ECE 604, Lecture 8

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1 Plane Waves in Lossy Conductive Media

In the previous section, we have derived the plane wave solution for a lossless homogeneous medium. The derivation can be generalized to a lossy conductive medium by invoking mathematical homomorphism. When conductive loss is present, $\sigma \neq 0$, and $\mathbf{J} = \sigma\mathbf{E}$. Then generalized Ampere's law becomes

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \sigma\mathbf{E} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\mathbf{E} \tag{1.1}$$

A complex permittivity can be defined as $\underset{\sim}{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega}$. Eq. (1.1) can be rewritten as

$$\nabla \times \mathbf{H} = j\omega\underset{\sim}{\varepsilon}\mathbf{E} \tag{1.2}$$

This equation is of the same form as source-free Ampere's law in the frequency domain for a lossless medium where ε is completely real. Using the same method as before, a plane-wave solution $\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$ will have the dispersion relation which is now given by

$$k_y^2 + k_y^2 + k_z^2 = \omega^2\mu\underset{\sim}{\varepsilon} \tag{1.3}$$

Since $\underset{\sim}{\varepsilon}$ is complex now, k_x, k_y, and k_z need not be all real. Equation (1.3) has been derived previously by assuming that $\mathbf{k}$ is a real vector. When $\mathbf{k} = \mathbf{k}' - j\mathbf{k}$" is a complex vector, some of the derivations may not be correct previously. It is also difficult to visualize a $\mathbf{k}$ vector that the wave is propagating in. Here, the wave can decay and oscillate in different directions.

So again, we look at the simplified case where

$$\mathbf{E} = \hat{x}E_x(z) \tag{1.4}$$

so that $\nabla \cdot \mathbf{E} = \partial_x E_x(z) = 0$, And let $\mathbf{k} = \hat{z}k = \hat{z}\omega\sqrt{\mu\underset{\sim}{\varepsilon}}$. In this manner, we are requiring that the wave decays and propagates (or oscillates) only in the z direction. For such a simple plane wave,

$$\mathbf{E} = \hat{x}\mathbf{E}_x(z) = \hat{x}E_0 e^{-jkz} \tag{1.5}$$

where $k = \omega\sqrt{\mu\underset{\sim}{\varepsilon}}$, since $\mathbf{k}\cdot\mathbf{k} = k^2 = \omega^2\mu\underset{\sim}{\varepsilon}$ is still true.

Faraday's law gives rise to

$$\mathbf{H} == \frac{\mathbf{k}\times\mathbf{E}}{\omega\mu} = \hat{y}\frac{kE_x(z)}{\omega\mu} = \hat{y}\sqrt{\frac{\underset{\sim}{\varepsilon}}{\mu}}E_x \tag{1.6}$$

or by letting $k = \omega\sqrt{\mu\underset{\sim}{\varepsilon}}$, then

$$E_x/H_y = \sqrt{\frac{\mu}{\underset{\sim}{\varepsilon}}} \tag{1.7}$$

When the medium is highly conductive, $\sigma \to \infty$, the following approximation can be made, namely,

$$k = \omega \sqrt{\mu \underset{\sim}{\varepsilon}} \simeq \omega \sqrt{-\mu \frac{j\sigma}{\omega}} = \sqrt{-j\omega\mu\sigma} \tag{1.8}$$

Taking $\sqrt{-j} = \frac{1}{\sqrt{2}}(1-j)$, we have

$$k = (1-j)\sqrt{\frac{\omega\mu\sigma}{2}} = k' - jk'' \tag{1.9}$$

For a plane wave, e^{-jkz}, it becomes

$$e^{-jkz} = e^{-jk'z - k''z} \tag{1.10}$$

This plane wave decays exponentially in the z direction. The penetration depth of this wave is then

$$\delta = \frac{1}{k''} = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{1.11}$$

this distance δ, the penetration depth, is called the skin depth of a plane wave propagating in a highly lossy conductive medium where conduction current dominates over displacement current, or that $\sigma \gg \omega\varepsilon$. This happens for radio wave propagating in the saline solution of the ocean, the Earth, or wave propagating in highly conductive metal, like your induction cooker.

