ECE 604 Electromagnetic Field Theory

Spring 2020

Homework No. 11. Due Date: May 1, 2020.

Read lecture notes 1-39.

1. This problem refers to lecture 37.

(i) Show that (37.1.14) expands to (37.1.12) after using (37.1.13). Hence, derive (37.1.18).

(ii) Show that (37.1.32) is indeed true. Give the physical meaning of (37.1.24).

(iii) Derive (37.1.40) and (37.1.41). What is the physical meaning of the extra δ in (37.1.40)?

2. This problem refers to lecture 38.

(i) Explain why in the short wavelength or high frequency limit, Equation (38.4.3) can be approximated by (38.4.2) using the spirit discussed in the paraxial wave approximation. Explain why this is similar to classical Hamiltonian theory.

(ii) Explain the physical meaning of the wave function $\psi(x,t)$ in Schrodinger equation.

(iii) What is the coordinate space representation of the momentum operator $\hat{p}$ and the position operator $\hat{x}$? Show that the commutator between these two operators is

$$\left[\hat{p},\hat{x}\right] = -i\hbar\hat{I}$$

where $\hat{I}$ is the identity operator. Show that the above identity (or equality) is representation independent. Namely, by defining a unitary transform $\hat{U}$ such that $\hat{U}^{\dagger}\hat{U} = \hat{I}$, then $\hat{p}$ and $\hat{x}$ are transformed to $\hat{p}'$ and $\hat{x}'$ and the above equality remains the same.

(iv) As shown in computational electromagnetics, the matrix representation of an operator $\wp$ is given by $P_{nm} = \left\langle \phi_{n,}^{*} \wp\, \phi_m \right\rangle$ where the inner product is $\left\langle f, g \right\rangle = \int dx f(x) g(x)$ or is the reaction inner product in electromagnetics. What is the matrix representation of operator $\wp$ is given by $P_{nm} = \left\langle \phi_{n,}^{*} \wp\, \phi_m \right\rangle$ in Dirac notation? Show that with this definition, if $\wp$ is Hermitian, then the matrix operator $\mathbf{P}$, such that $\left[\mathbf{P}\right]_{nm} = P_{nm}$, is also Hermitian. If now, ϕ_n's are the eigenfunctions of $\wp$, and they are orthonormal, what is the matrix $\mathbf{P}$?

3. This problem refers to lecture 39.

(i) The first two photon number states are given in text books, and they are $\psi_o(x) = \left(\dfrac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$ and $\psi_1(x) = \sqrt{2}\left(\dfrac{\alpha}{\pi}\right)^{1/4} \alpha^{1/2} x e^{-\alpha x^2/2}$ where $\alpha = \dfrac{m\omega_0}{\hbar}$. Rewrite these photon number states in the dimensionless coordinate $\xi = \sqrt{\alpha}x$. Note that the normalization constant has to be different because of the different normalization condition.

(ii) Verify that these first two photon number states satisfy (39.1.16) and (39.1.17) of the lecture notes.

(iii) In math, to find the exponentiation of a matrix operator, it is expressed as a power series. For instance, $e^{\bar{\mathbf{A}}} = \bar{\mathbf{I}} + \bar{\mathbf{A}} + \frac{1}{2!}\bar{\mathbf{A}} \bullet \bar{\mathbf{A}} + \frac{1}{3!}\bar{\mathbf{A}} \bullet \bar{\mathbf{A}} \bullet \bar{\mathbf{A}} + \ldots$ where this equality has been established using the Taylor series expansion of e^x. This power series can be easily evaluated when this operator expression above operates on the eigenstate of the matrix operator $\mathbf{A}$. Using this knowledge, derive the equality in (39.3.13).

(iv) Using the above knowledge, derive (39.3.16) and (39.3.17) of the lecture notes.