text{ m}. \tag{9}$$

$$c^2_m\left(\frac{1}{\rho_m}\Delta t \phi^1_{\rho,m}\right) - c_m(\phi^1_{\rho,m}) - \phi^1_{t,m} = 0, \tag{10}$$

$$c_m = \frac{\phi^1_{\rho,m} \pm \sqrt{(\phi^1_{\rho,m})^2 + 4\frac{\Delta t}{\rho_m}\phi^1_{t,m}\phi^1_{\rho,m}}}{2\frac{\Delta t}{\rho_m}\phi^1_{\rho,m}}. \tag{11}$$

§9.12

Consider the 1-D inhomogeneous profile shown in Figure 1 consisting of several dispersionless fine layers with equal travel time. That is,

$$d_1\sqrt{\mu_1\epsilon_1} = d_2\sqrt{\mu_2\epsilon_2} = \ldots = d_n\sqrt{\mu_n\epsilon_n} = \tau. \tag{1}$$

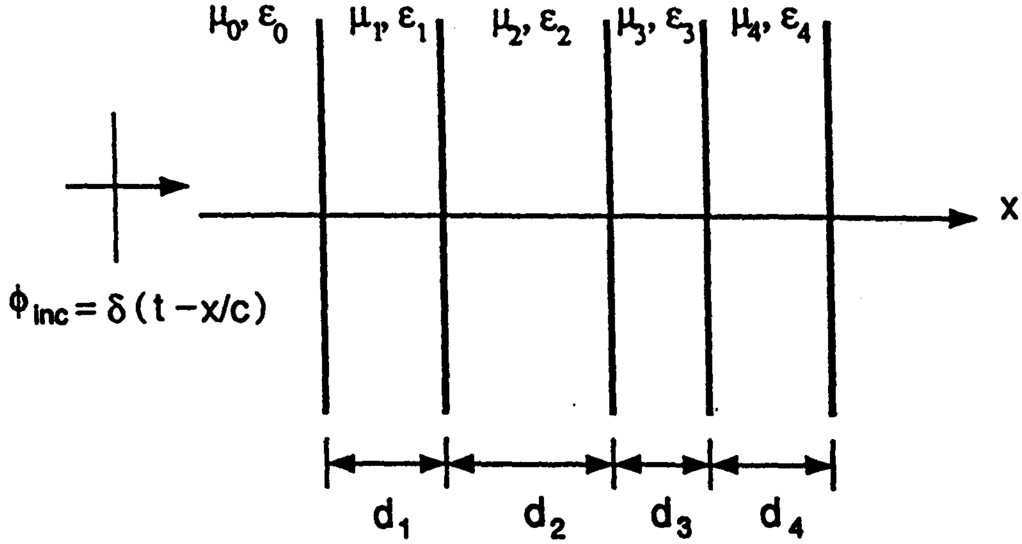

Figure 1 for Exercise Solution 9.12

Let the incident field consist of the impulse plane wave

$$\phi_{inc}(x,t) = \delta\left(t - \frac{x}{c_0}\right). \tag{2}$$

It is readily verified that the above is a solution of

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial z^2}\right)\phi(x,t) = 0 \tag{3}$$

Now since we assume that the layers are dispersionless, the reflection and transmission coefficients for each layer are independent of frequency. To see this, we write

$$R_{i,i+1} = \frac{\epsilon_{i+1}k_i - \epsilon_i k_{i+1}}{\epsilon_{i+1}k_i + \epsilon_i k_{i+1}} = \frac{\epsilon_{i+1}\sqrt{\mu_i\epsilon_i} - \epsilon_i\sqrt{\mu_{i+1}\epsilon_{i+1}}}{\epsilon_{i+1}\sqrt{\mu_i\epsilon_i} + \epsilon_i\sqrt{\mu_{i+1}\epsilon_{i+1}}} = \frac{\eta_i - \eta_{i+1}}{\eta_i + \eta_{i+1}} \tag{4}$$

and

$$T_{i,i+1} = \frac{2\epsilon_{i+1}k_i}{\epsilon_{i+1}k_i + \epsilon_i k_{i+1}} = \frac{2\eta_i}{\eta_i + \eta_{i+1}} \tag{5}$$

where

$$\eta_i = \sqrt{\frac{\mu_i}{\epsilon_i}}.$$

Assuming that the incident pulse strikes the first layer ($x = 0$) at time $t = 0$, the scattered field may be written using a geometric series as

$$\phi_{sca}(x,t) = A_1\delta\left(t + \frac{x}{c}\right) + A_2\delta\left(t + \frac{x}{c} - 2\tau\right) + A_3\delta\left(t + \frac{x}{c} - 4\tau\right) + A_4\delta\left(t + \frac{x}{c} - 6\tau\right) + \cdots \tag{6}$$

where

$$
\begin{aligned}
A_1 &= R_{01} \\
A_2 &= T_{01}R_{12}T_{10} \\
A_3 &= T_{01}R_{12}R_{10}R_{12}T_{10} + T_{01}T_{12}R_{23}T_{21}T_{10} \\
A_4 &= T_{01}R_{12}R_{10}R_{12}R_{10}R_{12}T_{10} + T_{01}R_{12}R_{10}T_{12}R_{23}T_{21}T_{10} \\
&\quad + T_{01}T_{12}R_{23}R_{21}R_{23}T_{21}T_{10} + T_{01}T_{12}T_{23}R_{34}T_{32}T_{21}T_{10}
\end{aligned}
$$

(b) Using time gating we can recover the coefficients $A_1, A_2, \ldots, A_n$ in Equation (6). From A_1 we know R_{01} and η_1 may be determined from Equation (4). Then T_{01} and T_{10} may be determined from Equation (5). Then from A_2 we may determine R_{12} and using Equations (4) and (5) we may solve for η_2, T_{12}, T_{21}. Then in A_3 the only unknown is R_{23}which determines η_3, T_{23}, T_{32}. Similarly in A_4 the only unknown is R_{34}. This process is repeated until the impedance of all of the layers are unraveled.

§9.13

From Equations (9.2.30) and (9.2.31) in the text, we have

$$\frac{\partial^2\psi(\zeta)}{\partial\zeta^2} + [\omega^2 - V(\zeta)]\psi(\zeta) = 0 \tag{1}$$

where

$$V(\zeta) = \eta^{-\frac{1}{2}}\frac{\partial^2}{\partial\zeta^2}\eta^{\frac{1}{2}}. \tag{2}$$

Now, however, the potential is frequency dependent since $\epsilon = \epsilon' + \frac{i\sigma}{\omega}$ is complex and we have

$$V(\zeta,\omega) = \left(\frac{\mu}{\epsilon' + \frac{i\sigma}{\omega}}\right)^{-\frac{1}{4}} \frac{\partial^2}{\partial\zeta^2}\left(\frac{\mu}{\epsilon' + \frac{i\sigma}{\omega}}\right)^{\frac{1}{4}} \tag{3}$$

where

$$\zeta = \int_{z_0}^{z} \sqrt{\mu(z')\left[\epsilon'(z') + \frac{i\sigma(z')}{\omega}\right]}\, dz'. \tag{4}$$