When the conductivity is low, namely, when the displacement current is larger than the conduction current, then $\frac{\sigma}{\omega\varepsilon} \ll 1$, we have

$$\begin{aligned} k &= \omega\sqrt{\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right)} = \omega\sqrt{\mu\varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right)} \\ &\approx \omega\sqrt{\mu\varepsilon}\left(1 - j\frac{1}{2}\frac{\sigma}{\omega\varepsilon}\right) = k' - jk'' \end{aligned} \tag{1.12}$$

The term $\frac{\sigma}{\omega\varepsilon}$ is called the loss tangent of a lossy medium.

In general, in a lossy medium $\varepsilon = \varepsilon' - j\varepsilon''$, $\varepsilon''/\varepsilon'$ is called the loss tangent of the medium. It is to be noted that in the optics and physics community, $e^{-i\omega t}$ time convention is preferred. In that case, we need to do the switch $j \to -i$, and a loss medium is denoted by $\varepsilon = \varepsilon' + i\varepsilon''$.

2 Lorentz Force Law

The Lorentz force law is given by

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{2.1}$$

It can be also written in terms of force density $\mathbf{f}$ to arrive at

$$\mathbf{f} = \varrho\mathbf{E} + \varrho\mathbf{v} \times \mathbf{B} = \varrho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{2.2}$$

where one can identified $\mathbf{J} = \varrho\mathbf{v}$.

Lorentz force law can also be derived from the integral form of Faraday's law, if one assumes that the law is applied to a moving loop intercepting a magnetic flux.

3 Drude-Lorentz-Sommerfeld Model

In the previous lecture, we have seen how loss can be introduced by having a conduction current flowing in a medium. Now that we have learnt the versatility of the frequency domain method, other loss mechanism can be easily introduced with the frequency-domain method.

First let us look at the simple constitutive relation where

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \tag{3.1}$$

We have a simple model where

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \tag{3.2}$$

where χ_e is the electric susceptibility. To see how $\chi_e(\omega)$ can be derived, we will study the Drude-Lorentz-Sommerfeld model. This is usually just known as the Drude model or the Lorentz model in many textbooks although Sommerfeld also contributed to it. This model can be unified in one equation as shall be shown.

We can start with a simple electron driven by an electric field $\mathbf{E}$ in the absence of a magnetic field $\mathbf{B}$. If the electron is free to move, then the force acting on it, from the Lorentz force law, is $-e\mathbf{E}$ where e is the charge of the electron. Then from Newton's law, assuming a one dimensional case, it follows that

$$m_e\frac{d^2x}{dt^2} = -eE \tag{3.3}$$

assuming that $\mathbf{E}$ points in the x-direction, and we neglect the vector nature of the electric field. Writing the above in the frequency domain, one gets

$$-\omega^2 m_e x = -eE \tag{3.4}$$

From this, we have

$$x = \frac{e}{\omega^2 m_e}E \tag{3.5}$$

This for instance, can happen in a plasma medium where the atoms are ionized, and the electrons are free to roam. Hence, we assume that the positive ions are more massive, and move very little compared to the electrons when an electric field is applied.

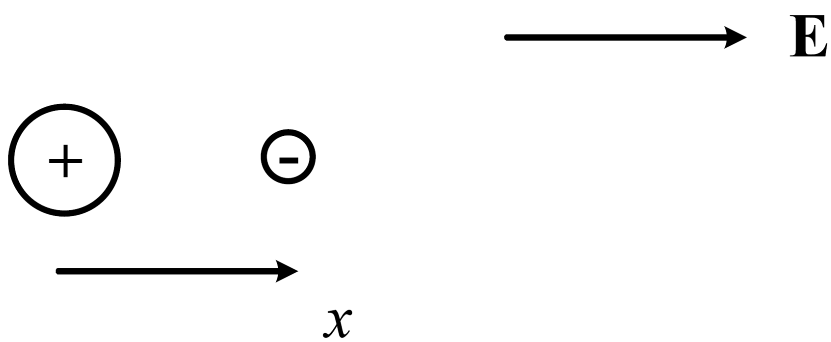

Figure 1:

The dipole moment formed by the displaced electron away from the ion is

$$p = -ex = -\frac{e^2}{\omega^2 m_e} E \tag{3.6}$$

for one electron. When there are N electrons per unit volume, the dipole density is given by

$$P = Np = -\frac{Ne^2}{\omega^2 m_e} E \tag{3.7}$$

In general,

$$\mathbf{P} = -\frac{Ne^2}{\omega^2 m_e}\mathbf{E} = -\frac{{\omega_p}^2 \varepsilon_0}{\omega^2}\mathbf{E} \tag{3.8}$$

where we have defined ${\omega_p}^2 = Ne^2/(m_e \varepsilon_0)$. Then,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \left(1 - \frac{{\omega_p}^2}{\omega^2}\right)\mathbf{E} \tag{3.9}$$

