APPENDIX A

Some Useful Mathematical Formulas

A.1 Useful Vector Identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \tag{A.1}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}), \tag{A.2}$$

$$\nabla \times \nabla \psi = 0, \tag{A.3}$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0, \tag{a.4}$$

$$\nabla \cdot (\psi \mathbf{A}) = \mathbf{A} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{A}, \tag{A.5}$$

$$\nabla \times (\psi \mathbf{A}) = \nabla \psi \times \mathbf{A} + \psi \nabla \times \mathbf{A}, \tag{A.6}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}, \tag{A.7}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{A}, \tag{A.8}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A}, \tag{A.9}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}. \tag{A.10}$$

In Cartesian coordinates, $\nabla^2 \mathbf{A}$ can be decomposed as

$$\nabla^2 \mathbf{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z, \tag{A.11}$$

because ∇^2 commutes with $\hat{x}$, $\hat{y}$, and $\hat{z}$, i.e., $\nabla^2 \hat{x} = \hat{x}\nabla^2$ and so on. This is not true in other curvilinear coordinates; hence, this decomposition is not allowed.

A.2 Gradient, Divergence, Curl, and Laplacian in Rectangular, Cylindrical, Spherical, and General Orthogonal Curvilinear Coordinate Systems

(a) Rectangular System; x, y, z:

$$\nabla \psi = \frac{\partial \psi}{\partial x}\hat{x} + \frac{\partial \psi}{\partial y}\hat{y} + \frac{\partial \psi}{\partial z}\hat{z}, \tag{A.12}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \tag{A.13}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}, \tag{A.14}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}. \tag{A.15}$$

(b) Cylindrical System; ρ, ϕ, z:

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{z}, \tag{A.16}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}, \tag{A.17}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right)\hat{z}, \tag{A.18}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}. \tag{A.19}$$

(c) Spherical System; r, θ, ϕ:

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\phi}, \tag{A.20}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}, \tag{A.21}$$

$$\begin{aligned}\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} \\ + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi},\end{aligned} \tag{A.22}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}. \tag{A.23}$$

(d) General Orthogonal Curvilinear Coordinate System; x_1, x_2, x_3:

The metric coefficients (h_1, h_2, h_3) in a general orthogonal curvilinear coordinate system are defined by

$$ds_i = h_i\,dx_i; \quad i = 1 \text{ or } 2, \text{ or } 3, \tag{A.24}$$

where ds_i denotes a differential length in the direction of dx_i. Moreover, the variable, x_i may not have the dimension of length. One way of finding the metric coefficients is to express the rectangular variables in terms of the variables of that system:

$$x = y(x_1, x_2, x_3),$$

$$y = y(x_1, x_2, x_3),$$

$$z = z(x_1, x_2, x_3).$$

Then

$$ds_i = \left[\left(\frac{\partial x}{\partial x_i}\right)^2 + \left(\frac{\partial y}{\partial x_i}\right)^2 + \left(\frac{\partial z}{\partial x_i}\right)^2\right]^{1/2} dx_i, \quad i = 1, 2, 3. \tag{A.25}$$

Hence,

$$h_i = \left[\left(\frac{\partial x}{\partial x_i}\right)^2 + \left(\frac{\partial y}{\partial x_i}\right)^2 + \left(\frac{\partial z}{\partial x_i}\right)^2\right]^{1/2}. \tag{A.26}$$

For instance, in an elliptical coordinate system,

$$x = c \cosh u \cos v, \tag{A.27}$$

$$y = c \sinh u \sin v. \tag{A.28}$$

If (x_1, x_2, x_3) represent (u, v, z), then by applying (26), we have

$$h_1 = h_2 = c(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v)^{1/2} = c(\cosh^2 u - \cos^2 v)^{1/2}, \tag{A.29}$$

$$h_3 = 1. \tag{A.30}$$

In general, for any orthogonal curvilinear coordinate system,

$$\nabla\psi = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial\psi}{\partial x_i}\hat{x}_i, \tag{A.31}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\Delta}\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\Delta A_i}{h_i}\right), \quad \Delta = h_1 h_2 h_3, \tag{A.32}$$

$$\nabla \times \mathbf{A} = \frac{1}{\Delta}\left|\begin{pmatrix} h_1\hat{x}_1 & h_2\hat{x}_2 & h_3\hat{x}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3, \end{pmatrix}\right|, \tag{A.33}$$

$$\nabla^2\psi = \frac{1}{\Delta}\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\Delta}{h_i^2}\frac{\partial\psi}{\partial x_i}\right). \tag{A.34}$$

A.3 Useful Integral Identities

In the following formulas, V is a volume bounded by a closed surface S. The unit vector $\hat{n}$ is normal to S and points outward.

(a) Gradient Identity:

$$\oint_V \nabla\phi \, dV = \oint_S \phi\hat{n} \, dS. \tag{A.35}$$