Note that although $\zeta = \zeta' + i\zeta''$ is now complex, if we assume ϵ', σ, μ to be analytic functions of space, then $\frac{\partial^2}{\partial\zeta^2} = \frac{\partial^2}{\partial(\zeta')^2}$. Alternatively, we may use the fact that $\frac{\partial^2}{\partial\zeta^2} = v^2\frac{\partial^2}{\partial z^2}$ and write

$$
\begin{aligned}
V(\zeta,\omega) =& [\mu(z_r)]^{-\frac{5}{4}}\left[\epsilon'(z_r) + i\sigma\frac{(z_r)}{\omega}\right]^{-\frac{3}{4}} \\
&\times \frac{d^2}{dz_r^2}\left[\frac{\mu(z_r)}{\epsilon'(z_r) + i\frac{\sigma(z_r)}{\omega}}\right]^{\frac{1}{4}}\Bigg|_{z_r=\Re e\{z(\zeta)\}}
\end{aligned} \tag{5}
$$

with

$$z(\zeta) = \int_{\zeta_0}^{\zeta} \left\{ \mu(\zeta') \left[\epsilon'(\zeta') + i\frac{\sigma(\zeta')}{\omega} \right] \right\}^{-\frac{1}{2}} \partial\zeta' \tag{6}$$

Equation (1) becomes

$$\frac{\partial^2 \psi(\zeta, \omega)}{\partial \zeta^2} + \omega^2 \psi(\zeta, \omega) - V(\zeta, \omega)\psi(\zeta, \omega) = 0. \tag{7}$$

Applying an inverse Fourier transform to Equation (7), we have

$$\frac{\partial^2 \psi(\zeta, t)}{\partial \zeta^2} - \frac{\partial^2 \psi(\zeta, t)}{\partial t^2} - V(\zeta, t) * \psi(\zeta, t) = 0 \tag{8}$$

where $V(\zeta, t)$ is the inverse transform of $V(\zeta, \omega)$ and $*$ indicates convolution in time.

§9.14

(a) The Schrodinger equation given by Equation (9.2.34) along with the boundary condition of Equation (9.2.35) may be related to the transmission line shown in Figure 1. Defining $v(0,t) = \phi(0,t)$, $i(0,t) = -\frac{\partial}{\partial\zeta}\phi(0,t)$, $Z_g = \frac{1}{h}$ and $i_g(t) = \delta'(t)$, we have

$$i(0,t) = i_g(t) - \frac{v(0,t)}{Z_g} \tag{1}$$

or

$$-\frac{\partial}{\partial\zeta}\phi(0,t) = \delta'(t) - h\phi(0,t), \tag{2}$$

which is the same as Equation (9.2.35) in the text. This is equivalent to having an impulsive source backed by an impedance boundary on the left. It is a sheet source because it is a 1-D problem and hence there is no variation in any other space dimension.

(b) The 1-D Schrodinger equation is given in Equation (9.2.34) of the text and repeated here as

$$\left[\frac{\partial^2}{\partial \zeta^2} - \frac{\partial^2}{\partial t^2} - V(\zeta) \right] \phi(\zeta, t) = 0. \tag{3}$$

We assume that the scattering potential $V(\zeta)$ has compact support and is identically zero for $|\zeta| > L$. In the frequency domain with $|\zeta| > L$. Equation (3) becomes

$$\left[\frac{\partial^2}{\partial \zeta^2} + \omega^2 \right] \phi(\zeta, \omega) = 0. \tag{4}$$

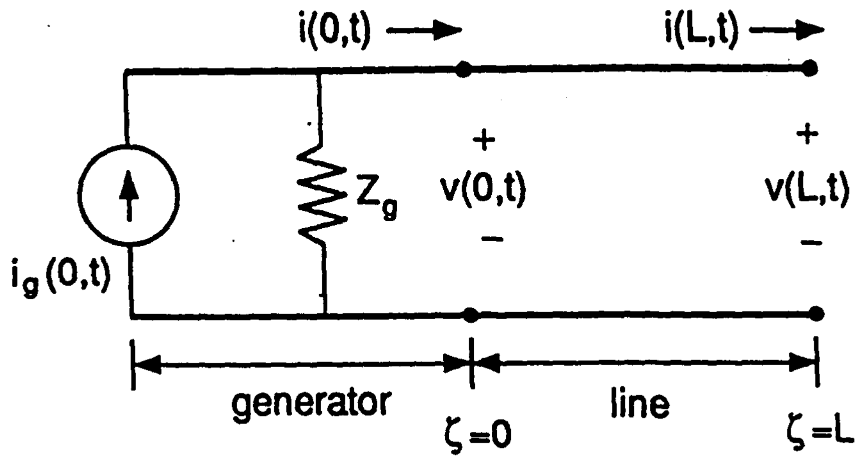

Figure 1 for Exercise Solution 9.14

A general solution to Equation (4) with $|\zeta| > L$ consists of the superposition of a plane wave travelling to the right ϕ_{out} corresponding to an outgoing wave and a plane wave travelling to the left ϕ_{in} corresponding to an incoming wave. We write

$$\phi(\zeta,\omega) = \phi_{out}(\zeta,\omega) + \phi_{in}(\zeta,\omega), \quad \zeta > L \tag{5}$$

$$= A(x)e^{i\omega\zeta} + B(\omega)e^{-i\omega\zeta}. \tag{6}$$

The radiation condition that $\phi(\zeta,\omega)$ is outgoing when $\zeta \to \infty$ corresponds to $B(\omega) = 0$. Then

$$\phi(\zeta,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, A(\omega)e^{-i\omega(t-\zeta)} = a(t-\zeta), \quad \zeta > L \tag{7}$$

where $a(t)$ is the inverse Fourier transform of $A(\omega)$, which is clearly a causal solution.

Finally, we note that the above principle is not restricted to 1-D and follows for the 2-D and 3-D cases easily. It also applies to the variable velocity acoustic and electromagnetic wave equations.

§9.15

Given the Cauchy data on $\zeta = 0$,

$$G(0,t) = 2\delta(t), \tag{1}$$

$$-\frac{\partial}{\partial\zeta}G(0,t) + hG(0,t) = 0, \tag{2}$$

we are asked to use the method of characteristics to show that the support of $G(\zeta,t)$ is as shown in Figure (9.2.5) of the text. From the Schrodinger equation (9.2.49) and Equation (9.2.1) we deduce that $A = 1$, $B = 0$, $C = -1$, and $\Phi = V(\zeta)G(\zeta,t)$. Then from Equation (9.2.6), $v_\pm = \pm 1$. Substituting these quantities into Equations (9.2.9) gives

$$dG(\zeta,t)_\zeta - dG(\zeta,t)_t = V(\zeta)G(\zeta,t)d\zeta \text{ on } C_+, \tag{3}$$

$$dG(\zeta,t)_\zeta + dG(\zeta,t)_t = V(\zeta)G(\zeta,t)d\zeta \text{ on } C_-. \tag{4}$$

Since $V_\pm = \pm 1$ and the Cauchy data consists of a single point on the $\zeta = 0$ axis, the function will fill out a triangular region in the (ζ,t) plane. To see this, we discretize the ζ and t axis and let $\zeta = n\Delta$, $t = l\Delta$. The impulse at $n = 0$, $l = 0$, first propagates to $l = \Delta$, $n = \pm l\Delta$. The point at $n = \Delta$, $l = \Delta$, can then propagate to both $n = 2\Delta$, $l = 0$, and $n = 2\Delta$, $l = 2\Delta$. This process repeats until the shaded triangular region in the Figure is filled out.