In this manner, we see that the effective permittivity is

$$\varepsilon = \varepsilon_0 \left(1 - \frac{{\omega_p}^2}{\omega^2}\right) \tag{3.10}$$

This gives the interesting result that in the frequency domain $\varepsilon < 0$ if

$$\omega < \omega_p = \sqrt{N/(m_e \varepsilon_0)} e$$

Here, ω_p is the plasma frequency. Since $k = \omega\sqrt{\mu\varepsilon}$, if ε is negative, k becomes pure imaginary, and a wave such as e^{-jkz} decays exponentially. In other words, the wave cannot propagate through such a medium: Our ionosphere is such a medium. So it was extremely fortuitous that Marconi, in 1901, was able to send a radio signal from Cornwall, England, to Newfoundland, Canada. Nay sayers thought his experiment would never succeed as the radio signal will propagate to outer space and never return. It is the presence of the ionosphere that bounces the radio wave back to Earth, making his experiment a resounding success. The experiment also heralds in the age of wireless communications.

The above model can be generalized to the case where the electron is bound to the ion, and the ion provides a restoring force, namely,

$$m_e \frac{d^2x}{dt^2} + \kappa x = -eE \tag{3.11}$$

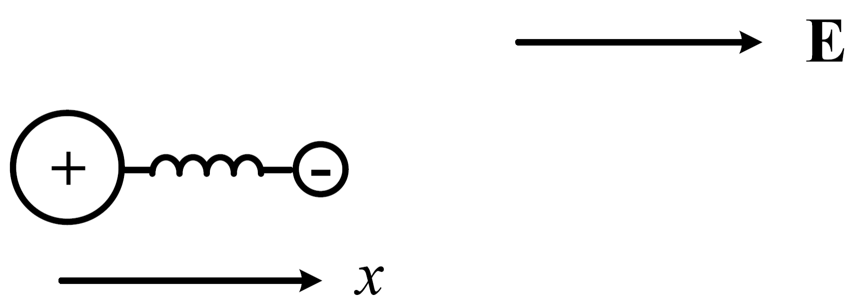

Figure 2:

We assume that ion provide a restoring force just like Hooke's law. Again, (3.11) can be solved easily in the frequency domain to yield

$$x = \frac{e}{(\omega^2 m_e - \kappa)} E == \frac{e}{(\omega^2 - {\omega_0}^2) m_e} E \tag{3.12}$$

where we define ${\omega_0}^2 m_e = \kappa$. Equation (3.11) can be generalized to the case when frictional or damping forces are involved, or that

$$m_e \frac{d^2x}{dt^2} + m_e \Gamma \frac{dx}{dt} + \kappa x = -eE \tag{3.13}$$

The second term is a force that is proportional to the velocity dx/dt of the electron. This is the hall-mark of a frictional force. Also, Γ has the unit of frequency, and for plasma, and conductor, it can be regarded as a collision frequency.

Solving the above in the frequency domain, one gets

$$x = \frac{e}{(\omega^2 - j\omega\Gamma - {\omega_0}^2) m_e} E \tag{3.14}$$

Following the same procedure in arriving at (3.7), we get

$$P = \frac{-Ne^2}{(\omega^2 - j\omega\Gamma - {\omega_0}^2)m_e} E \tag{3.15}$$

In this, one can identify that

$$\begin{aligned}\chi_e(\omega) &= \frac{-Ne^2}{(\omega^2 - j\omega\Gamma - {\omega_0}^2)m_e\varepsilon_0} \\ &= -\frac{{\omega_p}^2}{\omega^2 - j\omega\Gamma - {\omega_0}^2}\end{aligned} \tag{3.16}$$

where ω_p is as defined before. Function with the above frequency dependence is also called a Lorentzian function. It is the hallmark of a damped harmonic oscillator.

If $\Gamma = 0$ then when $\omega = \omega_0$, one sees a resonance peak exhibited by the DLS model. When Γ is small, but $\omega \approx \omega_0$, then

$$\chi_e \approx +\frac{{\omega_p}^2}{j\omega\Gamma} = -j\frac{{\omega_p}^2}{\omega\Gamma} \tag{3.17}$$

χ_e exhibits a large negative imaginary part, the hallmark of a dissipative medium.