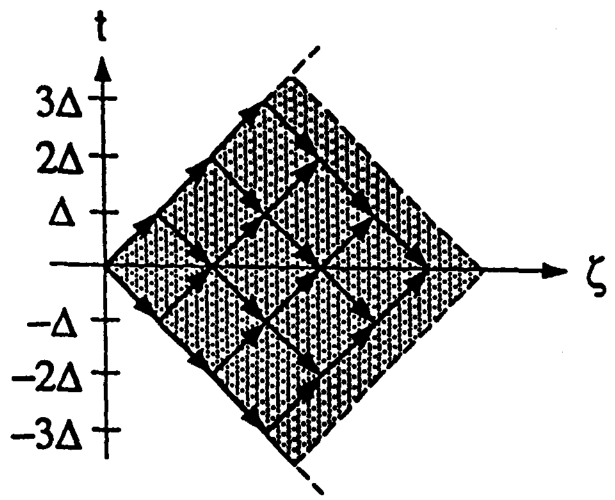

Figure for Exercise Solution 9.15

To see that the general solution to Equation (9.2.49) is of the form

$$G(\zeta,t) = \delta(\zeta - t) + \delta(\zeta + t) + K(\zeta,t),$$

we recall that the characteristics of Equation (9.2.49) are along $V = \pm 1$. By symmetry, the impulse $2\delta(t)$, at $\zeta = 0$ splits and travels along the characteristics. The symmetric kernel $K(\zeta, t)$ which is nonzero in the triangular region shown in Figure 1 accounts for the wake of the impulse as they propagate through the scatterer.

§9.16

The general form for $G(\zeta, t)$ is assumed to be

$$G(\zeta,t) = \delta(\zeta - t) + \delta(\zeta + t) + K(\zeta,t). \qquad (1)$$

Substituting this expression into Equation (9.2.49) leads to

$$\frac{\partial^2}{\partial t^2}K(\zeta,t) - \frac{\partial^2}{\partial \zeta^2}K(\zeta,t) + V(\zeta)[K(\zeta,t) + \delta(\zeta - t) + \delta(\zeta + t)] = 0. \qquad (2)$$

Substitution of Equation (1) into the boundary conditions of Equations (9.2.50) gives

$$-\frac{\partial}{\partial \zeta}K(\zeta,t)\Big|_{\zeta=0} + hK(0,t) + 2h\delta(t) = 0, \qquad (3)$$

$$K(0,t) = 0. \qquad (4)$$

Matching the most singular terms in Equation (2) gives

$$\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\partial^2}{\partial t^2}\right) K(\zeta,t) = V(\zeta)[\delta(\zeta + t) + \delta(\zeta - t)],\ \zeta \approx \pm t. \qquad (5)$$

This implies that $K(\zeta, t)$ must be discontinuous in the vicinity of $\zeta = \pm t$ and given the triangular support of $K(\zeta, t)$, it is fair to assume that

$$K(\zeta,t) = f(\zeta,t)[u(t+\zeta) - u(t-\zeta)]. \qquad (6)$$

Then

$$\begin{aligned}\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\partial^2}{\partial t^2}\right) K(\zeta,t) = &- \left[\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\partial^2}{\partial t^2}\right) f(\zeta,t)\right] u(t-\zeta) \\ &+ 2\left[\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial t}\right) f(\zeta,t)\right] \delta(t-\zeta) \\ &+ \left[\left(\frac{\partial^2}{\partial \zeta^2} - \frac{\partial^2}{\partial t^2}\right) f(\zeta,t)\right] u(t+\zeta) \\ &+ 2\left[\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\right) f(\zeta,t)\right] \delta(t-\zeta). \qquad (7)\end{aligned}$$

Substituting Equation (7) into Equation (5) and matching the most singular terms gives

$$2\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial t}\right) f(\zeta,t)\Big|_{\zeta=t} = 2\frac{d}{d\zeta}f(\zeta,\zeta) = 2\frac{d}{d\zeta}K(\zeta,\zeta) = V(\zeta), \qquad (8)$$

$$2\left(\frac{\partial}{\partial\zeta}-\frac{\partial}{\partial t}\right)f(\zeta,t)\bigg|_{\zeta=-t}=2\frac{d}{d\zeta}f(\zeta,-\zeta)=2\frac{d}{d\zeta}K(\zeta,-\zeta)=V(\zeta). \quad (9)$$

Combining Equations (8) and (9), we have

$$V(\zeta)=2\frac{d}{d\zeta}K(\zeta,\pm\zeta). \quad (10)$$

Then substituting Equation (6) into Equations (3) and (4) and matching the most singular terms gives

$$h=K(0,0). \quad (11)$$

§9.17

(a) The Gel'fand-Levitan equation is given as

$$0=f(\zeta,T)+\int_0^\zeta d\tau K(\zeta,\tau)f(\tau,T)+K(\zeta,T) \quad (1)$$

for $0<T<\zeta$. $f(\tau,T)$ is obtained from measurement data $\phi_1(\zeta,t)$ where

$$\phi_1(\zeta,t)=\delta(t-\zeta)-K_1(\zeta,t). \quad (2)$$

Using the relation

$$f(t,T)=-\frac{1}{2}\{K_1(0,|t-T|)+K_1(0,|t+T|)\}, \quad (3)$$

Equation (1) may be solved numerically for $K(\zeta,t)$ by discretizing the problem and representing $f(\zeta,t)$ and $K(\zeta,t)$ in terms of sub-domain basis functions. We let

$$f(\zeta,t)=\sum_{m=1}^{M}\sum_{n=1}^{N}a_{mn}f_{mn}(\zeta,t), \quad (4)$$

$$K(\zeta,t)=\sum_{p=1}^{P}\sum_{q=1}^{Q}b_{pq}k_{pq}(\zeta,t), \quad (5)$$

where we assume that the scatterer has compact support so that $f(\zeta,t)$ and $K(\zeta,t)$ may be represented in terms of a finite number of basis functions. The subdomain basis functions $f_{mn}(\zeta,t)$ and $k_{pq}(\zeta,t)$ may for example be 2-D pulse or chapeau functions and are chosen to model $f(\zeta,t)$ and $K(\zeta,t)$ to the desired level of accuracy.

Substituting Equations (4) and (5) in Equation (1) gives

$$0 = \sum_{m=1}^{m}\sum_{n=1}^{N} a_{mn} f_{mn}(\zeta, T) + \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{p=1}^{P}\sum_{q=1}^{Q} a_{mn} b_{pq} \int_0^{\zeta} d\tau k_{pq}(\zeta, t) f_{mn}(\tau, T)$$
$$+ \sum_{p=1}^{P}\sum_{q=1}^{Q} b_{pq} k_{pq}(\zeta, T) \tag{6}$$

Now let

$$g_{mnpq}(\zeta, T) = \int_0^{\zeta} d\tau k_{pq}(\zeta, \tau) f_{mn}(\tau, T) \tag{7}$$

and define the inner product

$$\langle u(\zeta, t), v(\zeta, t)\rangle = \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dt\ u(\zeta, t) v(\zeta, t\rangle. \tag{8}$$

Then taking the inner product of Equation (6) with the weighting vectors $w_{rs}(\zeta, t)$, we have

$$0 = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}\langle w_{rs}, f_{mn}\rangle + \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{p=1}^{P}\sum_{q=1}^{Q} a_{mn} b_{pq}\langle w_{rs}, g_{mnpq}\rangle$$
$$+ \sum_{p=1}^{P}\sum_{q=1}^{Q} b_{pq}\langle w_{rs}, k_{pq}\rangle. \tag{9}$$

The above may be written more conveniently by arranging the elements $\{a_{mn}\}$ in a single vector a of dimension MN and $\{b_{pq}\}$ in a vector **b** of dimension PQ. Similarly, $\langle w_{rs}, f_{mn}\rangle$ may be written as the $RS \times MN$ matrix $\overline{\mathbf{F}}$ and $\langle w_{rs}, k_{pq}\rangle$ may be expressed as the $RS \times PQ$ matrix $\overline{\mathbf{K}}$, where RS is the dimension of the space spanned by the set of weighting vectors $w_{rs}(\zeta, t)$. The inner product $\langle w_{rs}, g_{mnpq}\rangle$ may be written as a third rank tensor $\overline{\overline{\mathbf{G}}}$ with the property that

$$\overline{\overline{\mathbf{G}}} \cdot \mathbf{b} \cdot \mathbf{a} = G_{rsmnpq} b_{pq} a_{mn} \tag{10}$$

and summation is implied over repeated indices. Then we have

$$\overline{\mathbf{F}} \cdot \mathbf{a} + (\overline{\overline{\mathbf{G}}} \cdot \mathbf{b}) \cdot \mathbf{a} + \overline{\mathbf{K}} \cdot \mathbf{b} = 0. \tag{11}$$

In Equation (11) above, the matrices, $\overline{\mathbf{F}}$ and $\overline{\mathbf{K}}$ and the tensor $\overline{\overline{\mathbf{G}}}$ may be calculated for a given basis set. The vector **a** may be determined from the scattering data. The only unknown is the vector **b**. To simplify Equation (11), we define the rotation of $\overline{\overline{\mathbf{G}}}$ as

$$\overline{\overline{\mathbf{G}}}^r = (G_{rsmnpq})^r = G_{pqrsmn}. \tag{12}$$

Then Equation (11) becomes

$$\overline{F}\cdot\mathbf{a} + (\overline{\overline{\mathbf{G}}}^r\cdot\mathbf{a})^t\cdot\mathbf{b} + \overline{\mathbf{K}}\cdot\mathbf{b} = 0 \tag{13}$$

or

$$[(\overline{\overline{\mathbf{G}}}^r\cdot\mathbf{a})^t + \overline{\mathbf{K}}]\cdot\mathbf{b} = -\overline{\mathbf{F}}\cdot\mathbf{a} \tag{14}$$

where $(\cdot)^t$ is the matrix transpose operation. Then

$$\mathbf{b} = -[(\overline{\overline{\mathbf{G}}}^r\cdot\mathbf{a})^t + \overline{\mathbf{K}}]^{-1}\cdot\overline{\mathbf{F}}\cdot\mathbf{a}. \tag{15}$$

Note that the matrices involved are highly sparse and (15) may be inverted rapidly.

(b) The Marchenko integral equation is given as

$$-K(\zeta,t) = K_1(0,t+\zeta) + \int_{-t}^{\zeta} K(\zeta,\tau)K_1(0,t+\tau)d\tau,\ \zeta > t \tag{16}$$

where $K_1(\zeta,t)$ is the scattering data. Equation (16) may be solved by expanding $K(\zeta,t)$ in 2-D basis functions as before, but now we need only expand $K_1,(0,t)$ in a 1-D basis set. We write

$$K_1(0,t) = \sum_{m=1}^{M} a_m f_m(t) \tag{17}$$

$$K(\zeta,t) = \sum_{p=1}^{P}\sum_{q=1}^{Q} b_{pq}k_{pq}(\zeta,t). \tag{18}$$

Equation (16) becomes

$$-\sum_{p=1}^{P}\sum_{q=1}^{Q} b_{pq}k_{pq}(\zeta,t) = \sum_{m=1}^{M} a_m f_m(t+\zeta) + \sum_{m=1}^{M}\sum_{p=1}^{P}\sum_{q=1}^{Q} a_m b_{pq}\int_{-t}^{\zeta} d\tau f_m(t+\tau)k_{pq}(\zeta,t). \tag{19}$$

Now letting

$$g_{mpq}(\zeta,t) = \int_{-t}^{\zeta} d\tau f_m(t+\tau)k_{pq}(\zeta,t), \tag{20}$$

and taking the inner product with weighting vector $w_{rs}(\zeta,t)$, we have

$$\sum_{p=1}^{P}\sum_{q=1}^{Q} b_{pq}\langle w_{rs}(\zeta,t),k_{pq}(\zeta,t)\rangle + \sum_{m=1}^{M} a_m\langle w_{rs}(\zeta,t), f_m(t+\zeta)\rangle$$
$$+\sum_{m=1}^{M}\sum_{p=1}^{P}\sum_{q=1}^{Q} a_m b_{pq}\langle w_{rs}(\zeta,t), g_{mpq}(\zeta,t)\rangle = 0. \quad (21)$$

Then $\{b_{pq}\}$ may be written as the vector $\mathbf{b}$ of dimension PQ and $\{a_m\}$ as the vector $\mathbf{a}$ of dimension M. The inner products $\langle w_{rs}(\zeta,t), k_{pq}(\zeta,t)\rangle$ and $\langle w_{rs}(\zeta,t), f_m(t+\zeta)\rangle$ may be written as matrices $\overline{\mathbf{K}}$ and $\overline{\mathbf{F}}$ of dimension $RS\times PQ$ and $RS\times M$ while the inner product $\langle w_{rs}(\zeta,t), g_{mpq}(\zeta,t)\rangle$ may be written as the $RS\times M\times PQ$ tensor $\overline{\overline{\overline{\mathbf{G}}}} = G_{rsmpq}$. Then we have

$$\overline{\mathbf{K}}\cdot\mathbf{b}+\overline{\mathbf{F}}\cdot\mathbf{a}+\overline{\overline{\overline{\mathbf{G}}}}\cdot\mathbf{b}\cdot\mathbf{a}=0 \quad (22)$$

or

$$\overline{\mathbf{K}}\cdot\mathbf{b}+\overline{\mathbf{F}}\cdot\mathbf{a}+(\overline{\overline{\overline{\mathbf{G}}}}^{r}\cdot\mathbf{a})^{t}\cdot\mathbf{b}=0, \quad (23)$$

which becomes

$$\mathbf{b}=-[(\overline{\overline{\overline{\mathbf{G}}}}^{r}\cdot\mathbf{a})^{t}+\overline{\mathbf{K}}]^{-1}\cdot\overline{\mathbf{F}}\cdot\mathbf{a} \quad (24)$$

as before.