The DLS model is a wonderful model because it can capture phenomenologically the essence of the physics of many electromagnetic media. It can capture the resonance behavior of an atom absorbing energy from light excitation. When the light wave comes in at the correct frequency, it will excite electronic transition within an atom which can be approximately model as a resonance behavior. This electronic resonances will be radiationally damped, and the damped oscillation can be modeled by $\Gamma \neq 0$.

Moreover, the above model can also be used to model molecular vibrations. In this case, the mass of the electron will be replaced by the mass of the atom involved. The damping of the molecular vibration is caused by the hindered vibration of the molecule due to interaction with other molecules.

In the case of plasma, $\Gamma \neq 0$ can represents the collision frequency between the free electrons and the ions, giving rise to loss. In the case of a conductor, Γ represents the collision frequency between the conduction electrons in the conduction band with the lattice of the material.[1] Also, if there is no restoring force so that $\omega_0 = 0$, and for sufficiently low frequency, from (3.16)

$$\chi_e \approx -j\frac{{\omega_p}^2}{\omega\Gamma} \tag{3.18}$$

and

$$\varepsilon = \varepsilon_0(1 + \chi_e) = \varepsilon_0\left(1 - j\frac{{\omega_p}^2}{\omega\Gamma}\right) \tag{3.19}$$

[1] It is to be noted that electron has a different effective mass in a crystal lattice, and hence, the electron mass has to be changed accordingly in the DLS model.

We recall that for a conductive medium, we define a complex permittivity to be

$$\varepsilon = \varepsilon_0 \left(1 - j\frac{\sigma}{\omega\varepsilon_0}\right) \tag{3.20}$$

Comparing (3.19) and (3.20), we see that

$$\sigma = \varepsilon_0 \frac{{\omega_p}^2}{\Gamma} \tag{3.21}$$

The above formula for conductivity can be arrived at using collision frequency argument as is done in some textbooks.

Because the DLS is so powerful, it can be used to explain a wide range of phenomena from very low frequency to optical frequency.

The fact that $\varepsilon < 0$ can be used to explain many phenomena. The ionosphere is essentially a plasma medium described by

$$\varepsilon = \varepsilon_0 \left(1 - \frac{{\omega_p}^2}{\omega^2}\right) \tag{3.22}$$

Radio wave or microwave can only penetrate through this ionosphere when $\omega > \omega_p$, so that $\varepsilon > 0$.

Also, the Lorentz function is great for data fitting, as many experimentally observed resonances have finite Q and a line width. The Lorentz function models that well. If multiple resonances occur in a medium or an atom, then multi-species DLS model can be used. It is now clear that all media have to be frequency dispersive because of the finite mass of the electron and the inertial it has. In other words, there is no instantaneous response in a dielectric medium due to the finiteness of the electron mass.

Even at optical frequency, many metals, which has a sea of freely moving electrons in the conduction band, can be modeled approximately as a plasma. A metal consists of a sea of electrons in the conduction band which are not tightly bound to the ions or the lattice. Also, in optics, the inertial force due to the finiteness of the electron mass (in this case effective mass) can be sizeable compared to other forces. Then, $\omega_0 \ll \omega$ or that the restoring force is much smaller than the inertial force, in (3.16), and if Γ is small, $\chi_e(\omega)$ resembles that of a plasma, and ε of a metal can be negative. When a plasmonic nanoparticle made of gold is excited by light, its response is given by (see homework assignment)

$$\Phi_R = E_0 \frac{a^3 \cos\theta}{r^2} \frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s + 2\varepsilon_0} \tag{3.23}$$

when $\varepsilon_s = -2\varepsilon_0$, $\Phi_R \to \infty$. Therefore, when light interacts with such a particle, it can sparkle brighter than normal. This reminds us of the saying "All that glitters is not gold!" even though this saying has a different intended meaning.

Ancient Romans apparently knew about the potent effect of using gold and silver nanoparticles to enhance the reflection of light. These nanoparticles were impregnated in the glass or lacquer ware. By impregnating these particles in different media, the color of light will sparkle at different frequencies, and hence, the color of the glass emulsion can be changed (see website below).

https://www.smithsonianmag.com/history/this-1600-year-old-goblet-shows-that-the-romans-were-nanotechnology-pioneers-787224/

Figure 3: Courtesy of Smithsonian.com.