§9.18

(a) From Equations (9.2.74) with $r(t)=0$, we have

$$\phi(0,t)=\delta(t)+R(t) \quad (1)$$

$$\frac{\partial}{\partial\zeta}\phi(0,t)=-\delta'(t). \quad (2)$$

Using a transmission line analogy as in Exercise 9.14, we define $v(\zeta,t)=\phi(\zeta,t)$ and $i(\zeta,t)=-\frac{\partial}{\partial\zeta}\phi(\zeta,t)$. Then Equations (1) and (2) correspond to the model shown in the Figure consisting of an impulsive voltage source in series with a unit capacitance and diode D. Equating $\phi(\zeta,t)$ with electric field and $\frac{\partial\phi(\zeta,t)}{\partial\zeta}$ with magnetic field, this corresponds to a sheet source backed by a magnetic wall.

(b) Letting $r(t)=R(t)$ in Equations (9.2.74), we have

$$\phi(0,t)=\delta(t)+R(t) \quad (3)$$

$$-\frac{\partial}{\partial\zeta}\phi(0,t)=\delta'(t)-R'(t). \quad (4)$$

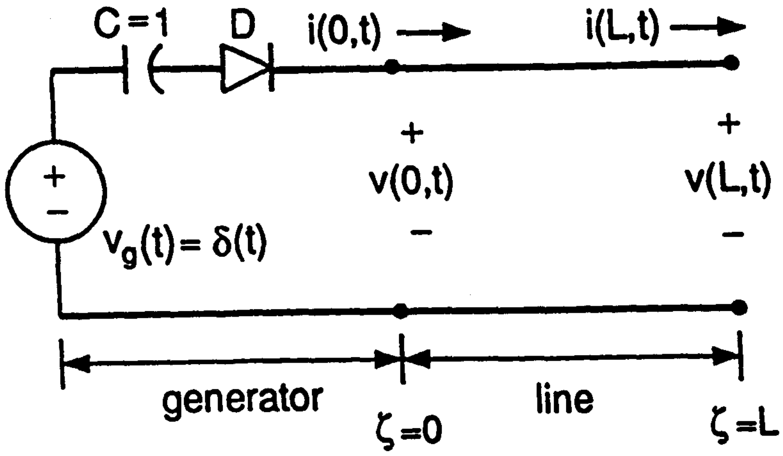

Figure 1 for Exercise Solution 9.18

The transmission line analogy of Equations (3) and (4) is shown in Figure 2 and in the same as that of Figure 1 without the diode. The voltage $v(0,t)$ consists of the impulse due to the generator and a time-varying reflection $R(t)$. The capacitor accounts for the fact that the current is the derivative of the voltage. This configuration is equivalent to probing a 1-D profile with an impulsive source where the scattering potential $V(\zeta)$ is zero for $\zeta < 0$.

§9.19

(a) For an arbitrary scalar field $\psi(\mathbf{r})$ with compact support in a homogeneous back ground medium k_b, the source that supports the scalar field may be derived directly from the scalar Helmholtz equation

$$(\nabla^2 + k_b^2)\psi(\mathbf{r}) = S(\mathbf{r}). \tag{1}$$

Since the differential operator $\nabla^2 + k_b^2$ is a local operator, the support of $S(\mathbf{r})$ is restricted to volume V, the support of $\psi(\mathbf{r})$.

As an example, consider the field

$$\psi(\mathbf{r}) = \cos\left(\frac{\pi r}{2}\right)[u(r+1) - u(r-1)] \tag{2}$$

where $u(r)$ is the unit step function. Then the volume V is a unit ball. Writing the Laplacian in spherical coordinates and noting that there is no θ or ϕ dependence, we have $\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r}$. Then

$$\begin{aligned} S(\mathbf{r}) &= (\nabla^2 + k_b^2)\psi(\mathbf{r}) \\ &= \left[\left(k_b^2 - \frac{\pi^2}{4}\right)\cos\left(\frac{\pi r}{2}\right) - \frac{\pi}{r}\sin\left(\frac{\pi r}{2}\right)\right][u(r+1) - u(r-1)] \\ &+ \left[\frac{2}{r}\cos\left(\frac{\pi r}{2}\right) - \pi\sin\left(\frac{\pi r}{2}\right)\right][\delta(r+1) - \delta(r-1)] \\ &+ \cos\left(\frac{\pi r}{2}\right)[\delta'(r+1) - \delta'(r-1)]. \end{aligned} \tag{3}$$

Note that $S(\mathbf{r})$ remains finite as $r \to 0$ since

$$\lim_{r\to 0}\frac{\sin\frac{\pi r}{2}}{r} = \frac{\pi}{2} \tag{4}$$

and that $S(\mathbf{r})$ has compact support on a unit ball.

(b) A general solution to Equation (1) that is valid for all $\mathbf{r}$ may be written as the superposition integral

$$\psi(\mathbf{r}) = \int d\mathbf{r}' g(\mathbf{r},\mathbf{r}')S(\mathbf{r}') \tag{5}$$

where

$$(\nabla^2 + k_b^2)g(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

is the point source response. But for $\mathbf{r} \notin V$, we have $\psi(\mathbf{r}) = 0$ and so

$$\int dr' g(\mathbf{r},\mathbf{r}')S(r') = 0 \ , \ \mathbf{r} \notin V. \tag{6}$$

This means that some nonzero $S(\mathbf{r}')$ gives rise to no field $\psi(\mathbf{r})$ for $\mathbf{r} \notin V$ and hence the above integral operator has a null space.

(c) Now consider an inhomogeneous scatterer $k(\mathbf{r})$ embedded in a homogeneous background k_b. The scalar Helmholtz equation becomes

$$[\nabla^2 + k^2(\mathbf{r})]\phi(\mathbf{r}) = S(\mathbf{r}), \tag{7}$$

which may be rewritten as

$$(\nabla^2 + k_b^2)\phi(\mathbf{r}) = S(\mathbf{r}) + [k_b^2 - k^2(\mathbf{r})]\phi(\mathbf{r}). \tag{8}$$

Using the superposition integral again and considering the term $[k_b^2 - k^2(\mathbf{r})]\phi(\mathbf{r})$ as an induced source, the total solution may be written as

$$\phi(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \phi_{sca}(\mathbf{r}), \tag{9}$$

where

$$\phi_{inc}(\mathbf{r}) = \int d\mathbf{r}' g(\mathbf{r},\mathbf{r}')S(\mathbf{r}'), \tag{10}$$

$$\phi_{sca}(\mathbf{r}) = \int d\mathbf{r}' g(\mathbf{r},\mathbf{r}')[k_b^2 - k^2(\mathbf{r}')]\phi(\mathbf{r}'). \tag{11}$$

Using the first Born approximation, we may write

$$\phi_{sca}(\mathbf{r}) \approx \int d\mathbf{r}' g(\mathbf{r},\mathbf{r}')[k_b^2 - k^2(\mathbf{r}')]\phi_{inc}(\mathbf{r}'). \tag{12}$$

(d) Assuming that $\phi_{inc}(\mathbf{r}') \neq 0$, the operator $\int d\mathbf{r}' g(\mathbf{r},\mathbf{r}')\phi_{inc}(\mathbf{r}')$ which operates on $k_b^2 - k^2(\mathbf{r})$ has a null space. To see this, we note that $\phi_{sca}(\mathbf{r})$ in Equation (12) is the field generated by the induced source $\phi_{inc}(\mathbf{r}')[k_b^2 - k^2(\mathbf{r}')]$. In part (b) we showed that the operator $\int d\mathbf{r}' g(\mathbf{r},\mathbf{r}')$ has a null space. For nonzero $\phi_{inc}(\mathbf{r}')$, it follows that $\int d\mathbf{r}' g(\mathbf{r},\mathbf{r}')\phi_{inc}(\mathbf{r}')$ must also have a null space. An example of a state vector that is in the null space of this operator is that of Equation (3) divided by $\phi_{inc}(\mathbf{r})$.

(e) From Equation (12), we see that the internal scattered field $\phi_{sca}(\mathbf{r})$ with $\mathbf{r} \in V$ generated by a non-radiating source for a given scatterer $k_b^2 - k^2(\mathbf{r})$ depends on the incident field $\phi_{inc}(\mathbf{r})$. Hence, different transmitter locations correspond to different nonradiating internal fields. By collecting data at many transmitter locations, the nonradiating internal fields can be squeezed to zero and the null space is reduced. This is because the non-radiating internal fields change with transmitter location, but the scattering object does not.

§9.20

(a) From Equation (9.3.11), we have

$$\begin{aligned}
I &= \delta_t \sum_{n,m} a_n a_m^* B_{nm} + \sum_k \left| \mathbf{E}_k - \sum_n a_n \overline{\mathbf{L}}_{kn} \right|^2 \\
&= \delta_t \sum_{n,m} a_n a_m^* B_{nm} + \sum_k \left(\mathbf{E}_k - \sum_n \overline{\mathbf{L}}_{kn} \right)^* \left(\mathbf{E}_k - \sum_n a_n \overline{\mathbf{L}}_{kn} \right) \\
&= \delta_t \sum_{n,m} a_n a_m^* B_{nm} \\
&\quad + \sum_k \left[\mathbf{E}_k^* \mathbf{E}_k - \sum_n a_n^* \overline{\mathbf{L}}_{kn}^* \mathbf{E}_k - \sum_n a_n \overline{\mathbf{L}}_{kn} \mathbf{E}_k^* - \sum_{n,m} a_n^* a_m \overline{\mathbf{L}}_{kn}^* \overline{\mathbf{L}}_{kn} \right] \\
&= \delta_t \mathbf{a}^\dagger \cdot \overline{\mathbf{B}} \cdot \mathbf{a} + \mathbf{a}^\dagger \cdot \overline{\mathbf{P}} \cdot \mathbf{a} - \mathbf{C} \cdot \mathbf{a} - (\mathbf{C} \cdot \mathbf{a})^* + \sum_k \mathbf{E}_k^* E_k
\end{aligned}$$

Taking the first variation with respect to a gives

$$\begin{aligned}
\delta I &= \delta_t \mathbf{a}^\dagger \cdot \overline{\mathbf{B}} \cdot (\delta \mathbf{a}) + \delta_t (\delta \mathbf{a})^\dagger \cdot \overline{\mathbf{B}} \cdot \mathbf{a} \\
&\quad + (\delta \mathbf{a})^\dagger \cdot \overline{\mathbf{P}} \cdot \mathbf{a} + \mathbf{a}^\dagger \cdot \overline{\mathbf{P}} \cdot \delta \mathbf{a} - \mathbf{C} \cdot \delta \mathbf{a} - [\mathbf{C} \cdot \delta \mathbf{a}]^* \\
&= (\delta \mathbf{a}^\dagger) \cdot (\delta_t \overline{\mathbf{B}} \cdot \mathbf{a} + \overline{\mathbf{P}} \cdot \mathbf{a} - \mathbf{C}) + [(\delta \mathbf{a})^\dagger \cdot (\delta_t \overline{\mathbf{B}} \cdot \mathbf{a} + \overline{\mathbf{P}} \cdot \mathbf{a} - \mathbf{C})]^* \\
&= 2Re[(\delta \mathbf{a})^\dagger \cdot (\delta_t \overline{\mathbf{B}} \cdot \mathbf{a} + \overline{\mathbf{P}} \cdot \mathbf{a} - \mathbf{C})]
\end{aligned}$$

since $\mathbf{a}^\dagger \cdot \overline{\mathbf{B}} \cdot \delta\mathbf{a} = [\delta\mathbf{a}^\dagger \cdot \overline{\mathbf{B}} \cdot \mathbf{a}]^*$ and $\mathbf{a}^\dagger \cdot \overline{\mathbf{P}} \cdot \delta\mathbf{a} = [\delta\mathbf{a}^\dagger \cdot \overline{\mathbf{P}} \cdot \mathbf{a}]^*$. Now since $\delta\mathbf{a}$ is arbitrary (and may be complex), we must have

$$\delta_t \overline{\mathbf{B}} \cdot \mathbf{a} + \overline{\mathbf{P}} \cdot \mathbf{a} - \mathbf{C} = 0$$

in order for I to be stationary.

(b) From (9.3.12a),

$$B_{nm} = \int_v d\mathbf{r}' b_n(\mathbf{r}') b_m^*(\mathbf{r}').$$

if we choose $\{b_n\}$ to be pulse functions, this integral is nonzero only for $n = m$ since the usual pulse basis functions do not overlap. Hence $\overline{\mathbf{B}} = [B_{nm}]$ is a diagonal matrix.

§9.21

$$I = \delta_t \int_V d\mathbf{r}' \left|\nabla'[k^2(\mathbf{r}') - k_b^2]\right|^2$$

$$+ \sum_{i,j} \left| \mathbf{E}_{sca}(\mathbf{r}_i, \mathbf{r}_j) - \int_V d\mathbf{r}' \mathbf{M}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}', \epsilon_b)[k^2(\mathbf{r}') - k_b^2] \right|^2$$

Now letting $k^2(\mathbf{r}) - k_b^2 = \sum_n a_n b_n(\mathbf{r})$, $\mathbf{E}_k = \mathbf{E}_{sca}(\mathbf{r}_i, \mathbf{r}_j)$,

$$D_{nm} = \int_V d\mathbf{r}' [\nabla' b_n(\mathbf{r}')][\nabla' b_m^*(\mathbf{r}')]$$

$$\overline{\mathbf{L}}_{kn} = \int d\mathbf{r}' \mathbf{M}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}', \epsilon_b) b_n(\mathbf{r}'),$$

we have

$$I = \delta_t \sum_{n,m} a_n a_m^* D_{nm} + \sum_k |\mathbf{E}_k - \sum_n a_n \overline{\mathbf{L}}_{kn}|^2.$$

Note that the above is nearly identical to Equation (9.3.11) and so we may write the optimal solution by inspection as

$$\mathbf{a} = [\mathbf{P} + \delta_t \mathbf{D}]^{-1} \cdot \mathbf{C}.$$

§9.22

(a) From Equation (9.3.21), the minimum norm solution is

$$\mathbf{a} = \mathbf{S} \cdot \overline{\overline{\lambda}}^{-1} \cdot \mathbf{S}^\dagger \cdot \mathbf{C}. \tag{1}$$

where S is a unitary eigenvector matrix and $\tilde{\lambda}^{-1}$ is the inverse (diagonal) eigenvalue matrix with the smallest elements removed. Since λ is diagonal, its inverse is just the element-wise multiplicative inverse of each of its diagonal elements. We let

$$\overline{\lambda} = \tilde{\overline{\lambda}} + \overline{\lambda}_n \tag{2}$$

where $\overline{\lambda}$ is the original eigenvalue matrix and $\overline{\lambda}_n$ contains only the near-zero elements that are to be removed. Then the solution of

$$\overline{\mathbf{P}} \cdot \mathbf{a} = \mathbf{C} \tag{3}$$

with

$$\overline{\mathbf{P}} = \mathbf{S} \cdot \overline{\lambda} \cdot \mathbf{S}^{\dagger} = \mathbf{S} \cdot \overline{\lambda} \cdot \mathbf{S} + \mathbf{S} \cdot \overline{\lambda}_n \cdot \mathbf{S} \tag{4}$$

becomes

$$\mathbf{a} = [\mathbf{S} \cdot \tilde{\overline{\lambda}} \cdot \mathbf{S} + \mathbf{S} \cdot \overline{\lambda_n} \cdot \mathbf{S}]^{-1} \cdot \mathbf{C} \tag{5}$$

$$= \mathbf{S} \cdot \tilde{\overline{\lambda}}^{-1} \cdot \mathbf{S} \cdot \mathbf{C} + \mathbf{S} \cdot \overline{\lambda}_n^{-1} \cdot \mathbf{S} \cdot \mathbf{C} \tag{6}$$

The norm of a is then obtained as

$$||\mathbf{a}|| = ||\mathbf{S} \cdot \tilde{\overline{\lambda}}^{-1} \cdot \mathbf{S} \cdot \mathbf{C}|| + ||\mathbf{S} \cdot \tilde{\overline{\lambda}}^{-1} \cdot \mathbf{S} \cdot \mathbf{C}|| \tag{7}$$

since the eigenvectors are orthogonal. Hence, Equation (1) gives the minimum norm solution.

(b) The regularization $\overline{\mathbf{P}'} = \overline{\mathbf{P}} + \delta_t \overline{\mathbf{B}}$ where $\overline{\mathbf{B}}$ is a well-conditioned matrix is equivalent to padding the small eigenvalues for the following reason. The matrices $\overline{\mathbf{P}}$ and $\overline{\mathbf{B}}$ must exist in the same vector space in order for their addition to be well-defined. Therefore, the eigenvectors S span both $\overline{\mathbf{P}}$ and $\overline{\mathbf{B}}$ and we may write

$$\overline{\mathbf{P}} = \mathbf{S} \cdot \overline{\lambda}_P \cdot \mathbf{S}^{\dagger}$$

$$\overline{\mathbf{B}} = \mathbf{S} \cdot \overline{\lambda}_B \cdot \mathbf{S}^{\dagger}$$

and

$$\overline{\mathbf{P}'} = \mathbf{S} \cdot (\overline{\lambda}_p + \delta_t \overline{\lambda}_B) \cdot \mathbf{S}^{\dagger}.$$

Usually δ_t is a small parameter and the large elements of $\overline{\lambda}_P$ are not affected. The small elements of $\overline{\lambda}_P$ however get a small boost from the well-conditioned matrix $\overline{\mathbf{B}}$.

§9.23

(a) Use of the first-order Taylor series expansion gives

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0) + O[(x - x_0)^2]$$

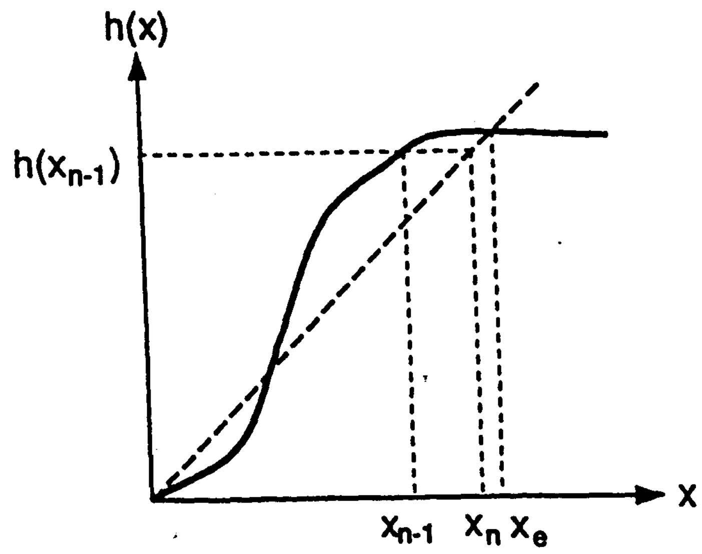

Figure for Exercise Solution 9.23

for $x \approx x_0$. Hence, the formula

$$x \approx x_0 - \frac{f(x_0)}{f'(x_0)}$$

has second-order error.

(b) From Equations (9.3.3) and (9.3.5),

$$\mathbf{E}(\mathbf{r}) \approx \mathbf{E}_{inc}(\mathbf{r}) + \int_V d\mathbf{r}' \mathbf{G}(\mathbf{r}, \mathbf{r}', \epsilon_b) \cdot [k^2(\mathbf{r}') - k_b^2] E_{inc}(\mathbf{r}').$$

The field $\mathbf{E}_{inc}(\mathbf{r})$ is actually the total field in the presence of the background k_b. Hence, this is equivalent to solving the Equation

$$m = m_0(\epsilon_b) + g(\epsilon_b)(\epsilon - \epsilon_b)\phi(\epsilon)$$

in multidimensions and is analogous to the Newton-Raphson method.

(c) For solving the equation $x = h(x)$, we write

$$h(x_e) = x_e \approx h(x_{n-1}) + (x_e - x_{n-1})h'(x_{n-1}) + \ldots$$

Then approximating $x_n \approx h(x_{n-1})$, we have

$$x_e - x_n \approx (x_e - x_{n-1})h'(x_{n-1}).$$

Hence, for the process to converge we must have

$$\frac{|x_e - x_n|}{|x_e - x_{n-1}|} \approx h'(x_{n-1}) < 1$$

or $h'(x_e) < 1$. This procedure is illustrated in the Figure.

(d) From Equation (9.3.24)

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \int_v d\mathbf{r}'\mathbf{G}_0(\mathbf{r},\mathbf{r}') \cdot [k^2(\mathbf{r}') - k_0^2]\mathbf{E}(\mathbf{r}') \tag{1}$$

it is apparent that the Born iterative method is equivalent to solving

$$\mathbf{x} = \mathbf{h}(\mathbf{x}) \tag{2}$$

where $\mathbf{x} = \mathbf{E}(\mathbf{r})$. Note that $\mathbf{h}(\cdot)$ is a non-linear integral function since $k^2(\mathbf{r})$ is related to $\mathbf{E}(\mathbf{r})$. For the N-dimensional Equation (2), we may use the same argument given in part (c) to show that a necessary condition for convergence is

$$||\nabla \mathbf{h}(\mathbf{x}_e)||_\infty < 1,$$

or that the maximum element of the gradient of $\mathbf{h}(\cdot)$ must be less than unity.

§9.24

(a) From (9.2.8),

$$\mathcal{E}\rangle = \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}} \cdot \mathcal{E}\rangle. \tag{1}$$

Taking the inner product with $\mathbf{k}\rangle$ and inserting the identity operator $\overline{\mathcal{I}} = \int d\mathbf{k}\mathbf{k}\rangle\langle\mathbf{k}$ appropriately gives

$$\begin{aligned}\langle\mathbf{k},\mathcal{E}\rangle &= \langle\mathbf{k},\mathcal{E}_{inc}\rangle \\ &\quad + \int d\mathbf{k}' \int d\mathbf{k}''\langle\mathbf{k},\overline{\mathcal{G}}_0,\mathbf{k}'\rangle\langle\mathbf{k}',\overline{\mathcal{O}},\mathbf{k}''\rangle\langle\mathbf{k}'',\mathcal{E}\rangle\end{aligned} \tag{2}$$

or

$$\mathbf{E}(\mathbf{k}) = \mathbf{E}_{inc}(\mathbf{k}) + \int d\mathbf{k}' \int d\mathbf{k}''\overline{\mathbf{G}}(k,k')O(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}'')\mathbf{E}(\mathbf{k}'') \tag{3}$$

$$= \mathbf{E}_{inc}(\mathbf{k}) + \int d\mathbf{k}'\overline{\mathbf{G}}(\mathbf{k},\mathbf{k}')O(\mathbf{k}')\mathbf{E}(\mathbf{k}') \tag{4}$$

Hence, in the spectral space $O(\mathbf{k})$ is again diagonal, but $\overline{\mathbf{G}}(\mathbf{k},\mathbf{k}')$ is not. This equation has the same form as its space-domain counterpart of Equation (9.3.24).

(b) Representation in the space spanned by the eigenfunctions $\mathcal{F}_n$ of $\mathcal{G}_0$:

$$\overline{\mathcal{G}}_0\mathcal{F}_n\rangle = \lambda\mathcal{F}_n\rangle. \tag{5}$$

Taking the inner product of Equation (1) with $\mathcal{F}_n\rangle$,

$$\langle\mathcal{F}_n,\mathcal{E}\rangle = \langle\mathcal{F}_n,\mathcal{E}_{inc}\rangle + \langle\mathcal{F}_n,\overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}} \cdot \mathcal{E}\rangle. \tag{6}$$

From the spatial and spectral representations of Equation (5), we know that the operators $\mathcal{G}$, $\mathcal{O}$, $\mathcal{E}$ are transitive. Hence, we may write

$$\begin{aligned}\mathcal{E}_n &= \mathcal{E}_{n,inc} + \langle \mathcal{E} \cdot \overline{\mathcal{O}} \cdot \overline{\mathcal{G}}_0, \mathcal{F}_n \rangle \\ &= \mathbf{E}_{n,inc} + \lambda \langle \mathcal{E} \cdot \overline{\mathcal{O}} \cdot \mathcal{F}_n \rangle,\end{aligned} \tag{7}$$

where $\mathbf{E}_n = \langle \mathcal{F}_n, \mathcal{E} \rangle$, $\mathbf{E}_{n,inc} = \langle \mathcal{F}_n, \mathcal{E}_{inc} \rangle$. In this case, the matrix representation of $\mathcal{G}_0$ is diagonal, but that of $\overline{\mathcal{O}}$ is not.

§9.25

(a) From Equation (9.3.40a),

$$\begin{aligned}\mathcal{E}_{b,inc}\rangle &= \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}}_b \cdot \mathcal{E}_{b,inc}\rangle \\ (\overline{\mathcal{I}} - \overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}}_b)\mathcal{E}_{b,inc}\rangle &= \mathcal{E}_{inc}\rangle \\ \mathcal{E}_{b,inc}\rangle &= (\overline{\mathcal{I}} - \overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}}_b)^{-1} \cdot \mathcal{E}_{inc}\rangle\end{aligned}$$

Similarly, from Equation (9.3.40b),

$$\begin{aligned}\overline{\mathcal{G}}_b\rangle &= \overline{\mathcal{G}}_0 + \overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}}_b \cdot \overline{\mathcal{G}}_b \\ (\overline{\mathcal{I}} - \overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}}_b) \cdot \overline{\mathcal{G}}_b &= \overline{\mathcal{G}}_0 \\ \overline{\mathcal{G}}_b &= (\overline{\mathcal{I}} - \overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}}_b)^{-1} \cdot \overline{\mathcal{G}}_0\end{aligned}$$

(b)

$$\mathcal{E}\rangle = \mathcal{E}_{b,inc}\rangle + \overline{\mathcal{G}}_b \cdot \delta\overline{\mathcal{O}} \cdot \mathcal{E}\rangle.$$

Substituting the above expressions

$$\begin{aligned}\mathcal{E}\rangle &= (\overline{\mathcal{I}} - \mathcal{G}_0 \cdot \overline{\mathcal{O}}_b)^{-1}[\mathcal{E}_{inc}\rangle + \mathcal{G}_0 \cdot \delta\overline{\mathcal{O}} \cdot \mathcal{E}\rangle] \\ \mathcal{E}\rangle + \overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}}_b \cdot \mathcal{E}\rangle &= \mathcal{E}_{inc}\rangle + \mathcal{G}_0 \cdot \delta\overline{\mathcal{O}} \cdot \mathcal{E}\rangle \\ \mathcal{E}\rangle &= \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}}_0 \cdot (\delta\overline{\mathcal{O}} - \overline{\mathcal{O}}_b) \cdot \mathcal{E}\rangle \\ &= \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}} \cdot \mathcal{E}\rangle\end{aligned}$$

(c) From Equation (9.3.28)

$$\mathcal{E}\rangle = \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}} \cdot \mathcal{E}\rangle$$

Let $\overline{\mathcal{O}} = \overline{\mathcal{O}}_b + \delta\overline{\mathcal{O}}$. We have

$$\begin{aligned}\mathcal{E}\rangle &= \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}}_b \cdot \mathcal{E}\rangle + \overline{\mathcal{G}} \cdot \delta\overline{\mathcal{O}} \cdot \mathcal{E}\rangle \\ &= \mathcal{E}_{b,inc}\rangle + \overline{\mathcal{G}} \cdot \delta\overline{\mathcal{O}} \cdot \mathcal{E}\rangle\end{aligned}$$

where

$$\mathcal{E}_{b,inc}\rangle = \mathcal{E}_{inc}\rangle + \overline{\mathcal{G}} \cdot \overline{\mathcal{O}}_b \cdot \mathcal{E}\rangle$$

(d) The operator $(\overline{\mathcal{I}} - \overline{\mathcal{G}}_0 \cdot \overline{\mathcal{O}})^{-1}$ appears in both Equations (9.3.38) and (9.3.43) indicating that the forward problem must be solved at each iteration. Hence, the computational complexity is the same.